The value function of an asymptotic exit-time optimal control problem

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Abstract. We consider a class of exit-time control problems for nonlinear systems with a nonnegative vanishing Lagrangian. In general, the associated PDE may have multiple solutions, and known regularity and stability properties do not hold. In this paper we obtain such properties and a uniqueness result under some explicit sufficient conditions. We briefly investigate also the infinite horizon problem.

Mathematics Subject Classification (2000). 49J15, 93C10, 49L20, 49L25, 93D20.

Keywords. Optimal control, Exit-time problems, Viscosity solutions, Asymptotic controllability.

1. Introduction

Among the hypotheses under which the boundary value problem (BVP),

$$\begin{cases} \mathcal{H}(x, Du(x)) = 0 \\ u = 0 \quad \text{on } \partial\mathcal{T}, \end{cases} \quad (1)$$

with

$$\mathcal{H}(x, u(x), Du(x)) \doteq \sup_{a \in A} \{ -\langle Du(x), f(x, a) \rangle - l(x, a) \} = 0 \quad (2)$$

and $\mathcal{T}$ a closed subset to $\mathbb{R}^n$ with compact boundary, has a unique solution, the condition that $l \geq c_0 > 0$ together with small time local controllability, STLC, around $\mathcal{T}$ plays a crucial role. In this case, under quite standard assumptions on the data, the solution to (1) is represented as the value function of an exit-time optimal control problem with target $\mathcal{T} \subset \mathbb{R}^n$, trajectories governed by a

This research is partially supported by the Marie Curie ITN SADCO, FP7-PEOPLE-2010-ITN n. 264735-SADCO and by the MIUR Grant PRIN 2009 “Metodi di viscosità, geometrici e di controllo per modelli diffusivi nonlineari” (2009KNZ5FK).

nonlinear control system

$$\dot{y}(t) = f(y(t), \alpha(t)), \quad y(0) = x \quad (x \in \mathbb{R}^n) \quad (3)$$

and payoff given by

$$\mathcal{J}(t, x, \alpha) = \int_0^t l(y(s), \alpha(s)) ds, \quad (4)$$

where the control $\alpha(t)$ belongs to the set $A \subset \mathbb{R}^m$ (assumed here to be compact). More precisely, for any $x \in \mathcal{T}^c \doteq \mathbb{R}^n \setminus \mathcal{T}$ the solution to (1) is given by the exit-time value function

$$\mathcal{V}^f(x) \doteq \inf_{\{\alpha \in \mathcal{A}: t_x(\alpha) < +\infty\}} \mathcal{J}(t_x(\alpha), x, \alpha) \quad (\leq +\infty), \quad (5)$$

where $\mathcal{A}$ is the set of measurable controls on A and, for any α ,

$$t_x(\alpha) \doteq \inf\{t \geq 0 : y_x(t, \alpha) \in \mathcal{T}\} \quad (\leq +\infty) \quad (6)$$

is the exit-time from $\mathcal{T}^c$.

In many interesting applications though, as for instance the Füller or the shape from shading problems, we have $l \geq 0$ and the set

$$\mathcal{Z} \doteq \{x : l(x, a) = 0 \text{ for some } a \in A\}, \quad (7)$$

is non empty. It is well known that in this case, without additional hypotheses, there is no hope to have a unique solution to (1), even among the continuous, nonnegative functions.

In this paper we consider an *exit-time* problem with l nonnegative and $\mathcal{T} \cap \mathcal{Z} \neq \emptyset$ and in Sect. 5 we will see that, in so doing, we can also cover some *undiscounted infinite horizon* problems like the LQR problem. Starting from the trivial remark that we might have minimizing trajectories approaching $\mathcal{T}$ in infinite time, we introduce for any $x \in \mathcal{T}^c$ the following *asymptotic* exit-time value function,

$$\mathcal{V}(x) \doteq \inf_{\alpha \in \mathcal{A}(x)} \mathcal{J}(t_x(\alpha), x, \alpha) \quad (\leq +\infty), \quad (8)$$

where

$$\mathcal{A}(x) \doteq \left\{ \alpha \in \mathcal{A} : \liminf_{t \rightarrow t_x^-(\alpha)} \text{dist}(y_x(t, \alpha), \mathcal{T}) = 0 \right\}.$$

Then we characterize $\mathcal{V}$ as the unique nonnegative solution to (1), under some global assumptions discussed below, and a *special local asymptotic controllability* hypothesis on the target, also involving the Lagrangian (see [14] and (LACL) in Sect. 2 below). Replacing the classical local small time controllability, STLC, on $\mathcal{T}$ by this *weaker* assumption implies that, while $\mathcal{V}$ will be well defined in a neighborhood of the target, in general $\mathcal{V}^f$ will not be finite there.

