Classical solvability of multidimensional two-phase Stefan problem for degenerate parabolic equations and Schauder's estimates for a degenerate parabolic problem with dynamic boundary conditions

To the memory of Professor B.V.Bazaliy

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Abstract. We consider multidimensional two-phase Stefan problem for degenerate parabolic equations of the porous medium type in classes of smooth functions. First we find a natural Hölder class for the Dirichlet boundary conditions in the initial boundary boundary problem for a degenerate parabolic equation of second order. This class then is used to obtain the Schauder estimates for a degenerate parabolic equation with dynamic boundary conditions. As a result we prove the existence locally in time of a smooth solution for Stefan problem for degenerate parabolic equations.

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1. Introduction and the main result

Classical solvability of Stefan problem for uniformly parabolic equations has been well studied—see, for example, papers [1–13] and the references therein. At the same time, as has long been known, the heat transfer model based on uniformly parabolic equations has some properties, which can not be observed in the reality. In particular, the infinite speed of propagation of disturbances takes place for uniformly parabolic equations. We also know that more accurate model of the heat transfer is the model on the base of degenerate parabolic equations such as the equations of the form

$$u_t(x, t) - \nabla(|u|^{m-1} \nabla u(x, t)) = f(x, t), \quad (1.1)$$

where $m > 1$. As it is known, a short formulation of classical Stefan problem for equation (1.1) is the equation

$$(\beta(u))_t - \nabla(|u|^{m-1} \nabla u(x, t)) = 0, \quad (1.2)$$

where $\beta(u)$ is the discontinuous function of the form

$$\beta(u) = \begin{cases} u, & u \leq 0, \\ u + k, & u > 0, \end{cases}$$

where $k > 0$ is the latent heat of fusion (crystallization) and Eq. (1.2) is considered in the sense of distributions. We stress that for quasilinear Eq. (1.2) the main unknown is, in fact, the interface $\{u = 0\}$. Outside of this interface the solution of (1.2) is smooth in view of well-known local theory of uniformly parabolic equations.

In its generalised formulation this problem was considered in a number of papers, from which we mention, for example, [14–20]. Note that the generalised formulation gives weak solutions of the problem and omit the question about properties of the interface $\{u = 0\}$.

As for the smooth solutions, in the most simple case of one spatial variable such problem for degenerate parabolic equations was considered in [21–25], where the existence of classical solutions was proved. The one-dimensional case is fundamentally different from the multidimensional setting. In the case of one spatial variable the interface $\{u = 0\}$ is reduced to a point at each moment of time. Therefore, there are no questions about its properties with respect to the spatial variables. At the same time the existence of the smooth interface for the solution of (1.2) in the multidimensional setting is still unknown.

However, this question is important, in our opinion, not only for studying of the melting-crystallisation mathematical models, where Stefan problem originates. It is well-known that Stefan problem arises as a mathematical model in very many fields of scientific investigations. We do not give broad examples of references to the relevant literature since they are very numerous. Let us mention however two areas of scientific research, where Stefan problem arises.

The first area is the mathematical biology. This includes the population theory, medicine, predator-pray models—see, for example [10, 26–29]. At the same time degenerate equations are increasingly used in models of mathematical biology [30, 31]. This leads to the consideration of Stefan type models for degenerate operators.

The second important area we mention is the area of mathematical finance including the mathematical models of stock markets. Here the optimal stopping problems for the corresponding stochastic processes are expressed by means of Stefan problem [32, 33]. It is known by now that many stochastic processes from mathematical finance obey degenerate stochastic differential equations. And again, the optimal stopping problems for such processes lead to the consideration of Stefan problem for degenerate operators.

Besides, there are two more points of interest to consider Stefan problem for degenerate equations in classes of smooth functions. In this paper we

are going to consider Stefan problem as a nonlinear equation in some Banach spaces and to apply a variant of classical inverse function theorem. So we need to consider the corresponding Frechet derivative of a nonlinear mapping, which is defined by Stefan problem. This leads to the investigation of a linear initial-boundary value problem with a dynamic boundary condition for linear degenerate parabolic equations. This linear problem is important by itself in our opinion. In recent years, the study of boundary value problems with dynamic boundary conditions is attracting increasing attention [34–41]. Note that to our knowledge there are no papers containing Schauder’s estimate of solutions to such problems for degenerate equations, which is the subject of our interest in this paper—Sect. 6, Theorem 6.2, Theorem 6.4. Finally, we use in this paper another new result—Theorem 2.7. Here we prove solvability and Schauder estimates for the solution of the inhomogeneous initial-boundary Dirichlet problem for a linear degenerate parabolic equation. Such a problem with zero boundary conditions has been studied previously in [42]. We essentially use the results of this paper.

Thus the aim of this paper is a proof of classical solvability of the Stefan problem of the type (1.2) for a degenerate equation in the multidimensional setting, that is, the proof of the existence of the smooth surface $\{u = 0\}$, and the proof of a smoothness of the solution up to the interface.

We now formulate the precise statement of Stefan problem in a more expanded than (1.2) form as it is custom in the theory of free boundary problems. Let Ω be a doubly connected domain in R^N , whose boundary consists of two smooth connected surfaces Γ^+ and Γ^- without self-intersections, $\partial\Omega = \Gamma^+ \cup \Gamma^-$. Suppose further, that Γ is a given smooth surface without self-intersections lying strictly between Γ^+ and Γ^- and separating the domain Ω into two doubly connected sub-domains Ω^+ and Ω^- , so that $\partial\Omega^\pm = \Gamma \cup \Gamma^\pm$. For a fixed $T > 0$ denote $\Omega_T = \Omega \times (0, T)$, $\Omega_T^\pm = \Omega^\pm \times (0, T)$, $\Gamma_T = \Gamma \times [0, T]$, $\Gamma_T^\pm = \Gamma^\pm \times [0, T]$.

Denote by S_T a smooth surface in the cylindrical domain Ω_T in space $(y, \tau) \in R^N \times [0, T]$ such that at $\tau = 0$ it coincides with Γ , $S_T \cap \{\tau = 0\} = \Gamma$. Let also S_T does not intersect surfaces $\Gamma_T^\pm$ and divides Ω_T into two sub-domains Q_T^+ and Q_T^- . The lateral boundaries of these domains consist of S_T and $\Gamma_T^\pm$ respectively. The surface S_T is unknown and has to be determined together with functions $u^+(y, \tau)$ and $u^-(y, \tau)$, which are defined in $Q_T^\pm$ respectively. The triple (S_T, u^+, u^-) must satisfy the following conditions (we denote the independent variables by (y, τ) in view of a subsequent change of variables):

$$\frac{\partial u^\pm}{\partial \tau} - \nabla_y(a^\pm |u^\pm|^{m-1} \nabla_y u^\pm) = 0, \quad (y, \tau) \in Q_T^\pm, \quad (1.3)$$

$$u^+(y, \tau) = u^-(y, \tau) = 0, \quad (y, \tau) \in S_T, \quad (1.4)$$

$$a^+ \sum_{i=1}^N \cos(\vec{N}, y_i) |u^+|^{m-1} u_{y_i}^+ - a^- \sum_{i=1}^N \cos(\vec{N}, y_i) |u^-|^{m-1} u_{y_i}^- = k \cos(\vec{N}, \tau), \quad (y, \tau) \in S_T, \quad (1.5)$$

$$u^\pm(y, \tau) = g^\pm(y, \tau), \quad (y, \tau) \in \Gamma_T^\pm, \quad (1.6)$$

$$u^\pm(y, 0) = u_0^\pm(y). \quad (1.7)$$

Here $m > 1$, $k > 0$, $a^\pm > 0$ are given constants, $\vec{N}$ is the unite normal to S_T with the direction into Q_T^+ , $g^+(y, \tau)$, $g^-(y, \tau)$, $u_0^+(y)$, $u_0^-(y)$ are given functions. We suppose that

$$\pm g^\pm(y, \tau) \geq \nu > 0, \quad (y, \tau) \in \Gamma_T^\pm; \quad \pm u_0^\pm(y) > 0, \quad y \in \Omega^\pm, \quad u_0^\pm(y) = 0, \quad y \in \Gamma, \quad (1.8)$$

where ν is some positive constant: here and below we denoted by the same symbols ν , μ , C all absolute constants or constants that depend only on the given data of the problem. Note that conditions (1.4), (1.5) arise from equation (1.2) and they are the three independent conditions at the unknown boundary S_T .

We introduce some weighted functional spaces to formulate the smoothness conditions on the data of the problem. First of all we use the standard Hölder spaces $H^{l+\delta}(\bar{\Omega}) \equiv C^{l+\delta}(\bar{\Omega})$, $\delta \in (0, 1)$, $l \in \mathbb{N} \cup \{0\}$ with the norm $|u|_{\bar{\Omega}}^{(l+\delta)}$. These spaces are introduced in [43]. We also use spaces $H^{l+\delta, \frac{l+\delta}{2}}(\bar{\Omega}_T) \equiv C^{l+\delta, \frac{l+\delta}{2}}(\bar{\Omega}_T)$ of functions of (y, τ) with the norm $|u|_{\bar{\Omega}_T}^{(l+\delta)}$. In [43] the surfaces of the corresponding classes are also defined. We assume that the surfaces Γ , $\Gamma^\pm$ belong to the following classes

$$\Gamma, \Gamma^\pm \in H^{4+\gamma} \quad (1.9)$$

with some $0 < \gamma < 1$. At the same time we suppose that the functions $g^\pm$ in (1.6) are such that

$$h^\pm(y, \tau) \equiv |g^\pm(y, \tau)|^{m-1} g^\pm(y, \tau) \in H^{4+\gamma, \frac{4+\gamma}{2}}(\Gamma_T^\pm). \quad (1.10)$$

Suppose further that $d^+(y)$ is a given function from $H^{2+\gamma}(\bar{\Omega}^+)$ such that it models the distance from a point $y \in \bar{\Omega}^+$ to the surface Γ , that is,

$$\nu \leq d^+(y)/\text{dist}(y, \Gamma) \leq \nu^{-1}. \quad (1.11)$$

Note that such function can be taken, for example, as the solution of the following problem

$$\begin{aligned} \Delta d^+(y) &= -1, \quad y \in \Omega^+, \\ d^+(y) &= 0, \quad y \in \Gamma, \quad d_+(y) = 1, \quad y \in \Gamma^+. \end{aligned}$$

Let $d^-(y)$ is an analogous function for the domain $\bar{\Omega}^-$.

Denote here and below

$$\alpha = \frac{m-1}{m} \in (0, 1), \quad (1.12)$$

where $m > 1$ is the exponent from equation (1.3).

We will use the spaces $C_s^{2+\gamma, \frac{2+\gamma}{2}}(\bar{\Omega}_T^\pm)$ from paper [42] (they are analogous to the corresponding spaces from [44]), where $0 < \gamma < \alpha$ is some exponent. Here we require

$$0 < \gamma < \alpha. \quad (1.13)$$

These spaces are defined in the following way. First we define the spaces $C_s^{2+\gamma}(R_{+T}^N)$ in the domain

$$R_{+T}^N = R_+^N \times [0, T], \quad R_+^N = \{x = (x', x_N) : x_N \geq 0, x' \in R^{N-1}\}. \quad (1.14)$$

Define a distance between points $x, \bar{x} \in R_+^N$ according to the following formula

$$s(x, \bar{x}) = \frac{|x - \bar{x}|}{x_N^{\alpha/2} + \bar{x}_N^{\alpha/2} + |x' - \bar{x}'|^{\alpha/2}}. \quad (1.15)$$

Define further a Hölder constant of a function $u(x, t)$ with respect to the variable x according to the distance from (1.15)

$$H_{s, R_{+T}^N}^\gamma(u) \equiv \sup_{(x, t), (\bar{x}, t) \in R_{+T}^N} \frac{|u(x, t) - u(\bar{x}, t)|}{s(x, \bar{x})^\gamma}. \quad (1.16)$$

Denote by $C_s^{\gamma, \gamma/2}(R_{+T}^N)$ the space of functions $u(x, t)$ with the finite norm

$$|u|_{C_s^{\gamma, \gamma/2}(R_{+T}^N)} \equiv |u|_{s, R_{+T}^N}^{(\gamma)} \equiv |u|_{R_{+T}^N}^{(0)} + H_{s, R_{+T}^N}^\gamma(u) + \langle u \rangle_{t, R_{+T}^N}^{(\frac{\gamma}{2})}, \quad (1.17)$$

where $\langle u \rangle_{t, R_{+T}^N}^{(\frac{\gamma}{2})}$ is the Hölder constant with respect to t with the exponent $\gamma/2$ of a function $u(x, t)$, $|u|_{R_{+T}^N}^{(0)} = \max_{R_{+T}^N} |u|$.

Define further the space $C_s^{2+\gamma, \frac{2+\gamma}{2}}(R_{+T}^N)$ as the Banach space of functions $u(x, t)$ with the finite norm

$$\begin{aligned} |u|_{C_s^{2+\gamma, \frac{2+\gamma}{2}}(R_{+T}^N)} &\equiv |u|_{s, R_{+T}^N}^{(2+\gamma)} \equiv |u|_{C_s^{\gamma, \gamma/2}(R_{+T}^N)} + \sum_{i=1}^N |u_{x_i}|_{C_s^{\gamma, \gamma/2}(R_{+T}^N)} \\ &+ |u_t|_{C_s^{\gamma, \gamma/2}(R_{+T}^N)} + \sum_{i,j=1}^N |x_N^\alpha u_{x_i x_j}|_{C_s^{\gamma, \gamma/2}(R_{+T}^N)}. \end{aligned} \quad (1.18)$$

Finally, the spaces $C_s^{\gamma, \gamma/2}(\bar{\Omega}_T^\pm)$ and $C_s^{2+\gamma, \frac{2+\gamma}{2}}(\bar{\Omega}_T^\pm)$ are defined as the spaces of functions $u(x, t)$ with the property, that in some neighborhood of Γ_T after the corresponding change of variables functions $u(x, t)$ belong to the space $C_s^{\gamma, \gamma/2}(R_{+T}^N)$ or to the space $C_s^{2+\gamma, \frac{2+\gamma}{2}}(R_{+T}^N)$ correspondingly. Out of some neighborhood of Γ_T such functions $u(x, t)$ belong to the standard spaces $C^{\gamma, \gamma/2}(\bar{\Omega}_T^\pm)$ or $C^{2+\gamma, \frac{2+\gamma}{2}}(\bar{\Omega}_T^\pm)$. In particular, for a function $u \in C_s^{2+\gamma, \frac{2+\gamma}{2}}(\bar{\Omega}_T^\pm)$ the following norm is finite

$$|(d^\pm)^\alpha u_{x_i x_j}|_{C_s^{\gamma, \gamma/2}(\bar{\Omega}_T^\pm)} < \infty, \quad i, j = \overline{1, N}. \quad (1.19)$$

In the case of functions $u(x)$ of the variable x only, $x \in \bar{\Omega}^\pm$, the spaces $C_s^\gamma(\bar{\Omega}^\pm)$ and $C_s^{2+\gamma}(\bar{\Omega}^\pm)$ are defined completely analogous.

We will use also some more general than the spaces $H^{l, l/2} \equiv C^{l, l/2}$ but standard anisotropic spaces of smooth functions. Namely, we will use the spaces

$$C^{l_1, l_2}(\bar{\Omega}_T^\pm), \quad C^{l_1, l_2}(\Gamma_T),$$

where l_1, l_2 are non-integer positive numbers. Such spaces are defined in [45], for example. They consist from functions that are smooth with respect to x and t up to the orders l_1 and l_2 respectively. This means that the following norm is finite

$$|u|_{C^{l_1, l_2}(\overline{\Omega}_T^\pm)} \equiv |u|_{\Omega_T^\pm}^{(l_1, l_2)} \equiv |u|_{\Omega_T^\pm}^{(0)} + \sum_{|\omega|=[l_1]} \langle \partial_x^\omega u \rangle_{x, \overline{\Omega}_T^\pm}^{(l_1 - [l_1])} + \langle \partial_t^{[l_2]} u \rangle_{t, \overline{\Omega}_T^\pm}^{(l_2 - [l_2])} < \infty,$$

where $\omega = (\omega_1, \dots, \omega_N)$ is a multiindex, $|\omega| = \omega_1 + \dots + \omega_N$,

$$\partial_x^\omega u = \frac{\partial^{|\omega|} u}{\partial x_1^{\omega_1} \dots \partial x_N^{\omega_N}}.$$

In fact, the functions from such spaces possess the property, that their derivatives with respect to x are smooth with respect to t and their derivatives with respect to t are smooth with respect to x . More precisely, if k_1 is a multiindex, that is, $k_1 = (k_{1,1}, k_{1,2}, \dots, k_{1,N})$ where $k_{1,i}$ are nonnegative integers, and k_2 is a nonnegative integer, then the function $\partial_x^{k_1} \partial_t^{k_2} u(x, t)$ belongs to the space $C^{m_1, m_2}(\overline{\Omega}_T^\pm)$ with the exponents $m_i = \mu l_i$, where

$$\mu = 1 - \frac{|k_1|}{l_1} - \frac{k_2}{l_2}$$

(see [45]), and also

$$|\partial_x^{k_1} \partial_t^{k_2} u|_{C^{m_1, m_2}(\overline{\Omega}_T^\pm)} \leq C |u|_{C^{l_1, l_2}(\overline{\Omega}_T^\pm)}. \quad (1.20)$$

In addition, we will use the spaces with zero at the bottom of their notation, that is, the spaces (compare [43], Chapter IV)

$$C_{0,s}^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm), C_0^{l_1, l_2}(\overline{\Omega}_T^\pm), C_{0,s}^{\gamma, \frac{\gamma}{2}}(\overline{\Omega}_T^\pm), C_0^{l_1, l_2}(\Gamma_T). \quad (1.21)$$

Such notation means the closed subspace of the corresponding space with the following property: all functions from this subspace vanish at $t = 0$ together with all their derivatives with respect to t up to the highest possible order in the corresponding space.

Let here and throughout below the exponent $\beta \in (0, 1)$ is connected to the exponent γ by the equality

$$\beta = \gamma \left(1 - \frac{\alpha}{2}\right). \quad (1.22)$$

In particular, we will use the spaces $C^{2+\beta-\alpha, 1+\frac{\gamma}{2}}(\overline{\Omega}_T^\pm)$ and $C^{2+\beta-\alpha, 1+\frac{\gamma}{2}}(\Gamma_T)$. These spaces consist of functions $u(x, t)$ with smoothness with respect to x up to the order $2 + \beta - \alpha$ and with smoothness with respect to t up to the order $1 + \gamma/2$, that is,

$$\sum_{i=1}^N \langle u_{x_i} \rangle_{x, \overline{\Omega}_T^\pm}^{(1+\beta-\alpha)} + \langle u_t \rangle_{t, \overline{\Omega}_T^\pm}^{(\frac{\gamma}{2})} < \infty. \quad (1.23)$$

In view of (1.22) it is easy to calculate that

$$1 + \frac{\gamma}{2} = \frac{2 + \beta - \alpha}{2 - \alpha}, \quad (1.24)$$

so we also use the following notation for the mentioned above spaces

$$\begin{aligned} C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\overline{\Omega}_T^\pm) &= C^{2+\beta-\alpha, 1+\frac{\gamma}{2}}(\overline{\Omega}_T^\pm), \\ C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T) &= C^{2+\beta-\alpha, 1+\frac{\gamma}{2}}(\Gamma_T). \end{aligned} \quad (1.25)$$

Further, we suppose that the initial conditions in (1.7) are such that

$$v_0^\pm(y) \equiv |u_0^\pm(y)|^{m-1} u_0^\pm(y) \in C_s^{2+\gamma'}(\overline{\Omega^\pm}), \quad (1.26)$$

where $\gamma' > \gamma$. Besides, we suppose that

$$\frac{\partial v_0^\pm(y)}{\partial \vec{n}} \geq \nu > 0, \quad y \in \Gamma, \quad (1.27)$$

where $\vec{n}$ is the normal vector to the surface Γ , which is directed into Ω^+ .

We will show below, that the free (unknown) boundary S_T in (1.4), (1.5) can be parameterized in terms of its deviation from the given surface $\Gamma_T = \Gamma \times [0, T]$. We follow to [2] to give the exact formulation. Let $\omega = (\omega_1, \dots, \omega_{N-1})$ is some local curvilinear coordinates in a domain Θ on Γ . In some small neighbourhood $\mathcal{N}$ in R^N of the surface Γ we introduce the coordinates (ω, λ) in the way that for any $x \in \mathcal{N}$ we have the following unique representation

$$x = x'(\omega) + \vec{n}(\omega)\lambda \equiv x(\omega) + \vec{n}(\omega)\lambda, \quad (1.28)$$

where $x'(\omega) = x(\omega)$ is the point in the domain Θ on the surface Γ with the coordinates ω , $\vec{n}(\omega)$ - the normal vector to Γ at the point $x(\omega)$ with the direction into Ω^+ . The coordinate $\lambda \in R$ means, in fact, the deviation of a point x from Γ , $\pm\lambda > 0$ for $x \in \Omega^\pm$. We assume that the neighbourhood $\mathcal{N}$ of the surface Γ is the set

$$\mathcal{N} = \{x \in \Omega : |\lambda(x)| < \gamma_0\},$$

where γ_0 is sufficiently small and will be chosen below.

Let $\rho(x', t) \equiv \rho(\omega, t)$ is a sufficiently small and regular function and $\rho(x', t) \equiv \rho(\omega, t)$ is defined on the surface Γ_T . Let us note that here and in what follows we use the notation $\rho(\omega, t)$ with the argument ω instead of $\rho(x', t)$ for all functions on the surface Γ if it does not cause ambiguity. We do that just for simplification of the notation, bearing in mind that in each local domain Θ on Γ we can introduce local coordinates ω . At the same time the coordinate λ in (1.28) does not depend on a choice of local coordinates ω .

