On a multidimensional moving boundary problem governed by anomalous diffusion: analytical and numerical study

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Abstract. We study the anomalous diffusion version of the quasistationary Stefan problem (the fractional quasistationary Stefan problem) in the multidimensional case $\Omega(t) \subset R^n$, $n \geq 2$. This free boundary problem is a mathematical model of a solute drug released from a polymer matrix ($n = 1, 3$). We prove the existence and uniqueness of the classical solution for this moving boundary problem locally in time. A numerical solution is constructed in the two-dimensional case.

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1. Introduction

Let $\Omega(t)$ be a “shell-like” bounded domain in R^n , $n \geq 2$, for each $t \in [0, T]$ with the boundary $\partial\Omega(t) = \Gamma(t) \cup \Gamma_0$, $\Gamma(t) \cap \Gamma_0 = \emptyset$.

In this paper, we will analyze the free boundary problem in the case of anomalous diffusion. The problem reads as follows: find a function $p(y, t)$, $y \in \Omega(t)$, $t \in [0, T]$, and a moving boundary $\Gamma(t)$ satisfying the conditions:

$$-\Delta_y p = 0 \quad \text{in } \Omega(t), \quad t \in [0, T]; \quad (1.1)$$

$$p = 0, \quad V_{n_t}^\nu = -\mu \frac{\partial p}{\partial n_t} \quad \text{on } \Gamma(t), \quad \nu \in (0, 1); \quad (1.2)$$

$$p = \varphi(y) \quad \text{on } \Gamma_0, \quad \Gamma_0 = \Gamma_0 \times [0, T]; \quad (1.3)$$

$$\Omega(t)|_{t=0} = \Omega, \quad \Gamma(t)|_{t=0} = \Gamma \quad \text{are given.} \quad (1.4)$$

Here $\Delta_y = \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2}$; μ is a given positive constant, $\varphi(y)$ is a given positive function; n_t is the unit normal to $\Gamma(t)$ directed outside $\Omega(t)$; $V_{n_t}^\nu$ is the frac-

tional velocity of the boundary $\Gamma(t)$ in the direction of the normal n_t and is represented by (see, e.g., [44]):

$$V_{n_t}^\nu = \langle \mathbf{D}_t^\nu \Gamma(t), n_t \rangle. \quad (1.5)$$

$\mathbf{D}_t^\nu$ denotes the Caputo fractional derivative with respect to t and is defined by (see (2.4.6) in [20])

$$\mathbf{D}_t^\nu w(\cdot, t) = \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_0^t \frac{w(\cdot, \tau) d\tau}{(t-\tau)^\nu} - \frac{w(\cdot, 0)}{\Gamma(1-\nu)t^\nu}, \quad \nu \in (0, 1), \quad (1.6)$$

where $\Gamma(\nu)$ is the Gamma function, $\langle \cdot, \cdot \rangle$ denotes the scalar product.

We recall that the anomalous diffusion can not be modelled as a standard Brownian motion [8, 35], and the mean square displacement of the diffusing species $\langle (\Delta y)^2 \rangle$ scales as a nonlinear power law in time, i.e. $\langle (\Delta y)^2 \rangle \sim t^\nu$ for some real number ν . In the case $\nu \in (0, 1)$, this is referred to as a subdiffusion.

If $n = 1, 3$ and Γ_0 is in the exterior of $\Gamma(t)$, problem (1.1)–(1.4) describes a solute released from an undissolved monolithic polymeric matrix in the quasistationary case [28, 29]. We recall that each matrix consists of two regions: the surface zone (or liquid phase) $x \in \Omega(t)$ and the core $B(t)$ within the interior of $\Gamma(t)$, and $B(t) \cap \Omega(t) = \emptyset$. The whole solute is dissolved in the surface zone, and the core contains undissolved solute. The two zones are separated by the diffusion front (liquid-solid interface) $\Gamma(t)$ which moves inward as time progresses. In other words, the domain $\Omega(t)$ expands, that is, $\Omega(t_1) \subset \Omega(t_2)$, $t_1 < t_2$.

Concentration of drug in the liquid phase is given by function $p(y, t)$, and function $\varphi(y)$ describes the drug concentration in the bulk liquid phase. Under conditions of practical significance, the amount of drug loaded is usually much bigger than the solubility of drug, and the time scale for the release process is long. Therefore, the rate of movement of the solid-liquid interface is relatively slow compared to that of drug diffusion.

Note that Liu and Xu [29] were the first who introduced the time-fractional diffusion equation to the drug release process. They obtained the well-known, semi-empirical formula in the controlled drug release system given by Ritger and Peppas.

Problem (1.1)–(1.4) can be called the fractional quasistationary Stefan problem which arises under consideration of the materials with memory [44]. If Eq. (1.1) in problem (1.1)–(1.4) is replaced by the subdiffusion equation, we will get the fractional Stefan problem formulated and studied by Atkinson [1] for the motion of planar, cylindrical and spherical boundaries.

If $\nu = 1$, free boundary problem (1.1)–(1.4) is interpreted as the mathematical model of the Hele-Shaw problem, which has a hydrodynamic sense and describes the evolution of interface of an immiscible incompressible fluid. The motion of fluid is governed by the Darcy law stating that the velocity of fluid is proportional to the pressure gradient. Note that the fractional Darcy law is formulated in [36]. As it is noted in [17], there is vast literature on the Hele-Shaw and related problems. This literature includes, in particular, well-posedness proofs of the Hele-Shaw problem in various physical settings.

However, we do not provide here a complete survey on the results related to problem (1.1)–(1.4) for $\nu = 1$, but rather present some of them. Weak and variational solutions for the one-phase Hele-Shaw problem were studied by Elliott and Ockendon [12], Di Benedetto and Friedman [6]. Using methods of the theory of complex variables, Hohlov and Howison [16] constructed some explicit solutions of the Hele-Shaw problem. Bazaliy [3] and Escher and Simonett [14] proved the existence of classical solutions for the Hele-Shaw problem with the regular initial data.

Presence of the fractional derivative in time in the second equation of (1.2) [that is $\nu \in (0, 1]$] essentially complicates studies of problem (1.1)–(1.4) since this condition is nonlocal and some useful properties of ordinary derivatives (e.g., product rules, chain rules) are not carried over to the case of the fractional derivative operator. To the best of our knowledge, some exact solutions of the fractional quasistationary Stefan problem (or the fractional Hele-Shaw problem) and the analogous one were constructed in [27, 30, 43, 44] if $\Omega(t) \subset R^1$, $\forall t \in [0, T]$. In the case of $\Omega(t) \subset R^2$, the classical solvability of the free boundary problem (1.1)–(1.4) for a small period of time has been proved in [24].

In general, this paper focuses on two main questions. The first one is the classical solvability of the fractional quasistationary multidimensional Stefan problem locally in time. To this end, we use the approach similar to the one used in [2, 3] that consists of:

- Reduction of a free boundary problem to a nonlinear problem defined in a fixed domain;
- Linearization of this nonlinear problem on an initial data;
- The proof of the one-valued solvability of the linear problem;
- Proving that the corresponding nonlinear mapping is a contraction, so that it has a unique fixed point.

The main analytical difficulties of the procedure listed above arises during the study of the nonclassical boundary value problem. Therefore, the most essential result of this paper is related to analysis of a boundary value problem with a fractional dynamic boundary condition in $R_{+T}^n = R_+^n \times (0, T)$. Note that this problem for $n = 2$ was studied in [23]. However, the case $n \geq 3$ is more complicated, and substantially differs from the two-dimensional analogue.

The second issue considered in the paper is the numerical solution of the problem (1.1)–(1.4) if $n = 2$. Solving for this problem requires tracking of the boundary. In the case of the classical Stefan problem different approaches exist which allow us to avoid explicit boundary tracking. For example, the level set method [9] and enthalpy formulation [42] can reduce the problem with the moving boundary to the problem in the fixed domain. However, these methods cannot be applied for the problems with the fractional derivatives. As it was noted in [44], the enthalpy formulation for a problem with the fractional derivative in general may describe a distinct physical process. The level set method, in turn, implies usage of the velocity from condition (1.2) which is given only for $\nu = 1$. Therefore, an explicit boundary tracking method is implemented in this study.

The relevant mathematical models of drug release from a polymeric matrix are powerful tools in studies of controlled-release drug systems. Therefore, the investigations of fractional calculus in moving boundary problems would be of great interest to both theoretical and experimental studies in the future.

The paper is organized as follows. In Sect. 2, we reduce the problem with unknown boundary to a problem in a fixed domain, define the “fractional” Hölder spaces and state the main result, Theorem 2.1. In Sect. 2.3, we represent the nonlinear problem in the form $\mathcal{A}\mathbf{z} = F(x, t) + \mathcal{F}^1(\mathbf{z})$, where $\mathbf{z} = (w, \sigma)$ and $\mathcal{A}$ is a linear operator, the vector $F(x, t)$ is constructed using the initial data and $\mathcal{F}^1$ is a nonlinear operator. Section 3 is devoted to investigation of the boundary value problem with the fractional derivative in time in the boundary condition. Then in Sect. 4, using the technique of the regularizer for parabolic systems [25] together with the results of Sect. 3, we prove the one-to-one solvability to the linear problem $\mathcal{A}\mathbf{z} = F(x, t)$, Theorem 4.1. In Section 5, we prove the main result using Theorem 4.1 and the fixed point theorem. Section 6 contains numerical investigations of free boundary problem (1.1)–(1.4) in the case $\Omega(t) \subset R^2$. Proofs of some auxiliary assertions which are applied in Section 3 are given in Appendix 7.

2. The nonlinear mapping and main result

2.1. Reducing problem (1.1)–(1.4) to a problem in the fixed domain

We use the Hanzawa method [18] to reduce the free boundary problem (1.1)–(1.4) to a problem in a fixed domain. Let $\Gamma_0, \Gamma \in C^{3+\alpha}$, $\alpha \in (0, 1)$, and $\omega = (\omega_1, \dots, \omega_{n-1})$ be some coordinates on Γ , $\omega \in W$. The position of a point on Γ we define as $\bar{m}(\omega)$. Let $\bar{n}(\omega)$ be the outward normal to Γ .

For sufficiently small $\gamma_0 > 0$, ω –surfaces: $\bar{m}(\omega) + \eta\bar{n}(\omega)$, $|\eta| < 2\gamma_0$, do not intersect each other and Γ_0 . The mapping $(\omega, \eta) \rightarrow y = y(\omega, \eta)$ defined by

$$y = (y_1, \dots, y_n) = \bar{m}(\omega) + \eta\bar{n}(\omega)$$

is a diffeomorphism from $M = W \times (-\gamma_0, \gamma_0)$ onto

$$N = \{y : y = \bar{m}(\omega) + \eta\bar{n}(\omega), (\omega, \eta) \in M\}.$$

The inverse mapping $\Sigma : N \rightarrow M$ is

$$\Sigma : y \rightarrow (\omega(y), \eta(y)).$$

We assume that the free boundary in problem (1.1)–(1.4) is represented as

$$\Gamma(t) = \{(y, t) : y(\omega, t) = \bar{m}(\omega) + \rho(\omega, t)\bar{n}(\omega), t \in [0, T]\}, \quad (2.1)$$

where $|\rho(\omega, t)| < \gamma_0/4$, $\rho(\omega, 0) = 0$. Thus, the free boundary equation is given by

$$\Phi_\rho(y, t) = \eta(y) - \rho(\omega(y), t) = 0, \quad (y, t) \in N \times [0, T]. \quad (2.2)$$

Using (1.5), (2.1) and (2.2), we can rewrite the second boundary condition in (1.2) as

$$\mathbf{D}_t^\nu \rho \frac{\langle \nabla_y \Phi_\rho(y, 0), \nabla_y \Phi_\rho(y, t) \rangle}{|\nabla_y \Phi_\rho(y, 0)| |\nabla_y \Phi_\rho(y, t)|} = -\mu \frac{\langle \nabla_y p, \pm \nabla_y \Phi_\rho(y, t) \rangle}{|\nabla_y \Phi_\rho(y, t)|}. \quad (2.3)$$

Here the sign “+/-” depends on the position of Γ_0 (“+” if Γ_0 lies inside of the domain bounded by Γ , and “-” otherwise).