Furthermore, we perturb the (BVP) with more regular problems having a unique solution, show that their limit, say $\mathcal{U}$, not coinciding in general with $\mathcal{V}$, can be represented as the value function of a constrained optimization problem. Moreover, we give sufficient conditions for the equality $\mathcal{U} \equiv \mathcal{V}$, which essentially require the continuity of either $\mathcal{U}$ or $\mathcal{V}$ on $\partial\mathcal{T}$.

In more detail, optimality principles obtained in [13,17] tell us that $\mathcal{V}$ is the maximal solution to (1) when it is continuous on $\partial\mathcal{T}$. The minimal nonnegative solution to (1) is represented by the value function:

$$\mathcal{V}^m(x) \doteq \inf_{\alpha \in \mathcal{A}} \mathcal{J}(t_x(\alpha), x, \alpha), \quad (9)$$

where the minimization is done over trajectories not necessarily steering to the target. Following this approach, we reduce the problem of uniqueness essentially to the control theoretical questions of whether $\mathcal{V}$ is continuous on $\partial\mathcal{T}$ and $\mathcal{V} \equiv \mathcal{V}^m$.

Exploiting some recent results by Motta and Rampazzo [14], we give suitable asymptotic controllability conditions implying the continuity of $\mathcal{U}$, introduced above, and of $\mathcal{V}$, on the target, but in general not in their whole domains. We also investigate the global continuity of $\mathcal{U}$ and $\mathcal{V}$ by introducing a kind of *turnpike condition*, that roughly states that trajectories not uniformly approaching the target, at least asymptotically, are unaffordable (see e.g. [19, 20]).

In order to have $\mathcal{V} \equiv \mathcal{V}^m$, we introduce some explicit sufficient conditions satisfied in many applications and generalizing several previous hypotheses. Let us remark that many of the assumptions existing in the literature ensuring uniqueness, *imply* that $\mathcal{V} \equiv \mathcal{V}^m$. We refer to Section 5 of [12,13] and Remark 2.2 below for an analysis of some of them.

Our uniqueness and stability results improve some previous research under several aspects. Substituting the STLC with the (LACL) and using the optimality principles instead of classical comparison theorems, allows us to have the uniqueness of a continuous solution among all the nonnegative solutions to (BVP). Moreover, owing to the (LACL), we can also extend our results to infinite horizon problems, where admissible trajectories approach asymptotically some set $\mathcal{T} \subset \mathcal{Z}$, noticeably the origin in the LQR problems. When uniqueness fails, we give sufficient conditions in order to characterize $\mathcal{V}$ as the limit of (unique) nonnegative solutions of perturbed boundary value problems. We point out that any stability result has to be proved directly, as it cannot rely on the standard viscosity approach, based on the uniqueness of the solution to (1).

Without aiming to be exhaustive, for the uniqueness issue and for an insight into many applications in which l is not strictly positive, we refer to [4,5,7,8,10,12,13,17], also concerning unbounded controls and infinite horizon problems. In particular, in [5,10] special Hamiltonians are considered; in [12] just the function $\mathcal{V}^f$ is characterized; in [4,7] uniqueness is obtained in the class of continuous functions with bounded subdifferential, and in [8] among the convex functions.

Finally, let us mention that, in the companion paper [15], we extend the research begun here to the case of a non compact control set and unbounded data.

The paper is organized as follows. In Sect. 2 we characterize $\mathcal{V}$ as unique nonnegative solution of the (BVP). Section 3 is devoted to the approximation

of $\mathcal{V}$. In Sect. 4 we give sufficient conditions for the continuity either of $\mathcal{V}$ in its domain, or of the limit function of the penalized problems, $\mathcal{U}$, on $\partial\mathcal{T}$.