We parameterize the unknown surface S_T with the help of the unknown function $\rho(\omega, t)$ as follows

$$S_T \equiv \Gamma_{\rho, T} = \{(x, t) \in \Omega_T : x = x' + \rho(x', t)\vec{n}(x') = x(\omega) + \rho(\omega, t)\vec{n}(\omega)\}, \quad (1.29)$$

where $x' \equiv x(\omega) \in \Gamma$. Note that this definition of the surface $S_T \equiv \Gamma_{\rho, T}$ does not depend on a choice of local coordinates ω in a particular local domain on Γ . Thus, the unknown function $\rho(\omega, t)$ means, in fact, deviation of the surface $\Gamma_{\rho, T} = S_T$ from the given surface Γ_T .

Along with Q_T^+ , Q_T^- in (1.3) we use the notation $\Omega_{\rho, T}^+ = Q_T^+$ and $\Omega_{\rho, T}^- = Q_T^-$ for the sub-domains of the domain Ω_T under consideration. Let further $\rho(x, t)$ is an extension of the function $\rho(\omega, t)$ from the surface Γ_T to the whole domain Ω_T to a function with support in the neighborhood $\mathcal{N}_T = \mathcal{N} \times [0, T]$ of the surface Γ_T , $\rho(x, t) = E\rho(\omega, t)$, where E is some fixed extension operator.

The way of such an extension will be described below and we will denote $\rho^\pm \equiv E\rho|_{\overline{\Omega_T^\pm}} \equiv E^\pm \rho$.

Define a mapping $e_\rho(x, t)$ from the domain $\overline{\Omega_T}$ on itself with the help of the formula $e_\rho : (x, t) \rightarrow (y, \tau)$, where, according to the notations of (1.28),

$$y = \begin{cases} x'(x) + \vec{n}(x'(x))(\lambda(x) + \rho(x, t)), & x \in \mathcal{N}, \\ x, & x \in \overline{\Omega} \setminus \mathcal{N}, \end{cases} \quad (1.30)$$

$$\tau = t,$$

or, with the help of the local coordinates ω ,

$$y = \begin{cases} x'(\omega(x)) + \vec{n}(\omega(x))(\lambda(x) + \rho(x, t)), & x \in \mathcal{N}, \\ x, & x \in \overline{\Omega} \setminus \mathcal{N}, \end{cases} \quad (1.31)$$

$$\tau = t.$$

Here $x'(x) \in \Gamma$, $\omega(x)$, $\lambda(x)$ are (ω, λ) -coordinates of a point x in the neighbourhood $\mathcal{N}$. Note that the definition of the mapping e_ρ does not depend on a choice of local coordinates ω on the surface Γ .

We choose γ_0 sufficiently small so that under the condition

$$|\rho|_{\Gamma}^{1+\beta} \leq 2\gamma_0 \quad (1.32)$$

the mapping e_ρ is a diffeomorphism of $\overline{\Omega_T}$ on itself and also the mapping e_ρ is a diffeomorphism of the domains $\overline{\Omega_T^\pm}$ on the domains $\overline{\Omega_{\rho, T}^\pm}$. Let us remark that the surface $\Gamma_{\rho, T}$ is exactly the image of the surface Γ_T under this mapping and the mapping $e_\rho(x, t)$ is the identical mapping out of the neighbourhood $\mathcal{N}_T$ of Γ_T .

About the exponents of the Hölder spaces under consideration we suppose that

$$0 < \gamma < \gamma' < 1, \quad \gamma < \min\{\alpha, 1 - \alpha\}. \quad (1.33)$$

Note that under our choice of γ the restriction $\gamma < \frac{\alpha}{1-\alpha/2}$ is also fulfilled. The last restriction was introduced in [42] at the studying of the homogeneous initial-boundary problem for degenerate equations. We need the restriction $\gamma < 1 - \alpha$ to have the inequality $1 + \beta - \alpha > 1 - \alpha > \gamma$ because this inequality implies that the first derivatives with respect to x of functions from the classes $C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}, C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega_T^\pm})$ are more smooth than $C^{\gamma, \gamma/2}, C_s^{\gamma, \gamma/2}$. At last, we need the restriction $\gamma < \alpha$ because under this restriction the expressions of the form $(d^\pm(x))^\alpha \eta(x, t)$ with a smooth function $\eta(x, t)$ belong to the corresponding space $C_s^{\gamma, \gamma/2}$.

We impose the following natural agreement conditions up to the first order at $\tau = 0$, $y \in \Gamma, \Gamma^\pm$ for problem (1.3)–(1.7). We suppose that the given functions $g^+(y, \tau)$, $g^-(y, \tau)$, $u_0^+(y)$, $u_0^-(y)$ satisfy

$$u_0^+(y)|_{\Gamma^+} = g^+(y, 0), \quad u_0^-(y)|_{\Gamma^-} = g^-(y, 0), \quad (1.34)$$

$$\nabla_y (a^\pm |u_0^\pm|^{m-1} \nabla_y u_0^\pm)|_{\Gamma^\pm} = \frac{\partial g^\pm(y, 0)}{\partial t}, \quad (1.35)$$

$$\frac{1}{k} \left(a^+ \frac{\partial v_0^+}{\partial \vec{n}} - a^- \frac{\partial v_0^-}{\partial \vec{n}} \right) \Big|_{\Gamma} = - (a^{\pm} |v_0^{\pm}|^{\alpha} \nabla^2 v_0^{\pm}) / \left(\frac{\partial v_0^{\pm}}{\partial \vec{n}} \right) \Big|_{\Gamma}, \quad (1.36)$$

where the functions $v_0^{\pm}$ are defined in (1.26) and satisfy (1.27). Note that condition (1.36) is necessary to agree conditions (1.3), (1.4), and (1.5). It is explained below in (5.1), (5.2).

Let us formulate now the main result.

Theorem 1.1. *Let conditions (1.8)–(1.10), (1.33) on the data of problem (1.3)–(1.7) and conditions (1.26), (1.27) are satisfied. Let also agreement conditions (1.34)–(1.36) at $\tau = 0$, $y \in \Gamma, \Gamma^{\pm}$ are satisfied. Then there exists $T > 0$ such that on the time interval $[0, T]$ problem (1.3)–(1.7) has the unique smooth solution. The unknown boundary can be represented as in (1.29) with a function $\rho(\omega, t)$ with the properties*

$$\begin{aligned} \rho(\omega, t) &\in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T), \quad \rho_t(\omega, t) \in C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T), \\ \rho^{\pm}(x, t) &= E^{\pm} \rho(\omega, t) \in C_s^{2+\gamma}(\bar{\Omega}_T^{\pm}), \end{aligned} \quad (1.37)$$

where $\rho(x, t) = E\rho(\omega, t)$ is an extension of the function $\rho(\omega, t)$ to the domain $\bar{\Omega}_T$. The functions $u^{\pm}(y, \tau)$ in (1.3) are such that

$$v^{\pm}(x, t) \equiv (|u^{\pm}(y, \tau)|^{m-1} u^{\pm}(y, \tau)) \circ e_{\rho}(x, t) \in C_s^{2+\gamma}(\bar{\Omega}_T^{\pm}). \quad (1.38)$$

Thus, all of the relations of problem (1.3)–(1.7) are satisfied in the classical sense.

For the proof of this theorem we use some simple assertion. This assertion is, in fact, a consequence of classical Newton's method of solving nonlinear equations in Banach spaces. Currently Newton's method itself is sufficiently custom in the free boundary problems theory. Previously this method was used in free boundary problems in a number of papers—see, for example [2–4, 6, 7, 46–49]. It seems paper [46] is the first application of Newton type iterations to a multidimensional free boundary problem. Note that the more contemporary methods of the investigation of such problems are associated with the use of semigroups theory for quasi-linear operators [50]. The semigroups method allows, in particular, to consider not only local but also global solvability of free boundary problems. However, in this paper we restrict ourselves to the local solvability of the problem and therefore we use more classical arguments. More precisely, in the contrast to [2–4, 6, 7, 46–49] we do not reproduce the whole scheme of the Newton's iterations explicitly with the corresponding estimates. Instead, we just use a variant of the classical inverse function theorem in Banach spaces (as a special case of a more general classical implicit function theorem). We formulate here these theorems. First we formulate some extended version of Theorem 1.2 from [51].

Theorem 1.2. ([51], Theorem 1.2 and its proof). *Let U be open in a Banach space H , and let $F : U \rightarrow Y$ be continuously differentiable on U , where Y is a Banach space. Let $x_0 \in U$ and assume that $F'(x_0) : H \rightarrow Y$ is a toplinear isomorphism (i.e. invertible as a continuous linear map). Then F is a local*

diffeomorphism at x_0 and there exists $d > 0$ such that the inverse mapping F^{-1} is defined on the ball $K_d = \{y \in Y : \|y - y_0\| \leq d\}$, $y_0 = F(x_0)$.

Here $d = d(M_1, M_2, \omega_0)$ depends only on M_1, M_2, ω_0 , where

$$M_1 = \|F'(x_0)\|_{H \rightarrow Y}, \quad M_2 = \|(F'(x_0))^{-1}\|_{Y \rightarrow H}, \\ \omega_0 = \sup_{x_1, x_2 \in U} \|F'(x_1) - F'(x_2)\|_{H \rightarrow Y}.$$

Due to this classical theorem we have the following very simple but fundamentally important for us assertion, where x_0 serves as an approximate solution to the equation $F(x) = 0$.

Corollary 1.3. *Let the conditions of Theorem 1.2 are satisfied. Then there exists $\varepsilon_0 = \varepsilon_0(M_1, M_2, \omega_0) > 0$ such that if $\|F(x_0)\|_Y = \|y_0\|_Y \leq \varepsilon_0$, then for some $x^* \in U$ we have $F(x^*) = 0$.*

Proof. This corollary immediately follows from Theorem 1.2 if we choose $\varepsilon_0 = d/2$ so that $0 \in K_d$. $\square$

Note that since we are going to use this corollary, the key ingredient of our proof of Theorem 1.1 is the proof of the fact that $f'(x_0)$ is invertible. In our case the nonlinear mapping $f(x)$ is defined by the left-hand-sides of relations (1.3)–(1.7), so we need to consider the Frechet derivative of such mapping. This is done in Sect. 6, Theorem 6.2.

The subsequent sections of the paper are devoted to the proof of Theorem 1.1 on the base of Corollary 1.3 according to the following plan.

In Sect. 2, we first introduce an equivalent norms in the spaces $C_s^{\gamma, \gamma/2}(\overline{\Omega}_T^\pm)$. This permits us to study the exact class of the traces on Γ_T of functions from the class $C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)$. This allows us to extend the results of [42] about the solvability of the homogeneous initial boundary value problem for a degenerate equation to the case of the inhomogeneous problem Theorem 2.7.

These results are then used in Sect. 3 to study a model Stefan problem for degenerate equations. This model problem is one of the central points of this paper. This model problem is an initial-boundary problem with dynamic boundary conditions for degenerate equations in half-spaces. To obtain Schauder's estimates for this problem the idea of paper [53] on the application of the classical maximum principle is used. In Sects. 4 and 5 the initial problem with the unknown boundary is reduced to nonlinear problem in a fixed domain. Then the corresponding operators of the problem are linearized (to find the Frechet derivative) on some functions that extend the initial data to the domain $t > 0$. These functions serve as an approximate solution x_0 in Corollary 1.3. Section 6 is devoted to the study of the linear problem with dynamic boundary conditions for degenerate equations in classes of smooth functions in arbitrary smooth domains. This gives the invertibility of the Frechet derivative of the nonlinear operator of our initial problem. The obtaining of Schauder's estimates is based on the results of Sect. 3. To prove the solvability of the linear problem, we use the parabolic regularization of the Stefan boundary condition. Note that the corresponding model problem in Sect. 3 is considered in the presence of the regularization. Finally, Sect. 7 completes the proof of Theorem 1.1 by Corollary 1.3.

2. Auxiliary results on the spaces $C_s^{\gamma, \gamma/2}(\overline{\Omega}_T^\pm)$, $C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)$, $C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(\Gamma_T)$

First we give some important inequalities for the spaces with zero in (1.21). Let $\gamma' > \gamma$, $l'_1 > l_1$, $l'_2 > l_2$. Let also

$$\begin{aligned} u_1 &\in C_{0,s}^{\gamma', \gamma'/2}(\overline{\Omega}_T^\pm), \quad u_2 \in C_{0,s}^{2+\gamma', \frac{2+\gamma'}{2}}(\overline{\Omega}_T^\pm), \\ u_3, v_3 &\in C_0^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm), \quad u_4 \in C_0^{l'_1, l'_2}(\overline{\Omega}_T^\pm). \end{aligned}$$

Then

$$|u_1|_{s, \overline{\Omega}_T^\pm}^{(\gamma)} \leq CT^\mu |u_1|_{s, \overline{\Omega}_T^\pm}^{(\gamma')}, \quad (2.1)$$

$$|u_2|_{s, \overline{\Omega}_T^\pm}^{(2+\gamma)} \leq CT^\mu |u_2|_{s, \overline{\Omega}_T^\pm}^{(2+\gamma')}, \quad (2.2)$$

$$|u_3 v_3|_{s, \overline{\Omega}_T^\pm}^{(\gamma)} \leq CT^\mu |u_3|_{s, \overline{\Omega}_T^\pm}^{(\gamma)} |v_3|_{s, \overline{\Omega}_T^\pm}^{(\gamma)}, \quad (2.3)$$

$$|u_4|_{\overline{\Omega}_T^\pm}^{(l_1, l_2)} \leq CT^\mu |u_4|_{\overline{\Omega}_T^\pm}^{(l'_1, l'_2)}, \quad (2.4)$$

where C, μ denote some positive constants. These constants depend on $\gamma, \gamma', l_i, l'_i$.

Inequalities (2.1)–(2.4) are well known for the spaces $C_0^{l_1, l_2}$ (see [43, 52]), and for the spaces $C_{0,s}^{\gamma, \gamma/2}$, $C_{0,s}^{2+\gamma, \frac{2+\gamma}{2}}$ they are completely analogous.

Before we proceed we need one more assertion about Hölder spaces. Let $l > 0$ be non-integer and let $h > 0$. Denote by $\Delta_h u(x) = u(x+h) - u(x)$ the operator of difference with the step $h > 0$. Let further $k > l$ be an integer and let $\Delta_h^k = (\Delta_h)^k$ be the k -th power of the operator Δ_h . For a function $u(x) \in C^l(R^1)$ define the following seminorms

$$\begin{aligned} \langle u \rangle_{R^1}^{(l)} &= \left\langle \frac{\partial^{[l]} u}{\partial x^{[l]}} \right\rangle_{x, R^1}^{(l-[l])}, \\ \langle \langle u \rangle \rangle_{R^1}^{(l)} &= \sup_{x \in R^1, h > 0} \frac{|\Delta_h^k u(x)|}{h^l}. \end{aligned}$$

Then it follows from [54] (see also [45]) that for any $k > l$ these two seminorms are equivalent

$$\nu_k \langle \langle u \rangle \rangle_{R^1}^{(l)} \leq \langle u \rangle_{R^1}^{(l)} \leq \nu_k^{-1} \langle \langle u \rangle \rangle_{R^1}^{(l)}. \quad (2.5)$$

2.1. An equivalent norm for the spaces $C_s^\gamma(\overline{\Omega}_T^\pm)$, $C_s^{2+\gamma}(\overline{\Omega}_T^\pm)$

Along with the seminorm $H_{s, R_{+T}^N}^\gamma$ from (1.16) we consider in R_{+T}^N the following weighted seminorm

$$H_{\alpha, R_{+T}^N}^\gamma(f) = \sup_{x, \bar{x} \in R_+^N} \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} + \sup_{x, \bar{x} \in R_+^N} \tilde{x}_N^{\frac{\alpha}{2}} \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\gamma}, \quad (2.6)$$

where $\tilde{x}_N = \max\{x_N, \bar{x}_N\}$, and here and throughout without loss of generality we assume that $\bar{x}_N \leq x_N$, so $\tilde{x}_N = x_N$.

Lemma 2.1. *The seminorms $H_{\alpha, R_{+T}^N}^\gamma(f)$ and $H_{s, R_{+T}^N}^\gamma(f)$ are equivalent.*

Proof. Let the seminorm $H_{s, R_{+T}^N}^\gamma(f)$ is finite. We show that

$$H_{\alpha, R_{+T}^N}^\gamma(f) \leq CH_{s, R_{+T}^N}^\gamma(f). \quad (2.7)$$

Let $\varepsilon_0 \in (0, 1)$ is small and fixed. Let $x = (x', x_N)$ and let first

$$|x' - \bar{x}'| \geq \varepsilon_0 x_N. \quad (2.8)$$

Then the more

$$|x - \bar{x}| \geq |x' - \bar{x}'| \geq \varepsilon_0 x_N. \quad (2.9)$$

Under this condition

$$\begin{aligned} s(x, \bar{x}) &= \frac{|x - \bar{x}|}{x_N^{\alpha/2} + \bar{x}_N^{\alpha/2} + |x' - \bar{x}'|^{\alpha/2}} \\ &\leq C \frac{|x - \bar{x}|}{|x - \bar{x}|^{\alpha/2} + \bar{x}_N^{\alpha/2} + |x' - \bar{x}'|^{\alpha/2}} \leq C|x - \bar{x}|^{1-\frac{\alpha}{2}}. \end{aligned}$$

Therefore, as $\beta = \gamma(1 - \frac{\alpha}{2})$,

$$\frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq C \frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq CH_{s, R_{+T}^N}^\gamma(f). \quad (2.10)$$

Besides, from (2.9) and then from (2.10),

$$x_N^{\gamma\frac{\alpha}{2}} \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\gamma} \leq \frac{x_N^{\gamma\frac{\alpha}{2}}}{(\varepsilon_0 x_N)^{\gamma\frac{\alpha}{2}}} \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^{\gamma(1-\alpha/2)}} \leq CH_{s, R_{+T}^N}^\gamma(f). \quad (2.11)$$

Let now

$$|x' - \bar{x}'| \leq \varepsilon_0 x_N. \quad (2.12)$$

Under this condition, as it easy to see,

$$s(x, \bar{x}) \sim Cx_N^{-\frac{\alpha}{2}} |x - \bar{x}|. \quad (2.13)$$

Consequently

$$x_N^{\gamma\frac{\alpha}{2}} \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\gamma} \leq C \frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq CH_{s, R_{+T}^N}^\gamma(f). \quad (2.14)$$

To estimate, further, the unweighted Hölder constant in the definition of $H_{\alpha, R_{+T}^N}^\gamma(f)$ we consider the following two cases.

If

$$|x_N - \bar{x}_N| \geq \varepsilon_0 x_N,$$

then

$$|x - \bar{x}| \geq |x_N - \bar{x}_N| \geq \varepsilon_0 x_N$$

and therefore, as it was above,

$$s(x, \bar{x}) \leq \frac{|x - \bar{x}|}{(|x - \bar{x}|/\varepsilon_0)^{\alpha/2}} \leq C|x - \bar{x}|^{1-\frac{\alpha}{2}},$$

so that, as above,

$$\frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq C \frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq CH_{s, R_{+T}^N}^\gamma(f). \quad (2.15)$$

If now under the condition (2.12) we have

$$|x_N - \bar{x}_N| \leq \varepsilon_0 x_N, \quad (2.16)$$

then in this case

$$|x - \bar{x}| \leq |x' - \bar{x}'| + |x_N - \bar{x}_N| \leq 2\varepsilon_0 x_N. \quad (2.17)$$

Therefore, in force of (2.13),

$$\begin{aligned} s(x, \bar{x}) &\leq Cx_N^{-\alpha/2}|x - \bar{x}| \\ &\leq Cx_N^{-\alpha/2}(2\varepsilon_0 x_N)^{\alpha/2}|x - \bar{x}|^{1-\alpha/2} = C|x - \bar{x}|^{1-\alpha/2}. \end{aligned} \quad (2.18)$$

Consequently, in this case

$$\frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq C \frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq CH_{s, R_{+T}^N}^\gamma(f). \quad (2.19)$$

Estimate (2.7) follows now from (2.10), (2.11), (2.14), (2.15), and (2.19).