Let $\chi(\lambda) \in C_0^\infty(R^1)$, $\chi(\lambda) = 1$ if $|\lambda| < \gamma_0/3$ and $\chi(\lambda) = 0$ if $|\lambda| > \gamma_0$, $|\chi'| \leq C/\gamma_0$, where C is a positive constant, $C < 2$. We use the coordinates (ω, η) to define the diffeomorphism

$$e_\rho : (x, t) \rightarrow (y, t)$$

from $X_T = R^n \times [0, T]$ onto $Y_T = R^n \times [0, T]$ by setting

$$\begin{cases} \omega(y) = \omega(x), \\ \eta(y) = \lambda(x) + \chi(\lambda(x))\rho(\omega(x), t), & \text{if } (\omega(x), \lambda(x)) \in N, \\ y = x, & \text{otherwise,} \end{cases} \quad (2.4)$$

such that the transformation e_ρ^{-1} maps $\Omega(t)$ onto $\Omega \times (0, T) = \Omega_T$ and $\Gamma(t)$ onto $\Gamma \times [0, T] = \Gamma_T$, and the free boundary is given by

$$e_\rho(\{\lambda(x) = 0\}). \quad (2.5)$$

Here $\omega(x)$, $\lambda(x)$ are the coordinates in X_T similar to the coordinates $\omega(y)$, $\eta(y)$ in Y_T . The change of variables gives the new desired functions

$$v(x_1, \dots, x_n, t) = p(y_1, \dots, y_n, t) \circ e_\rho(x, t), \quad (2.6)$$

which satisfy the equations

$$-\nabla_\rho^2 v(x, t) = 0 \quad \text{in } \Omega_T, \quad (2.7)$$

$$v(x, t) = 0 \quad \text{on } \Gamma_T, \quad (2.8)$$

$$v(x, t) = \varphi(x) \quad \text{on } \Gamma_{0T}, \quad (2.9)$$

where $\nabla_\rho = (E_\rho^*)^{-1} \nabla_x$, E_ρ is the Jacobi matrix of the mapping $y = e_\rho(x, t)$, $\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. Note that to get Eq. (2.9), we take into account that $y = x$ near Γ_{0T} . From (2.3) and (2.5), we can conclude that

$$\mathbf{D}_t^\nu \rho = -\mu \left[\pm S(\omega, \rho, \nabla_\omega \rho) \frac{\partial v}{\partial \lambda} \pm \sum_{i=1}^{n-1} S_i(\omega, \rho, \nabla_\omega \rho) \frac{\partial v}{\partial \omega_i} \right] \quad \text{on } \Gamma_T, \quad (2.10)$$

where $\nabla_\omega \rho = (\rho_{\omega_1}, \dots, \rho_{\omega_{n-1}})$, $S(\omega, \rho, \nabla_\omega \rho)$ and $S_i(\omega, \rho, \nabla_\omega \rho)$, $i = \overline{1, n-1}$, are some smooth functions [18] such that

$$S(\omega, 0, 0) = 1, \quad S_{\rho_{\omega_i}}(\omega, 0, 0) = 0, \quad S_i(\omega, 0, 0) = 0, \quad i = \overline{1, n-1}. \quad (2.11)$$

Thus, the initial free boundary problem (1.1)–(1.4) is reduced to the nonlinear problem in the fixed domain for the functions $v(x_1, \dots, x_n, t)$ and $\rho(\omega, t)$ that satisfy the following conditions [see Eqs. (2.7)–(2.10)]:

$$\begin{aligned} & -\nabla_\rho^2 v(x, t) = 0 \quad \text{in } \Omega_T, \quad v(x, t) = \varphi(x) \quad \text{on } \Gamma_{0T}, \quad \rho(\omega, 0) = 0; \\ & v(x, t)|_{\Gamma_T} = 0, \quad \mathbf{D}_t^\nu \rho = -\mu \left[\pm S(\omega, \rho, \nabla_\omega \rho) \frac{\partial v}{\partial \lambda} \pm \sum_{i=1}^{n-1} S_i(\omega, \rho, \nabla_\omega \rho) \frac{\partial v}{\partial \omega_i} \right] \\ & \quad \text{on } \Gamma_T. \end{aligned} \quad (2.12)$$

2.2. Functional spaces and the main result

Let D be a given domain in R^n , $D_T = D \times (0, T)$; $\bar{x}, x$ be any points in $\bar{D}$; $t, \tau \in [0, T]$; $\alpha, \beta \in (0, 1)$. Denote by

$$\begin{aligned}\langle u \rangle_{x, D_T}^{(\alpha)} &= \sup_{(x, t), (\bar{x}, t) \in D_T, x \neq \bar{x}} \frac{|u(x, t) - u(\bar{x}, t)|}{|x - \bar{x}|^\alpha}, \\ \langle u \rangle_{t, D_T}^{(\beta)} &= \sup_{(x, t), (x, \tau) \in D_T, t \neq \tau} \frac{|u(x, t) - u(x, \tau)|}{|t - \tau|^\beta}, \\ \langle u \rangle_{x, D}^{(\alpha)} &= \sup_{x, \bar{x} \in D, x \neq \bar{x}} \frac{|u(x) - u(\bar{x})|}{|x - \bar{x}|^\alpha}.\end{aligned}$$

In this paper we will use two types of functional spaces: $C([0, T], C^{k+\alpha}(\bar{D}))$ and $C^{k+\alpha, \frac{k+\alpha}{2}\nu}(\bar{D}_T)$, where k is integer, $k = \overline{0, 2}$, $\nu \in (0, 1)$. Recall that the spaces $C([0, T], C^{k+\alpha}(\bar{D}))$ used by many authors (see, e.g., [32] and references there); therefore, we only recall the definition of these classes.

Definition 2.1. The spaces $C([0, T], C^{k+\alpha}(\bar{D}))$ consist of k -times continuously differentiable in x functions $u(x, t)$, which are α -Hölder continuous in x , uniformly with respect to t . The norm in $C([0, T], C^{k+\alpha}(\bar{D}))$ is defined as

$$\|u\|_{C([0, T], C^{k+\alpha}(\bar{D}))} = \sum_{|j|=0}^k (\sup_{\bar{D}_T} |D_x^j u(x, t)| + \langle D_x^j u \rangle_{x, D_T}^{(\alpha)}).$$

We define the class $C([0, T], C^{k+\alpha}(\bar{D}))$ as the subspace of $C([0, T], C^{k+\alpha}(\bar{D}))$ such that $D_x^j u|_{t=0} = 0$, $|j| = \overline{0, k}$.

As for the spaces $C^{k+\alpha, \frac{k+\alpha}{2}\nu}(\bar{D}_T)$, $k = \overline{0, 2}$, these classes coincide with common Hölder spaces if $\nu = 1$, (see (1.10)–(1.12) in [25]). In the case of $D \subset R^1$, and $\nu \in (0, 1)$, the definition of $C^{k+\alpha, \frac{k+\alpha}{2}\nu}(\bar{D}_T)$, $k = \overline{0, 2}$, is given by (2.1) and (2.2) in [22].

Definition 2.2. The function $u(x, t) \in C^{k+\alpha, \frac{k+\alpha}{2}\nu}(\bar{D}_T)$, $k = \overline{0, 2}$, iff the following norms are finite:

$$\begin{aligned}\|u\|_{C^{k+\alpha, \frac{k+\alpha}{2}\nu}(\bar{D}_T)} &= \|u\|_{C([0, T], C^{k+\alpha}(\bar{D}))} + \sum_{|j|=0}^k \langle D_x^j u \rangle_{t, D_T}^{(\frac{k+\alpha-|j|}{2}\nu)}, \quad \text{if } [k/2] = 0; \\ \|u\|_{C^{k+\alpha, \frac{k+\alpha}{2}\nu}(\bar{D}_T)} &= \|u\|_{C([0, T], C^{k+\alpha}(\bar{D}))} + \|\mathbf{D}_t^\nu u\|_{C([0, T], C^\alpha(\bar{D}))} \\ &\quad + \sum_{|j|=0}^k \langle D_x^j u \rangle_{t, D_T}^{(\frac{k+\alpha-|j|}{2}\nu)} + \langle \mathbf{D}_t^\nu u \rangle_{t, D_T}^{(\frac{\alpha}{2}\nu)}, \quad \text{if } [k/2] = 1,\end{aligned}$$

where $[k/2]$ denotes the integer part of $k/2$.

In a similar way we introduce the spaces $C^{k+\alpha, \frac{k+\alpha}{2}\nu}(\partial D_T)$ and $C([0, T], C^{k+\alpha}(\partial D))$, where $\partial D_T = \partial D \times [0, T]$. Moreover, we will use the usual Hölder classes $C^{k+\alpha}(\bar{D})$ and $C^{k+\alpha}(\partial D)$, their definitions can be found, for instance, in [26].

We define the function $v_0(x)$ as a solution of the Dirichlet problem

$$\Delta_x v_0 = 0 \quad \text{in } \Omega; \quad v_0|_{\Gamma_0} = \varphi(x); \quad v_0|_{\Gamma} = 0. \quad (2.13)$$

We will assume that

$$\varphi(x) > 0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \varphi(x) \in C^{3+\alpha}(\Gamma_0). \quad (2.14)$$

According to the results of Chapter 3 in [26], there exists a unique solution v_0 of problem (2.13), and

$$\|v_0\|_{C^{3+\alpha}(\bar{\Omega})} \leq C\|\varphi\|_{C^{3+\alpha}(\Gamma_0)}, \quad \frac{\partial v_0}{\partial \bar{n}(\omega)}|_{\Gamma} < 0, \quad v|_{t=0} = v_0 \quad \text{in } \Omega, \quad (2.15)$$

where C is a positive constant.

Henceforward, the letter C will be used to denote different constants encountered in our formulae.

Theorem 2.1. *Let $\alpha, \nu \in (0, 1)$; Γ_0, Γ satisfy the assumptions mentioned in Sect. 1 and $\Gamma_0, \Gamma \in C^{3+\alpha}$; $\varphi(x)$ meet requirement (2.14); and the initial concentration $v(x, 0)$ be given with (2.13). Then for some small T , there exists a unique solution $(v(x, t), \rho(\omega, t))$ of nonlinear problem (2.12) for $t \in [0, T]$, such that*

$$v(x, t) \in C([0, T], C^{2+\alpha}(\bar{\Omega})), \quad \rho(\omega, t) \in C([0, T], C^{2+\alpha}(\Gamma)),$$

$$\mathbf{D}_t^\nu \rho(\omega, t) \in C([0, T], C^{1+\alpha}(\Gamma)).$$

2.3. A perturbation form of system (2.12)

In this subsection, we linearize system (2.12) on the initial data and rewrite it as a system $\mathcal{A}\mathbf{z} = \mathcal{F}\mathbf{z}$, where $\mathcal{A}$ is a linear operator and $\mathcal{F}$ is a nonlinear perturbation.

From (2.10) and (2.11), (2.15) for $t = 0$, it follows

$$\mathbf{D}_t^\nu \rho(\omega, 0) = -\mu \frac{\partial v_0}{\partial \bar{n}(\omega)}|_{\Gamma}, \quad (2.16)$$

here we take into account $\rho(\omega, 0) = 0$ and $\rho_{\omega_i}(\omega, 0) = 0, i = \overline{1, n-1}$.

Let a function $s(\omega, t)$ be such that

$$s(\omega, 0) = 0, \quad \mathbf{D}_t^\nu s(\omega, 0) = \mathbf{D}_t^\nu \rho(\omega, 0) \quad \text{on } \Gamma. \quad (2.17)$$

For example, the function $s(\omega, t)$ may be taken as

$$s(\omega, t) = -\mu \frac{t^\nu}{\Gamma(\nu+1)} \frac{\partial v_0}{\partial \bar{n}(\omega)}|_{\Gamma}. \quad (2.18)$$

Using the relation

$$\mathbf{D}_t^\nu \frac{t^\nu}{\Gamma(\nu+1)} = 1$$

together with (2.15) and representations (2.18), we can deduce the following result.

Corollary 2.1. *The function $s(\omega, t)$ given by (2.18) satisfies (2.17) and*

$$\|s\|_{C([0,T], C^{2+\alpha}(\Gamma))} + \|\mathbf{D}_t^\nu s\|_{C([0,T], C^{2+\alpha}(\Gamma))} \leq C\|\varphi\|_{C^{3+\alpha}(\bar{\Omega})}. \quad (2.19)$$

We introduce the new unknown functions $w(x, t)$ and $\sigma(\omega, t)$ in the following way:

$$\sigma(\omega, t) = \rho(\omega, t) - s(\omega, t); \quad (2.20)$$

$$w(x, t) = v(x, t) - v_0(x) - \langle \nabla_x v_0, \bar{e}_\sigma \rangle, \quad (2.21)$$

where

$$\bar{e}_\sigma = \frac{\partial x}{\partial \lambda} \chi(\lambda) \sigma(\omega, t), \quad x = (x_1, \dots, x_n). \quad (2.22)$$

Denote by

$$L_0 = -\nabla^2, \quad L_\rho = -\nabla_\rho^2,$$

such that $(L_0 u) \circ e_\rho = L_\rho(u \circ e_\rho)$. Then we can rewrite system (2.12) in the terms of the functions w, σ after some simple calculations:

$$\Delta_x w = \mathcal{F}_0(w, \sigma) \quad \text{in } \Omega_T; \quad (2.23)$$

$$w = 0 \quad \text{on } \Gamma_0 T; \quad \sigma(\omega, 0) = 0 \quad \text{on } \Gamma; \quad (2.24)$$

$$\mathbf{D}_t^\nu \sigma + \mu \frac{\partial w}{\partial \bar{n}(\omega)} = \mathcal{F}_1(w, \sigma) \quad \text{on } \Gamma_T; \quad (2.25)$$

$$w = -\frac{\partial v_0}{\partial \bar{n}(\omega)} \sigma \quad \text{on } \Gamma_T, \quad (2.26)$$

where

$$\begin{aligned} \mathcal{F}_0(w, \sigma) = & -[(L_0 v_0) \circ e_\rho - (L_0 v_0) \circ e_s] - L_s v_0 - (L_\rho - L_0)(v_0 - v_0 \circ e_\rho) \\ & + (L_s - L_0)(v_0 - v_0 \circ e_s) - L_0[v_0 + \langle \nabla v_0, \bar{e}_\rho \rangle - v_0 \circ e_\rho] \\ & + L_0[v_0 + \langle \nabla v_0, \bar{e}_s \rangle - v_0 \circ e_\rho] - (L_\rho - L_0)(w + \langle \nabla v_0, \bar{e}_\sigma \rangle); \end{aligned} \quad (2.27)$$

$$\begin{aligned} \mathcal{F}_1(w, \sigma) = & -\mathbf{D}_t^\nu s - \mu S(\omega, s, \nabla_\omega s) \frac{\partial v_0}{\partial \bar{n}(\omega)} + \mu[S(\omega, 0, 0) - S(\omega, \sigma + s, \nabla_\omega(\sigma + s))] \\ & \times \frac{\partial w}{\partial \bar{n}(\omega)} - \mu \frac{\partial v_0}{\partial \bar{n}(\omega)} \left[S(\omega, \sigma + s, \nabla_\omega \sigma + \nabla_\omega s) - S(\omega, s, \nabla_\omega s) - \sum_{i=1}^{n-1} \frac{\partial S(\omega, s, \nabla_\omega s)}{\partial s_{\omega_i}} \right. \\ & \times \sigma_{\omega_i} - \frac{\partial S(\omega, s, \nabla_\omega s)}{\partial s} \sigma \left. \right] - \mu \frac{\partial v_0}{\partial \bar{n}(\omega)} \left[\sum_{i=1}^{n-1} \left(\frac{\partial S(\omega, s, \nabla_\omega s)}{\partial s_{\omega_i}} - \frac{\partial S(\omega, 0, 0)}{\partial s_{\omega_i}} \right) \sigma_{\omega_i} \right. \\ & + \frac{\partial S(\omega, s, \nabla_\omega s)}{\partial s} \sigma \left. \right] + \sum_{i=1}^{n-1} \mu \frac{\partial w}{\partial \omega_i} \left[S_i(\omega, \sigma + s, \nabla_\omega \sigma + \nabla_\omega s) - S_i(\omega, s, \nabla_\omega s) \right. \\ & - \sum_{j=1}^{n-1} \frac{\partial S_i(\omega, s, \nabla_\omega s)}{\partial s_{\omega_j}} \sigma_{\omega_j} - \frac{\partial S_i(\omega, s, \nabla_\omega s)}{\partial s} \sigma \left. \right] - \mu S(\omega, \sigma + s, \nabla_\omega \sigma + \nabla_\omega s) \\ & \times \frac{\partial^2 v_0}{\partial \bar{n}(\omega)^2} \sigma - \mu \sum_{i=1}^{n-1} \left[\frac{\partial v_0}{\partial \bar{n}(\omega)} \sigma_{\omega_i} + \sigma \frac{\partial^2 v_0}{\partial \bar{n}(\omega) \partial \omega_i} \right] \left[S_i(\omega, \sigma + s, \nabla_\omega \sigma + \nabla_\omega s) \right. \\ & - S_i(\omega, s, \nabla_\omega s) - \sum_{j=1}^{n-1} \frac{\partial S_i(\omega, s, \nabla_\omega s)}{\partial s_{\omega_j}} \sigma_{\omega_j} - \frac{\partial S_i(\omega, s, \nabla_\omega s)}{\partial s} \sigma \left. \right]. \end{aligned} \quad (2.28)$$