Notations. Let $D \subset \mathbb{R}^N$ for some $N \in \mathbb{N}$. $\forall r > 0$ we denote by D_r the closed set $\overline{B(D, r)}$, while $D_r^c = \mathbb{R}^N \setminus D_r$. $\overset{\circ}{D}$ is the interior of D . Moreover, χ_D denotes the characteristic function of D , namely for any $x \in \mathbb{R}^N$ we set $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ if $x \notin D$. For any function $u : \mathbb{R}^n \setminus \overset{\circ}{\mathcal{T}} \rightarrow \mathbb{R} \cup \{+\infty\}$, we denote the set $\{x \in \mathbb{R}^n \setminus \overset{\circ}{\mathcal{T}} : u(x) < +\infty\}$ by $\text{Dom}(u)$. $[0, +\infty[\doteq \mathbb{R}_+$. A function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a *modulus* if: $\omega(\cdot, R)$ is increasing in a neighborhood of 0, continuous at 0, and $\omega(0, R) = 0$ for every $R > 0$; $\omega(r, \cdot)$ is increasing for every r . Let $\Omega \supset \mathcal{T}$ be an open set and let $U : \Omega \setminus \overset{\circ}{\mathcal{T}} \rightarrow \mathbb{R}_+$ be a locally Lipschitz function. Then $D^*U(x) \doteq \{p \in \mathbb{R}^n : p = \lim_k \nabla U(x_k), x_k \in \text{diff}(U) \setminus \{x\}, \lim_k x_k = x\}$ is the *set of limiting gradients* of U at x (here ∇ denotes the gradient operator and $\text{diff}(U)$ is the set of differentiability points of U). For the notion of locally semiconcave function and of viscosity solution we refer e.g. to [3, 6]. $\mathcal{KL}$ denotes the set of all continuous functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that: (1) $\beta(0, t) = 0$ and $\beta(\cdot, t)$ is strictly increasing and unbounded for each $t \geq 0$; (2) $\beta(r, \cdot)$ is decreasing for each $r \geq 0$; (3) $\beta(r, t) \rightarrow 0$ as $t \rightarrow +\infty$ for each $r \geq 0$.

2. Uniqueness

In this section we introduce sufficient conditions, under which we can characterize $\mathcal{V}$ as unique nonnegative solution to (1). The present assumptions generalize and in some sense unify several previous hypotheses, introduced either for exit-time or for undiscounted infinite horizon problems. In particular, our uniqueness result does not require neither the STLC around the target nor $\mathcal{Z} \subset \mathcal{T}$ (see also Remark 2.2 below). We end the section with two simple, illustrative examples.

Let us begin by stating the hypotheses assumed throughout the whole paper and the precise definition of the boundary value problem.

The control set $A \subset \mathbb{R}^m$ is compact and the target set $\mathcal{T} \subset \mathbb{R}^n$ is closed, with compact boundary. The function $l : \mathbb{R}^n \times A \rightarrow \mathbb{R}_+$ is continuous. Moreover, $f : \mathbb{R}^n \times A \rightarrow \mathbb{R}^n$ is continuous and there exist $M > 0$, and for any $R > 0$, there is some $L_R > 0$ such that

$$\begin{aligned} |f(x_1, a) - f(x_2, a)| &\leq L_R |x_1 - x_2|, \\ |l(x_1, a) - l(x_2, a)| &\leq L_R |x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}^n, \forall a \in A, \\ |f(x, a)| &\leq M(1 + |x|) \quad \forall x, \forall a \in A. \end{aligned} \quad (10)$$

Hence for any $x \in \mathbb{R}^n$ and for any measurable control $\alpha \in \mathcal{A}$, (3) admits just one solution, defined on the whole interval $\mathbb{R}_+$. We use $y_x(\cdot, \alpha)$ (or, when no confusion may arise, $y_x(\cdot)$) to denote such a solution.

Definition 2.1. (BVP) [13] Any function $u : \mathbb{R}^n \setminus \overset{\circ}{\mathcal{T}} \rightarrow \mathbb{R} \cup \{+\infty\}$ verifying $u_*(x) \geq 0$ on $\partial\mathcal{T}$ and such that u_* is a viscosity supersolution of

$$\mathcal{H}(x, Du(x)) = 0 \quad (11)$$

in $\mathbb{R}^n \setminus \mathcal{T}$, is called a supersolution to (BVP). Any pair (u, Ω) where $\Omega \supset \mathcal{T}$ is an open set and $u : \Omega \setminus \overset{\circ}{\mathcal{T}} \rightarrow \mathbb{R}$ is a locally bounded function verifying $u^*(x) \leq 0$ on $\partial\mathcal{T}$ and such that u^* is a viscosity subsolution of (11) in $\Omega \setminus \mathcal{T}$, is called a subsolution to (BVP) (in Ω).

Any pair (u, Ω) , where $u : \mathbb{R}^n \setminus \overset{\circ}{\mathcal{T}} \rightarrow \mathbb{R} \cup \{+\infty\}$ and Ω is an open set, $\Omega \supset \mathcal{T}$, is called a solution to (BVP) (in Ω) if u is a supersolution and (u, Ω) is a subsolution to (BVP).

We recall the optimality principles and some related results obtained in [13] (see Thms. 2.1, 4.2, 4.3). We refer also to the works [17, 18], where a strong formulation of the optimality principles has been first introduced and developed for this kind of problems.¹

Proposition 2.1. [13] *Let $\mathcal{W} \in \{\mathcal{V}, \mathcal{V}^f, \mathcal{V}^m\}$.*

- (i) *If $\mathcal{W}$ is locally bounded in $\text{Dom}(\mathcal{W})$, $\text{Dom}(\mathcal{W})$ is open and $\mathcal{W}^* \leq 0$ on $\partial\mathcal{T}$, then $\mathcal{W}$ is a subsolution to (BVP) in $\text{Dom}(\mathcal{W})$.*
- (ii) *$\mathcal{W}$ is a nonnegative supersolution to (BVP).*

In the next statement we will use the following convexity hypothesis.