Further, let now the seminorm $H_{\alpha, R_{+T}^N}^\gamma(f)$ is finite. Let us prove the following estimate

$$H_{s, R_{+T}^N}^\gamma(f) \leq CH_{\alpha, R_{+T}^N}^\gamma(f). \quad (2.20)$$

Let first

$$|x' - \bar{x}'| \leq \varepsilon_0 x_N, \quad x_N > 0. \quad (2.21)$$

Then

$$s(x, \bar{x}) \geq \nu \frac{|x - \bar{x}|}{x_N^{\alpha/2}},$$

and consequently

$$\frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq Cx_N^{\gamma\alpha/2} \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\gamma} \leq CH_{\alpha, R_{+T}^N}^\gamma(f). \quad (2.22)$$

In the particular case $x_N = 0$ we have $\bar{x}_N = 0$ and therefore

$$s(x, \bar{x}) = |x' - \bar{x}'|^{1-\alpha/2} = |x - \bar{x}|^{1-\alpha/2},$$

and so again

$$\frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} = \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq CH_{\alpha, R_{+T}^N}^\gamma(f). \quad (2.23)$$

Let now we have

$$|x' - \bar{x}'| \geq \varepsilon_0 x_N. \quad (2.24)$$

Then

$$s(x, \bar{x}) \geq \nu \frac{|x - \bar{x}|}{|x' - \bar{x}'|^{\alpha/2}} \geq \nu |x - \bar{x}|^{1-\alpha/2}, \quad (2.25)$$

and consequently

$$\frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq C \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq CH_{\alpha, R_{+T}^N}^\gamma(f). \quad (2.26)$$

Thus, (2.20) follows from (2.22), (2.23), (2.25). And so the equivalence of the seminorms $H_{s, R_{+T}^N}^\gamma(f)$ and $H_{\alpha, R_{+T}^N}^\gamma(f)$ is proved. $\square$

From this lemma it follows that the norm in the space $C_s^\gamma(R_{+T}^N)$ may be given in the form

$$|u|_{C_s^{\gamma, \gamma/2}(R_{+T}^N)} = |u|_{R_{+T}^N}^{(0)} + H_{\alpha, R_{+T}^N}^\gamma(u) + \langle u \rangle_{t, R_{+T}^N}^{\gamma/2}. \quad (2.27)$$

Bearing in mind the local straightening of the boundary Γ , for the case of arbitrary domains $\bar{\Omega}_T^\pm$, the norm in the space $C_s^{\gamma, \gamma/2}(\bar{\Omega}_T^\pm)$ may be explicitly written as

$$|u|_{C_s^\gamma(\bar{\Omega}_T^\pm)} = |u|_{\bar{\Omega}_T^\pm}^{(0)} + \langle u \rangle_{x, \bar{\Omega}_T^\pm}^{(\beta)} + \sup_{x, \bar{x} \in \bar{\Omega}^\pm} (\widetilde{d^\pm(x, \bar{x})})^{\gamma\alpha/2} \frac{|u(x, t) - u(\bar{x}, t)|}{|x - \bar{x}|^\gamma} + \langle u \rangle_{t, \bar{\Omega}_T^\pm}^{\gamma/2}, \quad (2.28)$$

where the functions $d^\pm(x)$ were introduced in the previous section in (1.11) and they model the distance to the boundary Γ , $\widetilde{d^\pm(x, \bar{x})} = \max\{d^\pm(x), d^\pm(\bar{x})\}$.

Quite similar, in terms of (2.28) and (1.19) we may explicitly define the norm (1.18) in the space $C_s^{2+\gamma}(\bar{\Omega}_T^\pm)$.

2.2. The traces of functions from $C_s^{2+\gamma, \frac{2+\gamma}{2}}(\bar{\Omega}_T^\pm)$ on Γ_T

Note first that the results of this and the next subsections are generalisations of those in [55] for elliptic case and for some another weighted spaces.

Note further that according to our assumptions the surface Γ is smooth, so we can use the local straightening of this surface to study the locally defined classes $C_s^{2+\gamma, \frac{2+\gamma}{2}}(\bar{\Omega}_T^\pm)$. Therefore, it is sufficient to consider the case of the half-space. So we consider a function $u(x, t)$ from the space $C_s^{2+\gamma, \frac{2+\gamma}{2}}(R_{+T}^N)$ with finite support in $R_{+T}^N = R_+^N \times [0, T]$ and study its trace at $x_N = 0$.

Lemma 2.2. *Let a function $u(x, t)$ has finite support and $u(x, t) \in C^{2+\gamma, \frac{2+\gamma}{2}}(R_{+T}^N)$, $0 < \gamma < \alpha$, $\beta = \gamma(1 - \alpha/2)$. Then the function $v(x', t) = u(x', 0, t) \in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(R_T^{N-1}) = C^{2+\beta-\alpha, \frac{2+\gamma}{2}}(R_T^{N-1})$ and*

$$|v(x', t)|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(R_T^{N-1})} = |u(x', 0, t)|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(R_T^{N-1})} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)}. \quad (2.29)$$

Besides,

$$|\nabla_{(x', x_N)} u(x', 0, t)|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(R_T^{N-1})} \leq C_T |u|_{s, R_{+T}^N}^{(2+\gamma)}. \quad (2.30)$$

Proof. It follows directly from the definition of the space $C^{2+\gamma, \frac{2+\gamma}{2}}(R_{+T}^N)$ in (1.18) and from Lemma 2.1 that $u_t(x', 0, t) \in C^{\beta, \gamma/2}(R_T^{N-1})$. Therefore, $v_t(x', 0) = u_t(x', 0, t) \in C^{\beta, \gamma/2}(R_T^{N-1})$ and in addition

$$|v_t|_{C^{\beta, \gamma/2}(R_T^{N-1})} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)}. \quad (2.31)$$

Therefore, in view of (1.20) (see [45]) it is sufficient to prove uniformly in t the following estimate

$$|v(\cdot, t)|_{R^{N-1}}^{(2+\beta-\alpha)} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)}. \quad (2.32)$$

And for this it is sufficient to prove that uniformly in t and in x_N we have for all $i = \overline{1, N}$ for the function $w = u_{x_i}$

$$\langle w \rangle_{x', R^{N-1}}^{(1+\beta-\alpha)} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)} \quad (2.33)$$

So let $w = u_{x_i}$, $i = \overline{1, N}$. From (2.5) (see [54]) it follows that to prove (2.33) it is sufficient to show that for arbitrary $h > 0$ the following inequality holds

$$\frac{|\Delta_{h, x'}^2 w(x_N)|}{h^{1+\beta-\alpha}} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)}. \quad (2.34)$$

Here $\Delta_{h, x'}^2 w(x_N) \equiv \Delta_{h, x'}^2 w \equiv \Delta_{h, x'}^2 w(x', x_N, t)$ is the second difference from the function w with respect to the variable x'_j , $j = \overline{1, N-1}$, with the step h , that is,

$$\Delta_{h, x'}^2 w(x_N) = u(x' + \vec{j}h, x_N, t) - 2u(x', x_N, t) + u(x' - \vec{j}h, x_N, t). \quad (2.35)$$

Consider the following two cases. Let first

$$h \leq x_N. \quad (2.36)$$

Then according to the mean value theorem we have with some $\theta_1, \theta_2 \in (0, 1)$

$$\begin{aligned} \frac{|\Delta_{h, x'}^2 w(x_N)|}{h^{1+\beta-\alpha}} &\leq \frac{|x_N^\alpha \Delta_{h, x'}^2 w(x_N)|}{h^{1+\beta}} \\ &= \left| x_N^\alpha \frac{w_{x_j}(x' + \vec{j}\theta_1 h, x_N, t) - w_{x_j}(x' - \vec{j}\theta_2 h, x_N, t)}{h^\beta} \right| \\ &= \frac{|x_N^\alpha u_{x_i x_j}(x' + \vec{j}\theta_1 h, x_N, t) - x_N^\alpha u_{x_i x_j}(x' - \vec{j}\theta_2 h, x_N, t)|}{h^\beta} \\ &\leq C \langle x_N^\alpha u_{x_i x_j} \rangle_{x', R_{+T}^N}^{(\beta)} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)}. \end{aligned} \quad (2.37)$$

Let now $h \geq x_N$. Write the difference $\Delta_{h, x'}^2 w(x_N)$ in the form

$$\begin{aligned} \Delta_{h, x'}^2 w(x_N) &= -\Delta_{h, x'}^2 (w(x', x_N + h, t) - w(x', x_N - h, t)) \\ &\quad + \Delta_{h, x'}^2 w(x', x_N + h, t) \equiv A_1 + A_2. \end{aligned} \quad (2.38)$$

In view of the fact that for the expression A_2 condition (2.36) holds, that is $h \leq x_N + h$, completely analogous to (2.37),

$$\frac{|A_2|}{h^{1+\beta-\alpha}} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)}. \quad (2.39)$$

To estimate the expression $|A_1|/h^{1+\beta-\alpha}$ we use the formula

$$w(x', x_N + h, t) - w(x', x_N, t) = h \int_0^1 w_{x_N}(x', x_N + \theta h, t) d\theta.$$

Consequently, in view of $w_{x_N} = u_{x_i x_N}$,

$$\begin{aligned} \frac{|A_1|}{h^{1+\beta-\alpha}} &\leq \int_0^1 \frac{h^\alpha}{(x_N + \theta h)^\alpha} \left| \frac{\Delta_{h, x'}^2 (x_N + \theta h)^\alpha u_{x_i x_N}(x', x_N + \theta h, t)}{h^\beta} \right| d\theta \\ &\leq C \langle x_N^\alpha u_{x_i x_N} \rangle_{x', R_{+T}^N}^{(\beta)} \int_0^1 (x_N/h + \theta)^{-\alpha} d\theta \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)}. \end{aligned} \quad (2.40)$$

Thus, from (2.39) and (2.40) we obtain (2.34), and so we have also (2.33) and (2.32). Together with (2.31) this completes the proof of (2.29).

We now show inequality (2.30). Note that for tangential derivatives u_{x_i} , $i = \overline{1, N-1}$ this inequality follows from above estimate (2.29) and from [45], (1.20). However, for u_{x_N} we need a distinct proof. We show (2.30) for u_{x_k} , $k = \overline{1, N}$.

According to (2.5) it is enough to show that for $h > 0$

$$|\Delta_{h, t}^2 u_{x_k}(x, t)| \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)} h^{\frac{1+\beta-\alpha}{2}}, \quad (2.41)$$

where

$$\Delta_{h, t}^2 v(x, t) = \Delta_{h, t}^2 v = v(x, t + 2h) - 2v(x, t + h) + v(x, t)$$

is the second difference of a function v with respect to the variable t with the step h .

Let first

$$h^{\frac{1}{2-\alpha}} \leq x_N. \quad (2.42)$$

Then we use the following interpolation inequality

$$|v_{x_k}|_{\Pi_T(x_N)}^{(0)} \leq C \left(|v|_{\Pi_T(x_N)}^{(0)} \right)^{1/2} \left(\langle v \rangle_{\Pi_T(x_N)}^{(2)} \right)^{1/2}. \quad (2.43)$$

Here we denote $\Pi_T(x_N) = \{(y, t) : x_N/2 < y_N < 3x_N/2, 0 < t < T\}$, and for an integer k

$$\langle v \rangle_{\Omega}^{(k)} = \sum_{|\omega|=k} |D_x^\omega v|_{\Omega}^{(0)}.$$

Note that the constant C in (2.43) does not depend on x_N as it follows from scaling arguments. Inequality (2.43) easily follows from the well-known interpolation inequality (see, for example, [45], [50], Ch.1)

$$\langle v \rangle_{\Pi_T(x_N)}^{(1)} \leq C \left(|v|_{\Pi_T(x_N)}^{(0)} \right)^{1/2} \left(\langle v \rangle_{\Pi_T(x_N)}^{(2)} \right)^{1/2}, \quad (2.44)$$

where for an integer k

$$|v|_{\Omega}^{(k)} = \sum_{|\omega| \leq k} |D_x^\omega v|_{\Omega}^{(0)}.$$

To obtain (2.43) from (2.44) it is enough to apply first to the right-hand side of (2.44) the Young inequality with $\varepsilon > 0$ and get

$$\langle v \rangle_{\Pi_T(x_N)}^{(1)} \leq \varepsilon \left(|v|_{\Pi_T(x_N)}^{(0)} + \langle v \rangle_{\Pi_T(x_N)}^{(1)} + \langle v \rangle_{\Pi_T(x_N)}^{(2)} \right) + C \frac{1}{\varepsilon} |v|_{\Pi_T(x_N)}^{(0)}.$$

From this inequality it follows that for $\varepsilon < 1/2$ and with some another constant C

$$\langle v \rangle_{\Pi_T(x_N)}^{(1)} \leq \varepsilon \langle v \rangle_{\Pi_T(x_N)}^{(2)} + C \frac{1}{\varepsilon} |v|_{\Pi_T(x_N)}^{(0)}.$$

Optimising this inequality with respect to ε in usual way we get (2.43).

From (2.43) we obtain for $\Delta_{h,t}^2 u_{x_k}(x, t)$

$$|\Delta_{h,t}^2 u_{x_k}(x, t)| \leq C \left(|\Delta_{h,t}^2 u|_{\Pi_T(x_N)}^{(0)} \right)^{1/2} \left(|\Delta_{h,t}^2 u|_{\Pi_T(x_N)}^{(2)} \right)^{1/2}. \quad (2.45)$$

In view of (2.5) and the properties of the space $C_s^{2+\gamma, \frac{2+\gamma}{2}}(R_{+T}^N)$ (see (1.20), [45, 54]) we have

$$|\Delta_{h,t}^2 u|_{\Pi_T(x_N)}^{(0)} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)} h^{\frac{2+\gamma}{2}}, \quad (2.46)$$

$$|\Delta_{h,t}^2 u|_{\Pi_T(x_N)}^{(2)} \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)} x_N^{-\alpha} h^{\frac{\gamma}{2}}. \quad (2.47)$$

From (2.46) and (2.47), in view of (2.42), we obtain

$$|\Delta_{h,t}^2 u_{x_k}(x, t)| \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)} h^{\frac{2+\gamma}{4} + \frac{\gamma}{4} - \frac{\alpha}{2(2-\alpha)}} = C |u|_{s, R_{+T}^N}^{(2+\gamma)} h^{\frac{1+\beta-\alpha}{(2-\alpha)}}, \quad (2.48)$$

that is inequality (2.41).

Let now

$$h^{\frac{1}{2-\alpha}} \geq x_N. \quad (2.49)$$

Write $\Delta_{h,t}^2 u_{x_k}(x, t)$ as

$$\begin{aligned} \Delta_{h,t}^2 u_{x_k}(x, t) &= -\Delta_{h,t}^2 \left[u_{x_k}(x', x_N + h^{\frac{1}{2-\alpha}}, t) - u_{x_k}(x, t) \right] \\ &\quad + \Delta_{h,t}^2 u_{x_k}(x', x_N + h^{\frac{1}{2-\alpha}}, t) \equiv A_1 + A_2, \end{aligned} \quad (2.50)$$

and for A_2 , by the above case (2.42), the estimate

$$|A_2| \leq C |u|_{s, R_{+T}^N}^{(2+\gamma)} h^{\frac{1+\beta-\alpha}{(2-\alpha)}}$$

is valid.

To estimate the expression A_1 write it as

$$A_1 = -h^{\frac{1}{2-\alpha}} \int_0^1 \Delta_{h,t}^2 u_{x_k x_N}(x', x_N + \theta h^{\frac{1}{2-\alpha}}, t) d\theta.$$

Thus, we have for A_1

$$\begin{aligned} |A_1| &\leq C|u|_{s, R_{+T}^N}^{(2+\gamma)} h^{\frac{1}{2-\alpha}} h^{\frac{\gamma}{2}} \int_0^1 \left(x_N + \theta h^{\frac{1}{2-\alpha}}\right)^{-\alpha} d\theta \\ &\leq C|u|_{s, R_{+T}^N}^{(2+\gamma)} h^{\frac{1}{2-\alpha} + \frac{\gamma}{2} - \frac{\alpha}{2-\alpha}} \int_0^1 \theta^{-\alpha} d\theta = C|u|_{s, R_{+T}^N}^{(2+\gamma)} h^{\frac{1+\beta-\alpha}{2-\alpha}}, \end{aligned}$$

that is, again inequality (2.41).

The lemma is proved. $\square$

Thus, due to the possibility of the local straightening of the boundary, the following is true.

Lemma 2.3. *Let functions $u^\pm(x, t)$ belong to the spaces $C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)$. Then the functions $v^\pm(x, t) = u^\pm(x, t)|_{x \in \Gamma}$ belong to the space $C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)$, and*

$$|v^\pm|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)} \leq C|u|_{s, \overline{\Omega}_T^\pm}^{(2+\gamma)}. \quad (2.51)$$

Besides,

$$|\nabla u^\pm|_{\Gamma_T}|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T)} \leq C|u^\pm|_{s, \overline{\Omega}_T^\pm}^{(2+\gamma)}. \quad (2.52)$$

2.3. An extension from the surface Γ_T of functions from the space $C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)$

In this section we prove the converse of Lemma 2.3, that is, we show that any function from the class $C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T) = C^{2+\beta-\alpha, 1+\gamma/2}(\Gamma_T)$ can be extended to all domains $\overline{\Omega}_T^\pm$ up to functions of the class $C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)$, and the extension operator is bounded (here, as above $\beta = \gamma(1 - \alpha/2)$). Such an extension operator is constructed in the standard way by applying a sufficiently small partition of unity in the neighborhood of Γ and by the local straightening of the boundary Γ , see [43]. In this case, it is enough to require the $C^{2+\gamma}$ -smoothness of the boundary Γ . Therefore, the existence of the extension operator follows in the standard way from the following lemma.

Lemma 2.4. *Let $f(x', t)$ be a function from the class $C^{2+\beta-\alpha, 1+\gamma/2}(R_T^{N-1})$ with finite support in R_{+T}^N at $x_N = 0$. Then f can be extended to the domain $x_N > 0$ up to a function $u(x, t)$ from the class $C_s^{2+\gamma, \frac{2+\gamma}{2}}(R_{+T}^N)$, and*

$$|u|_{s, R_{+T}^N}^{(2+\gamma)} \leq C|f|_{C^{2+\beta-\alpha, 1+\gamma/2}(R_T^{N-1})}. \quad (2.53)$$

Proof. Let $u(x, t)$ be the solution of the following Dirichlet problem with the parameter $t \in [0, T]$:

$$\Delta u = 0, \quad x \in R_+^N \quad (x_N > 0), \quad (2.54)$$

$$u|_{x_N=0} = f(x', t), \quad (2.55)$$

$$u \rightarrow 0, \quad |x| \rightarrow \infty. \quad (2.56)$$

As it is well known, the solution of (2.54)–(2.56) is given by the potential of the double layer, defined by the Newton potential.

Note first, that for problem (2.54)–(2.56) we have the following maximum principle

$$|u|_{R_+^N}^{(0)} \leq |f|_{R^{N-1}}^{(0)}. \quad (2.57)$$

Indeed, by (2.56) we can choose $K > 0$ so large that $|u| \leq |f|_{R_T^{N-1}}^{(0)}/2$ for $|x| \geq K$. By the properties of the double layer potential and the finiteness of f a constant K can be chosen independent of t . Now consider in the domain $B_K = R_+^N \cap \{|x| < K\}$ the functions $v^\pm = \pm u + |f|_{R^{N-1}}^{(0)}$. Within this domain we have

$$\Delta v^\pm = 0, \quad x \in B_K, \quad (2.58)$$

and on the boundary $\partial B_K = \{x_N = 0, |x'| \leq K\} \cup \{x_N > 0, |x'| = K\}$ the inequality

$$v^\pm \geq 0, \quad x \in \partial B_K \quad (2.59)$$

holds. It follows from (2.58), (2.59) and from the maximum principle that $v^\pm \geq 0$ in $\overline{B}_K$ for all t and thus,

$$|u|_{B_K}^{(0)} \leq |f|_{R^{N-1}}^{(0)} \quad t \in [0, T].$$

Due to the choice of K we have inequality (2.57) on the whole domain R_{+T}^N .

Consider first the properties of the function $u(x, t)$ with respect to t . Let $v(x, t)$ be the solution of (2.54)–(2.56) with the boundary condition

$$v|_{x_N=0} = f_t(x', t) \quad (2.60)$$

instead of (2.55). Consider also for $h > 0$ the following function

$$u_h(x, t) = \frac{u(x, t+h) - u(x, t)}{h},$$

which satisfies problem (2.54)–(2.56) with the boundary condition

$$u_h|_{x_N=0} = f_h(x', t) \equiv \frac{f(x', t+h) - f(x', t)}{h}. \quad (2.61)$$

Let further $w(x, t) = u_h(x, t) - v(x, t)$ and the function $w(x, t)$ also satisfies problem (2.54)–(2.56) but with the boundary condition

$$w|_{x_N=0} = f_h(x', t) - f_t(x', t) \equiv \varphi_h(x', t). \quad (2.62)$$

Due to the properties of the function $f(x', t)$ we have with some $\theta(x', t, h) \in (0, 1)$ according to the mean value theorem

$$\begin{aligned} |\varphi_h(x', t)| &= |f_h(x', t) - f_t(x', t)| \\ &= |f_t(x', t + \theta h) - f_t(x', t)| \leq \langle f_t(x', t) \rangle_{t, R_T^{N-1}}^{(\gamma/2)} h^{\gamma/2} \rightarrow 0, h \rightarrow 0. \end{aligned} \quad (2.63)$$

Consequently, on the base of (2.57),

$$|u_h - v|_{R_{+T}^N}^{(0)} \leq Ch^{\gamma/2} \rightarrow 0, \quad h \rightarrow 0. \quad (2.64)$$

This means that the function $u(x, t)$ has the derivative with respect to the variable t for $x \in R_{+T}^N$ and $u_t(x, t) = v(x, t)$, that is $u_t(x, t)$ satisfies problem (2.54)–(2.56) with boundary condition (2.60).