Note that to get representation (2.28) we used (2.26) and the last condition in (2.13), which mean

$$\frac{\partial v_0}{\partial \omega_i} \Big|_{\Gamma} = 0, \quad \text{and} \quad - \frac{\partial w}{\partial \omega_i} \Big|_{\Gamma_T} = \sigma \frac{\partial^2 v_0}{\partial \bar{n}(\omega) \partial \omega_i} + \sigma_{\omega_i} \frac{\partial v_0}{\partial \bar{n}(\omega)}, \quad i = \overline{1, n-1}. \quad (2.29)$$

Thus, system (2.12) is written as

$$\mathcal{A}\mathbf{z} = \mathcal{F}\mathbf{z}, \quad \text{where} \quad \mathbf{z} = (w, \sigma). \quad (2.30)$$

Based on representations (2.27) and (2.28), properties (2.11) and (2.13), and Corollary 2.1, we infer the following result.

Corollary 2.2. *The functions $\mathcal{F}_i(w, \sigma)$, $i = 0, 1$, contain the higher derivatives of $w(x, t)$ and $\sigma(\omega, t)$ with coefficients that tend to zero as $t \rightarrow 0$, the “quadratic” terms with respect to $w(x, t)$ and $\sigma(\omega, t)$, and their derivatives, and the terms of minor differential orders of unknown functions. Moreover,*

$$\mathcal{F}_i(w, \sigma)|_{t=0} = 0. \quad (2.31)$$

The next step of our consideration is proof of the boundedness of the linear operator $\mathcal{A}$ in the corresponding functional spaces. To this end, we freeze the functional arguments in the functions $\mathcal{F}_0(w, \sigma)$ and $\mathcal{F}_1(w, \sigma)$ that leads to a linear system with variable coefficients (2.30) or (2.23)–(2.26) studied in detail in Sect. 4. Note that the investigation of this linear system is based on the research of the corresponding model problem with a fractional dynamic boundary condition.

3. Model problem with a fractional dynamic boundary condition

Let a and b be some positive constants,

$$R_+^n = \{(x', x_n) : x' \in R^{n-1}, x_n > 0\}, \quad x' = (x_1, \dots, x_{n-1});$$

$$R_{+T}^n = R_+^n \times (0, T); \quad R_T^{n-1} = R^{n-1} \times (0, T).$$

We look for solution (u, σ) of the following boundary problem:

$$\Delta_x u = g_0(x, t) \quad \text{in} \quad R_{+T}^n; \quad (3.1)$$

$$u(x, t) = a\sigma(x', t) + g_1(x', t) \quad \text{on} \quad \bar{R}_T^{n-1}; \quad (3.2)$$

$$\mathbf{D}_t^\nu \sigma(x', t) - b \frac{\partial u}{\partial x_n} = g_2(x', t) \quad \text{on} \quad \bar{R}_T^{n-1}; \quad (3.3)$$

$$\sigma(x', 0) = 0 \quad \text{in} \quad R^{n-1}, \quad (3.4)$$

where g_i , $i = \overline{0, 2}$, are given functions,

$$g_i \equiv 0 \quad \text{if} \quad \text{either} \quad t \leq 0 \quad \text{or} \quad |x| > R_0, \quad (3.5)$$

for a positive number R_0 .

Theorem 3.1. *Let $\alpha, \nu \in (0, 1)$, condition (3.5) hold,*

$$\begin{aligned} g_0 &\in C([0, T], C^\alpha(\bar{R}_+^n)), & g_1 &\in C([0, T], C^{2+\alpha}(R^{n-1})), \\ g_2 &\in C([0, T], C^{1+\alpha}(R^{n-1})). \end{aligned} \quad (3.6)$$

Then there exists a unique solution $(u(x, t), \sigma(x', t))$ of (3.1)–(3.4):

$$\begin{aligned} &\|u\|_{C([0, T], C^{2+\alpha}(\bar{R}_+^n))} + \|\mathbf{D}_t^\nu \sigma\|_{C([0, T], C^{1+\alpha}(R^{n-1}))} + \|\sigma\|_{C([0, T], C^{2+\alpha}(R^{n-1}))} \\ &\leq C_1 [\|g_0\|_{C([0, T], C^\alpha(\bar{R}_+^n))} + \|g_1\|_{C([0, T], C^{2+\alpha}(R^{n-1}))} + \|g_2\|_{C([0, T], C^{1+\alpha}(R^{n-1}))}]. \end{aligned} \quad (3.7)$$

If the right-hand sides in (3.1)–(3.4) meet the requirements

$$g_0, g_1 \equiv 0; \quad (3.8)$$

$$g_2 \in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_T^{n-1}), \quad (3.9)$$

then there is a unique solution (u, σ) of (3.1)–(3.4), $u \in C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\bar{R}_{+T}^n)$, $\sigma \in C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\bar{R}_T^{n-1})$

$$\begin{aligned} &\|u\|_{C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\bar{R}_{+T}^n)} + \|\mathbf{D}_t^\nu u\|_{C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_{+T}^n)} + \|\mathbf{D}_t^\nu \sigma\|_{C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_T^{n-1})} \\ &+ \|\sigma\|_{C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\bar{R}_T^{n-1})} \leq C_2 \|g_2\|_{C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_T^{n-1})}, \end{aligned} \quad (3.10)$$

where C_i , $i = 1, 2$, are positive constants independent of the right-hand sides of problem (3.1)–(3.4).

Note that conditions (3.6) or (3.8) and (3.9) together with (3.5) and (3.2) lead to

$$u|_{t=0} = 0 \quad \text{in } \bar{R}_+^n. \quad (3.11)$$

3.1. Integral representation of the solution to problem (3.1)–(3.4) in the case of (3.8)

First of all, we prove Theorem 3.1 for g_i , $i = \overline{0, 2}$, satisfying conditions (3.8) and (3.9). To this end, we construct the integral representation for σ and u and then obtain estimates (3.10).

Let $\xi = (\xi_1, \dots, \xi_{n-1})$ and $|\xi| = \left(\sum_{k=1}^{n-1} \xi_k^2\right)^{1/2}$. We denote by $\tilde{u}(\xi, x_n, t)$ the Fourier transform of $u(x', x_n, t)$, and by $\hat{u}(\cdot, p)$ the Laplace transform of $u(\cdot, t)$, and use the notation “ $*$ ” instead of “ $\widehat{}$ ”.

Then, applying the Laplace and Fourier transformations to (3.1)–(3.4), we get

$$u^*(\xi, x_n, p) = a \exp(-2\pi|\xi|x_n) \sigma^*(\xi, p), \quad (3.12)$$

$$\sigma^*(\xi, p) = \frac{g_2^*(\xi, p)}{p^\nu + 2\pi ab|\xi|}. \quad (3.13)$$

Note that we used formula (2.2.38) from [20]:

$$\widehat{\mathbf{D}_t^\nu w}(\cdot, t) = p^\nu \hat{w}(\cdot, p) - p^{\nu-1} \hat{w}(\cdot, 0), \quad (3.14)$$

to obtain (3.13). Denote by

$$G^*(\xi, p) := (p^\nu + 2\pi ab|\xi|)^{-1}. \quad (3.15)$$

One can easily check that

$$G^*(\xi, p) = \int_0^\infty \exp[-(p^\nu + 2\pi ab|\xi|)\varrho] d\varrho \equiv \int_0^\infty e^{-\varrho p^\nu} K^*(\xi, \varrho) d\varrho, \quad (3.16)$$

where

$$K^*(\xi, \varrho) = e^{-2\pi ab|\xi|\varrho}. \quad (3.17)$$

Let

$$G_1^*(\xi, p, x_n) = \int_0^\infty e^{-\varrho p^\nu} K^*(\xi, \varrho + x_n/ab), \quad (3.18)$$

then relations (3.12)–(3.18) deduce to

$$\sigma^*(\xi, p) = G^*(\xi, p)g_2^*(\xi, p), \quad u^*(\xi, x_n, p) = aG_1^*(\xi, p, x_n)g_2^*(\xi, p). \quad (3.19)$$

To obtain the representation for the functions u and σ , we use formula (3.2.7) from [37]:

$$\widehat{e^{-cp^\nu}} = t^{-1}W(-ct^{-\nu}; -\nu, 0), \quad c > 0. \quad (3.20)$$

Here $W(z; \beta, \gamma)$ is the Wright function which is defined for $z, \beta, \gamma \in \mathbb{C}$ as (see, e.g., (1.8.1(27)) in [13])

$$W(z; \beta, \gamma) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\beta k + \gamma)}. \quad (3.21)$$

Note that the main properties of the Wright functions are described in Chapter 18.1 [13], Chapter 1.11 [20], Chapter 2 [37], and in [19].

Thus, applying the inverse Laplace and Fourier transformations in (3.19), we obtain

$$\sigma(x', t) = \int_0^t d\tau \int_{R^{n-1}} G(y', \tau) g_2(x' - y', t - \tau) dy', \quad y' = (y_1, \dots, y_{n-1}), \quad (3.22)$$

$$\begin{aligned} u(x', x_n, t) &= a \int_{R^{n-1}} dy' \sigma(x' - y', t) K(y', x_n/ab) \\ &= a \int_0^t d\tau \int_{R^{n-1}} dy' G_1(y', \tau, x_n) g_2(x' - y', t - \tau), \end{aligned} \quad (3.23)$$

where

$$G(y', \tau) = \int_0^\infty d\varrho \tau^{-1} W(-\varrho \tau^{-\nu}; -\nu, 0) K(y', \varrho), \quad (3.24)$$

$$G_1(y', \tau, x_n) = \int_0^\infty d\varrho \tau^{-1} W(-\varrho \tau^{-\nu}; -\nu, 0) K(y', \varrho + x_n/ab), \quad (3.25)$$

$$K(y', z) = \int_{R^{n-1}} \exp\{-2\pi ab|\xi|z + 2\pi i \langle \xi, y' \rangle\} d\xi, \quad \langle \xi, y' \rangle = \sum_{k=1}^{n-1} \xi_k y_k. \quad (3.26)$$

As usual in the potential theory, to estimate the functions u and σ , it is necessary to describe well the properties of the kernels $G(y', t)$ and $K(y', z)$.

For $n = 2$ integral in (3.26) may be calculated, and the function $K(y', z)$ has the exact representation (see (3.11) in [23])

$$K(y', z) = K(y_1, z) = \frac{2abz}{4\pi^2 y_1^2 + (abz)^2}. \quad (3.27)$$

In the case of $n \geq 3$, there is not an exact view for the inverse Fourier transformation of the function $e^{-ab|\xi|z}$. Hence, we can not construct a simple formula for the kernel $K(y', z)$. Therefore, in contrast to the two-dimensional case, description of the properties of the function $K(y', z)$ demands additional investigations (see Lemma 3.1 below). These studies differ essentially from the case $n = 2$ (compare, Lemma 3.1 [23] with Lemma 3.1 here). Nevertheless, our method of investigation and results obtained hold true for any $n \geq 2$.

For the sake of simplicity, we consider $a = b = 1$.

3.2. Estimates of the constructed solution $(u(x, t), \sigma(x', t))$

At the beginning, we define the fractional Riemann-Liouville integral and derivative of a function $g(\cdot, t)$ with respect to t (see, e.g., (2.1.1) and (2.1.8) [20]) as:

$$I_t^\theta g(\cdot, t) := \frac{1}{\Gamma(\theta)} \int_0^t \frac{g(\cdot, \tau) d\tau}{(t - \tau)^{1-\theta}}, \quad \theta > 0, \quad t > 0; \quad (3.28)$$

$$\partial_t^\theta g(\cdot, t) := \frac{1}{\Gamma(1 - \theta)} \frac{\partial}{\partial t} \int_0^t \frac{g(\cdot, \tau) d\tau}{(t - \tau)^\theta}, \quad \theta \in (0, 1). \quad (3.29)$$

The next lemma describes the main properties of the kernels $G(y', t)$, $K(y', z)$ which are necessary to prove estimates (3.7) and (3.10).