(CV) *For each $x \in \mathbb{R}^n$, the following set is convex:*

$$\mathcal{L}(x) \doteq \{(\mu, \gamma) \in \mathbb{R}^{n+1} : \exists a \in A \text{ s. t. } \mu = f(x, a), l(x, a) \leq \gamma\}. \quad (12)$$

Proposition 2.2. [13]

- (i) *We have $\mathcal{V}^m \leq u$ for any nonnegative and continuous supersolution u to (BVP). If we assume (CV), then $\mathcal{V}^m$ is l.s.c and it is the minimal nonnegative supersolution to (BVP).*
- (ii) *If $\mathcal{V}$ is continuous on $\partial\mathcal{T}$, then $(\mathcal{V}, \text{Dom}(\mathcal{V}))$ is the maximal subsolution to (BVP) among the pairs $(u, \text{Dom}(\mathcal{V}))$.*²

Remark 2.1. As usual, if (CV) does not hold the above result remains true if we replace $\mathcal{V}^m$ by the corresponding value function, say $\mathcal{V}_r^m$, obtained by taking the infimum over relaxed controls (see [13]).

In [18], there is a formally similar characterization of the maximal subsolution and minimal supersolution to (BVP) for a discontinuous Lagrangian. However, in the undiscounted case considered here, those results are proved when the Lagrangian is bounded below by a positive constant.

We point out that (BVP) is a *free-boundary* value problem, and that the exit-time value functions do not satisfy, in general, the boundary condition

$$\lim_{x \rightarrow \bar{x}} u(x) = +\infty \quad \forall \bar{x} \in \partial\text{Dom}(u). \quad (13)$$

¹ We remark that in [13], l is also satisfying $l(x, a) \leq M(1 + |x|) \quad \forall (x, a) \in \mathbb{R}^n \times A$, for some $M > 0$, but this sublinear growth condition can be removed, as in [18].

² We recall that $\text{Dom}(\mathcal{V})$ is an open set, $\mathcal{V}$ is locally bounded and upper semicontinuous in view of Proposition 2.5 below.

In the sequel, improving the results of [13], we characterize *the pair* $(\mathcal{V}, \text{Dom}(\mathcal{V}))$ as unique solution of (BVP), among the solutions verifying the boundary condition (13). Disregarding such a restriction, we could still prove the uniqueness of $\mathcal{V}$ among the pairs (u, Ω) with $\Omega = \text{Dom}(\mathcal{V})$.

Let us now state the following continuity and uniqueness result, whose proof follows from Theorem 2.2 below. This general, but quite theoretical statement, is the starting point for handier results, given in the sequel.

Theorem 2.1. *Let $\mathcal{V} \equiv \mathcal{V}^m$.*

- (i) *If $\mathcal{V}$ is continuous in $\text{Dom}(\mathcal{V})$ and satisfies the boundary condition (13), then $\mathcal{V}$ is the unique nonnegative viscosity solution to (BVP) among the pairs (u, Ω) , where u is continuous in Ω and satisfies (13).*
- (ii) *If (CV) holds and $\mathcal{V}$ is continuous on $\partial\mathcal{T}$, then $(\mathcal{V}, \text{Dom}(\mathcal{V}))$ is the unique nonnegative viscosity solution to (BVP) among the pairs (u, Ω) , where u satisfies (13). Moreover, $\mathcal{V}$ is continuous.³*

One might wonder why the maximal subsolution of (BVP) is $\mathcal{V}$ and not $\mathcal{V}^f$, obviously larger. This is not a contradiction, because, if $\mathcal{V}^f$ is continuous on $\partial\mathcal{T}$, so that it solves (BVP), then $\mathcal{V}^f \equiv \mathcal{V}$ by Theorem 3.2 below. Let us stress however, that, using the above uniqueness result we can characterize the solution to (BVP) also in situations where $\mathcal{V} < \mathcal{V}^f$, as shown by Examples 2.1, 2.2 at the end of the section.