Further, considering the function

$$v_h(x, t) = \frac{u_t(x, t+h) - u_t(x, t)}{h^{\gamma/2}},$$

we see that it satisfies the same problem with the boundary condition

$$v_h(x, t)|_{x_N=0} = \frac{f_t(x', t+h) - f_t(x', t)}{h^{\gamma/2}} \equiv f_{th}(x', t). \quad (2.65)$$

Thus, on the base of (2.57) again

$$\left| \frac{u_t(x, t+h) - u_t(x, t)}{h^{\gamma/2}} \right|_{R_{+T}^N}^{(0)} \leq C \left| \frac{f_t(x', t+h) - f_t(x', t)}{h^{\gamma/2}} \right|_{R_T^{N-1}}^{(0)}. \quad (2.66)$$

As h is arbitrary this means that

$$\langle u_t(x, t) \rangle_{t, R_{+T}^N}^{(\gamma/2)} \leq C \langle f_t(x', t) \rangle_{t, R_T^{N-1}}^{(\gamma/2)}. \quad (2.67)$$

Consider now the properties of the function $u(x, t)$ with respect to the variables x . First, it follows from the results of [56, 57] that for each $t \in [0, T]$ we have $u \in C_x^{2+\beta-\alpha}(R_{+T}^N)$ due to $f \in C_x^{2+\beta-\alpha}(R_T^{N-1})$ and

$$|u|_{C_x^{2+\beta-\alpha}(R_{+T}^N)} \leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})}. \quad (2.68)$$

Here the symbol x at the bottom of the space notation means that we consider the smoothness only with respect to x and

$$\begin{aligned} |u|_{C_x^{2+\beta-\alpha}(R_{+T}^N)} &= \max_{t \in [0, T]} |u(\cdot, t)|_{C^{2+\beta-\alpha}(R_{+T}^N)}, \\ |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})} &= \max_{t \in [0, T]} |f(\cdot, t)|_{C^{2+\beta-\alpha}(R_T^{N-1})}. \end{aligned}$$

Show that the following estimates

$$\langle x_N^\alpha D^2 u \rangle_{x, R_{+T}^N}^{(\beta)} \leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})}, \quad (2.69)$$

$$\left\langle \tilde{x}_N^{\gamma\alpha/2} (x_N^\alpha D^2 u) \right\rangle_{x, R_{+T}^N}^{(\gamma)} \leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})} \quad (2.70)$$

are valid, that is,

$$H_\alpha^\gamma(x_N^\alpha D^2 u) \leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})}. \quad (2.71)$$

We will use the fact that, as it follows from [58], Chapter 5.4, the condition $f \in C_x^l(R^{N-1})$ in (2.55), $l \in (0, 2)$, is equivalent to the condition

$$|D_x^k u| \leq C_k x_N^{-k+l} |f|_{C_x^l(R^{N-1})}, \quad k \geq 2, \quad (2.72)$$

where here and below $D_x^k u = D^k u$ means a derivative of the k -th order with respect to x of the function $u(x, t)$. Since it is important to prove (2.69) for $x_N < 1$ only (for $x_N > 1$, such the estimate follows from the local estimates

and is well-known), we consider only the case $x_N < 1$. We also use the well-known interpolation inequality

$$\langle v(x) \rangle_{x, \Omega}^{(\beta)} \leq C \left(|v|_{\Omega}^{(0)} \right)^{1-\beta} \left(\langle v \rangle_{\Omega}^{(1)} \right)^{\beta}, \quad (2.73)$$

which is valid for functions $v(x) \in C^1(\overline{\Omega})$, where Ω is a domain (possibly unbounded) with sufficiently smooth boundary (see, for example [50] Chapter 1). This inequality is analogous to (2.43) and it is important that the constant C does not depend on the size of the domain Ω under scaling. In addition, at the proof of (2.69) without loss of generality we prove smoothness of the function $x_N^\alpha D^2 u$ with respect to x' and with respect to x_N separately. So we obtain estimate (2.69) separately for these two cases.

Let first x_N be fixed. Then by (2.73) and (2.72),

$$\begin{aligned} \langle x_N^\alpha D^2 u(\cdot, x_N) \rangle_{x', R^{N-1}}^{(\beta)} &\leq C \left(|x_N^\alpha D^2 u|_{x', R^{N-1}}^{(0)} \right)^{1-\beta} \left(|\nabla_{x'} x_N^\alpha D^2 u|_{x', R^{N-1}}^{(0)} \right)^{\beta} \\ &\leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})} \left(x_N^\alpha x_N^{-2+(2+\beta-\alpha)} \right)^{1-\beta} \left(x_N^\alpha x_N^{-3+(2+\beta-\alpha)} \right)^{\beta} \\ &= C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})}, \end{aligned} \quad (2.74)$$

that is, estimate (2.69) with respect to x' .

Analogously, using (2.73) and (2.72), we prove (2.70) with respect to x' :

$$\begin{aligned} x_N^{\gamma\alpha/2} \langle x_N^\alpha D^2 u(\cdot, x_N) \rangle_{x', R^{N-1}}^{(\gamma)} &\leq C x_N^{\gamma\alpha/2} \left(|x_N^\alpha D^2 u|_{x', R^{N-1}}^{(0)} \right)^{1-\gamma} \left(|x_N^\alpha D^2 u|_{x', R^{N-1}}^{(1)} \right)^{\gamma} \\ &\leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})} x_N^{\gamma\alpha/2} \left(x_N^\alpha x_N^{-2+(2+\beta-\alpha)} \right)^{1-\gamma} \left(x_N^\alpha x_N^{-3+(2+\beta-\alpha)} \right)^{\gamma} \\ &= C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})} x_N^{\gamma\alpha/2+\beta-\gamma} = C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})}. \end{aligned} \quad (2.75)$$

We prove now relations (2.69), (2.70) with respect to the variable x_N . For this we fix some $\varepsilon_0 \in (0, 1/16)$ and consider the two cases, assuming without loss of generality that $\bar{x}_N \leq x_N$.

Let first

$$|x_N - \bar{x}_N| = (x_N - \bar{x}_N) \geq \varepsilon_0 x_N. \quad (2.76)$$

Then

$$\frac{|x_N^\alpha D^2 u(x, t) - \bar{x}_N^\alpha D^2 u(\bar{x}, t)|}{|x_N - \bar{x}_N|^{\beta}} \leq C \left(|x_N^{\alpha-\beta} D^2 u(x, t)| + |\bar{x}_N^{\alpha-\beta} D^2 u(\bar{x}, t)| \right). \quad (2.77)$$

In this case, as above

$$|x_N^{\alpha-\beta} D^2 u(x, t)| \leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})} x_N^{\alpha-\beta} x_N^{-2+(2+\beta-\alpha)} = C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})}, \quad (2.78)$$

and similarly for $|\bar{x}_N^{\alpha-\beta} D^2 u(\bar{x}, t)|$. In the same way

$$x_N^{\gamma\alpha/2} \frac{|x_N^\alpha D^2 u(x, t) - \bar{x}_N^\alpha D^2 u(\bar{x}, t)|}{|x_N - \bar{x}_N|^\gamma} \leq C \left(|x_N^{\alpha-\beta} D^2 u(x, t)| + |\bar{x}_N^{\alpha-\beta} D^2 u(\bar{x}, t)| \right) \quad (2.79)$$

and then proceeding as in (2.78).

Let now

$$0 < (x_N - \bar{x}_N) \leq \varepsilon_0 x_N, \quad (2.80)$$

and let also

$$\Pi(x_N) = \{y \in R_+^N : x_N - 2\varepsilon_0 x_N \leq y_N \leq x_N + 2\varepsilon_0 x_N\},$$

$$\Pi_T(x_N) = \Pi(x_N) \times [0, T]. \quad (2.81)$$

Then, taking into account that on $\Pi_T(x_N)$ we have $y_N \sim x_N$, as in the previous case

$$\begin{aligned} \frac{|x_N^\alpha D^2 u(x, t) - \bar{x}_N^\alpha D^2 u(\bar{x}, t)|}{|x_N - \bar{x}_N|^\beta} &\leq \langle y_N^\alpha D^2 u(y, t) \rangle_{y, \Pi_T(x_N)}^{(\beta)} \\ &\leq C \left(|y_N^{\alpha-\beta} D^2 u|_{\Pi_T(x_N)}^{(0)} + x_N^\alpha \langle D^2 u(y, t) \rangle_{y, \Pi_T(x_N)}^{(\beta)} \right) \equiv A_1 + A_2 \end{aligned} \quad (2.82)$$

Here A_1 is estimated in the same way as in (2.78), and A_2 —as well as in (2.74). This gives

$$\frac{|x_N^\alpha D^2 u(x, t) - \bar{x}_N^\alpha D^2 u(\bar{x}, t)|}{|x_N - \bar{x}_N|^\beta} \leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})}. \quad (2.83)$$

The estimate

$$x_N^{\gamma\alpha/2} \frac{|x_N^\alpha D^2 u(x, t) - \bar{x}_N^\alpha D^2 u(\bar{x}, t)|}{|x_N - \bar{x}_N|^\gamma} \leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})} \quad (2.84)$$

is quite similar.

This completes the proof of (2.71).

Similarly, we obtain the properties with respect to the variables x of the derivative u_t , that is,

$$H_\alpha^\gamma(u_t) \leq C |f|_{C_x^{2+\beta-\alpha}(R_T^{N-1})} \quad (2.85)$$

because u_t satisfies problem (2.54)–(2.56) with boundary condition (2.60). Indeed, since $u_t|_{x_N=0} = f_t$,

$$\langle u_t \rangle_{x, R_{+T}^N}^{(\beta)} \leq C \langle f_t \rangle_{x, R_T^{N-1}}^{(\beta)} \leq C |f|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(R_T^{N-1})}. \quad (2.86)$$

Further, for $x, \bar{x} \in R_+^N$, $x_N \geq \bar{x}_N$ consider the difference

$$\Delta(x, \bar{x}) u_t = x_N^{\gamma\alpha/2} \frac{|u_t(x, t) - u_t(\bar{x}, t)|}{|x - \bar{x}|^\gamma}. \quad (2.87)$$

If $|x - \bar{x}| \geq \varepsilon_0 x_N$, then

$$\begin{aligned} \Delta(x, \bar{x})u_t &\leq C \frac{x_N^{\gamma\alpha/2}}{x_N^{\gamma\alpha/2}} \frac{|u_t(x, t) - u_t(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq C \langle u_t \rangle_{x, R_{+T}^N}^{(\beta)} \\ &\leq C |f|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(R_T^{N-1})}. \end{aligned} \quad (2.88)$$

If now $|x - \bar{x}| \leq \varepsilon_0 x_N$, then $x_N \sim \bar{x}_N$ and using (2.72) we obtain

$$\begin{aligned} \Delta(x, \bar{x})u_t &\leq C x_N^{\gamma\alpha/2} \frac{|u_t(x, t) - u_t(\bar{x}, t)|}{|x - \bar{x}|} |x - \bar{x}|^{1-\gamma} \\ &\leq C x_N^{\gamma\alpha/2} |\nabla_x u_t(x, t)|_{\Pi_T(x_N)}^{(0)} x_N^{1-\gamma} \\ &\leq C |f_t|_{C_x^\beta(R_T^{N-1})} x_N^{\gamma\alpha/2+1-\gamma} x_N^{-1+\beta} \\ &= C |f_t|_{C_x^\beta(R_T^{N-1})}. \end{aligned} \quad (2.89)$$

Now (2.85) follows from (2.88) and (2.89).

Let us show now the smoothness of the function $x_N^\alpha D^2 u(x, t)$ with respect to the variable t , that is show that

$$\langle x_N^\alpha D^2 u \rangle_{t, R_{+T}^N}^{(\gamma/2)} \leq C |f|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(R_T^{N-1})} = C |f|_{C^{2+\beta-\alpha, 1+\gamma/2}(R_T^{N-1})}. \quad (2.90)$$

For this we fix some $h > 0$ and consider the function

$$v_h(x, t) = \frac{u(x, t+h) - u(x, t)}{h^{\gamma/2}}, \quad (2.91)$$

which satisfies problem (2.54)–(2.56) with the following boundary condition

$$v_h(x, t)|_{x_N=0} = \varphi_h(x, t) \equiv \frac{f(x, t+h) - f(x, t)}{h^{\gamma/2}}. \quad (2.92)$$

It follows from the results of [45], (1.20) that $\varphi_h(x, t) \in C_x^{2-\alpha}(R^{N-1})$ with respect to the variables x uniformly with respect to the variable t and

$$\max_{t \in [0, T]} |\varphi_h(\cdot, t)|_{R^{N-1}}^{(2-\alpha)} \leq C |f|_{C^{2+\beta-\alpha, 1+\gamma/2}(R_T^{N-1})}. \quad (2.93)$$

Now note that

$$\frac{x_N^\alpha D^2 u(x, t+h) - x_N^\alpha D^2 u(x, t)}{h^{\gamma/2}} = x_N^\alpha D^2 v_h(x, t).$$

Consequently, it follows from (2.72) that

$$|x_N^\alpha D^2 v_h(x, t)| \leq C x_N^\alpha \max_{t \in [0, T]} |\varphi_h(\cdot, t)|_{R^{N-1}}^{(2-\alpha)} x_N^{-2+(2-\alpha)} = C \max_{t \in [0, T]} |\varphi_h(\cdot, t)|_{R^{N-1}}^{(2-\alpha)}. \quad (2.94)$$

So (2.90) follows from (2.94) and (2.93) in view of the definition of $v(x, t)$.

Multiplying now the function $u(x, t)$ by a smooth cut-of function $\eta(x)$ with finite support that is equal to one in a neighborhood of support of $f(x', t)$, we get an extension with finite support of $f(x', t)$ of desired class and the estimate (2.53) is valid.

Lemma 2.4 is proved. $\square$

From this lemma in the standard way (see [43]) as it was described in the beginning of this section we get the following assertion.

Lemma 2.5. *There exist bounded extension operators E^+ and E^- , such that*

$$\rho \in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(\Gamma_T) \rightarrow E\rho \equiv E^\pm \rho \equiv \rho^\pm \in C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm), \quad (2.95)$$

$$|E^\pm \rho|_{C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)} \leq C |\rho|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(\Gamma_T)}, \quad (2.96)$$

and we can assume that the supports of the extended functions $\rho^\pm$ are included in the sufficiently small neighbourhood $\mathcal{N}_T$ of the surface Γ_T .

We will denote the extended functions $\rho \equiv E\rho \equiv E^\pm \rho$ by the same symbol ρ to not to overload the notation, that is,

$$\rho|_{\overline{\Omega}_T^\pm} \equiv \rho^\pm \equiv E^\pm \rho \equiv E\rho|_{\overline{\Omega}_T^\pm}. \quad (2.97)$$

Due to Lemma 2.4 and to the results of [42] we have the following assertion.

Lemma 2.6. *Let functions f and g are finite and*

$$f(x', t) \in C_0^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(R_T^{N-1}), \quad g(x, t) \in C_{s,0}^\gamma(R_{+T}^N).$$

Then the problem

$$\frac{\partial u}{\partial t} - x_N^\alpha \Delta u = g(x, t), \quad (x, t) \in R_{+T}^N, \quad (2.98)$$

$$u(x', 0, t) = f(x', t), \quad x_N = 0, t \in [0, T], \quad (2.99)$$

$$u(x, 0) = 0, \quad x \in R_+^N \quad (2.100)$$

has the unique solution $u(x, t)$, which satisfies the estimate

$$|u|_{s, R_{+T}^N}^{(2+\gamma)} \leq C \left(|f|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(R_T^{N-1})} + |g|_{s, R_{+T}^N}^{(\gamma)} \right). \quad (2.101)$$

For the case $f = 0$ this lemma was proved in [42] and general case is reduced to such case by Lemma 2.4.

Consider now the initial-boundary value problem for the degenerate equations of the form

$$\frac{\partial u^\pm}{\partial t} - (d^\pm(x))^\alpha B^\pm(x, t) \Delta u^\pm = g^\pm(x, t), \quad (x, t) \in \Omega_T^\pm, \quad (2.102)$$

$$u^\pm|_{\Gamma_T} = f^\pm(x, t), \quad (2.103)$$

$$u^\pm|_{\Gamma_T^\pm} = h^\pm(x, t), \quad (2.104)$$

$$u^\pm(x, 0) = 0, \quad (2.105)$$

where the functions $d^\pm(x)$ are introduced in (1.11),

$$\begin{aligned} B^\pm(x, t) &\in C^{\gamma, \gamma/2}(\overline{\Omega}_T^\pm), \quad \nu \leq B^\pm \leq \nu^{-1}, \\ g^\pm &\in C_{0,s}^{\gamma, \gamma/2}(\overline{\Omega}_T^\pm), \quad f^\pm \in C_0^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(\Gamma_T), \quad h^\pm \in C_0^{2+\gamma, \frac{2+\gamma}{2}}(\Gamma_T^\pm). \end{aligned} \quad (2.106)$$

Theorem 2.7. *Problem (2.102)–(2.105) has the unique solution from the space $C_{0,s}^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)$ and the following estimate is valid*

$$|u^\pm|_{s, \overline{\Omega}_T^\pm}^{(2+\gamma)} \leq C \left(|g^\pm|_{s, \overline{\Omega}_T^\pm}^{(\gamma)} + |f^\pm|_{\Gamma_T}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2})} + |h^\pm|_{\Gamma_T^\pm}^{(2+\gamma, \frac{2+\gamma}{2})} \right). \quad (2.107)$$

This theorem was proved in [42] for the case $f^\pm = 0$ and from Lemma 2.5 we get this assertion for general case.

3. A model problem with dynamic boundary conditions for degenerate equations

In this section we consider the simplest problem in half-spaces, which corresponds to the linearization of the original nonlinear Stefan problem. This model problem is a parabolic problem with dynamic boundary conditions. It is convenient in this section to consider the domains with $t \geq -a$ as it will allow us to consider the points $(x, 0)$ with $t = 0$ as interior points of a general position. This will facilitate the further notation and preserve the notation of paper [53].

Let $a \geq 0$ be a fixed number. Denote $Q_\pm^N = \{(x, t) : x \in R_\pm^N, t \geq -a\}$, $Q^{N-1} = \{(x', t) : x' \in R^{N-1}, t \geq -a\}$. Denote further $R_{\pm, T}^{N,a} = Q_\pm^N \cap \{t \leq T\}$, $R_T^{N-1,a} = Q^{N-1} \cap \{t \leq T\}$. Similar to (1.21) we agree that zero at the bottom of the designation of functional spaces means the subspace of the corresponding space whose elements vanish at $t = -a$ together with all its derivatives with respect to t . Let a function $f(x', t)$ is defined in Q^{N-1} , has finite with respect to x support, and let $f(x', t)$ is such that

$$f(x', -a) \equiv 0, \quad f \in C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(\overline{Q}^{N-1}). \quad (3.1)$$

This allows us to consider f as a function, which is defined for $t \in (-\infty, \infty)$ and which is identical zero in the domain $t < -a$ with the preservation of the smoothness. Let further

$$f_1^\pm(x, t) \in C_{s,0}^{\gamma, \gamma/2}(\overline{Q}_\pm^N), \quad f_2^\pm(x', t) \in C_0^{2+\beta-\alpha, 1+\gamma/2}(\overline{Q}^{N-1}) \quad (3.2)$$

are given functions, which are also extended by identical zero in the domain $t < -a$ with the preservation of the classes and have finite support.

Consider the following model problem for the triple of unknown functions $u^\pm(x, t)$ and $\rho(x', t)$, which are defined on $\overline{Q}_\pm^N$ and on $\overline{Q}^{N-1}$ correspondingly:

$$\frac{\partial u^\pm}{\partial t} - (\pm x_N)^\alpha \Delta u^\pm = f_1^\pm, \quad (x, t) \in Q_\pm^N, \quad (3.3)$$

$$u^\pm + A^\pm \rho = f_2^\pm, \quad x_N = 0, (x', t) \in Q^{N-1}, \quad (3.4)$$

$$\rho_t - \varepsilon \Delta_{x'} \rho + b^+ \frac{\partial u^+}{\partial x_N} - b^- \frac{\partial u^-}{\partial x_N} = f(x', t), \quad x_N = 0, (x', t) \in Q^{N-1}, \quad (3.5)$$

$$u^\pm(x, -a) = 0, \quad \rho(x', -a) = 0, \quad (3.6)$$

$$u^\pm \in C_{s,0}^{2+\gamma, \frac{2+\gamma}{2}}(\overline{Q}_\pm^N), \quad \rho \in C_0^{2+\beta-\alpha, 1+\gamma/2}(\overline{Q}^{N-1}), \quad \rho_t \in C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\overline{Q}^{N-1}). \quad (3.7)$$

Here $A^\pm$, $b^\pm$, ε are given positive constants and $\varepsilon \in (0, 1)$. Note that the term with ε in (3.5) does not apply directly to Stefan problem and serves as a regularization of the problem that will be needed in the proof of the solvability of the corresponding linearized Stefan problem in an arbitrary domain. To the author's knowledge, this regularization of the boundary condition in the Stefan problem was first used in paper [1]. Let us remark also that problem (3.3)–(3.7) is a parabolic problem for degenerate equations with dynamic boundary conditions. To see this it is enough to express the unknown ρ from relation (3.4) as $\rho = -u^\pm/A^\pm + f_2^\pm/A^\pm$ and then to substitute it in relation (3.5).

Below we prove the following a priori estimate of a solution of problem (3.3)–(3.7).

Theorem 3.1. *Let $u^\pm(x, t) \in C_{s,0}^{2+\gamma, \frac{2+\gamma}{2}}(\overline{Q}_\pm^N)$, $\rho \in C_0^{3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha}}(\overline{Q}^{N-1})$ give a solution with finite support of problem (3.3)–(3.7). Then for arbitrary $T > 0$ the following estimate is valid*

$$\begin{aligned} \mathcal{U}(T) &\equiv |u^+|_{s, R_{+T}^{N,a}}^{(2+\gamma)} + |u^-|_{s, R_{-T}^{N,a}}^{(2+\gamma)} + \varepsilon \sum_{i,j=1}^{N-1} |\rho_{x_i x_j}|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} \\ &\quad + |\rho|_{R_T^{N-1,a}}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha})} + |\rho_t|_{R_T^{N-1,a}}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} \\ &\leq C_T \left(|f_1^+|_{s, R_{+T}^{N,a}}^{(\gamma)} + |f_1^-|_{s, R_{-T}^{N,a}}^{(\gamma)} + |f_2^+|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} + \right. \\ &\quad \left. + |f_2^-|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} + |f|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} \right) \equiv C_T \mathcal{M}(T), \end{aligned} \quad (3.8)$$

where the constant C_T in (3.8) does not depend on $\varepsilon \in (0, 1)$.

Subsequent content of this section is devoted to the proof of Theorem 3.1.