Lemma 3.1. *Let $\alpha, \nu \in (0, 1)$; $y' \in R^{n-1}$; $y'' \in R^{n-2}$, $y'' = (y_1, \dots, y_{l-1}, y_{l+1}, \dots, y_{n-1})$, $l = \overline{1, n-1}$, $z \in (0, +\infty)$; ε and A be positive numbers. Then functions $K(y', z)$ and $G(y', t)$ which are given by (3.23) and (3.26) satisfy the following inequalities:*

i

$$|D_{y'}^m K(y', z)| \leq C \frac{\exp\left(-A \sum_{j=1}^{n-1} |y_j|/z\right)}{z^{n-1+|m|}}, \quad |m| = \sum_{i=1}^{n-1} m_i, \quad |m| = 0, 1, 2, \dots; \quad (3.30)$$

ii

$$|\partial K(y', z)/\partial z| \leq C \frac{\exp\left(-A \sum_{j=1}^{n-1} |y_j|/z\right)}{z^n}; \quad (3.31)$$

iii

$$\int_{R^{n-1}} K(y', z) dy' = 1; \quad \int_{-\infty}^{\infty} \frac{\partial K}{\partial y_i}(y', z) dy_i = 0, \quad i = \overline{1, n-1}; \quad (3.32)$$

iv

$$K(y', 0) = \Pi_{j=1}^{n-1} \delta(-y_j), \quad (3.33)$$

where $\delta(y)$ is the Dirac delta function;

v

$$\int_{R^{n-1}} G(y', t) dy' = \frac{t^{\nu-1}}{\Gamma(\nu)}; \quad (3.34)$$

$$\int_0^t d\tau \int_{R_+^{n-1}} |G(y', \tau)| dy' \leq Ct^\nu; \quad (3.35)$$

vi

$$\int_0^t d\tau \int_{R_+^{n-1}} y_j^\alpha \left| \frac{\partial G(y', \tau)}{\partial y_i} \right| dy' \leq Ct^{\alpha\nu}, \quad i, j = \overline{1, n-1}; \quad (3.36)$$

vii

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_0^\varepsilon y_j^\alpha \left| \frac{\partial G(y', \tau)}{\partial y_i} \right| dy_l \leq C\varepsilon^\alpha, \quad i, j = \overline{1, n-1}; \quad (3.37)$$

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_\varepsilon^{+\infty} \left| \frac{\partial G(y', \tau)}{\partial y_i} \right| dy_l \leq C, \quad i \neq l; \quad (3.38)$$

viii

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_\varepsilon^{+\infty} y_j^\alpha \left| \frac{\partial^2 G(y', \tau)}{\partial y_i \partial y_l} \right| dy_l \leq C\varepsilon^{\alpha-1}, \quad i, j = \overline{1, n-1}; \quad (3.39)$$

ix

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \left| \int_{|y_j| \geq \varepsilon} \frac{\partial G(y', \tau)}{\partial y_j} dy_j \right| = 0, \quad j = \overline{1, n-1}. \quad (3.40)$$

Proof. From (3.26) it follows that

$$D_{y'}^m K(y', z) = C \int_{R^{n-1}} \xi_1^{m_1} \dots \xi_{n-1}^{m_{n-1}} \exp\{-2\pi|\xi|z + 2\pi i\langle \xi, y' \rangle\} d\xi, \\ |m| = 0, 1, 2, \dots \quad (3.41)$$

Then we do the following change of variables: $\xi_k z = \mu_k$, $k = \overline{1, n-1}$, in (3.41) and get

$$D_{y'}^m K(y', z) = \frac{const.}{z^{n-1+|m|}} \int_{R^{n-1}} \mu_1^{m_1} \dots \mu_{n-1}^{m_{n-1}} \exp\{-2\pi|\mu| + 2\pi i\langle \mu, y' \rangle/z\} d\mu. \quad (3.42)$$

Since the function $\mu_1^{m_1} \dots \mu_{n-1}^{m_{n-1}} \exp\{-2\pi|\mu| + 2\pi i\langle \mu, y' \rangle/z\}$ vanishes if $|\mu_k| \rightarrow 0$, $k = \overline{1, n-1}$; and one does not have any poles, we can calculate the integrals in (3.42) along the shifted contours: $\mu_k = \lambda_k \pm i\check{A}$, $k = \overline{1, n-1}$. Here the sign “+” is chosen if $y_k > 0$, and “−” if $y_k < 0$. Then we obtain

$$D_{y'}^m K(y', z) = \frac{C \exp(-A \sum_{k=1}^{n-1} |y_k|/z)}{z^{n-1+|m|}} \int_{R^{n-1}} (\lambda_1 \pm i\check{A})^{m_1} \dots (\lambda_{n-1} \pm i\check{A})^{m_{n-1}} \\ \times \exp\{-2\pi|\lambda \pm i\check{A}| + 2\pi i\langle \lambda, y' \rangle/z\} d\lambda, \quad A = \check{A}2\pi. \quad (3.43)$$

Finally, it is easy to check that the module of the integrals in (3.43) is bounded uniformly in y' and z . Thus, statement (i) follows from (3.43).

As for estimate (3.31), it follows immediately from representation

$$\frac{\partial K}{\partial z}(y', z) = - \int_{R^{n-1}} 2\pi |\xi| \exp\{-2\pi |\xi| z + 2\pi i \langle \xi, y' \rangle\} d\xi, \quad (3.44)$$

and the arguments above for the function $D_{y'}^m K(y', z)$.

To get the first equality in (3.32), we use the following relation from Lemma 5.3 [4]:

$$\operatorname{Re} \int_0^\infty e^{ixy} dy = \pi \delta(-x), \quad (3.45)$$

where $\delta(x)$ is the Dirac delta function. Then

$$\begin{aligned} \int_{R^{n-1}} K(y', z) dy' &= \int_{R^{n-1}} e^{-2\pi |\xi| z} d\xi \int_{R^{n-1}} e^{2i\pi \langle y', \xi \rangle} dy' \\ &= \pi^{1-n} \int_{R^{n-1}} e^{-2\pi |\xi| z} d\xi \operatorname{Re} \int_{R_+^{n-1}} e^{i \langle y', \xi \rangle} dy'. \end{aligned}$$

After that, using (3.45) to calculate the last integral, we obtain the first equality in (iii). Note that the second equality in (3.32) follows immediately from the representation for $\frac{\partial K}{\partial y_i}$ (see (3.41) with $|m| = 1$) and (3.45). Namely,

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\partial K}{\partial y_i} dy_i &= 2i\pi \int_{R^{n-1}} \xi_i \exp\{-2\pi |\xi| z + 2i\pi \sum_{k=1, k \neq i}^{n-2} y_k \xi_k\} d\xi \int_{-\infty}^{+\infty} e^{2i\pi \langle y_i \xi_i \rangle} dy_i \\ &= 2i\pi \int_{R^{n-1}} \xi_i \exp\{-2\pi |\xi| z + 2i\pi \sum_{k=1, k \neq i}^{n-2} y_k \xi_k\} \delta(-\xi_i) d\xi = 0. \end{aligned}$$

One can easily check that equality (3.33) is a simple consequence of (3.45).

Applying the well-known equality for the Wright function (see, e.g., (F7) [33]):

$$W(-z; -\beta, 0) = \beta z W(-z; -\beta, 1 - \beta), \quad (3.46)$$

and the first equality in (3.32), we can deduce that

$$\begin{aligned} \int_{R^{n-1}} G(y', t) dy' &= \nu \int_0^{+\infty} t^{-1-\nu} \varrho W(-\varrho t^{-\nu}; -\nu, 1 - \nu) d\varrho = \nu t^{\nu-1} \\ &\quad \times \int_0^{+\infty} \eta W(-\eta; -\nu, 1 - \nu) d\eta, \quad \varrho t^{-\nu} = \eta. \end{aligned} \quad (3.47)$$

Then applying equality (10) from [38]

$$\int_0^{+\infty} x^{l-1} W(-x; -\beta, \delta) dx = \frac{\Gamma(l)}{\Gamma(\beta l + \delta)}, \quad (3.48)$$

where either $l > 0$ if $\beta > 0$, or $l > -1$ if $\beta = 0$; $\delta \in (0, 1)$, we get relation (3.34).

As for inequality (3.35), we use estimate (3.30) for $|m| = 0$ and $z = \varrho$ together with equality (3.46) to get

$$\int_0^t d\tau \int_{R_+^{n-1}} |G(y', \tau)| dy' \leq C \int_0^t d\tau \int_0^\infty d\varrho \int_{R_+^{n-1}} W(-\varrho\tau^{-\nu}; -\nu, 1-\nu) \\ \times \varrho^{2-n}\tau^{-1-\nu} \exp\left(-A \sum_{k=1}^{n-1} \frac{y_k}{\varrho}\right) dy'. \quad (3.49)$$

Here, we take into account the positiveness of the Wright function $W(-z; \beta, \gamma)$ if $z, \gamma \geq 0$ (see Lemma 2.2.4 [37]). After that, performing the consecutive change of variables:

$$y_k = \varrho x_k, \quad k = \overline{1, n-1}; \quad \varrho = \tau^\nu \eta, \quad (3.50)$$

in the right-hand side of (3.49), we obtain

$$\int_0^t d\tau \int_{R_+^{n-1}} |G(y', \tau)| dy' \leq Ct^\nu \int_0^{+\infty} \eta W(-\eta; -\nu, 1-\nu) d\eta. \quad (3.51)$$

Equality (3.48) ensures the boundedness of the integral in the right-hand side of (3.51). Hence, (3.35) is proved.

For inequality (3.36), we can conclude that

$$\int_0^t d\tau \int_{R_+^{n-1}} y_j^\alpha \left| \frac{\partial G}{\partial y_i} \right| dy' = \int_0^t d\tau \int_{R_+^{n-1}} dy' y_j^\alpha \int_0^\infty W(-\varrho\tau^{-\nu}; -\nu, 1-\nu) \\ \times \tau^{-1-\nu} \varrho \left| \frac{\partial K(y', \varrho)}{\partial y_i} \right| d\varrho. \quad (3.52)$$

In this case we used (3.46) and the positiveness of the Wright function. Then, estimate (3.30) together with the change of variables (3.50) and equality (3.48) lead to (3.36).

Let us prove (3.37) if $l \neq j$. In a similar way as for (3.36), we can deduce that

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_0^\varepsilon y_j^\alpha \left| \frac{\partial G}{\partial y_i} \right| dy_l \leq C \int_{R_+^{n-2}} dx'' x_j^\alpha \exp\left(-A \sum_{k=1, k \neq l}^{n-1} x_k\right) \int_0^\varepsilon dy_l \\ \int_0^{+\infty} W(-\eta; \nu, 1-\nu) d\eta \int_0^{+\infty} \tau^{-1-\nu+\alpha\nu} \eta^{\alpha-1} \exp\left(-A \frac{y_l}{\eta\tau^\nu}\right) d\tau, \quad (3.53)$$

where $x'' = (x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_{n-1})$. Then the change of variable:

$$\tau = \left(\frac{y_l}{\eta r}\right)^{1/\nu}, \quad (3.54)$$

in the last integral in (3.53) provides

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_0^\varepsilon y_j^\alpha \left| \frac{\partial G}{\partial y_i} \right| dy_l \leq C \int_0^\varepsilon y_l^{\alpha-1} dy_l \int_0^\infty W(-\eta; -\nu, 1-\nu) d\eta \\ \times \int_0^\infty e^{-Ar} r^{-\alpha} dr. \quad (3.55)$$

Therefore, inequality (3.55) together with equality (3.48) lead to estimate (3.37) if $l \neq j$. In the case $l = j$, inequality (3.37) is proved in the same way.

To get (3.38), we use again inequality (3.30) with $|m| = 1$ and change of variables: (3.50) and $\varrho\tau^{-\nu} := p$, $\tau^{-\nu} := p/\varrho$. Thus, we have after some simple calculations:

$$\int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_{\varepsilon}^{+\infty} \left| \frac{\partial G}{\partial y_i} \right| dy_l \leq C \int_0^{+\infty} d\varrho \frac{e^{-A\varepsilon/\varrho}}{\varrho} \int_{\varrho t^{-\nu}}^{+\infty} W(-p; -\nu, 1-\nu) dp. \quad (3.56)$$

Then applying the following inequality from [34]:

$$W(-p; -\nu, 1-\nu) \leq a(\nu) p^{\frac{\nu-1/2}{1-\nu}} \exp[-b(\nu) p^{\frac{1}{1-\nu}}], \quad a(\nu), b(\nu), p > 0,$$

to the right-hand side of (3.56) leads to

$$\begin{aligned} \int_0^t d\tau \int_{R_+^{n-2}} dy'' \int_{\varepsilon}^{+\infty} \left| \frac{\partial G}{\partial y_i} \right| dy_l &\leq C \int_0^{+\infty} d\varrho \frac{e^{-A\varepsilon/\varrho}}{\varrho} \int_{\varrho t^{-\nu}}^{+\infty} a(\nu) \exp[-b(\nu) p^{\frac{1}{1-\nu}}] \\ &\times p^{\frac{\nu-1/2}{1-\nu}} dp \leq C \int_0^{+\infty} d\varrho \varrho^{-1-\frac{1}{2(1-\nu)}} e^{-A\varepsilon/\varrho} \int_0^{+\infty} \exp[-b(\nu) p^{\frac{1}{1-\nu}}] p^{\frac{\nu}{1-\nu}} dp \leq C. \end{aligned}$$

This completes the proof of (3.38).

Note that to obtain estimate (3.39), it is necessary to repeat the proof of the statement (vii) using inequality (3.30) with $|m| = 2$.