Remark 2.2. In the literature many uniqueness results for the solution of (BVP) concerning the function $\mathcal{V}^f$, are proved under hypotheses that imply $\mathcal{V} \equiv \mathcal{V}^m$. For instance, this is easily seen in [11], where, for all $x \in \mathcal{T}^c$ and $\alpha \in \mathcal{A}$, one supposes that

$$\int_0^{+\infty} l(y_x(t, \alpha), \alpha(t)) dt < +\infty \implies \lim_{t \rightarrow +\infty} y_x(t, \alpha) \in \mathcal{T}.$$

Also the following hypotheses, used in [12],

- (a) the trajectories that have a finite cost must stay in a bounded set,
- (b) $\forall t > 0$ and $\forall \alpha \in \mathcal{A}$ one has $\int_0^t l(y_x(s, \alpha), \alpha(s)) ds > 0$, together with (CV), with a little bit of work can be shown to imply $\mathcal{V} \equiv \mathcal{V}^m$.

As discussed in Remark 5.3 in Sect. 5, in many undiscounted infinite horizon problems, seen as asymptotic exit-time problems for a suitable target (as the LQR problem), condition $\mathcal{V} \equiv \mathcal{V}^m$ is naturally verified.

Since (BVP) is a free boundary problem, for any solution pair (u, Ω) we introduce the Kruzkov transform $W(x) \doteq \Psi(u(x)) \doteq 1 - e^{-uv(x)}$, leading to another boundary value problem in $\mathbb{R}^n \setminus \mathcal{T}$, whose solution, when unique, simultaneously gives both u and $\Omega \doteq \text{Dom}(u)$.

More precisely, the Hamiltonian associated to W is

$$\mathcal{K}(x, u, p) \doteq \sup_{a \in A} \{-\langle p, f(x, a) \rangle - l(x, a) + l(x, a)u\} \quad (14)$$

³ Since $\mathcal{V}^m$ is lsc and $\mathcal{V}$ is usc, when $\mathcal{V} \equiv \mathcal{V}^m$ (13) is trivially satisfied.

and we consider the following boundary value problem, in short, (BVPK),

$$\begin{cases} \mathcal{K}(x, W(x), DW(x)) = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{T} \\ W(x) = 0 & \text{on } \partial\mathcal{T}, \end{cases} \quad (15)$$

where super- and subsolutions are defined analogously to Definition 2.1.

Remark 2.3. From [13], the statements of Propositions 2.1 and 2.2 can be reformulated in terms of the Kruzkov transforms of the exit-time value functions, in the *whole* space $\mathbb{R}^n \setminus \mathcal{T}$ (see Cor. 3.1 and Thm. 4.3 in [13]). Incidentally, these results hold without assuming the boundary condition (13). Hence they are not trivial, since, given a subsolution (u, Ω) to (BVP), $\Psi(u)$ is not in general a subsolution to (BVPK) in $\mathbb{R}^n \setminus \mathcal{T}$ but just in $\Omega \setminus \mathcal{T}$.

We have the following uniqueness result in $\mathbb{R}^n$.

Theorem 2.2. *Let $\mathcal{V} \equiv \mathcal{V}^m$.*

- (i) *If $\mathcal{V}$ is continuous in $\text{Dom}(\mathcal{V})$ and satisfies the boundary condition (13), then there is a unique continuous, nonnegative viscosity solution W to (BVPK). Moreover, $\mathcal{V} \equiv \Psi^{-1}(W) = -\log(1 - W)$ and $\text{Dom}(\mathcal{V}) = \{x : W(x) < 1\}$.*
- (ii) *If (CV) holds and $\mathcal{V}$ is continuous on $\partial\mathcal{T}$, then there is a unique nonnegative viscosity solution W to (BVPK) which turns out to be continuous. Moreover, $\mathcal{V} \equiv \Psi^{-1}(W) = -\log(1 - W)$ and $\text{Dom}(\mathcal{V}) = \{x : W(x) < 1\}$.*

Proof. We prove just (ii), the proof of (i) being similar and actually simpler. By Proposition 2.2 and Remark 2.3, for any solution W to (BVPK) we get

$$\Psi(\mathcal{V}^m)(x) \leq W_*(x) \leq W(x) \leq W^*(x) \leq \Psi(\mathcal{V})(x) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{T}.$$

Thesis (ii) follows now easily. $\square$

Explicit sufficient conditions for the equality $\mathcal{V} = \mathcal{V}^m$ are hypotheses (SC1), (SC2) below.

- (SC1) *There exists a Lyapunov function $U : \mathbb{R}^n \setminus \overset{\circ}{\mathcal{T}} \rightarrow \mathbb{R}_+$, C^1 in $\mathbb{R}^n \setminus \overset{\circ}{\mathcal{T}}$, positive definite, proper on $\mathcal{T}^c$ and such that $\forall x \in \mathcal{T}^c$,*

$$\sup_{a \in A} \{ \langle \nabla U(x), f(x, a) \rangle \} \leq -m(\mathbf{d}(x)) \quad (16)$$

for some continuous, increasing function $m :]0, +\infty[\rightarrow]0, +\infty[$.