Note first that by Lemmas 2.6 and 2.2 we can without loss of generality assume that

$$f_1^\pm \equiv 0, \quad f_2^\pm \equiv 0, \quad (3.9)$$

since the general case can be reduced to the specified one by the change of the unknown functions $u^\pm = v^\pm + w^\pm$, where $v^\pm$ are the new unknowns, and $w^\pm$ satisfy (3.3) with the boundary conditions

$$w^\pm|_{x_N=0} = f_2^\pm(x', t). \quad (3.10)$$

Thus, further we assume that only the function $f(x', t)$ is nonzero in the right-hand sides of (3.3)–(3.5).

In addition, because the right sides of relations (3.3)–(3.5) belong to the classes with zero at the bottom and because of conditions (3.6), (3.7) we can

consider that relation (3.3)–(3.5) are valid for $t < -a$ assuming that all the functions are extended by zero to this domain.

An important point of proving (3.8) is to prove the following a priori estimate.

Lemma 3.2. *Under the conditions of Theorem 3.1 and under condition (3.9) the following estimate is valid*

$$\begin{aligned} \langle \nabla_{x'} \rho \rangle_{x', R_T^{N-1, a}}^{(1+\beta-\alpha)} &\leq C_T \left(|\nabla_{x'} u^+|_{R_{+T}^{N, a}}^{(0)} + |\nabla_{x'} u^-|_{R_{-T}^{N, a}}^{(0)} + |\nabla_{x'} \rho|_{R_T^{N-1, a}}^{(0)} + \mathcal{M}(T) \right) \\ &\equiv C_T (\mathcal{N}(T) + \mathcal{M}(T)) \leq C_T (T + a)^\delta \mathcal{U}(T) + C_T \mathcal{M}(T), \end{aligned} \quad (3.11)$$

where

$$\mathcal{N}(T) \equiv |\nabla_{x'} u^+|_{R_{+T}^{N, a}}^{(0)} + |\nabla_{x'} u^-|_{R_{-T}^{N, a}}^{(0)} + |\nabla_{x'} \rho|_{R_T^{N-1, a}}^{(0)}.$$

To obtain the last inequality in (3.11) we use estimates (2.1)–(2.4).

Proof. Denote for brevity $l = 1 + \beta - \alpha$ and fix a point (x'_0, t_0) in the set $R_T^{N-1, a}$. We are going to apply the method of paper [53]. So in order to maintain the succession of the notation with this paper we will assume without loss of generality that $(x'_0, t_0) = (0, 0)$. This choice is not important as can be seen from the following proof. Suppose further that $O = (x' = 0, x_N = 0, t = 0)$ is the corresponding point in $R_{+T}^{N, a}$. We show that for every $h \in (0, 1)$ and for any $i, j = \overline{1, N-1}$ we have the following inequality

$$|\rho_{x_i}(\vec{e}_j h, 0) - \rho_{x_i}(-\vec{e}_j h, 0)| \leq C_T (\mathcal{N}(T) + \mathcal{M}(T)) h^l, \quad (3.12)$$

where $\vec{e}_j$ is the unit vector of the Ox_j -axis. Since the point O and the step $h \in (0, 1)$ in relation (3.12) are arbitrary, estimate (3.11) of the lemma follows from estimate (3.12).

Let $y_1, y_2 \in [0, 1]$, $y \equiv (y_1, y_2)$ and let also $i, j \in \{1, 2, \dots, N-1\}$ are fixed. Consider the differences

$$\begin{aligned} v^\pm(x, t, y_1, y_2) &= \Delta_{i, y_1} \Delta_{j, y_2} u^\pm(x, t) \\ &= u^\pm(x + y_1 \vec{e}_i + y_2 \vec{e}_j, t) - u^\pm(x - y_1 \vec{e}_i + y_2 \vec{e}_j, t) \\ &\quad - u^\pm(x + y_1 \vec{e}_i - y_2 \vec{e}_j, t) + u^\pm(x + y_1 \vec{e}_i + y_2 \vec{e}_j, t), \end{aligned} \quad (3.13)$$

where

$$\Delta_{k, h} u(x, t) \equiv u(x + h \vec{e}_k, t) - u(x - h \vec{e}_k, t). \quad (3.14)$$

Denote also

$$r(x', t, y_1, y_2) = \Delta_{i, y_1} \Delta_{j, y_2} \rho(x', t). \quad (3.15)$$

Note that

$$\frac{\partial^2 v^\pm}{\partial x_i^2} - \frac{\partial^2 v^\pm}{\partial y_1^2} = 0, \quad \frac{\partial^2 v^\pm}{\partial x_j^2} - \frac{\partial^2 v^\pm}{\partial y_2^2} = 0. \quad (3.16)$$

Therefore in the domains $R_{\pm}^N \times \{-\infty < t < T\} \times \{0 < y_1 < 1\} \times \{0 < y_2 < 1\}$ the functions $v^{\pm}(x, t, y)$ satisfy the equations

$$L^*v^{\pm} \equiv \frac{\partial v^{\pm}}{\partial t} - (\pm x_N)^{\alpha} \left(\sum_{k \neq i, j} \frac{\partial^2 v^{\pm}}{\partial x_k^2} + \frac{3}{4} \frac{\partial^2 v^{\pm}}{\partial x_i^2} + \frac{3}{4} \frac{\partial^2 v^{\pm}}{\partial x_j^2} + \frac{1}{4} \frac{\partial^2 v^{\pm}}{\partial y_1^2} + \frac{1}{4} \frac{\partial^2 v^{\pm}}{\partial y_2^2} \right) = 0, \quad (3.17)$$

Also note that

$$\begin{aligned} |v^{\pm}| &= |\Delta_{i, y_1} \Delta_{j, y_2} u^{\pm}(x, t)| \\ &= y_2 \left| \Delta_{i, y_1} \int_{-1}^1 u_{x_j}^{\pm}(x + \omega y_2 \vec{e}_j) d\omega \right| \leq 4y_2 \mathcal{N}(T). \end{aligned}$$

Exactly the same way

$$|v^{\pm}| \leq 4y_1 \mathcal{N}(T),$$

and therefore

$$|v^{\pm}| \leq 4y_{\min} \mathcal{N}(T), \quad (3.18)$$

where

$$y_{\min} = \min \{y_1, y_2\}. \quad (3.19)$$

Similarly, we have

$$|r| \leq 4y_{\min} \mathcal{N}(T), \quad (3.20)$$

Denote

$$y = (y_1, y_2), \quad P^{\pm} = \{(x, t, y) : |x_m| < 1, m = \overline{1, N-1}, 0 < \pm x_N < 1, -1 < t < 0, 0 < y_k < 1, k = 1, 2\}. \quad (3.21)$$

Denote also

$$\Sigma^{\pm} = \partial P^{\pm} \setminus (\{t = 0\} \cup \{x_N = 0\}), \quad \Sigma_0 = \partial P^{\pm} \cap \{x_N = 0\}, \quad (3.22)$$

that is, $\Sigma^{\pm}$ are parabolic boundaries of the parallelepipeds $P^{\pm}$ without their common part $\{x_N = 0\}$. This common part will be denoted by Σ_0 .

In the parallelepipeds $P^{\pm}$ the functions $v^{\pm}$ and r satisfy the following problem

$$L^*v^{\pm} = 0, \quad (x, t, y) \in P^{\pm}, \quad (3.23)$$

$$v^{\pm}|_{\Sigma^{\pm}} = g^{\pm}(x, t, y), \quad (3.24)$$

$$v^{\pm} + A^{\pm}r = 0, \quad x_N = 0, \quad (3.25)$$

$$r_t - \varepsilon \Delta_{x'} r + b^+ \frac{\partial v^+}{\partial x_N} - b^- \frac{\partial v^-}{\partial x_N} = F(x, y, t), \quad x_N = 0, \quad (3.26)$$

where

$$F(x, t, y) = \Delta_{i, y_1} \Delta_{j, y_2} f, \quad g^\pm(x, t, y) = \Delta_{i, y_1} \Delta_{j, y_2} u^\pm. \quad (3.27)$$

Note that in view of assumptions (3.1)

$$|F(x, t, y)| \leq 2 \langle f \rangle_{x_j}^{(l)} (2y_2)^l.$$

A similar inequality with replacing j by i and y_2 by y_1 gives similar to (3.18)

$$|F(x, t, y)| \leq C\mathcal{M}(T)y_{\min}^l. \quad (3.28)$$

Also note that by (3.18)

$$|g^\pm(x, t, y)| \leq C\mathcal{N}(T)y_{\min}. \quad (3.29)$$

To estimate $v^\pm$ and r we're going to apply to the problem (3.23)–(3.26) the maximum principle in the following form.

Lemma 3.3. *Let functions $H^\pm(x, t, y) \in C^{2,1}(P^\pm) \cap C^{1,0}(\overline{P}^\pm)$, $S(x', t, y) \in C^{2,1}(\Sigma_0)$ satisfy the conditions*

$$L^*H^\pm \geq 0, \quad (x, t, y) \in P^\pm, \quad (3.30)$$

$$H^\pm|_{\Sigma^\pm} \geq 0, \quad (3.31)$$

$$H^\pm + A^\pm S = 0, \quad x_N = 0, \quad (3.32)$$

$$S_t - \varepsilon \Delta_{x'} S + b^+ \frac{\partial H^+}{\partial x_N} - b^- \frac{\partial H^-}{\partial x_N} \leq 0, \quad x_N = 0. \quad (3.33)$$

Then

$$H^\pm \geq 0, \quad (x, t, y) \in \overline{P}^\pm; \quad S \leq 0, \quad (x', t, y) \in \Sigma_0. \quad (3.34)$$

We do not give a detailed proof of this lemma since it uses standard arguments. We only note that the functions $H^\pm$ can not reach a negative minimum at $\{x_N = 0\}$. In this case by (3.32) they would reached a negative minimum simultaneously and corresponding point would be, again by (3.32), a point of a positive maximum of the function S . All this together in the standard way contradicts boundary condition (3.33).

We shall need the auxiliary functions $w^\pm(x, t)$ defined on

$$\Pi^\pm = \{|x_m| \leq 1, m = \overline{1, N-1}, 0 \leq \pm x_N \leq 1, -1 \leq t \leq 0\}$$

correspondingly and such that

$$L_{xt}^* w^\pm = \frac{\partial w^\pm}{\partial t} - (\pm x_N)^\alpha \left(\sum_{k \neq i, j} \frac{\partial^2 w^\pm}{\partial x_k^2} + \frac{3}{4} \frac{\partial^2 w^\pm}{\partial x_i^2} + \frac{3}{4} \frac{\partial^2 w^\pm}{\partial x_j^2} \right) = 0, \quad x_N \neq 0, \quad (3.35)$$

$$w^\pm|_{\{|x_k|=1\} \cup \{t=-1\}} \geq \nu > 0, \quad (3.36)$$

$$w^\pm(0, 0) = 0, \quad w^\pm|_{\Pi^\pm} \geq 0, \quad (3.37)$$

$$w^\pm(x, t) \in C^{2,1}(\Pi^\pm \cap \{x_N = 0\}). \quad (3.38)$$

Such functions can be constructed as follows. Consider, for example, $w^+(x, t)$. Let $G^+(x, t)$ be a function from C^∞ in $\overline{R_+^N} \times (-\infty, \infty)$ such that $G^+ \equiv 0$ for $|x| + |t| \leq 1/4$ and for $t \leq -2$, $|x| \geq 2$ and $G^+ > 0$ in the other points of $\overline{R_+^N} \times (-\infty, \infty)$. Let $w^+(x, t)$ be the solution of the following initial boundary value problem in half-space

$$L_{x,t}^* w^+ = 0, \quad x_N > 0, \quad t > -2,$$

$$w^+|_{x_N=0} = G^+(x, t) \in C^\infty,$$

$$w^+|_{t=-2} = 0.$$

Lemma 2.6 implies that the function w^+ exists in the appropriate class and

$$|w^+|_{s, \overline{R_+^N} \times [-2, 0]}^{(2+\gamma)} \leq C(G^+). \quad (3.39)$$

Because of the properties of $G^+(x, t)$ and by the strong maximum principle (see [59]) the function w^+ has all desirable properties including (3.36).

Now consider the following comparison functions defined on $\overline{P}^\pm$. Denote

$$\varphi(y) = y_1 y_2 (y_1^l + y_2^l)^{-\frac{1}{l}+1}, \quad (3.40)$$

$$\psi^\pm(x_N, y) = y_1 y_2 [(y_1 \pm x_N)^l + (y_2 \pm x_N)^l]^{-\frac{1}{l}+1}, \quad \pm x_N \geq 0, \quad (3.41)$$

$$\theta^\pm(x, t, y) = \begin{cases} (y_1^{-1} + y_2^{-1})^{-1} w^\pm(x, t), & y_{\min} > 0, \\ 0, & y_{\min} = 0. \end{cases} \quad (3.42)$$

The direct verification shows (cf. [53]), that the functions φ and $\psi^\pm$ possess properties

$$\pm \frac{\partial \psi^\pm}{\partial x_N} \Big|_{x_N=0} \leq -\nu y_{\min}^l, \quad (3.43)$$

$$|L^* \psi^\pm| \leq C |x_N|^\alpha y_{\min}^{-1+l}, \quad (3.44)$$

$$\frac{\partial \varphi}{\partial x_N} \Big|_{x_N=0} = 0, \quad (3.45)$$

$$L^* \varphi \geq \nu |x_N|^\alpha y_{\min}^{-1+l}, \quad (3.46)$$

$$\varphi|_{y_k=1} \geq \nu y_{\min}. \quad (3.47)$$

Thus, if we choose a sufficiently large constant $K > 0$ the functions

$$h^\pm \equiv \psi^\pm + K\varphi \quad (3.48)$$

will have the properties

$$\pm \frac{\partial h^\pm}{\partial x_N} \Big|_{x_N=0} \leq -\nu y_{\min}^l, \quad (3.49)$$

$$L^* h^\pm \geq \nu |x_N|^\alpha y_{\min}^{-1+l} > 0, \quad (x, t, y) \in P^\pm, \quad (3.50)$$

$$h^\pm|_{y_k=1} \geq \nu y_{\min}. \quad (3.51)$$

At the same time the functions $\theta^\pm(x, t, y)$ have the properties

$$L^* \theta^\pm \geq 0, \quad (x, t, y) \in P^\pm, \quad (3.52)$$

$$\left| \frac{\partial \theta^\pm}{\partial x_N} \right|_{x_N=0} \leq C y_{\min}, \quad (3.53)$$

$$\theta^\pm|_{\cup_k \{|x_k|=1\} \cup \{t=-1\}} \geq \nu y_{\min}. \quad (3.54)$$

Consider now the following comparison functions

$$H^\pm(x, t, y) \equiv A^\pm [L_1 \theta^\pm(x, t, y) + L_2 h^\pm(x_N, y)] (N(T) + M(T)), \quad (3.55)$$

$$S(x', t, y) = - [L_1 \theta^\pm(x', 0, t, y) + L_2 h^\pm(0, y)] (N(T) + M(T)), \quad (3.56)$$

where L_1 and L_2 are some positive constants.

Choosing first L_1 and then L_2 sufficiently large and using on one hand (3.28), (3.29), and on the other hand (3.49)–(3.54), we see that the triple of the functions

$$H^+ \pm v^+(x, t, y), \quad H^- \pm v^-(x, t, y), \quad S \pm r(x', t, y)$$

satisfies in $P^\pm$ the conditions of Lemma 3.3. Hence,

$$H^+ \pm v^+ \geq 0, \quad H^- \pm v^- \geq 0, \quad S \pm r \leq 0,$$

that is,

$$|v^\pm| \leq CH^\pm, \quad |r| \leq C|S|, \quad (x, t, y) \in \bar{P}^\pm. \quad (3.57)$$

Take in (3.57) $x = 0$, $t = 0$. Then in view of $\theta^\pm(0, 0, y) = 0$ we obtain for $v^+(0, 0, y_1, y_2)$ similarly [53]

$$|v^+(0, 0, y_1, y_2)| = |\Delta_{i, y_1} \Delta_{j, y_2} u^+(0, 0)| \leq C [\mathcal{N}(T) + \mathcal{M}(T)] y_1 y_2 (y_1^l + y_2^l)^{-\frac{1}{l}+1}.$$

Dividing both sides of this relation by y_1 and taking the limit as $y_1 \rightarrow 0$ we obtain

$$\left| \frac{\partial u^+}{\partial x_i}(0 + y_2 \vec{e}_j, 0) - \frac{\partial u^+}{\partial x_i}(0 - y_2 \vec{e}_j, 0) \right| \leq C [\mathcal{N}(T) + \mathcal{M}(T)] y_2^l, \quad (3.58)$$

and similarly

$$\left| \frac{\partial \rho}{\partial x_i}(0 + y_2 \vec{e}_j, 0) - \frac{\partial \rho}{\partial x_i}(0 - y_2 \vec{e}_j, 0) \right| \leq C [\mathcal{N}(T) + \mathcal{M}(T)] y_2^l. \quad (3.59)$$

Since all of the above arguments are valid for any $(x'_0, t_0) \in R_T^{N-1, a}$, by the same token estimate (3.11) and Lemma 3.2 are proved. $\square$

We continue the proof of the theorem. From (3.5) it follows that

$$\rho_t - \varepsilon \Delta_{x'} \rho = F(x', t) \equiv -b^+ \frac{\partial u^+}{\partial x_N} + b^- \frac{\partial u^-}{\partial x_N} + f. \quad (3.60)$$

Moreover, in view of inequalities (1.33), (2.1), (2.4)

$$\begin{aligned} |F|_{R_T^{N-1}}^{(\gamma)} &\leq C \left(|\nabla u^+|_{R_T^{N-1,a}}^{(\gamma)} + |\nabla u^-|_{R_T^{N-1,a}}^{(\gamma)} \right) + C\mathcal{M}(T) \\ &\leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T), \end{aligned} \quad (3.61)$$

and

$$F(x', 0) \equiv 0.$$

Making in problem (3.60), (3.6) the change of variables $x' = \varepsilon^{1/2}y$ we obtain the problem

$$\rho_t - \Delta_y \rho = \tilde{F}(y, t), \quad (y, t) \in R_T^{N-1,a}, \quad \rho(y, -a) = 0, \quad (3.62)$$

and

$$\left\langle \tilde{F}(y, t) \right\rangle_{t, R_T^{N-1,a}}^{(\gamma/2)} = \left\langle F(x', t) \right\rangle_{t, R_T^{N-1,a}}^{(\gamma/2)}, \quad (3.63)$$

$$\left\langle \tilde{F}(y, t) \right\rangle_{y, R_T^{N-1,a}}^{(\gamma)} = \varepsilon^{\gamma/2} \left\langle F(x', t) \right\rangle_{x', R_T^{N-1,a}}^{(\gamma)}.$$

From the arguments of [43], Chapter IV it follows that

$$\left\langle \rho_{y_i y_j} \right\rangle_{y, R_T^{N-1,a}}^{(\gamma)} \leq C \left\langle \tilde{F}(y, t) \right\rangle_{y, R_T^{N-1,a}}^{(\gamma)}, \quad (3.64)$$

$$\left\langle \rho_t \right\rangle_{t, R_T^{N-1,a}}^{(\gamma/2)} \leq C \left(\left\langle \tilde{F}(y, t) \right\rangle_{y, R_T^{N-1,a}}^{(\gamma)} + \left\langle \tilde{F}(y, t) \right\rangle_{t, R_T^{N-1,a}}^{(\gamma/2)} \right). \quad (3.65)$$

Making in (3.64), (3.65) the inverse change of variables in view of (3.63) we obtain

$$\left\langle \rho_t(x', t) \right\rangle_{t, R_T^{N-1,a}}^{(\gamma/2)} + \varepsilon \sum_{i,j} \left\langle \rho_{x_i x_j} \right\rangle_{x', R_T^{N-1,a}}^{(\gamma)} \leq C |F(x', t)|_{R_T^{N-1,a}}^{(\gamma)}. \quad (3.66)$$

Thus, in view of estimate (3.11) of Lemma 3.2 it is proved that

$$\left\langle \rho_t(x', t) \right\rangle_{t, R_T^{N-1,a}}^{(\gamma/2)} + \left\langle \nabla_{x'} \rho \right\rangle_{x', R_T^{N-1,a}}^{(1+\beta-\alpha)} + \varepsilon |\rho|_{R_T^{N-1,a}}^{(2+\gamma)} \leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T),$$

or, in view of (1.20) ([45]) and due to the finiteness of ρ

$$|\rho|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} + \varepsilon |\rho|_{R_T^{N-1,a}}^{(2+\gamma)} \leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T), \quad (3.67)$$

where the constant C does not depend on $\varepsilon > 0$.