Finally, we prove statement (ix). Similarly to the proof of the second equality in (3.32), we conclude that

$$\int_{-\varepsilon}^{\varepsilon} \frac{\partial K}{\partial y_i}(y', z) dy_i = 0. \quad (3.57)$$

Thus, equality (3.32) together with (3.57) ensure (3.40). $\square$

Lemma 3.2. *Let $\alpha, \nu \in (0, 1)$, condition (3.8) hold and $g_2(x', t) \in C([0, T], C^{1+\alpha}(R^{n-1}))$. Then there are the following inequalities:*

$$|\sigma(x', t)| \leq C t^{\nu} \sup_{\bar{R}_T^{n-1}} |g_2(x', t)|; \quad (3.58)$$

$$\sup_{\bar{R}_T^{n-1}} |D_{x'}^m \sigma| + \langle D_{x'}^m \sigma \rangle_{x, \bar{R}_T^{n-1}}^{(\alpha)} \leq C \|g_2\|_{C([0, T], C^{1+\alpha}(R^{n-1}))}, \quad |m| = 1, 2, \quad (3.59)$$

where σ is given by (3.22)

Proof. Inequality (3.58) is a simple consequence of estimate (3.35). Let us get inequality (3.59) in the case of $|m| = 2$. Taking into account the second equality in (3.32), we can represent $\sigma_{x_i x_j}$, $i, j = \overline{1, n-1}$, as

$$\begin{aligned} \sigma_{x_i x_j} = & \int_0^t d\tau \int_{R^{n-1}} \frac{\partial G}{\partial y_i}(y', \tau) \left[\frac{\partial g_2}{\partial y_j}(x_1 - y_1, \dots, x_j - y_j, \dots, x_{n-1} - y_{n-1}, t - \tau) \right. \\ & \left. - \frac{\partial g_2}{\partial y_j}(x_1 - y_1, \dots, x_j, \dots, x_{n-1} - y_{n-1}, t - \tau) \right] dy', \quad i, j = \overline{1, n-1}. \end{aligned} \quad (3.60)$$

After that, inequality (3.36) together with the property of the function $g_2(x', t)$ lead to

$$\sup_{R_T^{n-1}} |\sigma_{x_i x_j}| \leq CT^{\alpha\nu} \left\langle \frac{\partial g_2}{\partial x_j} \right\rangle_{x, R_T^{n-1}}^{(\alpha)}. \quad (3.61)$$

Denote by

$$\Delta_{x_l} \sigma_{x_i x_j} = \sigma(x_1, \dots, x_{l-1}, x_l, x_{l+1}, \dots, x_{n-1}, t) - \sigma(x_1, \dots, x_{l-1}, \bar{x}_l, x_{l+1}, \dots, x_{n-1}, t); \quad \Delta x_l = x_l - \bar{x}_l; \quad x'' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}), \quad j, l = \overline{1, n-1}.$$

First, we consider the case $l = j$, and estimate the Hölder seminorm of $\sigma_{x_i x_j}$ with respect to x . To this end, we use the following representation for $\Delta_{x_j} \sigma_{x_i x_j}$ (the analogous one may be found in Chapter 4 [25]):

$$\begin{aligned} \Delta_{x_j} \sigma_{x_i x_j} = & \int_0^t d\tau \int_{R^{n-2}} dy'' \int_{|x_j - y_j| \leq 2|\Delta x_j|} \left[\frac{\partial g_2}{\partial y_j}(y'', y_j, t - \tau) - \frac{\partial g_2}{\partial x_j}(y'', x_j, t - \tau) \right] \\ & \times \frac{\partial G}{\partial y_i}(x' - y', \tau) dy_j + \int_0^t d\tau \int_{R^{n-2}} dy'' \int_{|x_j - y_j| \leq 2|\Delta x_j|} \frac{\partial G}{\partial y_i}(x'' - y'', \bar{x}_j - y_j, \tau) \\ & \times \left[\frac{\partial g_2}{\partial y_j}(y'', y_j, t - \tau) - \frac{\partial g_2}{\partial \bar{x}_j}(y'', \bar{x}_j, t - \tau) \right] dy_j \\ & + \int_0^t d\tau \int_{R^{n-2}} dy'' \int_{|x_j - y_j| \geq 2|\Delta x_j|} \left[\frac{\partial G}{\partial y_i}(x'' - y'', x_j - y_j, \tau) \right. \\ & \left. - \frac{\partial G}{\partial y_i}(x'' - y'', \bar{x}_j - y_j, \tau) \right] \times \left[\frac{\partial g_2}{\partial y_j}(y'', y_j, t - \tau) - \frac{\partial g_2}{\partial x_j}(y'', x_j, t - \tau) \right] dy_j \\ & + \int_0^t d\tau \int_{R^{n-2}} dy'' \left[\frac{\partial g_2}{\partial x_j}(y'', \bar{x}_j, t - \tau) - \frac{\partial g_2}{\partial x_j}(y'', x_j, t - \tau) \right] \\ & \times \int_{|x_j - y_j| \geq 2|\Delta x_j|} \frac{\partial G}{\partial y_i}(x'' - y'', \bar{x}_j - y_j, \tau) dy_j \equiv \sum_{j=1}^4 \mathfrak{B}_j. \end{aligned} \quad (3.62)$$

Note that estimates of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are the simplest and follow from (3.37) where $\varepsilon = |\Delta x_j|$:

$$|\mathfrak{B}_1| + |\mathfrak{B}_2| \leq C \left\langle \frac{\partial g_2}{\partial x_j} \right\rangle_{x, R_T^{n-1}}^{(\alpha)} |\Delta x_j|^\alpha. \quad (3.63)$$

To evaluate the term $\mathfrak{B}_3$, it is enough to apply the mean value theorem to the difference $[\frac{\partial G}{\partial y_i}(x'' - y'', x_j - y_j, \tau) - \frac{\partial G}{\partial y_i}(x'' - y'', \bar{x}_j - y_j, \tau)]$ and to use inequality (3.39) with $\varepsilon = |\Delta x_j|$, so that

$$|\mathfrak{B}_3| \leq C \left\langle \frac{\partial g_2}{\partial x_j} \right\rangle_{x, R_T^{n-1}}^{(\alpha)} |\Delta x_j|^\alpha. \quad (3.64)$$

Finally, equality (3.40) together with inequality (3.38) allow us to infer

$$\mathfrak{B}_4 = 0, \quad \text{if } i = j; \quad |\mathfrak{B}_4| \leq C \left\langle \frac{\partial g_2}{\partial x_j} \right\rangle_{x, R_T^{n-1}}^{(\alpha)} |\Delta x_j|^\alpha, \quad \text{if } i \neq j. \quad (3.65)$$

Hence, representation (3.62) together with estimates (3.63)–(3.65) lead to

$$|\Delta x_j \sigma_{x_i x_j}| \leq C \left\langle \frac{\partial g_2}{\partial x_j} \right\rangle_{x, R_T^{n-1}}^{(\alpha)}. \quad (3.66)$$

In the same way, it is possible to get estimate (3.66) if $l \neq j$.

Thus, inequality (3.59) for $|m| = 2$ follows from (3.61) and (3.66). To obtain (3.59) in the case $|m| = 1$, we apply interpolation inequality, estimates (3.58) and (3.59) for $|m| = 2$. That completes the proof of Lemma 3.2. $\square$

We will use the following results to estimate the functions $\mathbf{D}_t^\nu \sigma(x', t)$ and $u(x, t)$ given by (3.23).

Proposition 3.1. *Let conditions of Lemma 3.2 hold. Then the functions $\sigma(x', t)$ and $u(x, t)$, which are represented by (3.22) and (3.23), satisfy boundary conditions (3.2) and (3.3) with $g_1 \equiv 0$.*

Since the proof of this result is technically tedious, we provide it in the Appendix 7.

Based on Lemma 3.2 and Proposition 3.1 and results of Chapter 3 [26], we can conclude the following.

Proposition 3.2. *Let conditions of Lemma 3.2 hold. Then the function $u(x, t)$ and $\mathbf{D}_t^\nu \sigma(x', t)$ satisfy the following inequality:*

$$\begin{aligned} & \|u\|_{C([0, T], C^{2+\alpha}(\bar{R}_+^n))} + \|\mathbf{D}_t^\nu u\|_{C([0, T], C^{1+\alpha}(\bar{R}_+^n))} + \|\mathbf{D}_t^\nu \sigma\|_{C([0, T], C^{1+\alpha}(R^{n-1}))} \\ & \leq C \|g_2\|_{C([0, T], C^{1+\alpha}(R^{n-1}))}. \end{aligned} \quad (3.67)$$

The next step in our consideration is obtaining the corresponding estimates for the functions $u(x, t)$ and $\sigma(x', t)$ with respect to t .

Proposition 3.3. *Let conditions (3.8) and (3.9) be satisfied. Then the following estimates hold:*

$$\sum_{|k|=1}^2 \left[\langle D_x^k \sigma \rangle_{t, R_T^{n-1}}^{(\frac{2-|k|+\alpha}{2}\nu)} + \langle D_x^k u \rangle_{t, R_{+T}^n}^{(\frac{2-|k|+\alpha}{2}\nu)} \right] \leq C \|g_2\|_{C([0, T], C^{1+\alpha}(R^{n-1}))}; \quad (3.68)$$

$$\langle \mathbf{D}_t^\nu D_{x'} \sigma \rangle_{t, R_T^{n-1}}^{(\frac{\alpha}{2}\nu)} + \langle \mathbf{D}_t^\nu \sigma \rangle_{t, R_T^{n-1}}^{(\frac{1+\alpha}{2}\nu)} \leq C \|g_2\|_{C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_T^{n-1})}. \quad (3.69)$$

Proof. First, following the arguments of Theorem 3.1 [39], one can prove the next statements:

Let $\psi, \mathbf{D}_t^\nu \psi \in C([0, T], C^\alpha(\bar{\Omega}))$, then

$$|\psi(\cdot, t_1) - \psi(\cdot, t_2)| \leq C |t_1 - t_2|^\nu \sup_{\Omega_T} |\mathbf{D}_t^\nu \psi|; \quad (3.70)$$

$$\langle \psi(\cdot, t_1) - \psi(\cdot, t_2) \rangle_{x, \Omega_T}^{(\alpha)} \leq C \langle \mathbf{D}_t^\nu \psi \rangle_{x, \Omega_T}^{(\alpha)} |t_1 - t_2|^\nu \quad \forall t_1, t_2 \in [0, T]. \quad (3.71)$$

Based on estimates (3.70) and (3.67), we deduce that

$$\left| \frac{\partial \sigma}{\partial x_i}(x', t_1) - \frac{\partial \sigma}{\partial x_i}(x', t_2) \right| \leq C |t_1 - t_2|^\nu \sup_{\bar{R}_T^{n-1}} \left| \mathbf{D}_t^\nu \frac{\partial \sigma}{\partial x_i} \right| \leq C |t_1 - t_2|^\nu \times \|g_2\|_{C([0,T], C^{1+\alpha}(R^{n-1}))}, \quad i = \overline{1, n-1}. \quad (3.72)$$

Because of $\frac{1+\alpha}{2}\nu < \nu$, we assert from (3.72) that

$$\left\langle \frac{\partial \sigma}{\partial x_i} \right\rangle_{t, R_T^{n-1}}^{(\frac{1+\alpha}{2}\nu)} \leq C \|g_2\|_{C([0,T], C^{1+\alpha}(R^{n-1}))}, \quad i = \overline{1, n-1}. \quad (3.73)$$

In the same way, we get

$$\left\langle \frac{\partial u}{\partial x_i} \right\rangle_{t, R_{+T}^n}^{(\frac{1+\alpha}{2}\nu)} \leq C \|g_2\|_{C([0,T], C^{1+\alpha}(R^{n-1}))} \quad i = \overline{1, n}. \quad (3.74)$$

To evaluate $\left\langle \frac{\partial^2 \sigma}{\partial x_i \partial x_j} \right\rangle_{t, R_T^{n-1}}^{(\frac{\alpha}{2}\nu)}$, we use the next interpolation inequality (see Lemma 3.1 [40] and Corollary 1.2.18 [32]):

$$\|V\|_{C^{l_0}(Q)} \leq C \|V\|_{C^{l_2}(Q)}^\delta \|V\|_{C^{l_1}(Q)}^{1-\delta}, \quad (3.75)$$

where $\delta \in (0, 1)$, $0 \leq l_1 < l_2$, $l_0 = l_1 + \delta(l_2 - l_1)$. Let $V := \sigma(x', t_1) - \sigma(x', t_2)$, $l_0 := 2$, $l_2 := 2 + \alpha$, $l_1 := \alpha$, $\delta := 1 - \frac{\alpha}{2}$ in (3.75), then

$$\sup_{R^{n-1}} |D_{x'}^2 \sigma(x', t_1) - D_{x'}^2 \sigma(x', t_2)| \leq C \|\sigma(\cdot, t_1) - \sigma(\cdot, t_2)\|_{C^{2+\alpha}(R^{n-1})}^{1-\frac{\alpha}{2}} \times \|\sigma(\cdot, t_1) - \sigma(\cdot, t_2)\|_{C^\alpha(R^{n-1})}^{\frac{\alpha}{2}}. \quad (3.76)$$

Next, we apply inequalities (3.70) and (3.71) to the term $\|\sigma(\cdot, t_1) - \sigma(\cdot, t_2)\|_{C^\alpha(R^{n-1})}$ in the right-hand side of (3.76) and get

$$\langle D_{x'}^2 \sigma \rangle_{t, R_T^{n-1}}^{(\frac{\alpha}{2}\nu)} \leq C \|\sigma\|_{C([0,T], C^{2+\alpha}(R^{n-1}))}^{1-\frac{\alpha}{2}} \|\mathbf{D}_t^\nu \sigma\|_{C([0,T], C^\alpha(R^{n-1}))}^{\alpha/2}. \quad (3.77)$$

Finally, estimate (3.67) and Lemma 3.2 allow us to obtain from (3.77)

$$\langle D_{x'}^2 \sigma \rangle_{t, R_T^{n-1}}^{(\frac{\alpha}{2}\nu)} \leq C \|g_2\|_{C([0,T], C^{1+\alpha}(R^{n-1}))}. \quad (3.78)$$

It is easy to see that the same estimate holds for $D_x^2 u(x, t)$ [see representation (3.23)]. Thus, inequality (3.68) follows from (3.73), (3.74) and (3.78).

To prove (3.69) for $\langle \mathbf{D}_t^\nu \sigma \rangle_{t, R_T^{n-1}}^{(\frac{1+\alpha}{2}\nu)}$, we use Proposition 3.1 and inequality (3.68):

$$\langle \mathbf{D}_t^\nu \sigma \rangle_{t, R_T^{n-1}}^{(\frac{1+\alpha}{2}\nu)} \leq \langle g_2 \rangle_{t, R_T^{n-1}}^{(\frac{1+\alpha}{2}\nu)} + \left\langle \frac{\partial u}{\partial x_n} \right\rangle_{t, R_{+T}^n}^{(\frac{1+\alpha}{2}\nu)} \leq C \|g_2\|_{C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_T^{n-1})}. \quad (3.79)$$

The corresponding estimate for $\langle \mathbf{D}_t^\nu D_{x'} \sigma \rangle_{t, R_T^{n-1}}^{(\frac{\alpha}{2}\nu)}$ follows from interpolation inequality (3.75) with

$$V = \mathbf{D}_t^\nu \sigma(\cdot, t_1) - \mathbf{D}_t^\nu \sigma(\cdot, t_2), \quad l_0 = 1, \quad l_2 = 1 + \alpha, \quad l_1 = 0, \quad \delta = \frac{1}{1 + \alpha},$$

and (3.79), (3.67). $\square$

3.3. Proof of Theorem 3.1

If conditions (3.8) and (3.9) hold, then the results of Propositions 3.2 and 3.3 and Lemma 3.2 prove estimate (3.10). Note that the uniqueness of the constructed solution $(u(x, t), \sigma(x', t))$ follows from coercive estimate (3.10). Moreover, representation (3.23) together with inequality (3.10) mean that $u = K \star \sigma$, where $\sigma \in C^{2+\alpha}(R^{n-1})$ for each $t \in [0, T]$. Thus, arguments from Chapter 3 [26] allow us to show that $u(x, t)$ satisfies equation (3.1) with $g_0 \equiv 0$. Inequality (3.58) assures that $\sigma(x', t)$ meets requirement (3.4). Hence, all written in the previous section together with Proposition 3.1 imply that the functions $u(x, t)$ and $\sigma(x', t)$ [see (3.22) and (3.23)] satisfy (3.1)–(3.4). That proves Theorem 3.1 in the case of (3.8) and (3.9).