- (SC2) *there exists a continuous, increasing function $c_2 :]0, +\infty[\rightarrow]0, +\infty[$ such that*

$$l(x, a) \geq c_2(\mathbf{d}(x)) \quad \forall (x, a) \in \mathcal{T}^c \times A. \quad (17)$$

Condition (SC1) implies that the control system (3) is *uniformly globally asymptotically stable, in short UGAS, in $\mathcal{T}^c$* , which roughly means that *all* trajectories with initial condition in a compact set, approach $\mathcal{T}$ uniformly (see e.g. [2] and the references therein). We point out that (SC1) allows the Lagrangian to be zero outside the target.

Hypothesis (SC2), involving just the cost, implies instead that the set $\mathcal{Z}$ (see (7)) is a subset of $\mathcal{T}$. It is the simplest generalization of a strictly positive Lagrangian, considered e.g. in [16], in the framework of differential

games. Assumption (SC2) for $\mathcal{T} \equiv \{0\}$ is also satisfied in LQR problems, where $l(x, a) = x^T Q x + a^T R a$ and the matrices Q and R are symmetric and positive definite.

Proposition 2.3. *If either (SC1) or (SC2) holds, then $\mathcal{V} = \mathcal{V}^m$.*

Proof. It is immediate to see that condition (SC2) yields the equality $\mathcal{V} \equiv \mathcal{V}^m$, since $\mathcal{V} \equiv \mathcal{V}^m$ is equivalent to

$$\forall x \in \mathcal{T}^c : \int_0^{+\infty} l(y_x(t), \alpha(t)) dt \geq \mathcal{V}(x) \quad \forall \alpha \in \mathcal{A} \setminus \mathcal{A}(x), \quad (18)$$

and (SC2) implies that

$$\int_0^{+\infty} l(y_x(t), \alpha(t)) dt = +\infty \quad \forall x \in \mathcal{T}^c, \forall \alpha \in \mathcal{A} \setminus \mathcal{A}(x).$$

Condition (SC1) instead, yields (18) because $\mathcal{A} \setminus \mathcal{A}(x) = \emptyset$ for all $x \in \mathcal{T}^c$. $\square$

In order to state a special local asymptotic controllability condition, in short (LACL), introduced in [14] and sufficient to obtain the continuity of $\mathcal{V}$ on partial $\mathcal{T}$, let us recall the notion of (local) Minimum Restraint Function from [14]. For some terminology borrowed from nonsmooth analysis we refer to the Notation. Here $h : \mathbb{R}^n \times A \rightarrow \mathbb{R}_+$ is an arbitrary continuous Lagrangian.

Definition 2.2. [14] Given an open set $\Omega \subset \mathbb{R}^n$, $\Omega \supset \mathcal{T}$ we say that $U : \Omega \setminus \overset{\circ}{\mathcal{T}} \rightarrow \mathbb{R}_+$ is a local Minimum Restraint Function, in short, a local MRF for h , if U is continuous on $\Omega \setminus \overset{\circ}{\mathcal{T}}$, locally semiconcave, positive definite, proper⁴ on $\Omega \setminus \mathcal{T}$, $\exists U_0 \in]0, +\infty]$ such that

$$\lim_{x \rightarrow x_0, x \in \Omega} U(x) = U_0 \quad \forall x_0 \in \partial\Omega; \quad U(x) < U_0 \quad \forall x \in \Omega \setminus \overset{\circ}{\mathcal{T}},$$

and, moreover, $\exists k > 0$ such that, for every $x \in \Omega \setminus \mathcal{T}$,

$$\min_{a \in A} \{ \langle p, f(x, a) \rangle + k h(x, a) \} < 0 \quad \forall p \in D^*U(x), \quad (19)$$

where $D^*U(x)$ is the set of limiting gradients of U at x .

Let us observe that any MRF is a Control Lyapunov function for the system w.r.t. $\mathcal{T}$, which yields local asymptotic controllability to $\mathcal{T}$. In the sequel, fixed a continuous function $h \geq 0$, we will often use the following hypothesis.

(LACL) There exists a local MRF U for h , as introduced in Definition 2.2.

By Theorem 1.1 in [14], we have

Proposition 2.4. *If there exists a local MRF U for l , then $\mathcal{V}(x) \leq U(x)/k$ in a neighborhood of the target. Hence $\mathcal{V}$ is continuous on $\partial\mathcal{T}$.*

⁴ U is said *positive definite* on $\Omega \setminus \mathcal{T}$ if $U(x) > 0 \quad \forall x \in \Omega \setminus \mathcal{T}$ and $U(x) = 0 \quad \forall x \in \partial\mathcal{T}$. U is called *proper* on $\Omega \setminus \mathcal{T}$ if $U^{-1}(K)$ is compact for every compact set $K \subset \mathbb{R}_+$.