Consider now $u^\pm(x, t)$ as the solution of initial-boundary value problem (3.3), (3.4), (3.6). By Lemma 2.6 and estimate (3.67) we conclude that

$$|u^+|_{s, R_{+T}^{N,a}}^{(2+\gamma)} + |u^-|_{s, R_{-T}^{N,a}}^{(2+\gamma)} \leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T). \quad (3.68)$$

It follows that in condition (3.5)

$$\begin{aligned} \left| \frac{\partial u^+}{\partial x_N} \right|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} + \left| \frac{\partial u^-}{\partial x_N} \right|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} \\ \leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T) \end{aligned}$$

Thus, the function $\rho(x', t)$ satisfies the Cauchy problem (3.60), (3.6) with the right hand side F and the last has the property

$$F(x', -a) = 0, \quad |F|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(R_T^{N-1, a})} \leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T). \quad (3.69)$$

Making again in (3.60), (3.6) the change of variables $x' = \varepsilon^{1/2}y$, we arrive at the problem of the form (3.62) with $\tilde{F}$, where $\tilde{F}$ is such that

$$\langle \tilde{F} \rangle_{t, R_T^{N-1, a}}^{(\frac{1+\beta-\alpha}{2-\alpha})} = \langle F \rangle_{t, R_T^{N-1, a}}^{(\frac{1+\beta-\alpha}{2-\alpha})}, \quad \langle \tilde{F} \rangle_{y, R_T^{N-1, a}}^{(1+\beta-\alpha)} = \varepsilon^{\frac{1+\beta-\alpha}{2}} \langle F \rangle_{x', R_T^{N-1, a}}^{(1+\beta-\alpha)}. \quad (3.70)$$

As above, completely similar to [43], Chapter IV, we have the estimates for solutions of problem (3.62)

$$\langle \rho_t \rangle_{t, R_T^{N-1, a}}^{(\frac{1+\beta-\alpha}{2-\alpha})} \leq C \langle \tilde{F} \rangle_{t, R_T^{N-1, a}}^{(\frac{1+\beta-\alpha}{2-\alpha})} \leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T), \quad (3.71)$$

$$\sum_{i,j=1}^{N-1} \langle \rho_{y_i y_j} \rangle_{y, R_T^{N-1, a}}^{(1+\beta-\alpha)} \leq C \langle \tilde{F} \rangle_{y, R_T^{N-1, a}}^{(1+\beta-\alpha)}. \quad (3.72)$$

Note that in obtaining estimate (3.71) condition $F(x', -a) = 0$ is important.

Proceeding as before and going back to the variables x' we find from (3.72) and (3.70) that

$$\varepsilon \sum_{i,j=1}^{N-1} \langle \rho_{x_i x_j} \rangle_{x', R_T^{N-1, a}}^{(1+\beta-\alpha)} \leq C \langle F \rangle_{x', R_T^{N-1, a}}^{(1+\beta-\alpha)} \leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T). \quad (3.73)$$

Now combining estimates (3.73), (3.71), (3.68) and (3.67) we find that

$$\mathcal{U}(T) \leq C(T+a)^\mu \mathcal{U}(T) + C\mathcal{M}(T). \quad (3.74)$$

Taking now in (3.74) $T = T_0$ so that the value of $T_0 + a > 0$ is sufficiently small we obtain estimate (3.8) on the interval $[-a, T_0]$. We can consider now problem (3.3)–(3.7) on the interval $[-a + (a + T_0)/2, -a + 3(a + T_0)/2]$ and remove the initial data with the help of the known functions. This permits us to move up along the Ot -axis exactly as in [43], Chapter IV, and we obtain the assertion of Theorem 3.1 on an arbitrary time interval $[-a, T]$.

Thus, Theorem 3.1 is proved. $\square$

As a consequence of Theorem 3.1 we have also an estimate for a solution of the following model problem with the dynamic boundary condition. Let a function $u(x, t)$ with finite support satisfies the conditions

$$\frac{\partial u}{\partial t} - (x_N)^\alpha \Delta u = f_1, \quad (x, t) \in Q_+^N, \quad (3.75)$$

$$u_t - \varepsilon \Delta_{x'} u - b \frac{\partial u}{\partial x_N} = f(x', t), \quad x_N = 0, (x', t) \in Q^{N-1}, \quad (3.76)$$

$$u(x, -a) = 0, \quad u(x, t) \in C_{s,0}^{2+\gamma, \frac{2+\gamma}{2}}(\overline{Q}_+^N), \quad u_t(x', 0, t) \in C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\overline{Q}^{N-1}), \quad (3.77)$$

where $\varepsilon, b > 0$,

$$f_1(x, t) \in C_{s,0}^{\gamma, \frac{\gamma}{2}}(\overline{Q}_+^N), \quad f(x', t) \in C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\overline{Q}^{N-1}). \quad (3.78)$$

Problem (3.75)–(3.77) is obtained from problem (3.3)–(3.7) by a simplification. We drop in (3.3)–(3.7) the unknown $u^-(x, t)$, denote $u(x, t) \equiv u^+(x, t)$, and put $f_2^+ \equiv 0$. Then we express $\rho(x', t)$ from relation (3.4) as $\rho(x', t) = -\frac{1}{A^+}u^+ = -\frac{1}{A^+}u(x', 0, t)$ and substitute this expression in (3.5). Thus the proof of Theorem 3.1 gives the following assertion.

Theorem 3.4. *Let $u(x, t)$ be a solution with finite support of problem (3.75)–(3.77). Then for arbitrary $T > 0$ the following estimate is valid*

$$\begin{aligned} |u|_{s, R_{+T}^{N,a}}^{(2+\gamma)} + \varepsilon \sum_{i,j=1}^{N-1} |u(x', 0, t)_{x_i x_j}|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} + |u_t(x', 0, t)|_{R_T^{N-1,a}}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} \\ \leq C_T \left(|f_1|_{s, R_{+T}^{N,a}}^{(\gamma)} + |f|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(R_T^{N-1,a})} \right), \end{aligned} \quad (3.79)$$

where the constant C_T in (3.79) does not depend on $\varepsilon \in (0, 1)$.

4. Reduction of problem (1.3)–(1.7) to a problem in a fixed domain

Let $\rho(\omega, t)$ be the unknown function defined in Sect. 1, which gives the parameterisation of unknown (free) boundary $\Gamma_{\rho, T}$ ($\rho(\omega, 0) \equiv 0$). Let also $\rho(x, t) = E\rho(\omega, t)$ be the extension of this function to the whole domain $\overline{\Omega}_T$ by the extension operator E from (2.97). We pass in problem (1.3)–(1.7) from the unknown functions $u^\pm(y, \tau)$ to the unknowns $v^\pm(y, \tau) = |u^\pm|^{m-1}u^\pm(y, \tau)$. Then relations (1.3)–(1.7) take the form

$$L_0(v^\pm)v^\pm \equiv \frac{\partial v^\pm}{\partial \tau} - a^\pm |v^\pm|^\alpha \nabla_y^2 v^\pm(y, \tau) = 0, \quad (y, \tau) \in \Omega_{\rho, T}^\pm, \quad (4.1)$$

$$v^+(y, \tau) = v^-(y, \tau) = 0, \quad (y, \tau) \in \Gamma_{\rho, T}, \quad (4.2)$$

$$a^+ \sum_{i=1}^N \cos(\vec{N}, y_i) v_{y_i}^+ - a^- \sum_{i=1}^N \cos(\vec{N}, y_i) v_{y_i}^- = k \cos(\vec{N}, \tau), \quad (y, \tau) \in \Gamma_{\rho, T}, \quad (4.3)$$

$$v^\pm(y, \tau) = |g^\pm|^{m-1} g^\pm(y, \tau) \equiv h^\pm(y, \tau), \quad (y, \tau) \in \Gamma_T^\pm, \quad (4.4)$$

$$v^\pm(y, 0) = |u_0^\pm|^{m-1} u_0^\pm(y) \equiv v_0^\pm(y), \quad y \in \overline{\Omega}^\pm. \quad (4.5)$$

We make in problem (4.1)–(4.5) the change of variables $(y, \tau) = e_\rho(x, t)$, which is defined in (1.31). Denote for simplicity by the same symbols $v^\pm(x, t)$ the unknown functions after this change of variables, that is,

$$v^\pm(x, t) \equiv v^\pm(y, \tau) \circ e_\rho(x, t).$$

In view of the properties of $(y, \tau) = e_\rho(x, t)$ problem (4.1)–(4.5) in the variables (x, t) reduces to the following problem in the known fixed domains $\Omega_T^\pm$ for the

unknown functions v^+ , v^- , ρ (besides x -variables we use the corresponding coordinates (ω, λ) , which are introduced in (1.28))

$$L_\rho(v^\pm)v^\pm \equiv \frac{\partial v^\pm}{\partial t} - h_\rho^\pm \rho_t - |v^\pm|^\alpha \nabla_\rho^2 v^\pm(x, t) = 0, \quad (x, t) \in \Omega_T^\pm, \quad (4.6)$$

$$v^+(x, t) = v^-(x, t) = 0, \quad (x, t) \in \Gamma_T, \quad (4.7)$$

$$\left(1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \rho) \rho_{\omega_i} \rho_{\omega_j}\right) \left(a^+ \frac{\partial v^+}{\partial \lambda} - a^- \frac{\partial v^-}{\partial \lambda}\right) + k \rho_t (1 + \rho_\lambda) = 0, \quad (x, t) \in \Gamma_T, \quad (4.8)$$

$$v^\pm(x, t) - h^\pm(x, t) = 0, \quad (x, t) \in \Gamma_T^\pm, \quad (4.9)$$

$$v^\pm(x, 0) - v_0^\pm(x) = 0, \quad x \in \overline{\Omega^\pm}, \quad \rho(\omega, 0) \equiv 0, \quad (4.10)$$

$$\rho(x, t) = E\rho(\omega, t). \quad (4.11)$$

Here $\nabla_\rho \equiv \mathcal{E}_\rho \nabla_x$ and the matrix $\mathcal{E}_\rho$ is the conjugate and inverse to the Jacobi matrix of mapping (1.31) for $t = \text{const}$, $m_{ij}(x, \rho)$ are some given smooth functions of their arguments, and

$$h_\rho^\pm(x, t) \equiv \frac{\partial v^\pm}{\partial \lambda} \frac{1}{1 + \rho_\lambda}. \quad (4.12)$$

Note that the last definition is legitimate since the function $\rho(x, t)$ is not identically zero only if $x \in \mathcal{N}$, where the coordinates $(\omega, \lambda) \equiv (\omega_x, \lambda_x)$ of a point x are defined. Recall also that the coordinate λ is independent of a choice of local coordinates ω . We use the notation with index (ω_x, λ_x) to distinguish these coordinates for a point x from the corresponding coordinates (ω_y, λ_y) for a point y .

Below we explain the derivation of relations (4.6)–(4.10) but here we note the following. Relation (4.8) contains the expression

$$S_\rho \equiv S_\rho(\omega, \rho, \rho_\omega) \equiv \left(1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \rho) \rho_{\omega_i} \rho_{\omega_j}\right),$$

which is explicitly expressed in the local coordinates ω . But in fact the expression S_ρ is a function of points of the surface Γ_T only and its values at points of Γ_T do not depend on a choice of local coordinates ω . Indeed, first, for any choice of local coordinates ω condition (4.8) is equivalent to (4.3) and the last is independent of a choice of local coordinates. And secondly, all the other factors and the terms but S_ρ in relation (4.8) are invariant with respect to a choice of ω and they are functions of a point of the surface Γ_T only. Hence, the expression S_ρ as a function of a point of the surface Γ_T is invariant on a choice of local coordinates ω as well. And thus, the map $\rho \rightarrow S_\rho(\omega, \rho, \rho_\omega)$ defines a nonlinear operator acting on functions defined on Γ_T . This operator is invariant under choice of local coordinates ω , it acts in the space of functions

on Γ_T and has a certain expression $S_\rho(\omega, \rho, \rho_\omega)$ for every particular choice of the local coordinates ω .

Further, the expression $\frac{\partial v^\pm}{\partial t} - h_\rho^\pm \rho_t$ is recalculated in the variables (x, t) derivative $\frac{\partial v^\pm}{\partial \tau}$ after change of variables (1.31):

$$\frac{\partial v^\pm}{\partial \tau} = \frac{\partial v^\pm}{\partial t} \frac{\partial t}{\partial \tau} + \sum_{i=1}^{N-1} \frac{\partial v^\pm}{\partial \omega_{xi}} \frac{\partial \omega_{xi}}{\partial \tau} + \frac{\partial v^\pm}{\partial \lambda_x} \frac{\partial \lambda_x}{\partial \tau}.$$

Here in fact

$$\frac{\partial t}{\partial \tau} = 1, \quad \frac{\partial \omega_{xi}}{\partial \tau} = 0. \quad (4.13)$$

And for the value of $\frac{\partial \lambda_x}{\partial \tau}$ due to the relation

$$\lambda_x = \lambda_y - \rho(x, t) \circ e_\rho^{-1},$$

and taking into account (4.13) we have

$$\begin{aligned} \frac{\partial \lambda_x}{\partial \tau} &= -\frac{\partial}{\partial \tau} [\rho(x, t) \circ e_\rho(x, t)^{-1}] \\ &= -\frac{\partial \rho}{\partial t} \frac{\partial t}{\partial \tau} - \frac{\partial \rho}{\partial \lambda_x} \frac{\partial \lambda_x}{\partial \tau} - \sum_{i=1}^{N-1} \frac{\partial \rho}{\partial \omega_{xi}} \frac{\partial \omega_{xi}}{\partial \tau} = -\rho_t - \rho_{\lambda_x} \frac{\partial \lambda_x}{\partial \tau}. \end{aligned}$$

So in the variables x and t

$$\frac{\partial \lambda_x}{\partial \tau} = -\rho_t / (1 + \rho_{\lambda_x}). \quad (4.14)$$

Thus, from (4.13) and (4.14) it follows that

$$\frac{\partial v^\pm}{\partial \tau} \circ e_\rho = \frac{\partial v^\pm}{\partial t} - \left[\frac{\partial v^\pm}{\partial \lambda} / (1 + \rho_\lambda) \right] \rho_t = \frac{\partial v^\pm}{\partial t} - h_\rho^\pm \rho_t.$$

We explain further the transition from condition (4.3) to condition (4.8) under change of variables (1.31), as we shall need in the future the exact explicit form of this condition. Define in the neighborhood $\mathcal{N}_T$ of the surface Γ_T the function

$$\Phi_\rho(y, \tau) = \lambda_x \circ e_\rho^{-1}(y, \tau) = \lambda_y - \rho(x, t) \circ e_\rho^{-1}(y, \tau) = \lambda(y) - \rho(y, \tau), \quad (4.15)$$

where for simplicity we have retained for the function $\rho(x, t) \circ e_\rho^{-1}(y, \tau)$ the same notation $\rho(y, \tau)$. By the definition we have $\pm \Phi_\rho(y, \tau) > 0$ for $(y, \tau) \in \Omega_{\rho, T}^\pm$ and $\Phi_\rho(y, \tau) = 0$ for $(y, \tau) \in \Gamma_{\rho, T}$. Hence, in (4.3)

$$\cos(\vec{N}, y_i) = \frac{\Phi_{\rho y_i}}{|\nabla_{(y, \tau)} \Phi_\rho|}, \quad \cos(\vec{N}, \tau) = \frac{\Phi_{\rho \tau}}{|\nabla_{(y, \tau)} \Phi_\rho|}.$$

Therefore, relation (4.3) can be written as follows

$$a^+(\nabla_y v^+, \nabla_y \Phi_\rho) - a^-(\nabla_y v^-, \nabla_y \Phi_\rho) = k \Phi_{\rho \tau}. \quad (4.16)$$

Under change of variables (1.31) the right hand side of (4.16) due to the definition of Φ_ρ takes the form

$$k \Phi_{\rho \tau} = k \frac{\partial \lambda_x}{\partial \tau} = -k \rho_t / (1 + \rho_{\lambda_x}), \quad (4.17)$$

owing to (4.14).

On the other hand, under change of variables (1.31)

$$(\nabla_y v^\pm, \nabla_y \Phi_\rho) \circ e_\rho(x, t) = (\nabla_\rho v^\pm, \nabla_\rho \lambda_x). \quad (4.18)$$

Denote by $\Lambda(x)$ the transition matrix from the gradient with respect to the variables x to the gradient with respect to the variables (ω_x, λ_x) , that is,

$$\nabla_x = \Lambda(x) \nabla_{(\lambda_x, \omega_x)} \quad (\nabla_y = \Lambda(y) \nabla_{(\lambda_y, \omega_y)}), \quad (4.19)$$

where

$$\Lambda(x) = \begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \omega_1}{\partial x_1} & \cdots & \frac{\partial \omega_{N-1}}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \lambda}{\partial x_N} & \frac{\partial \omega_1}{\partial x_N} & \cdots & \frac{\partial \omega_{N-1}}{\partial x_N} \end{pmatrix}, \quad (4.20)$$

and similarly for the variables y . Then in the variables (x, t)

$$(\nabla_\rho v^\pm, \nabla_\rho \lambda_x) = (\mathcal{E}_\rho \Lambda \nabla_{(\lambda, \omega)} v^\pm, \mathcal{E}_\rho \Lambda \nabla_{(\lambda, \omega)} \lambda_x).$$

Note that $\nabla_{(\lambda_x, \omega_x)} \lambda_x = \{1, 0, \dots, 0\}$, and also $v^\pm \equiv 0$ on Γ , hence $\partial v^\pm / \partial \omega_i = 0$, and therefore

$$\nabla_{(\lambda_x, \omega_x)} v^\pm = \left\{ \frac{\partial v^\pm}{\partial \lambda_x}, 0, \dots, 0 \right\} = \frac{\partial v^\pm}{\partial \lambda_x} \{1, 0, \dots, 0\} = \frac{\partial v^\pm}{\partial \lambda_x} \nabla_{(\lambda_x, \omega_x)} \lambda_x.$$

Thus we obtain

$$(\nabla_y v^\pm, \nabla_y \Phi_\rho) \circ e_\rho(x, t) = (\nabla_\rho v^\pm, \nabla_\rho \lambda_x) = \frac{\partial v^\pm}{\partial \lambda_x} (\nabla_\rho \lambda_x, \nabla_\rho \lambda_x). \quad (4.21)$$

On the other hand, due to the definition of $\Phi_\rho(y, \tau)$

$$(\nabla_\rho \lambda_x, \nabla_\rho \lambda_x) = (\nabla_y (\lambda_x \circ e_\rho^{-1}), \nabla_y (\lambda_x \circ e_\rho^{-1})) \circ e_\rho = (\nabla_y \Phi_\rho, \nabla_y \Phi_\rho) \circ e_\rho. \quad (4.22)$$

Using introduced in (4.19) matrix $\Lambda(y)$, we have

$$\begin{aligned} (\nabla_y \Phi_\rho, \nabla_y \Phi_\rho) &= (\Lambda(y) \nabla_{(\lambda_y, \omega_y)} \Phi_\rho, \Lambda(y) \nabla_{(\lambda_y, \omega_y)} \Phi_\rho) \\ &= (\nabla_{(\lambda_y, \omega_y)} \Phi_\rho, \Lambda(y)^* \Lambda(y) \nabla_{(\lambda_y, \omega_y)} \Phi_\rho). \end{aligned} \quad (4.23)$$

First, by the definition of Φ_ρ

$$\begin{aligned} \frac{\partial \Phi_\rho}{\partial \lambda_y} &= \frac{\partial}{\partial \lambda_y} (\lambda_y - \rho(y, \tau)) = 1 - \rho_{\lambda_y}, \\ \frac{\partial \Phi_\rho}{\partial \omega_{yi}} &= \frac{\partial}{\partial \omega_{yi}} (\lambda_y - \rho(y, \tau)) = -\rho_{\omega_{yi}}. \end{aligned} \quad (4.24)$$

In addition, since the coordinate λ_y is counted along the normal to Γ , and ω_{yi} are coordinates on the surface Γ , we have

$$(\nabla_y \lambda(y), \nabla_y \lambda(y)) = 1, \quad (\nabla_y \lambda(y), \nabla_y \omega_i(y)) = 0, \quad i = 1, \dots, N-1.$$

Therefore the matrix $\Lambda^*(y) \Lambda(y)$ has the form

$$\Lambda^*(y) \Lambda(y) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & m_{11} & m_{12} & \cdots & m_{1(N-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & m_{(N-1)1} & m_{(N-1)2} & \cdots & m_{(N-1)(N-1)} \end{pmatrix}, \quad (4.25)$$

where

$$m_{ij} = m_{ji} = (\nabla_y \omega_i(y), \nabla_y \omega_j(y)) - \quad (4.26)$$

are some smooth functions. Thus,

$$(\nabla_{(\lambda_y, \omega_y)} \Phi_\rho, \Lambda^*(y) \Lambda(y) \nabla_{(\lambda_y, \omega_y)} \Phi_\rho) = (1 - \rho_{\lambda_y})^2 + \sum_{i,j=1}^{N-1} m_{ij}(y) \rho_{\omega_{yi}} \rho_{\omega_{yj}}. \quad (4.27)$$

Make now in (4.27) change of variables (1.31), and recalculate the derivatives of ρ with respect to (λ_y, ω_y) in terms of the derivatives with respect to (λ_x, ω_x) . We have

$$\rho_{\lambda_y} \circ e_\rho = \rho_t \frac{\partial t}{\partial \lambda_y} + \rho_{\lambda_x} \frac{\partial \lambda_x}{\partial \lambda_y} + \sum_{i=1}^{N-1} \rho_{\omega_{xi}} \frac{\partial \omega_{xi}}{\partial \lambda_y}. \quad (4.28)$$

From the definition of the mapping e_ρ it follows that

$$\frac{\partial t}{\partial \lambda_y} = 0, \quad \frac{\partial \omega_{xi}}{\partial \lambda_y} = 0. \quad (4.29)$$

At the same time by (4.28), (4.29)

$$\frac{\partial \lambda_x}{\partial \lambda_y} = 1 - \rho_{\lambda_y} = 1 - \rho_{\lambda_x} \frac{\partial \lambda_x}{\partial \lambda_y},$$

that is,

$$\frac{\partial \lambda_x}{\partial \lambda_y} = \frac{1}{1 + \rho_{\lambda_x}}. \quad (4.30)$$

Therefore, by (4.28), (4.29) and (4.30)

$$\rho_{\lambda_y} \circ e_\rho = \frac{\rho_{\lambda_x}}{1 + \rho_{\lambda_x}}. \quad (4.31)$$

Further,

$$\rho_{\omega_{yi}} \circ e_\rho = \rho_t \frac{\partial t}{\partial \omega_{yi}} + \rho_{\lambda_x} \frac{\partial \lambda_x}{\partial \omega_{yi}} + \sum_{j=1}^{N-1} \rho_{\omega_{xj}} \frac{\partial \omega_{xj}}{\partial \omega_{yi}}, \quad (4.32)$$

and

$$\frac{\partial t}{\partial \omega_{yi}} = 0, \quad \frac{\partial \omega_{xj}}{\partial \omega_{yi}} = \delta_{ij}, \quad i, j = 1, \dots, N-1. \quad (4.33)$$

At the same time

$$\frac{\partial(\lambda_x \circ e_\rho)}{\partial \omega_{yi}} = \left[\frac{\partial}{\partial \omega_{yi}} (\lambda_y - \rho(y, \tau)) \right] \circ e_\rho = -\rho_{\omega_{yi}} \circ e_\rho,$$

That is, by virtue of (4.32) and (4.33),

$$\rho_{\omega_{yi}} \circ e_\rho = \rho_{\lambda_x} (-\rho_{\omega_{yi}} \circ e_\rho) + \rho_{\omega_{xi}}, \quad (4.34)$$

hence by (4.34),

$$\rho_{\omega_{yi}} \circ e_\rho = \frac{\rho_{\omega_{xi}}}{1 + \rho_{\lambda_x}}. \quad (4.35)$$

Thus, from (4.21), (4.27), (4.31) and (4.35) it follows that in (4.21)

$$(\nabla_\rho \lambda_x, \nabla_\rho \lambda_x) = \frac{1}{(1 + \rho_{\lambda_x})^2} \left[1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \rho) \rho_{\omega_{xi}} \rho_{\omega_{xj}} \right]. \quad (4.36)$$

Finally, relation (4.8) follows from relations (4.16), (4.17), (4.21) and (4.36).