Next we get Theorem 3.1 if condition (3.9) is replaced by (3.6). To this end, we repeat the arguments from Sect. 4 [23]. Let $\{g_{2n}\}_{n=1}^\infty$ be the sequence such that:

$$g_{2n} \in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_T^{n-1}), \quad g_{2n} \xrightarrow{n \rightarrow \infty} g_2 \quad \text{in } C([0, T], C^{1+\alpha}(R^{n-1})). \quad (3.80)$$

We find the functions $u_n(x, t)$ and $\sigma_n(x', t)$ as a solution of the problem:

$$\begin{aligned} \Delta u_n &= 0 \quad \text{in } R_{+T}^n; & \sigma_n(x', 0) &= 0 \quad \text{in } R^{n-1}; \\ u_n &= a\sigma_n \quad \text{and} \quad \mathbf{D}_t^\nu \sigma_n - b \frac{\partial u_n}{\partial x_n} &= g_{2n} \quad \text{on } \bar{R}_T^{n-1}. \end{aligned} \quad (3.81)$$

Note that the right-hand sides of (3.81) satisfy condition (3.8) and (3.9). Therefore, we can apply Theorem 3.1 to (3.81) and get the unique solution of problem (3.81):

$$\begin{aligned} u_n &\in C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\bar{R}_{+T}^n), & \sigma_n &\in C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\bar{R}_T^{n-1}), \\ \mathbf{D}_t^\nu \sigma_n &\in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_T^{n-1}), & \mathbf{D}_t^\nu u_n &\in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{R}_{+T}^n), \end{aligned} \quad (3.82)$$

for which estimate (3.10) holds. Embedding (3.82) leads to

$$\begin{aligned} u_n &\in C([0, T], C^{2+\alpha}(\bar{R}_+^n)), & \sigma_n &\in C([0, T], C^{2+\alpha}(R^{n-1})), \\ \mathbf{D}_t^\nu \sigma_n &\in C([0, T], C^{1+\alpha}(R^{n-1})), & \mathbf{D}_t^\nu u_n &\in C([0, T], C^{1+\alpha}(\bar{R}_+^n)). \end{aligned}$$

Taking into account the results of Proposition 3.2 and Lemma 3.2, and the completeness of the spaces $C([0, T], C^{k+\alpha}(\bar{D}))$, we assert the existence of the functions $u \in C([0, T], C^{2+\alpha}(\bar{R}_+^n))$, $\sigma \in C([0, T], C^{2+\alpha}(R^{n-1}))$, such that

$$u_n \rightarrow u \quad \text{in } C([0, T], C^{2+\alpha}(\bar{R}_+^n)), \quad \sigma_n \rightarrow \sigma \quad \text{in } C([0, T], C^{2+\alpha}(R^{n-1})),$$

$$\begin{aligned} \mathbf{D}_t^\nu \sigma_n &\rightarrow \mathbf{D}_t^\nu \sigma \quad \text{in } C([0, T], C^{1+\alpha}(R^{n-1})) \\ \mathbf{D}_t^\nu u_n &\rightarrow \mathbf{D}_t^\nu u \quad \text{in } C([0, T], C^{1+\alpha}(\bar{R}_+^n)). \end{aligned} \quad (3.83)$$

Moreover, the functions u and σ satisfy to (3.1)–(3.4) and (3.7) with $g_0, g_1 \equiv 0$. Thus, Theorem 3.1 in the case of (3.6) and (3.8) is proved.

To get Theorem 3.1 in the general case, we search the solution of problem (3.1)–(3.4) in the form

$$u(x, t) = u_1(x, t) + u_2(x, t),$$

where $u_1(x, t)$ is a solution of the Dirichlet problem:

$$\Delta_x u_1 = g_0(x, t) \text{ in } R_{+T}^n; \quad u_1 = g_1(x', t) \text{ on } \bar{R}_T^{n-1}; \quad (3.84)$$

and the function u_2 satisfies (3.1)–(3.4) with $g_0, g_1 \equiv 0$ and the new right-hand side $g_2 := g_2 + b \frac{\partial u_1}{\partial x_n}$.

Finally, applying of the results of Chapter 3 [26] to (3.84) and of Theorem 3.1 in the case of (3.8) to u_2 completes the proof of Theorem 3.1 in the general case.

4. The solvability of the linear problem

As it follows from (2.23)–(2.26), the linear system corresponding to the non-linear system (2.30), where the right-hand sides are some fixed functions, has the form:

$$\Delta_x w = f_0(x, t) \text{ in } \Omega_T; \quad (4.1)$$

$$w = 0 \text{ on } \Gamma_{0T}, \quad \sigma(\omega, 0) = 0 \text{ on } \Gamma; \quad (4.2)$$

$$\mathbf{D}_t^\nu \sigma + \mu \frac{\partial w}{\partial \bar{n}(\omega)} = f_1(x', t) \text{ on } \Gamma_T; \quad (4.3)$$

$$w = a(x)\sigma \text{ on } \Gamma_T, \quad (4.4)$$

where $a(x)$, $f_0(x, t)$, $f_1(x', t)$ are given functions such that [see Corollary 2.2 and (2.15)],

$$f_0(x, 0), f_1(x', 0) = 0; \quad (4.5)$$

$$a(x) \in C^{2+\alpha}(\bar{\Omega}), \quad a(x) > 0. \quad (4.6)$$

One can easily conclude from (4.5) and (4.1)–(4.4) that

$$w(x, 0) = 0 \text{ in } \bar{\Omega}. \quad (4.7)$$

Theorem 4.1. *Let $\alpha, \nu \in (0, 1)$; $a(x)$, $f_0(x, t)$, $f_1(x', t)$ satisfy conditions (4.5) and (4.6); $f_0 \in C([0, T], C^\alpha(\bar{\Omega}))$, $f_1 \in C([0, T], C^{1+\alpha}(\Gamma))$; Γ and Γ_0 meet the requirements of Theorem 2.1; μ be a positive constant. Then for some small T , there exists a unique solution (w, σ) of problem (4.1)–(4.4) such that*

$$\begin{aligned} & \|w\|_{C([0, T], C^{2+\alpha}(\bar{\Omega}))} + \|\mathbf{D}_t^\nu \sigma\|_{C([0, T], C^{1+\alpha}(\Gamma))} + \|\sigma\|_{C([0, T], C^{2+\alpha}(\Gamma))} \\ & \leq C[\|f_0\|_{C([0, T], C^\alpha(\bar{\Omega}))} + \|f_1\|_{C([0, T], C^{1+\alpha}(\Gamma))}], \end{aligned} \quad (4.8)$$

where the positive constant C depends on the measure of Ω , $\|a\|_{C^{2+\alpha}(\bar{\Omega})}$, and is independent of the functions f_0 and f_1 .

Using the results of [26], we can reduce problem (4.1)–(4.4) to the similar one with $f_0 = 0$ and a new function $f_1 \in C([0, T], C^{1+\alpha}(\Gamma))$. Indeed, let

$$w(x, t) = w_1(x, t) + w_2(x, t), \quad (4.9)$$

where $w_1(x, t)$ is the solution of the Dirichlet problem

$$\Delta_x w_1 = f_0 \quad \text{in } \Omega_T, \quad w_1|_{\Gamma_{0T}} = 0, \quad w_1|_{\Gamma_T} = 0. \quad (4.10)$$

Problem (4.10), as it follows from Chapter 3 [26], has a unique solution:

$$\|w_1\|_{C([0, T], C^{2+\alpha}(\bar{\Omega}))} \leq C\|f_0\|_{C([0, T], C^\alpha(\bar{\Omega}))}; \quad w_1(x, 0) = 0 \quad \text{in } \bar{\Omega}. \quad (4.11)$$

As for the function $w_2(x, t)$, we have

$$\Delta_x w_2 = 0 \quad \text{in } \Omega_T; \quad w_2|_{\Gamma_{0T}} = 0, \quad w_2|_{\Gamma_T} = a(x)\sigma; \quad (4.12)$$

$$\mathbf{D}_t^\nu \sigma + \mu \frac{\partial w_2}{\partial \bar{n}_\omega} = f_1(x', t) - \mu \frac{\partial w_1}{\partial \bar{n}_\omega} \equiv \bar{f}_1 \quad \text{on } \Gamma_T; \quad \sigma(x', 0) = 0 \quad \text{on } \Gamma. \quad (4.13)$$

Because of (4.11) we can deduce:

$$\|\bar{f}_1\|_{C([0, T], C^{1+\alpha}(\bar{\Gamma}))} \leq C[\|f_0\|_{C([0, T], C^\alpha(\bar{\Omega}))} + \|f_1\|_{C([0, T], C^{1+\alpha}(\Gamma))}], \quad \bar{f}_1(x, 0) = 0. \quad (4.14)$$

Hence, it is enough to prove Theorem 4.1 for problem (4.12) and (4.13). This proof consists of two parts. The first of them is a solvability of problem (4.12) and (4.13), which is done using the regularizer for parabolic systems taken from [25]. The uniqueness of the solution which is the second part of the proof is deduced from a priori estimates.

To prove the solvability of system (4.12), (4.13), we reduce one similarly [5] to the nonlocal equation:

$$\mathcal{L}\sigma = \bar{f}_1 \quad \text{on } \Gamma_T, \quad \sigma|_{t=0} = 0 \quad \text{on } \Gamma. \quad (4.15)$$

The operator $\mathcal{L}$ is built as follows. Let σ be given in Eq. (4.12). Then equations in (4.12) formulate the Dirichlet problem for the functions w_2 . The solution of this Dirichlet problem can be used in Eq. (4.13). Thus, $\mathcal{L}\sigma$ is given by the left-hand side of (4.13).

A nonlocal character of system (4.15) causes difficulties requiring technical tricks related to a suitable localization. As suggested in either Chapter 4 from [25] or Section 4 from [5], we introduce the two collections of open sets: $\{\omega^m\}$ and $\{\Omega^m\}$, such that

$$\bar{\omega}^m \subset \Omega^m \subset \bar{\Omega}, \quad \cup_m \omega^m = \cup_m \Omega^m = \bar{\Omega}, \\ \omega^m = B_{r/2}(x^m) \cap \bar{\Omega}; \quad \Omega^m = B_r(x^m) \cap \bar{\Omega}$$

with $m = 1, \dots, N_0$; and $B_r(x^m)$, $B_{r/2}(x^m)$ are balls with the center in x^m and the radii of r and $r/2$, correspondingly. Denote by $\Gamma^m = \Gamma \cap B_r(x^m)$, and $\Gamma_0^m = \Gamma_0 \cap B_r(x^m)$. The index m belongs to one of two sets:

$$m \in M \quad \text{if } \bar{\Omega}^m \cap \Gamma = \emptyset \quad \text{and} \quad m \in N \quad \text{if } \bar{\omega}^m \cap \Gamma \neq \emptyset.$$

The covering $\{\omega^m\}$ and $\{\Omega^m\}$ define a partition of unity for the domain Ω . Let $\xi^m : \Omega \rightarrow [0, 1]$ be a smooth function such that

$$\xi^m = 1, \quad \text{if } x \in \omega^m, \quad \xi^m = 0, \quad \text{if } x \in \Omega \setminus \Omega^m, \quad \xi^m \in (0, 1), \quad \text{if } x \in \Omega^m \setminus \omega^m,$$

and $|\nabla^{|l|}\xi^m| \leq \text{const.}r^{-|l|}$, $1 \leq \sum_m (\xi^m)^2 \leq M_0$. Using the functions ξ^m , we define the function

$$\eta^m = \frac{\xi^m}{\sum_j (\xi^j)^2}.$$

From the properties of the functions ξ^m it follows that the functions η^m vanish for $x \in \Omega \setminus \Omega^m$; and in addition, $|\nabla^{|l|}\eta^m| \leq \text{const.}r^{-|l|}$.

The functions $\eta^m \xi^m$ define the partition of unity by the following formula

$$\sum_m \eta^m \xi^m = 1.$$

For each $m \in N$, we pick out one point $x^m \in \omega^m \cap \Gamma$ which will be the origin of the local coordinate system. The description of this system can be found in Chapter 4 [25].

We define local coordinate systems connected with each point x^m , $m \in N$. Let the curve Γ be described with $y_n = \Psi^m(y_1, \dots, y_{n-1})$ in a small vicinity of every point x^m , $m \in N$, and

$$y = B^{(m)}(x - x^m), \quad \left| \frac{\partial \Psi^m}{\partial y_i} \right| \leq \text{const. } r, \quad i = \overline{1, n-1}, \quad (4.16)$$

where $B^{(m)} = (B_{ij}^{(m)})_{i,j=\overline{1,n}}$ is an orthogonal matrix. To obtain the local “flatness” of the boundary, we change the variables as

$$z_i = y_i, \quad z_n = y_n - \Psi^m(y_1, \dots, y_{n-1}), \quad i = \overline{1, n-1}. \quad (4.17)$$

Thus, the variables $(x_1, \dots, x_n)$ are connected with $(z_1, \dots, z_n)$ by the mappings Z_m [see (4.16), (4.17)], such that

$$x = Z_m(z), \quad \text{and} \quad z = Z_m^{-1}(x), \quad m \in N.$$

First, we prove solvability (4.15) in the more smooth classes.