Because of the degeneracy of l , the continuity of $\mathcal{V}$ on $\partial\mathcal{T}$ does not imply, in general, the continuity in its whole domain. Using a standard dynamic programming argument, it is not difficult to prove that in this case $\mathcal{V}$ is upper semicontinuous.

Proposition 2.5. *Let $\mathcal{V}$ be continuous on $\partial\mathcal{T}$. Then $\text{Dom}(\mathcal{V})$ is an open set and $\mathcal{V}$ is locally bounded and upper semicontinuous in it.*

A general condition, sufficient for the propagation of the continuity of $\mathcal{V}$, will be given in Sect. 4.1. In order to state the following explicit result, let us anticipate that both (SC1) and (SC2) imply such a propagation and the boundary condition (13) (see Proposition 4.1).

Corollary 2.1. *Assume (CV), (LACL) for l and either (SC1) or (SC2). Then*

- (i) *there is a unique nonnegative viscosity solution W to (BVPK), which turns out to be continuous. Moreover, $\mathcal{V} \equiv \Psi^{-1}(W) = -\log(1 - W)$ and $\text{Dom}(\mathcal{V}) = \{x : W(x) < 1\}$;*
- (ii) *$(\mathcal{V}, \text{Dom}(\mathcal{V}))$ is the unique nonnegative viscosity solution to (BVP) among the pairs (u, Ω) , where u satisfies (13). Moreover, $\mathcal{V}$ is continuous.*

If (CV) is not satisfied, the above uniqueness results hold just among the continuous functions.

In the following examples satisfying either (SC2) or (SC1), respectively, we have uniqueness without the STLC on the target.

Example 2.1. Let $\mathcal{T} \doteq \{0\}$, consider the scalar control system

$$y' = -y a \quad \forall t > 0, \quad y(0) = x, \quad a \in [0, 1],$$

and define

$$\mathcal{J}(t, x, \alpha) \doteq \int_0^t |y_x(t, \alpha)| dt.$$

Since $y(t) = x e^{-\int_0^t |\alpha(s)| ds}$, for every $x \neq 0$ a control α belongs to $\mathcal{A}(x)$ if and only if it verifies $\int_0^{+\infty} |\alpha(s)| ds = +\infty$. Hence $\mathcal{V}^f(x) = +\infty$, while setting $\alpha \equiv 1$ we can easily deduce that $\mathcal{V}(x) = |x|$.

Notice that the system is asymptotically controllable to $\{0\}$, $\{0\}$ is the unique zero of the Lagrangian and actually $l(x, a) = |x|$ satisfies (SC2). Moreover, (CV) holds and $\mathcal{V}$ is continuous everywhere. Incidentally, for any $k \in]0, 1[$ the value function itself is a MRF for l . Therefore, in view of Corollary 2.1, $\mathcal{V}(x) = |x|$ is the unique nonnegative solution of the boundary value problem

$$\max_{a \in [0, 1]} \{ \langle Du, x a \rangle - |x| \} = 0, \quad u(0) = 0.$$

Example 2.2. For any $x \in \mathbb{R}^2$ and any measurable function $\alpha : \mathbb{R}_+ \rightarrow [0, 1]$, consider the control system

$$\begin{cases} y'(t) = -y(t) - y(t) \alpha(t) & \forall t > 0, \\ y(0) = (y_1, y_2)(0) = (x_1, x_2) = x, \end{cases}$$

and the payoff

$$\mathcal{J}(t, x, \alpha) \doteq \int_0^t y_1^2(s, \alpha) ds \quad \forall t > 0,$$

with the target $\mathcal{T} \doteq \{(0, 0)\}$. Since $y_x(t, \alpha) = x e^{\int_0^t (-\alpha(s)-1) ds}$ in correspondence to any control α , it is not difficult to prove that, for every $x \neq 0$, $\mathcal{V}^f(x) = +\infty$ and

$$\mathcal{V}(x) \doteq \inf_{\alpha \in \mathcal{A}(x)} \mathcal{J}(t_x(\alpha), x, \alpha) = \frac{x_1^2}{4}.$$

Notice that the zero level set of the Lagrangian $l(x, a) = x_1^2$ is given by the unbounded set $\mathcal{Z} = \{(0, x_2) : x_2 \in \mathbb{R}\}$, but the control system is UGAS in $\mathbb{R}^2 \setminus \{(0, 0)\}$, so that condition (SC1) is verified. Moreover, (CV) holds and it is easy to see that the following function

$$U(x) \doteq \frac{x_1^2 + x_2^2}{4}$$

is a MRF function for $l(x, a) = x_1^2$ that verifies (19) for any $k \in]0, 1[$.