5. Properties of the nonlinear operator of problem (4.6)–(4.11) and it's Frechet derivative

Our goal in this section is to calculate the Frechet derivative of the nonlinear operator of problem (4.6)–(4.11). This operator is defined by the left-hand sides of relations (4.6)–(4.11). The domain of definition of this operator will be a small neighbourhood of some approximate solution to problem (4.6)–(4.11). So we first construct such an approximate solution. This will be done by extending the initial data $(v_0^+, v_0^-, 0)$ from $t = 0$ to the region $t \in [0, T]$ up to some functions (w^+, w^-, σ) in a way to ensure conditions (4.6)–(4.11) at $t = 0$. Then for sufficiently small $T > 0$ the triple (w^+, w^-, σ) will give an approximate solution to (4.6)–(4.11).

Note that from equations (4.6), (4.8) and from initial data (4.10) we can calculate the derivatives with respect to time $\partial v^\pm / \partial t$ and $\partial \rho / \partial t$ at $t = 0$:

$$\frac{\partial \rho}{\partial t}(\omega, 0) = \rho_1(\omega) \equiv \frac{1}{k} \left(a^+ \frac{\partial v_0^+}{\partial \lambda} - a^- \frac{\partial v_0^-}{\partial \lambda} \right) \Big|_\Gamma, \quad (5.1)$$

$$\frac{\partial v^\pm}{\partial t}(x, 0) = v_1^\pm(x) \equiv \frac{\partial v_0^\pm}{\partial \lambda} \rho_1 + a^\pm |v_0^\pm(x)|^\alpha \nabla^2 v_0^\pm(x), \quad (5.2)$$

and in view of assumptions (1.10), (1.26)

$$\rho_1(\omega) \in C^{1+\beta'-\alpha}(\Gamma), \beta' \equiv \gamma'(1 - \alpha/2) > \beta, \quad v_1^\pm(x) \in C_s^{\gamma'}(\overline{\Omega^\pm}). \quad (5.3)$$

Note that in view of conditions (1.36) and (4.7) these definitions are agreed at Γ .

Completely analogous to [43], Chapter IV, on the base of the results of [42] on the solvability of the initial-boundary value problem for degenerate equations (or on the base of Theorem 2.7) we construct functions $w^\pm(x, t) \in C_s^{2+\gamma', 1+\gamma'/2}(\overline{\Omega_T^\pm})$ such that

$$|w^\pm|_{s, \overline{\Omega_T^\pm}}^{(2+\gamma', 1+\gamma'/2)} \leq C \left(|v_0^\pm|_{s, \overline{\Omega^\pm}}^{(2+\gamma')} + |v_1^\pm|_{s, \overline{\Omega^\pm}}^{(\gamma')} \right) \leq C |v_0^\pm|_{s, \overline{\Omega^\pm}}^{(2+\gamma')} \quad (5.4)$$

and

$$w^\pm(x, 0) = v_0^\pm(x), \quad \frac{\partial w^\pm}{\partial t}(x, 0) = v_1^\pm(x), \quad w^\pm(x, t)|_{\Gamma_T} = 0, \quad w^\pm(x, t)|_{\Gamma^\pm} = h^\mp. \quad (5.5)$$

In addition, just as described in [43], Chapter 4, there is a function $\sigma(\omega, t) \in C^{3+\beta'-\alpha, 1+\frac{1+\beta'-\alpha}{2}}(\Gamma_T)$ such that

$$|\sigma|_{\Gamma_T}^{(3+\beta'-\alpha, \frac{3+\beta'-\alpha}{2})} \leq C \left(|v_0^+|_{s, \overline{\Omega^+}}^{(2+\gamma')} + |v_0^-|_{s, \overline{\Omega^-}}^{(2+\gamma')} \right) \quad (5.6)$$

and

$$\sigma(\omega, 0) = \rho(\omega, 0) = 0, \quad \frac{\partial \sigma}{\partial t}(\omega, 0) = \rho_1(\omega). \quad (5.7)$$

Moreover, by the method described in [43], Chapter IV, the function $\sigma(\omega, t)$ can be extended with the class and with inequality (5.6) to a function defined in $\overline{\Omega_T}$ with support in the neighborhood $\mathcal{N} \times [0, T]$ of the surface Γ_T .

We denote the space

$$P^{2+\beta-\alpha}(\Gamma_T) = \{\rho : \rho \in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(\Gamma_T), \rho_t \in C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(\Gamma_T)\} \quad (5.8)$$

with the norm

$$|\rho|_{P^{2+\beta-\alpha}(\Gamma_T)} \equiv |\rho|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2}}(\Gamma_T)} + |\rho_t|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(\Gamma_T)}. \quad (5.9)$$

Denote also

$$\begin{aligned} \psi = (v^+, v^-, \rho) \in \mathcal{H} &\equiv C_s^{2+\gamma, 1+\gamma/2}(\overline{\Omega_T^+}) \times C_s^{2+\gamma, 1+\gamma/2}(\overline{\Omega_T^-}) \times P^{2+\beta-\alpha}(\Gamma_T), \\ \psi_0 &= (w^+, w^-, \sigma), \end{aligned} \quad (5.10)$$

and represent relations (4.6)–(4.11) as

$$\tilde{F}(\psi)(\tilde{F}^+(\psi), \tilde{F}^-(\psi), \tilde{F}_1^+(\psi), \tilde{F}_1^-(\psi), \tilde{F}^0(\psi), \tilde{F}_2^+(\psi), \tilde{F}_2^-(\psi), \tilde{F}_3^+(\psi), \tilde{F}_3^-(\psi)) = 0 \quad (5.11)$$

with the non-linear operator of ψ , which is defined by the left-hand sides of problem (4.6)–(4.11). Here $\tilde{F}^+(\psi), \tilde{F}^-(\psi)$ are defined by (4.6), $\tilde{F}_1^+(\psi), \tilde{F}_1^-(\psi)$ are defined by (4.7), $\tilde{F}^0(\psi)$ is defined by (4.8), $\tilde{F}_2^+(\psi), \tilde{F}_2^-(\psi)$ are defined by (4.9), and $\tilde{F}_3^+(\psi), \tilde{F}_3^-(\psi)$ are defined by (4.10). Denote further the following space with the elements $\varphi = (\theta^+, \theta^-, \delta)$

$$\begin{aligned} \mathcal{H}_0 &\equiv \left\{ \varphi \in C_{0,s}^{2+\gamma, 1+\gamma/2}(\overline{\Omega_T^+}) \times C_{0,s}^{2+\gamma, 1+\gamma/2}(\overline{\Omega_T^-}) \times P_0^{2+\beta-\alpha}(\Gamma_T) : \right. \\ &\quad \left. \theta^\pm|_{\Gamma_T} \equiv 0, \quad \theta^\pm|_{\Gamma_T^\pm} \equiv 0 \right\}. \end{aligned} \quad (5.12)$$

Note that if $\varphi \in \mathcal{H}_0$ then the triple $\psi_0 + \varphi$ satisfies initial conditions (4.10) and boundary conditions (4.7), (4.9). This permits instead of the operator $\tilde{F}$ in (5.11) to consider the operator

$$F(\varphi) = (F^+(\varphi), F^-(\varphi), F^0(\varphi)) = (\tilde{F}^+(\varphi + \psi_0), \tilde{F}^-(\varphi + \psi_0), \tilde{F}^0(\varphi + \psi_0)). \quad (5.13)$$

This operator is defined on $\mathcal{H}_0$ by the left-hand sides of relations (4.6), (4.8) with the values in the space

$$Y = C_{0,s}^{\gamma, \gamma/2}(\overline{\Omega_T^+}) \times C_{0,s}^{\gamma, \gamma/2}(\overline{\Omega_T^-}) \times C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(\Gamma_T). \quad (5.14)$$

Let us remark that F maps $\mathcal{H}_0$ to spaces “with zeros” by the construction of the functions $\psi_0 = (w^+, w^-, \sigma)$. These functions were constructed exactly in the way to ensure that the conditions (4.6), (4.8) are valid at $t = 0$. Let

$$\mathcal{B}_r = \mathcal{B}_r(0) = \{\varphi \in \mathcal{H}_0 : \|\varphi\| \leq r\} \subset \mathcal{H}_0, \quad r < \gamma_0/2, \quad (5.15)$$

be a ball in $\mathcal{H}_0$ of radius $r < \gamma_0/2$ with center at zero. We claim that if the radius r is sufficiently small, then the mapping $F(\varphi)$ is continuously differentiable on $\mathcal{B}_r$. Consider first the mapping $F^0(\varphi)$, which corresponds to (4.8). The smoothness of the functions $m_{ij}(x, \rho)$ with respect to their arguments is defined solely by the smoothness of the surface Γ . As this surface is sufficiently smooth, the functions $m_{ij}(x, \rho)$ are continuously differentiable as operators in ρ and consequently they are continuously differentiable as operators in φ from $\mathcal{H}_0$ to $C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T)$. Other terms in the left-hand side of (4.8) are evidently continuously differentiable as operators in φ from $\mathcal{H}_0$ to $C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T)$.

Consider now, for example, the operator $F^+(\varphi) = \tilde{F}^+(\varphi + \psi_0)$, which is defined by the left-hand side of (4.6). Write it in the form (recall that $\nabla_\rho = E_\rho \nabla$, $E_\rho = (e_{ij}(x, \rho, \nabla \rho)|_{i,j=\overline{1,N}})$)

$$\begin{aligned} F^+(\varphi) &= \frac{\partial v^+}{\partial t} - \left(\frac{\partial v^+}{\partial \lambda} \frac{1}{1 + \rho_\lambda} \right) \frac{\partial \rho}{\partial t} - (v^+)^\alpha \nabla_\rho^2 v^+ \\ &= \frac{\partial v^+}{\partial t} - \left(\frac{\partial v^+}{\partial \lambda} \frac{1}{1 + \rho_\lambda} \right) \frac{\partial \rho}{\partial t} \\ &\quad - \left(\frac{v^+}{\lambda(x)} \right)^\alpha \sum_{i,j=1}^N \left(\sum_{k,l=1}^N e_{li}(x, \rho, \nabla \rho) e_{kj}(x, \rho, \nabla \rho) \right) \left(\lambda(x) \frac{\partial^2 v^+}{\partial x_i \partial x_j} \right) \\ &\quad - \left(\frac{v^+}{\lambda(x)} \right)^\alpha \sum_{i,l=1}^N \left(\sum_{k,j=1}^N e_{li} \frac{\partial e_{kj}(x, \rho, \nabla \rho)}{\partial \rho_{x_l}} \frac{\partial v^+}{\partial x_j} \right) \left(\lambda(x) \frac{\partial^2 \rho}{\partial x_i \partial x_l} \right) \\ &\quad - \left(\frac{v^+}{\lambda(x)} \right)^\alpha \lambda(x)^\alpha \sum_{i,l,k,j=1}^N \left(e_{li} \frac{\partial e_{kj}(x, \rho, \nabla \rho)}{\partial \rho} \frac{\partial \rho}{\partial x_i} + e_{li} \frac{\partial e_{kj}(x, \rho, \nabla \rho)}{\partial x_i} \right) \frac{\partial v^+}{\partial x_j}. \end{aligned} \quad (5.16)$$

Here $\lambda(x)$ - is extension of the λ - coordinate with the preservation of the class from the neighborhood $\mathcal{N}$ of the surface Γ on all $\overline{\Omega}$ to a function satisfying the conditions

$$\nu d^+(x) \leq \lambda(x) \leq \nu^{-1} d^+(x), \quad (5.17)$$

retaining for her the same notation. Again, the functions $e_{ij}(x, \rho, \nabla \rho)$, $\frac{\partial e_{kj}(x, \rho, \nabla \rho)}{\partial \rho_{x_l}}$, $\frac{\partial e_{kj}(x, \rho, \nabla \rho)}{\partial \rho}$, and $\frac{\partial e_{kj}(x, \rho, \nabla \rho)}{\partial x_i}$ are sufficiently smooth with respect to their arguments because of the smoothness of the surface Γ in view of the definition of the mapping $e_\rho(x, t)$ in (1.30), (1.31). Moreover, the expression for $F^+(\varphi)$ in (5.16) is linear with respect to weighted second derivatives $\lambda(x) \frac{\partial^2 v^+}{\partial x_i \partial x_j}$ and $\lambda(x) \frac{\partial^2 \rho}{\partial x_i \partial x_l}$. So $F^+(\varphi)$ is smooth with respect to these arguments as a mapping from $\mathcal{H}_0$ to $C_{0,s}^{\gamma, \gamma/2}(\overline{\Omega_T^+})$. Thus it is enough to consider the smoothness of the mapping

$$\varphi \rightarrow \left(\frac{v^+}{\lambda(x)} \right)^\alpha = \left(\frac{w^+(x, t) + \theta^+(x, t)}{\lambda(x)} \right)^\alpha. \quad (5.18)$$

Since $v_0, v^+ = 0$ for $x \in \Gamma$, then for $x \in \mathcal{N}$

$$\frac{v^+(x, t)}{\lambda(x)} = \int_0^1 \frac{\partial v^+}{\partial \lambda}(\lambda s, \omega, t) ds, \quad \frac{v_0^+}{\lambda} = \int_0^1 \frac{\partial v_0^+}{\partial \lambda}(\lambda s, \omega) ds, \quad (5.19)$$

where we assume that the neighborhood $\mathcal{N}$ is so small that

$$\frac{\partial v_0^+}{\partial \lambda} \geq \nu > 0, \quad x \in \mathcal{N}.$$

In addition, we assume that $T \leq T_0$ is so small that

$$\frac{\partial w^+}{\partial \lambda}(x, t) \geq \nu > 0, \quad (x, t) \in \mathcal{N} \times [0, T], \quad T \leq T_0.$$

Assuming now that the radius $r = r(\nu)$ of the ball $\mathcal{B}_r \ni \varphi$ is sufficiently small $r \leq r_0(\nu)$, we can assume that for $\varphi = (\theta^+, \theta^-, \delta) \in \mathcal{B}_r$

$$\frac{\partial v^+}{\partial \lambda}(x, t) = \frac{\partial w^+}{\partial \lambda}(x, t) + \frac{\partial \theta^+}{\partial \lambda}(x, t) \geq \nu > 0, \quad (x, t) \in N \times [0, T], r \leq r_0(\nu), \quad T \leq T_0. \quad (5.20)$$

At the same time outside the neighborhood $\mathcal{N}_T$ holds

$$v_0^+(x) \geq \nu > 0, \quad x \in \overline{\Omega^+} \setminus \mathcal{N}.$$

Therefore, assuming as above T and r sufficiently small, we can assume that

$$v^+(x, t) = w^+(x, t) + \theta^+(x, t) \geq \nu > 0, \quad x \in \overline{\Omega^+} \setminus N, \quad r \leq r_0(\nu), T \leq T_0. \quad (5.21)$$

Thus we have the representation

$$\frac{v^+(x, t)}{\lambda(x)} \equiv A^+(x, t, \varphi) = \begin{cases} \int_0^1 \frac{\partial v^+}{\partial \lambda}(\lambda s, \omega, t) ds \geq \nu, & x \in \mathcal{N}, \\ \frac{v_0^+}{\lambda}, & x \in \overline{\Omega^+} \setminus \mathcal{N}, \end{cases} \quad (5.22)$$

and

$$|A^+(x, t, \varphi)|_{s, \Omega_T^+}^{(\gamma)} \leq C \left(\left| \frac{\partial v^+}{\partial \lambda} \right|_{s, \overline{\Omega_T^+}}^{(\gamma)} + |v^+|_{s, \Omega_T^+}^{(\gamma)} \right) \leq C(\|\varphi\|_{\mathcal{H}}), \quad (5.23)$$

and also

$$A^+(x, t, \varphi) \geq \nu > 0. \quad (5.24)$$

From (5.22)–(5.24) it follows that the mapping

$$\varphi \rightarrow \left(\frac{v^+(x, t)}{\lambda(x)} \right)^\alpha = \left(\frac{w^+(x, t) + \theta^+(x, t)}{\lambda(x)} \right)^\alpha = (A^+(x, t, \varphi))^\alpha \quad (5.25)$$

is a continuously differentiable mapping from $\mathcal{B}_r$ to $C_{0,s}^{\gamma, \gamma/2}(\overline{\Omega_T^-})$ for $r \leq r_0(\nu), T \leq T_0$. Besides, the following quantities do not depend on $T \in (0, T_0)$ for a fixed $r \leq r_0(\nu)$

$$M_1 = \sup_{\varphi \in \mathcal{B}_r} \|F'(\varphi)\|_{\mathcal{B}_r \rightarrow Y}, \quad M_2 = \sup_{\varphi_1, \varphi_2 \in \mathcal{B}_r} \|F'(\varphi_2) - F'(\varphi_1)\|_{\mathcal{B}_r \rightarrow Y}, \quad T \leq T_0. \quad (5.26)$$

Moreover, since the functions $(w^+, w^-, \sigma) \in C_s^{2+\gamma', 1+\gamma'/2}(\overline{\Omega_T^+}) \times C_s^{2+\gamma, 1+\gamma/2}(\overline{\Omega_T^-}) \times C^{3+\beta'-\alpha, 1+\frac{1+\beta'-\alpha}{2}}(\Gamma_T)$ with $\gamma' > \gamma$, $\beta' > \beta$, from their definition and (5.14) it follows that $(\psi_0 = (w^+, w^-, \sigma))$

$$F(0) = (\tilde{F}^+(\psi_0), \tilde{F}^-(\psi_0), \tilde{F}^0(\psi_0)) \in Y',$$

$$Y' \equiv C_{0,s}^{\gamma', \gamma'/2}(\overline{\Omega_T^+}) \times C_{0,s}^{\gamma', \gamma'/2}(\overline{\Omega_T^-}) \times C_0^{1+\beta'-\alpha, \frac{1+\beta'-\alpha}{2}}(\Gamma_T).$$

Using inequalities (2.1)–(2.4) we have with some $\mu > 0$

$$\|F(0)\|_Y \leq C(u_0^+, u_0^-, h^+, h^-)T^\mu, \quad r \leq r_0(\nu), T \leq T_0. \quad (5.27)$$

Thus the following lemma holds.

Lemma 5.1. *Let $r_0(\nu)$, T_0 are from (5.21). Let $r \leq r_0(\nu)$ be fixed and let $T \in (0, T_0)$. Then the mapping $F(\varphi)$ from (5.13) is a continuous differentiable mapping from $\mathcal{B}_r$ to the space Y from (5.14), estimate (5.27) is valid, and M_1, M_2 in (5.26) do not depend on $T \in (0, T_0)$.*

Now we give the explicit form of the Frechet derivative of $F(\varphi)$ at $\varphi = 0$, that is, $F'(0)$.

Lemma 5.2. *The Frechet derivative $F'(0)$ of the mapping $F(\varphi)$ at $\varphi = 0$ is the linear mapping from $\mathcal{H}_0$ to Y of the form $F'(0)\varphi = (F^{+\prime}(0)\varphi, F^{-\prime}(0)\varphi, F^{0\prime}(0)\varphi)$, where $(\varphi = (\theta^+, \theta^-, \delta))$*

$$F^{\pm\prime}(0)\varphi = \frac{\partial\theta^\pm}{\partial t} - |w^\pm|^\alpha \nabla_\sigma^2 \theta^\pm - B(x, t) \frac{\partial w^\pm}{\partial \lambda} \left(\frac{\partial \delta}{\partial t} - |w^\pm|^\alpha \nabla_\sigma^2 \delta \right) \quad (5.28)$$

$$+ \sum_{i=0}^N (a_i^\pm(x, t)\theta_{x_i}^\pm + b_i^\pm(x, t)\delta_{x_i}) + a_0^\pm(x, t)\theta^\pm + b_0^\pm(x, t)\delta + D^\pm(x, t) \frac{\theta^\pm(x, t)}{\lambda(x)},$$

$$F^{0\prime}(0)\varphi = k\delta_t + [a^+ \frac{\partial\theta^+}{\partial \lambda} - a^- \frac{\partial\theta^-}{\partial \lambda}] \Big|_{\Gamma_T} - \delta_\lambda [a^+ \frac{\partial w^+}{\partial \lambda} - a^- \frac{\partial w^-}{\partial \lambda}] \Big|_{\Gamma_T} + c_0^\pm(x, t)\theta^\pm + d_0^\pm(x, t)\delta, \quad (5.29)$$

where $B(x, t)$, $a_i^\pm(x, t)$, $b_i^\pm(x, t)$, $a_0^\pm(x, t)$, $b_0^\pm(x, t)$, $D(x, t)$, $c_0^\pm(x, t)$, $d_0^\pm(x, t)$ are some given coefficients at the lower order terms in (5.28), (5.29), $\lambda(x)$ is from (5.17), and

$$B(x, t), a_i^\pm, b_i^\pm, a_0^\pm, b_0^\pm, D^\pm(x, t) \in C_s^{\gamma, \gamma/2}(\overline{\Omega_T^\pm}), \quad c_0^\pm, d_0^\pm \in C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(\Gamma_T), \quad (5.30)$$

$$B(x, 0) \equiv 1, \quad x \in \overline{\Omega}^\pm. \quad (5.31)$$

Note that the higher order parts of differential operators acting on $\theta^\pm$ and δ in (5.28), (5.29) are proportional to each other. Moreover, the proportionality factors are the same for $t = 0$ as it follows from (5.31). This important fact will be used below.