Definition 4.1. An operator

$$\mathcal{R} : C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\Gamma_T) \rightarrow C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\Gamma_T) \times C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\Gamma_T),$$

such that

$$\mathcal{R}\bar{f}_1 = \sum_{m \in N} \eta^m \sigma^m, \quad (4.18)$$

is called a regularizer, where $\sigma^m = \bar{\sigma}^m(Z_m^{-1}(x), t)$ is a solution of the following problems:

$$\Delta_z \bar{u}^m = 0 \quad \text{in } R_{+T}^n, \quad \bar{u}^m(z, t) = a^m \bar{\sigma}^m \quad \text{on } R_T^{n-1};$$

$$\mathbf{D}_t^\nu \bar{\sigma}^m + \mu \frac{\partial \bar{u}^m}{\partial \hat{n}} = \bar{f}_1^m(z, t) \quad \text{on } R_T^{n-1}, \quad \bar{\sigma}^m(z, 0) = 0 \quad \text{in } R^{n-1}, \quad (4.19)$$

where $\hat{n} := \hat{n}(z)$ is the unit vector with the coordinates $\{0, \dots, 0, -1\}$ in the plane R_+^n , i.e. $\frac{\partial}{\partial \hat{n}} = -\frac{\partial}{\partial z_n}$. Here

$$\bar{f}_1^m(z, t) = \bar{f}_1^m(Z_m(z), t), \quad \bar{f}_1^m(x, t) = \bar{f}_1(x, t) \xi^m(x), \quad a^m = a(x^m) > 0.$$

Note that problem (4.19) is studied in Sect. 3.

The operator $\mathcal{R}$ enables to construct an inverse operator to $\mathcal{L}$ by the methods from Section 4 [25] or Section 4 [5]. First of all, we need the following result.

Lemma 4.1. *Let α , ν and Γ_0 , Γ satisfy conditions of Theorem 4.1; (4.5) and (4.6) hold; $f_0 \equiv 0$, $f_1 \in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\Gamma_T)$. Then*

$$\mathcal{L}\mathcal{R}\bar{f}_1 = \bar{f}_1 + \mathcal{T}\bar{f}_1,$$

and the norm of the operator $\mathcal{T}$ is small and controlled by a quantity $C(r, T)$. If the time interval and r tend to zero, $C(r, T)$ vanishes.

Note that conditions of Lemma 4.1 mean that $f_1 \equiv \bar{f}_1$.

Since the proof of Lemma 4.1 is standard (see Section 4 [5]), although with extended calculations and based on the results of Theorem 3.1, we omit these arguments here.

The results of Lemma 4.1 provides existence of an element σ which satisfies (4.15) and $\sigma \in C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\Gamma_T)$, $\mathbf{D}_t^\nu \sigma \in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\Gamma_T)$. Then the existence of the functions $w_2(x, t)$ from (4.12) follows from Chapter 3 [26]:

$$\|w_2(\cdot, t)\|_{C^{2+\alpha}(\bar{\Omega})} \leq \text{const.} \|\sigma(\cdot, t)\|_{C^{2+\alpha}(\Gamma)}, \quad \forall t \in [0, T]; \quad (4.20)$$

$$\sum_{|l|=1}^2 \langle D_x^l w_2 \rangle_{t, \Gamma_T}^{(\frac{\alpha}{2}\nu)} \leq C \|\sigma\|_{C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\Gamma_T)}. \quad (4.21)$$

Moreover, due to existence of $\mathbf{D}_t^\nu \sigma$ and the last condition in (4.12), we can consider the auxiliary Dirichlet problem for the function $\mathbf{D}_t^\nu w_2$:

$$\Delta_x \mathbf{D}_t^\nu w_2 = 0 \quad \text{in } \Omega_T; \quad \mathbf{D}_t^\nu w_2|_{\Gamma_0 T} = 0, \quad \mathbf{D}_t^\nu w_2|_{\Gamma_T} = a(x) \mathbf{D}_t^\nu \sigma. \quad (4.22)$$

Finally, we apply Theorems 8.33 and 8.34 [15] to problems (4.12) and (4.22) and conclude

$$\begin{aligned} \|w_2(\cdot, t)\|_{C^{1+\alpha}(\bar{\Omega})} &\leq C \|\sigma(\cdot, t)\|_{C^{1+\alpha}(\Gamma)}, \\ \|\mathbf{D}_t^\nu w_2(\cdot, t)\|_{C^{1+\alpha}(\bar{\Omega})} &\leq C \|\mathbf{D}_t^\nu \sigma(\cdot, t)\|_{C^{1+\alpha}(\Gamma)} \quad \forall t \in [0, T]. \end{aligned} \quad (4.23)$$

Thus, we can infer from (4.20), (4.21) and (4.23) that

$$\begin{aligned} \|w_2\|_{C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\bar{\Omega}_T)} + \|\mathbf{D}_t^\nu w_2\|_{C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{\Omega}_T)} &\leq C \left[\|\sigma\|_{C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\Gamma_T)} \right. \\ &\quad \left. + \|\mathbf{D}_t^\nu \sigma\|_{C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\Gamma_T)} \right]. \end{aligned} \quad (4.24)$$

Hence, Lemma 4.1 and estimate (4.24) lead to the following result.

Lemma 4.2. *Let the conditions of Lemma 4.1 hold; then there is a solution (w_2, σ) to problem (4.12), (4.13), and $w_2 \in C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\bar{\Omega}_T)$, $\mathbf{D}_t^\nu w_2 \in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{\Omega}_T)$, and $\sigma \in C^{2+\alpha, \frac{2+\alpha}{2}\nu}(\Gamma_T)$, $\mathbf{D}_t^\nu \sigma \in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\Gamma_T)$.*

To get the same results in the space $C([0, T], C^{k+\alpha}(\bar{D}_T))$, we repeat the arguments from Subsection 3.3 [see (3.80)–(3.83)]. In other words, we consider the sequence

$$\bar{f}_{1n} \in C^{1+\alpha, \frac{1+\alpha}{2}\nu}(\bar{\Gamma}_T) : \bar{f}_{1n} \rightarrow \bar{f}_1 \text{ in } C([0, T], C^{1+\alpha}(\Gamma)) \quad \bar{f}_{1n}|_{t=0} = 0.$$

Because of Lemmas 4.1 and 4.2, we can construct the corresponding solution (w_{2n}, σ_n) :

$$w_{2n} \rightarrow w_2 \text{ in } C([0, T], C^{2+\alpha}(\bar{\Omega})); \quad \sigma_n \rightarrow \sigma \text{ in } C([0, T], C^{2+\alpha}(\Gamma));$$

$$\mathbf{D}_t^\nu \sigma_n \rightarrow \mathbf{D}_t^\nu \sigma \text{ in } C([0, T], C^{1+\alpha}(\Gamma)),$$

where (w_2, σ) is a solution of problem (4.12), (4.13).

Further we get the coercive estimate of the solution (w_2, σ) that provides the one-to-one solvability of problem (4.12) and (4.13). Repeating the standard Schauder technique and using Theorem 3.1 in the case $g_0 \neq 0$, we obtain the following result.

Lemma 4.3. *Let conditions of Lemma 4.1 hold. Then*

$$\begin{aligned} & \|w_2\|_{C([0, T], C^{2+\alpha}(\bar{\Omega}))} + \|\mathbf{D}_t^\nu \sigma\|_{C([0, T], C^{1+\alpha}(\Gamma))} + \|\sigma\|_{C([0, T], C^{2+\alpha}(\Gamma))} \\ & \leq C \|\bar{f}_1\|_{C([0, T], C^{1+\alpha}(\Gamma))}, \end{aligned}$$

with the constant independent of $\bar{f}_1$ and dependent on measures Ω , Γ , Γ_0 and $\|a\|_{C^{2+\alpha}(\Gamma)}$.

Thus, the proof of Theorem 4.1 is deduced from the results of Lemmas 4.1–4.3.

5. The nonlinear problem: proof of Theorem 2.1

We introduce the functional spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ (such that $\mathbf{z} \in \mathcal{H}_1$ and $\mathcal{F}\mathbf{z} \in \mathcal{H}_2$),

$$\begin{aligned} \mathcal{H}_1 &= C_0([0, T], C^{2+\alpha}(\bar{\Omega})) \times C_0([0, T], C^{2+\alpha}(\Gamma)) \times C_0([0, T], C^{1+\alpha}(\Gamma)); \\ \mathcal{H}_2 &= C_0([0, T], C^\alpha(\bar{\Omega})) \times C_0([0, T], C^{2+\alpha}(\Gamma_0)) \times C_0([0, T], C^{1+\alpha}(\Gamma)) \\ &\quad \times C_0([0, T], C^{2+\alpha}(\Gamma)); \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{z}\|_{\mathcal{H}_1} &= \|(w_2, \sigma)\|_{\mathcal{H}_1} = \|w_2\|_{C([0, T], C^{2+\alpha}(\bar{\Omega}))} + \|\sigma\|_{C([0, T], C^{2+\alpha}(\Gamma))} \\ &\quad + \|\mathbf{D}_t^\nu \sigma\|_{C([0, T], C^{1+\alpha}(\Gamma))}; \quad \|\mathcal{F}\mathbf{z}\|_{\mathcal{H}_2} = \|(\mathcal{F}_0(\mathbf{z}), 0, \mathcal{F}_1(\mathbf{z}), 0)\|_{\mathcal{H}_2} \\ &= \|\mathcal{F}_0(\mathbf{z})\|_{C([0, T], C^\alpha(\bar{\Omega}))} + \|\mathcal{F}_1(\mathbf{z})\|_{C([0, T], C^{1+\alpha}(\Gamma))}. \end{aligned}$$

Based on (2.30) and representations (2.27) and (2.28), we can rewrite problem (2.23)–(2.26) in the form

$$\mathcal{A}\mathbf{z} = \mathcal{F}\mathbf{z} = F(x, t) + \mathcal{F}^1(\mathbf{z}), \quad (5.1)$$

where $\mathbf{z} = (w, \sigma)$, $\mathcal{A}$ is the linear operator which has been studied in Sect. 4, $\mathcal{A} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$; the vector $F(x, t)$ is constructed with initial data; $\mathcal{F}^1(\mathbf{z})$ contains the elements described in Corollary 2.2. Moreover, as it follows from (2.31)

$$F(x, 0) = 0, \quad \mathcal{F}^1(\mathbf{z})|_{t=0} = 0. \quad (5.2)$$

As the operator $\mathcal{A}$ satisfies the conditions of Theorem 4.1, nonlinear problem (5.1) can be represented as

$$\mathbf{z} = \mathcal{A}^{-1}F(x, t) + \mathcal{A}^{-1}\mathcal{F}^1(\mathbf{z}) \equiv \mathcal{P}(\mathbf{z}).$$

Lemma 5.1. *Let $B_d, B_d \subset \mathcal{H}_1$, be a ball with the center located in the origin and the radius of d . For $\mathbf{z} \in B_d$ the following estimates hold*

$$\|\mathcal{F}(0)\|_{\mathcal{H}_2} \leq C_1(T); \quad (5.3)$$

$$\|\mathcal{F}(\mathbf{z}_1) - \mathcal{F}(\mathbf{z}_2)\|_{\mathcal{H}_2} \leq C_2(T, d)\|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathcal{H}_1}, \quad (5.4)$$

where constants $C_1(T)$ and $C_2(T, d)$ vanish if $T, d \rightarrow 0$.

The proof of Lemma 5.1 is based on the results of Theorem 4.1, Corollary 2.2, relations (5.2) and representations (2.27) and (2.28); and it repeats all the arguments from Section 5 [41].

Inequalities (5.3) and (5.4) imply that for sufficiently small T and d the nonlinear operator $\mathcal{P}(\mathbf{z})$ satisfies conditions of the fixed point theorem for a contraction operator. Hence, the fixed point of the operator is the solution of problem (2.12), and this completes the proof of Theorem 2.1.

In Theorem 2.1 there is a loss of regularity: while Γ is assumed from $C^{3+\alpha}$, the free boundary $\Gamma(t)$ is only in $C^{2+\alpha}$ for $t > 0$. This is caused by parametrization of the free boundary $\Gamma(t)$ [see (2.1)] as a graph in the normal direction over the initial surface Γ . Note that these requirements on the smoothness of the boundaries can be weakened. To this end, it is enough to choose parametrization of the free boundary in form (2.4), (2.1) from [7], and repeat all the arguments from Sects. 2–5. Thus, one can get the following result.

Remark 5.1. Let $\Gamma_0, \Gamma \in C^{2+\alpha}$, $\varphi(x) \in C^{2+\alpha}(\Gamma_0)$, $\varphi(x) > 0$, the initial concentration $v(x, 0)$ be given by (2.13). Then the results of Theorem 2.1 hold.

6. Numerical solution

6.1. Solving scheme

In this section we present and analyze numerical results for the problem (1.1)–(1.4) in R^2 . Note that it may be considered as two separate problems. To illustrate this assertion, we rewrite (1.1)–(1.4) in the form

$$\Delta_y p = 0 \quad \text{in } \Omega(t), \quad t \in [0, T], \quad \Omega(t)|_{t=0} = \Omega; \quad (6.1)$$

$$p = 0 \quad \text{on } \Gamma(t); \quad (6.2)$$

$$p = \varphi(y) \quad \text{on } \Gamma_0 T = \Gamma_0 \times [0, T]; \quad (6.3)$$

$$\langle \mathbf{D}_t^\nu \Gamma(t), n_t \rangle = -\mu \frac{\partial p}{\partial n_t} \quad \text{on } \Gamma(t), \quad \nu \in (0, 1); \quad (6.4)$$

$$\Gamma(t)|_{t=0} = \Gamma. \quad (6.5)$$

The first problem is the elliptic one defined on $\Omega(t)$, and it consists of (6.1) with the boundary conditions (6.2), (6.3), while the second problem is the initial

value problem (6.4) on $\Gamma(t)$ with the initial condition (6.5). These problems are connected through the reciprocal usage of their solutions. Thus, the numerical approach for solving quasistationary problem (1.1)–(1.4) may be constructed as the splitting scheme, that is, at each time point problems (6.1)–(6.3) and (6.4), (6.5) are solved separately with the mutual exchange of their solutions on the next time step.