We do not give a detailed proof of this lemma. Relation (5.29) easy follows from (4.8) and the very detailed proof of relation (5.28) is contained, in fact,

in [6] (see [6], formula (2.26); see also [3] for another way to find the Frechet derivative). The only difference from [6] is the presence of the degenerate factor $|v^\pm|^\alpha$ in equations (4.6). This factor is treated exactly as in (5.22)–(5.25) above. We have, for example, for v^+

$$\begin{aligned} G(\varphi) &\equiv (v^+)^{\alpha} [\nabla_{\rho}^2 v^+] = (w^+ + \theta^+)^{\alpha} \nabla_{\sigma+\delta}^2 (w^+ + \theta^+) \\ &= \left[\left(\frac{w^+}{\lambda(x)} + \frac{\theta^+}{\lambda(x)} \right)^{\alpha} \right] [\lambda^{\alpha}(x) \nabla_{\sigma+\delta}^2 (w^+ + \theta^+)] \equiv G_1(\varphi) G_2(\varphi) \end{aligned}$$

The Frechet derivative of $G_1(\varphi)$ at $\varphi = 0$ is

$$G'_1(0)\varphi = \alpha \left(\frac{w^+}{\lambda(x)} \right)^{\alpha-1} \left(\frac{\theta^+}{\lambda(x)} \right).$$

And the Frechet derivative of $G_2(\varphi)$ at $\varphi = 0$ is found in [6]. So

$$G'(0)\varphi = [G'_1(0)\varphi] G_2(0) + G_1(0) [G'_2(0)\varphi],$$

where

$$[G'_1(0)\varphi] G_2(0) = \left[\alpha \left(\frac{w^+}{\lambda(x)} \right)^{\alpha-1} \left(\frac{\theta^+}{\lambda(x)} \right) \right] [\lambda^{\alpha}(x) \nabla_{\sigma}^2 w^+]$$

This gives the terms $D(x, t) \frac{\theta^{\pm}(x, t)}{\lambda(x)}$ in relations (5.29). It is important that in view of the representation (see (5.22)–(5.25))

$$\frac{\theta^+(x, t)}{\lambda(x)} = \begin{cases} \int_0^1 \frac{\partial \theta^+}{\partial \lambda}(\lambda s, \omega, t) ds \geq \nu, & x \in \mathcal{N}, \\ \frac{\theta^+}{\lambda}, & x \in \bar{\Omega}^+ \setminus \mathcal{N}, \end{cases} \quad (5.32)$$

this term can be considered in the usual way as a lower order term. This actually completes the proof of the lemma.

6. An initial-boundary value problem with dynamic boundary condition: the invertibility of the Frechet derivative of the operator of problem (4.6)–(4.11)

In this section we consider the invertibility of the Frechet derivative $F'(0)$ of the operator $F(\varphi)$ from (5.13). As in the previous section, we assume that $\lambda(x)$ is extended to all $\bar{\Omega}$ to a smooth function of the class $C^{3+\gamma}$,

$$\nu \leq \lambda(x), d^{\pm}(x) \leq \nu^{-1}, \quad x \in \bar{\Omega} \setminus \mathcal{N}. \quad (6.1)$$

According to the definition of the space $\mathcal{H}_0$ in (5.12) and to relations (5.28), (5.29) of Lemma 5.2 we must prove solvability and estimates of the solution to the following problem for the unknown functions $\theta^{\pm}(x, t)$, $\delta(\omega, t)$ defined on the domains $\bar{\Omega}_T^{\pm}$ and on Γ_T respectively (we replace $|w^{\pm}|^{\alpha}$ in (5.28), (5.29)

with $\lambda(x)^\alpha e^\pm(x, t)$ and $\partial w^\pm / \partial \lambda$ with $A^\pm(x, t)$)

$$\begin{aligned} & \frac{\partial \theta^\pm}{\partial t} - \lambda(x)^\alpha e^\pm(x, t) \nabla_\sigma^2 \theta^\pm - B(x, t) A^\pm(x, t) \left(\frac{\partial \delta}{\partial t} - \lambda(x)^\alpha e^\pm(x, t) \nabla_\sigma^2 \delta \right) \\ & + \sum_{i=0}^N (a_i^\pm(x, t) \theta_{x_i}^\pm + b_i^\pm(x, t) \delta_{x_i}) + a_0^\pm(x, t) \theta^\pm \\ & + b_0^\pm(x, t) \delta + D^\pm(x, t) \frac{\theta^\pm(x, t)}{\lambda(x)} \\ & = f_1^\pm(x, t), \quad (x, t) \in \Omega_T^\pm, \end{aligned} \quad (6.2)$$

$$\theta^+(x, t) = \theta^-(x, t) = 0, \quad (x, t) \in \Gamma_T, \quad (6.3)$$

$$\begin{aligned} & k \delta_t - \varepsilon \Delta_\Gamma \delta + \left(a^+ \frac{\partial \theta^+}{\partial \lambda} - a^- \frac{\partial \theta^-}{\partial \lambda} \right) - \delta_\lambda (a^+ A^+(x, t) - a^- A^-(x, t)) \\ & + c_0^\pm(x, t) \theta^\pm + d_0^\pm(x, t) \delta = f_2(x, t), \quad (x, t) \in \Gamma_T, \end{aligned} \quad (6.4)$$

$$\theta^\pm(x, t) = 0, \quad (x, t) \in \Gamma_T^\pm, \quad (6.5)$$

$$\theta^\pm(x, 0) = 0, \quad \delta(x, 0) = 0, \quad x \in \overline{\Omega^\pm}, \quad (6.6)$$

$$\delta(x, t) = E\delta(\omega, t), \quad (6.7)$$

where the extension operator E is defined in Sect. 2, Δ_Γ is the Laplace-Beltrami operator on the surface Γ . The term $\varepsilon \Delta_\Gamma$ is a regularisation of the problem and we need this term to prove the solvability of the problem. We assume that

$$f_1^\pm(x, t) \in C_{0,s}^{\gamma, \gamma/2}(\overline{\Omega_T^\pm}), \quad f_2(x, t) \in C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T), \quad (6.8)$$

$\varepsilon, a^\pm, k$ are given positive constants,

$$\nu \leq k, a^\pm, e^\pm(x, t), B(x, t), \quad A^\pm(x, t) \leq \nu^{-1}, \quad B(x, 0) \equiv 1, \quad (6.9)$$

$$B(x, t), D^\pm(x, t) \in C_s^{\gamma, \gamma/2}(\overline{\Omega_T^\pm}), \quad A^\pm(x, t) \in C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\overline{\Omega_T^\pm}), \quad (6.10)$$

and the functions $a_i^\pm, b_i^\pm, a_0^\pm, b_0^\pm, D^\pm(x, t), c_0^\pm, d_0^\pm$ satisfy (5.30).

We can obtain the Schauder a priori estimates of the solution for problem (6.2)–(6.7) by the standard method. This method consists in the dropping the lower order terms in (6.2), (6.4), the freezing coefficients, and multiplying by smooth cutting functions. This can be done completely similar to [43] (or [6] in the case of Stefan problem). The principal model problem obtained by the freezing of the coefficients at points of the boundary Γ at $t = 0$ (with the subsequent local rectification of the boundary) is studied in Sect. 3. At considering such a model problem, the functions $e^\pm(x, t)$ and $A^\pm(x, t)$ are replaced by the constants $e^\pm \equiv e^\pm(x_0, 0)$ and $A^\pm \equiv A^\pm(x_0, 0)$, $x_0 \in \Gamma$. After this, in view of the condition $B(x, 0) \equiv 1$, the change of the unknown function

$$u^\pm(x, t) = \theta^\pm(x, t) - A^\pm \delta \quad (6.11)$$

reduces problem (6.2)–(6.7) with the frozen coefficients and with the flat boundary exactly to problem (3.3)–(3.7). From these model problems associated with the boundary Γ we get an estimate of the function $\delta(x, t)|_\Gamma$ and estimates of the functions $\theta^\pm(x, t)$ near the boundary Γ_T . The rest of the model problems associated with strictly interior points of $\Omega^\pm$ are standard because of the condition (6.7) and due to the absence of degeneracy of the equations at these points—see [43]. Thus we have the following assertion.

Lemma 6.1. *Suppose that conditions (6.8)–(6.10) are satisfied. Then there exists sufficiently small $T_0 = T_0(B)$ such that for $T \leq T_0$ for the solution of problem (6.2)–(6.7) from the class $\theta^\pm \in C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)$, $\delta \in C^{3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T)$ the following a priori estimate is valid*

$$\begin{aligned} & |\theta^\pm|_{s, \overline{\Omega}_T^\pm}^{(2+\gamma)} + |\delta|_{\Gamma_T}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha})} + |\delta_t|_{\Gamma_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} + \varepsilon \sum_{i,j=1}^{N-1} |\delta_{\omega_i \omega_j}|_{\Gamma_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} \\ & \leq C_T \left(|f_1^+|_{s, \overline{\Omega}_T^+}^{(\gamma)} + |f_1^-|_{s, \overline{\Omega}_T^-}^{(\gamma)} + |f_2|_{\Gamma_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} \right) \equiv C_T \mathcal{M}(T), \end{aligned} \quad (6.12)$$

where the constant C_T in (6.12) does not depend on $\varepsilon \in (0, 1)$.

If $B(x, t) \equiv 1$ then $T_0 = \infty$.

Let us remark that the condition $B(x, t) \equiv 1$ permits us to make change of the unknown functions (6.11) for any $t = t_0 > 0$ not only for $t_0 = 0$. So we have the estimate of the lemma for any $T > 0$.

We now show the solvability of problem (6.2)–(6.7).

Theorem 6.2. *Suppose that conditions (6.8)–(6.10) are satisfied. Then for $T \leq T_0$, $\varepsilon \in (0, 1)$ problem (6.2)–(6.7) is solvable in the space $\theta^\pm \in C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)$, $\delta \in C^{3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T)$ and estimate (6.12) of the solution is valid.*

When $\varepsilon = 0$ problem (6.2)–(6.7) is solvable in the space $\theta^\pm \in C_s^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T^\pm)$, $\delta \in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)$, $\delta_t \in C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T)$ and estimate (6.12) without the term with ε is valid.

If $B(x, t) \equiv 1$ then the above assertions are valid for any $T > 0$.

Proof. Define a linear operator $M : \delta \rightarrow \theta^\pm \rightarrow M\delta$ in the following way. Let M first maps a function $\delta \in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)$ to the functions $\theta^\pm$ as the solution of the problem (6.2)–(6.6) with the given function δ in (6.2). And then the operator M maps the functions $\theta^\pm$ to the function $M\delta$ and the last is determined from the condition (6.4) with the given $\theta^\pm$, δ_λ and δ in lower order term. That is, the function $M\delta$ is the solution of the problem

$$k(M\delta)_t - \varepsilon \Delta_\Gamma(M\delta) = f_2 - \left(a^+ \frac{\partial \theta^+}{\partial \lambda} - a^- \frac{\partial \theta^-}{\partial \lambda} \right) + \delta_\lambda (a^+ A^+ - a^- A^-) - \quad (6.13)$$

$$-(c_0^\pm(x, t)\theta^\pm + d_0^\pm(x, t)\delta),$$

$$M\delta(\omega, 0) = 0.$$

By Theorem 2.7, this operator is well defined. By the known properties of the problem (6.13) we have for $\varepsilon > 0$

$$|M\delta|_{\Gamma_T}^{(3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha})} \leq C_{\varepsilon, T} \left(|\delta|_{\Gamma_T}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha})} + \mathcal{M}(T) \right), \quad (6.14)$$

$$|M\delta_2 - M\delta_1|_{\Gamma_T}^{(3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha})} \leq C_{\varepsilon, T} |\delta_2 - \delta_1|_{\Gamma_T}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha})}. \quad (6.15)$$

Consequently, by (2.4)

$$\begin{aligned} |M\delta_2 - M\delta_1|_{\Gamma_T}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha})} &\leq CT^\mu |M\delta_2 - M\delta_1|_{\Gamma_T}^{(3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha})} \leq \\ &\leq C_{\varepsilon, T} T^\mu |\delta_2 - \delta_1|_{\Gamma_T}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha})}. \end{aligned} \quad (6.16)$$

Thus for a sufficiently small $T = T_\varepsilon$ the operator M is a contraction on $C_0^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)$ and therefore has a unique fixed point. By (6.14) this fixed point belongs also to the space $C_0^{3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T)$ and together with the corresponding $\theta^\pm$ gives the solution of the problem. The estimate of the solution is given by Lemma 6.1. Moving now step by step up the axis Ot as in [43], we obtain the theorem with $\varepsilon > 0$ for any $T \leq T_0$.

Consider now the sequence of the solutions $v_\varepsilon^\pm$, δ_ε , $\varepsilon \rightarrow 0$. From estimate (6.12) it follows that this sequence is a compact set in the spaces $C_{0,s}^{2+\bar{\gamma}, \frac{2+\bar{\gamma}}{2}}(\bar{\Omega}_T^\pm)$ and $C_0^{2+\bar{\beta}-\alpha, \frac{2+\bar{\beta}-\alpha}{2-\alpha}}(\Gamma_T)$ correspondingly for any $\bar{\gamma} < \gamma$, $\bar{\beta} = \bar{\gamma}(1 - \alpha/2)$. The passing to the limit of this sequence in the spaces $C_{0,s}^{2+\bar{\gamma}, \frac{2+\bar{\gamma}}{2}}(\bar{\Omega}_T^\pm)$ and $C_0^{2+\bar{\beta}-\alpha, \frac{2+\bar{\beta}-\alpha}{2-\alpha}}(\Gamma_T)$, $\delta_t \in C_0^{2+\bar{\beta}-\alpha, \frac{2+\bar{\beta}-\alpha}{2-\alpha}}(\Gamma_T)$ gives the solution of problem (6.2)–(6.7) for $\varepsilon = 0$. Besides, from uniform in ε estimate (6.12) it follows that the limit functions $v^\pm$ and δ belong to the spaces $C_{0,s}^{2+\gamma, \frac{2+\gamma}{2}}(\bar{\Omega}_T^\pm)$ and $C_0^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)$, $\delta_t \in C_0^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)$ correspondingly.

Thus Theorem 6.2 is proved. $\square$

From this theorem, Lemma 5.2, and the definition of the space $\mathcal{H}_0$ we obtain the following assertion.

Corollary 6.3. *Let the mapping $F(\varphi)$ is defined in (5.13) and T_0 is defined in Lemma 5.1. The Frechet derivative $F'(0)$ is invertible operator and*

$$\|(F'(0))^{-1}\|_{Y \rightarrow \mathcal{H}_0} \leq C_{T_0},$$

where the constant C_{T_0} does not depend on $T \leq T_0$.

As a corollary of Theorem 6.2, Lemma 6.1 and their proofs we can obtain the following assertion. Let V be a domain in R^N with boundary S of the class $C^{2+\gamma}$, $V_T = V \times [0, T]$, $S_T = S \times [0, T]$. Consider the following initial-boundary value problem with a dynamic boundary condition for a unknown function $u(x, t)$

$$\frac{\partial u(x, t)}{\partial t} - d^\alpha(x) \nabla^2 u(x, t) = f_1(x, t), \quad (x, t) \in V_T, \quad (6.17)$$

$$\frac{\partial u(x, t)}{\partial t} - \varepsilon \Delta_S u(x, t) + \frac{\partial u(x, t)}{\partial \vec{n}} = f_2(x, t), \quad (x, t) \in S_T, \quad (6.18)$$

$$u(x, 0) = u_0(x). \quad (6.19)$$

Here $\varepsilon \in [0, 1)$, Δ_S is the Laplace-Beltrami operator on S , $d(x)$ satisfies conditions

$$\nu \operatorname{dist}(x, S) \leq d(x) \leq \nu^{-1} \operatorname{dist}(x, S), \quad \nu > 0, \quad (6.20)$$

and

$$d(x) \in C^{1+\gamma}(\bar{V}),$$

$$f_1 \in C_s^{\gamma, \gamma/2}(\bar{V}_T), f_2 \in C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(S_T), u_0 \in C^{2+\gamma}(\bar{V}), u_0|_S \in C^{3+\gamma}(\bar{V}). \quad (6.21)$$

Theorem 6.4. *Let conditions (6.20), (6.21) are satisfied. Then for $\varepsilon > 0$ problem (6.17)–(6.19) has a unique solution $u(x, t)$ from the class $u \in C_s^{2+\gamma, \frac{2+\gamma}{2}}(\bar{V}_T)$, $u|_{S_T} \in C^{3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2}}(\Gamma_T)$ and the estimate*

$$\begin{aligned} & |u|_{s, V_T}^{(2+\gamma)} + |u_t|_{S_T}|_{S_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2})} + \varepsilon \sum_{i,j=1}^{N-1} \left| (u|_{S_T})_{\omega_i \omega_j} \right|_{S_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2})} \leq \\ & \leq C_T \left(|f_1|_{s, \bar{V}_T}^{(\gamma)} + |f_2|_{S_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2})} \right) \end{aligned} \quad (6.22)$$

is valid. When $\varepsilon = 0$ problem (6.17)–(6.19) is solvable in the space $u \in C_s^{2+\gamma, \frac{2+\gamma}{2}}(\bar{\Omega}_T^\pm)$, $u_t|_{S_T} \in C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(\Gamma_T)$ and estimate (6.22) without the term with ε is valid.

We do not give the proof of this theorem as it exactly coincides with the proofs of Lemma 6.1 and Theorem 6.2 with the help of Theorem 3.4. The difference is the presence of the nonzero initial data in (6.19). This case is reduced to the case of the zero initial data and to the spaces with zeros in (6.21) for f_1, f_2 by results of paper [42]. Besides, to obtain estimate (6.22) one should use Theorem 3.4 as a simplified version of Theorem 3.1.

7. Completion of the proof of Theorem 1.1

We complete the proof of Theorem 1.1 by Corollary 1.3. Consider the mapping $F(\varphi)$ on $\mathcal{B}_r$, where $F(\varphi)$ is defined in (5.13) and $\mathcal{B}_r$ is defined in (5.15). From Lemma 5.1 and Corollary 6.3 it follows that the conditions of Corollary 1.3 are satisfied except may be for the condition $\|F(0)\|_Y \leq \varepsilon_0$. But again from Lemma 5.1 we have

$$\|F(0)\|_Y \leq CT^\mu, \quad T \leq T_0,$$

where T_0 is defined in Corollary 6.3. So by decreasing the value of T_0 we can obtain all conditions of Corollary 1.3. Consequently, there exists $\varphi^* = (\theta^{+*}, \theta^{-*}, \delta^*) \in \mathcal{B}_r$ such that $F(\varphi^*) = 0$. Then the triple $(v^+, v^-, \rho) = (w^+ +$

$\theta^{+*}, w^{-} + \theta^{-*}, \sigma + \delta^*$) gives a solution of problem (4.6)–(4.11) and so a solution of the original nonlinear problem with free boundary.

Let us explain now the uniqueness of the solution of the problem for a sufficiently small time interval $[0, T]$. From Theorem 1.2 it follows that the ball $\mathcal{B}_r$ contains only one solution of the equation $F(\varphi) = 0$ if its radius r is sufficiently small. Let $\overline{\mathcal{H}}_0$ and $\overline{Y}$ are the spaces $\mathcal{H}_0, Y$ with exponents $\overline{\gamma} < \gamma, \overline{\beta} < \beta$ instead of γ, β . Let $(v_i^+, v_i^-, \rho_i), i = 1, 2$ are two solutions of (4.6)–(4.11) from the desired classes. Let also $\varphi_i, i = 1, 2$ are the two corresponding elements of $\mathcal{H}_0$. From (2.1)–(2.4) it follows that

$$\|\varphi_i\|_{\overline{\mathcal{H}}_0} \leq CT^\mu \|\varphi_i\|_{\mathcal{H}_0}.$$

This means that for arbitrary small $T > 0$ both solutions φ_1, φ_2 belong to the ball $\overline{\mathcal{B}}_r(0) \subset \overline{\mathcal{H}}_0$ of arbitrary small radius r . Consider now the mapping $F(\varphi)$ as a mapping from the space $\overline{\mathcal{H}}_0$ to $\overline{Y}$. Evidently, all the proved above properties of $F(\varphi)$ are preserved for such mapping. Particularly, we must have $\varphi_1 = \varphi_2$ if the radius r of $\overline{\mathcal{B}}_r(0)$ is sufficiently small, which is the case if the time interval $[0, T]$ is sufficiently small.

Thus Theorem 1.1 is proved. $\square$

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