The elliptic problem (6.1)–(6.3) is solved by the finite element method implemented using the finite element library DOLFIN [31]. At each time point, t_i , we built a new mesh in accordance with the solution of (6.4), (6.5) which provides $\Gamma(t)|_{t=t_i}$.

To solve the problem (6.4), (6.5), we start from reducing the moving boundary $\Gamma(t)$ to the fixed domain through the mapping (2.1) using the initial boundary, $\Gamma = y_0(\omega)$, and the increment, $\rho(\omega, t)$, where $y_0(\omega) = (y_1^0(\omega), y_2^0(\omega))$, $\omega = [0, 1]$:

$$\Gamma(t) = y_0(\omega) + \rho(\omega, t)n_0(\omega), \quad n_0(\omega) = (n_1^0, n_2^0). \quad (6.6)$$

Here $n_0(\omega)$ denotes the outward normal vector to the boundary Γ . Therefore, using (6.6), the problem (6.4), (6.5) is transformed to the form

$$\mathbf{D}_t^\nu \rho(\omega, t) = -\frac{\mu}{\langle n_0(\omega), n_t \rangle} \frac{\partial \rho}{\partial n_t} \quad \omega \in [0, 1], \quad \nu \in (0, 1); \quad (6.7)$$

$$\rho(\omega, 0) = 0. \quad (6.8)$$

Thus, instead of solving (6.4), (6.5) we seek the solution of (6.7), (6.8) on a fixed spatial domain $\omega \in [0, 1]$.

Solving for (6.7), (6.8) is performed using the predictor-corrector scheme suggested in [10]. However, on the predictor stage we use analytical approximation similar to (2.18):

$$\rho^{pr}(\omega, t_i) = -\frac{t_i^\nu}{\Gamma(\nu + 1)} \frac{\partial p_0}{\partial n_0(\omega)}. \quad (6.9)$$

The correction step is performed as it is given in [11].

The solving algorithm for the coupled problem (6.1)–(6.3), and (6.7), (6.8) is listed below.

Algorithm

Given Ω , Γ , μ , $\phi(y)$, $n_0(\omega)$, the temporal grid $t_i = i\Delta t : 0, \overline{N}$, $\Delta t = T/N$.

(1) solve for the problem (6.1)–(6.3) and compute $\frac{\partial p_0}{\partial n_0(\omega)}$

(2) for each $i = \overline{1, N}$ perform

- compute $\rho^{pr}(\omega, t_i)$ from (6.9)
- update Γ (and Ω) according to (6.6) using $\rho^{pr}(\omega, t_i)$
- solve for the problem (6.1)–(6.3) and compute $\frac{\partial p_i^{pr}}{\partial n_{t_i}^{pr}}$
- compute corrected $\rho(\omega, t_i)$ as

$$\rho(\omega, t_i) = -\frac{\mu}{\langle n_0, n_{t_i}^{pr} \rangle} \frac{\Delta t^\nu}{\Gamma(2 + \nu)} \left(\sum_{k=0}^{i-1} c_{k,i} \frac{\partial p_k}{\partial n_{t_k}} + \frac{\partial p_i^{pr}}{\partial n_{t_i}^{pr}} \right), \quad (6.10)$$

$$c_{k,i} = \begin{cases} (1+\nu)i^\nu - i^{1+\nu} + (i-1)^{1+\nu}, & \text{if } k=0; \\ (i-k+1)^{1+\nu} - 2(i-k)^{1+\nu} + (i-k-1)^{1+\nu}, & \text{if } 0 < k < i-1; \\ 1, & \text{if } k=i; \end{cases} \quad (6.11)$$

- update Γ (and Ω) according to (6.6) using $\rho(\omega, t_i)$
- solve for the problem (6.1)–(6.3) and compute $\frac{\partial p_i}{\partial n_{t_i}}$

6.2. Numerical tests

Here we present results of the simulations with different $\nu \in (0, 1)$, but using the same initial spatial domain. The problem is specified as follows. Let $\Omega(t)|_{t=0}$ be bounded by two elliptic boundaries $\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} - 1 = 0$ where $a = a_0$, $b = b_0$ for the external boundary Γ_0 , and $a = a_{t_0}$, $b = b_{t_0}$ for the internal boundary $\Gamma(t)|_{t=0}$. Results presented are obtained using $a_0 = 5.0$, $b_0 = 6.5$, $a_{t=0} = 2.0$, $b_{t=0} = 2.6$. Let $\mu = 1$ on $\Omega(t)$ and $p = 1$ on Γ_0 . Because of the symmetry of the problem, it is solved on one quarter of the domain $\Omega(t)$ corresponding to the second quadrant. Note that computation on the normal derivatives is performed in the vicinity of the boundary that is also supported by the theoretical studies [7].

Numerical experiments are performed for three different values of ν , namely, $\nu_1 = 0.25$, $\nu_2 = 0.5$, $\nu_3 = 0.75$. Since we aim to analyze the dynamics of the moving boundary for different values of ν , we have chosen time intervals which allow us to obtain similar final solutions (see Fig. 1). Naturally, smaller values of ν require longer time to change the boundary. Therefore, we consider

FIGURE 1. Final states for the internal moving boundary calculated for different ν

(a) $\nu = 0.25$ (b) $\nu = 0.5$ (c) $\nu = 0.75$

FIGURE 2. Dynamics of the free boundary for **a** $\nu_1 = 0.25$, **b** $\nu_2 = 0.5$, **c** $\nu_3 = 0.75$. Time points normed by the final time T_i for each ν_i , that is, $T_1 = 16.5$, $T_2 = 4.2$, $T_3 = 2.85$, correspond to $0.0 T_i$, $0.01 T_i$, $0.1 T_i$, $0.25 T_i$, $0.5 T_i$, $0.75 T_i$, T_i

normed time which relates true time to the final time T_i for each ν_i . Considering normed time for all three cases, Fig. 2 demonstrates that ν controls dynamics of the free boundary $\Gamma(t)$ in the following way. On the initial phase proximity of ν to 0 causes more rapid movement of the boundary than for ν close to 1. However, for more distant time points the effect is opposite. The boundary moves slowly for small ν and much faster for ν approaching 1.

7. Appendix: Proof of Proposition 3.1

Let $a = b = 1$. One can easily check using (3.23) and (3.33) that

$$u(x', x_n, t)|_{x_n=0} = \int_{R^{n-1}} dy' \sigma(x' - y', t) K(y', 0) = \sigma(x', t). \quad (7.1)$$

Thus, we have shown that the functions u and σ satisfy boundary condition (3.2).

Next, we calculate $\frac{\partial u}{\partial x_n}|_{x_n=0}$. Using (3.23), we can obtain

$$\begin{aligned} \frac{\partial u}{\partial x_n}|_{x_n=0} &= \int_0^t d\tau \int_{R^{n-1}} dy' \frac{\partial G_1}{\partial x_n}(y', \tau, x_n)|_{x_n=0} [g_2(x' - y', t - \tau) - g_2(x', t - \tau)] \\ &\quad + \int_0^t d\tau g_2(x', t - \tau) \int_{R^{n-1}} dy' \frac{\partial G_1}{\partial x_n}(y', \tau, x_n)|_{x_n=0}. \end{aligned} \quad (7.2)$$

Representation (3.25) together with (3.46) lead to

$$\frac{\partial G_1}{\partial x_n}(y', \tau, x_n)|_{x_n=0} = \nu \int_0^\infty d\varrho \tau^{-1-\nu} W(-\varrho \tau^{-\nu}; -\nu, 1 - \nu) \frac{\partial K}{\partial \varrho}(y', \varrho). \quad (7.3)$$

After that we can calculate $\int_{R^{n-1}} \frac{\partial G_1}{\partial x_n}(y', \tau, x_n)|_{x_n=0} dy'$. To this end, we use representation (7.3), (3.44) and equality (3.45) which provide

$$\int_{R^{n-1}} \frac{\partial G_1}{\partial x_n}(y', \tau, x_n)|_{x_n=0} dy' = 0. \quad (7.4)$$

Combining (7.2)–(7.4), we can represent $\frac{\partial u}{\partial x_n}|_{x_n=0}$ as

$$\begin{aligned} \frac{\partial u}{\partial x_n}|_{x_n=0} &= \nu \int_0^t d\tau \int_{R^{n-1}} dy' [g_2(x' - y', t - \tau) - g_2(x', t - \tau)] \int_0^\infty d\varrho \tau^{-1-\nu} \\ &\quad \times \frac{\partial K}{\partial \varrho}(y', \varrho) W(-\varrho \tau^{-\nu}; -\nu, 1 - \nu). \end{aligned} \quad (7.5)$$

To get representation for $\mathbf{D}_t^\nu \sigma(x', t)$, we use the following properties of the fractional derivative (see Lemma 2.10 and formula (2.4.10) in [20]):

i

$$\mathbf{D}_t^\nu \varphi(t) = \partial_t^\nu \varphi(t) \quad \text{if} \quad \varphi(0) = 0, \quad \nu \in (0, 1), \quad (7.6)$$

where $\partial_t^\nu \varphi(t)$ is given by (3.29).

ii If the functions $\varphi_i(t)$, $i = 1, 2$, are bounded in $[0, T]$, and $\partial_t^\nu \varphi_1(t) \in L^1[0, T]$, then

$$\begin{aligned} \partial_t^\nu \int_0^t \varphi_1(t - \tau) \varphi_2(\tau) d\tau &= \int_0^t \varphi_2(t - \tau) \partial_t^\nu \varphi_1(\tau) d\tau \\ &\quad + \varphi_2(t) \lim_{z \rightarrow 0} I_z^{1-\alpha} \varphi_1(z), \quad \forall t \in [0, T], \end{aligned} \quad (7.7)$$

where $I_t^\nu \varphi_1(t)$ is defined in (3.28).

One can check that (7.6) and (7.7) hold if the functions φ and φ_i , $i = 1, 2$, depend on x' too. Because of property (3.34), we can rewrite the function $\sigma(x', t)$ as

$$\sigma(x', t) = \int_0^t d\tau \int_{R^{n-1}} [g_2(x' - y', t - \tau) - g_2(x', t - \tau)] G(y', \tau) dy' + I_t^\nu g_2(x', t). \quad (7.8)$$

Note that, inequality (3.35) means that $\sigma(x', 0) = 0$. Thus, equalities (7.6), (7.8) allow us to represent $\mathbf{D}_t^\nu \sigma(x', t)$ as

$$\mathbf{D}_t^\nu \sigma(x', t) = g_2(x', t) + \int_0^t d\tau \int_{R^{n-1}} [g_2(x' - y', t - \tau) - g_2(x', t - \tau)] \partial_\tau^\nu G(y', \tau) dy' + \int_{R^{n-1}} [g_2(x' - y', t) - g_2(x', t)] \lim_{z \rightarrow 0} I_z^{1-\nu} G(y', z) dy' \equiv \sum_{i=1}^3 \mathfrak{L}_i. \quad (7.9)$$

Further, we use the well-known formulae for differentiation and integration of the Wright functions (see, e.g. (8) and (9) in [38] or (11) in [21]):

$$\frac{d}{dx} W(x; -\beta, \delta) = W(x; -\beta, \delta - \beta); \quad (7.10)$$

$$\partial_t^\gamma [t^{\delta-1} W(-ct^{-\beta}; -\beta, \delta)] = t^{\delta-1-\gamma} W(-ct^{-\beta}; -\beta, \delta - \gamma); \quad (7.11)$$

$$I_t^\gamma [t^{\delta-1} W(-ct^{-\beta}; -\beta, \delta)] = t^{\delta-1+\gamma} W(-ct^{-\beta}; -\beta, \delta + \gamma), \quad \delta \geq 0, \quad \gamma > 0. \quad (7.12)$$

First of all, we calculate $\partial_\tau^\nu G(y', \tau)$. Equalities (7.10) and (7.11) give that

$$\partial_\tau^\nu G(y', \tau) = -\nu \int_0^\infty \tau^{-1} \frac{\partial}{\partial \varrho} [\varrho \tau^{-\nu} W(-\varrho \tau^{-\nu}; -\nu, 1 - \nu)] K(y', \varrho) d\varrho. \quad (7.13)$$

Then, we integrate by parts in (7.13) and take into account equality (3.33). Thus, we obtain

$$\begin{aligned} \partial_\tau^\nu G(y', \tau) &= \nu \int_0^\infty \tau^{-1-\nu} \varrho W(-\varrho \tau^{-\nu}; -\nu, 1 - \nu) \frac{\partial}{\partial \varrho} K(y', \varrho) d\varrho \\ &\quad + \nu \tau^{-1-\nu} \Pi_{j=1}^{n-1} \delta(-y_j) \lim_{\varrho \rightarrow 0} \varrho W(-\varrho \tau^{-\nu}; -\nu, 1 - \nu). \end{aligned} \quad (7.14)$$

Therefore, equalities (7.14) and (7.5) lead to

$$\mathfrak{L}_2 = \frac{\partial u}{\partial x_n} \Big|_{x_n=0}. \quad (7.15)$$

Applying (7.12) to the term $\mathfrak{L}_3$, we can get

$$I_t^{1-\nu} G(y', t) = \int_0^\infty t^{-\nu} W(-\varrho t^{-\nu}; -\nu, 1 - \nu) K(y', \varrho) d\varrho. \quad (7.16)$$

Let us consider the following integral:

$$d_j \equiv \int_{R_+^{n-2}} dy'' \int_0^{+\infty} y_j^\alpha \lim_{z \rightarrow 0} I_z^{1-\nu} G(y', z) | dy_j.$$

Representation (7.16) together with estimate (3.30) and change of variables (3.50) allow us to derive the estimate

$$0 \leq d_j \leq \lim_{z \rightarrow 0} z^\alpha \int_{R_+^{n-2}} dx'' \int_0^{+\infty} x_j^\alpha e^{-A \sum_{k=1}^{n-1} x_k} dx_j \int_0^\infty \eta^\alpha \times W(-\eta^\nu; -\nu, 1 - \nu) d\eta = 0. \quad (7.17)$$

Thus, inequality (7.17) together with the properties of the function $g_2(x', t)$ give

$$\mathfrak{L}_3 = 0. \quad (7.18)$$

Equalities (7.9), (7.15) and (7.18) imply that the functions u and σ given by (3.22) and (3.23) satisfy boundary condition (3.3). That completes the proof of Proposition 3.1.

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