Hence in view of Corollary 2.1, $\mathcal{V}(x) = x_1^2/4$ is the unique nonnegative viscosity solution of the following (BVP):

$$\max_{\alpha \in [0, 1]} \{-\langle Du(x), -x - xa \rangle - x_1^2\} = 0 \quad \forall x \in \mathbb{R}^2 \setminus \{(0, 0)\}, \quad u(0, 0) = 0.$$

When $\mathcal{V} \neq \mathcal{V}^m$, owing to Proposition 2.2 we can still characterize $\mathcal{V}$ as maximal subsolution of (BVP). Following an alternative approach, we can obtain $\mathcal{V}$ as limit of perturbed boundary value problems (having a unique solution). This is the topic of the next section.

3. Approximation results

This section is mainly devoted to discuss whether and when the exit-time problem (8) can be approximated by some perturbed problems, giving also *uniform* convergence conditions. These results can be seen as stability properties for the (BVP), in the sense that $\mathcal{V}$ can be selected as the solution which is the limit of value functions, themselves characterized as unique solution of suitable boundary value problems.

3.1. Penalized problems

Let $\rho : \mathbb{R}^n \times A \rightarrow \mathbb{R}_+$ be a continuous function. Fix $\varepsilon > 0$. For every $x \in \mathcal{T}^c$, $\alpha \in \mathcal{A}(x)$, let us define the ε -penalized payoff,

$$\mathcal{J}_\varepsilon^\rho(t, x, \alpha) \doteq \int_0^t [l(y_x(\tau, \alpha), \alpha(\tau)) + \varepsilon \rho(y_x(\tau, \alpha), \alpha(\tau))] d\tau \quad (20)$$

and the corresponding ε -penalized value function,

$$\mathcal{V}_\varepsilon^\rho(x) \doteq \inf_{\alpha \in \mathcal{A}(x)} \mathcal{J}_\varepsilon(t_x(\alpha), x, \alpha). \quad (21)$$

We introduce also the *penalized value function*

$$\mathcal{U}^\rho(x) \doteq \inf_{\alpha \in \mathcal{A}_\rho(x)} \mathcal{J}(t_x(\alpha), x, \alpha) \quad (\leq +\infty), \quad (22)$$

where

$$\mathcal{A}_\rho(x) \doteq \left\{ \alpha \in \mathcal{A}(x) : \int_0^{t_x(\alpha)} \rho(y_x(t, \alpha), \alpha(t)) dt < +\infty \right\}, \quad (23)$$

which will play a crucial role in the sequel.

Clearly, $\mathcal{V}(x) \leq \mathcal{U}^\rho(x) \leq \mathcal{V}_\varepsilon^\rho(x)$ and the inequalities may be strict, as one can easily see in Example 2.1 choosing $\rho(x, a) \doteq |a|$ for every (x, a) .

For any $\varepsilon > 0$, let $\mathcal{K}_\varepsilon(x, u, p)$ denote the Hamiltonian defined as $\mathcal{K}$ in (14) with l replaced by $l + \varepsilon\rho$. In view of Theorem 2.2 and Corollary 2.1, Theorem 3.3 below implies the following stability result.

Theorem 3.1. *Let $\rho : \mathbb{R}^n \times A \rightarrow \mathbb{R}_+$ be a continuous function. Assume (CV), (LACL) for $l + \rho$ ⁵ and either (SC1) or (SC2) for l replaced by $l + \rho$. Then for any $\varepsilon \in]0, 1]$ there exists a unique nonnegative solution W_ε to*

$$\begin{cases} \mathcal{K}_\varepsilon(x, W(x), DW(x)) = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{T} \\ W(x) = 0 & \text{on } \partial\mathcal{T}. \end{cases} \quad (24)$$

Moreover, as $\varepsilon \rightarrow 0^+$ the W_ε converge to a function W such that $\mathcal{V} \equiv -\log(1 - W)$ and $\text{Dom}(\mathcal{V}) = \{x : W(x) < 1\}$. If W is continuous in $\mathbb{R}^n \setminus \overset{\circ}{\mathcal{T}}$, then the convergence is locally uniform.

If (CV) is not satisfied, for any $\varepsilon \in]0, 1]$, the function W_ε is the unique solution to (24) just among the continuous functions.

Notice that, if (SC1) holds, owing to Proposition 4.1, $\mathcal{V}$ is continuous in its domain. When instead (SC2) for $l + \rho$ (not implying, in general, (SC2) for l) is assumed, the global continuity of $\mathcal{V}$ is not guaranteed a priori.

Choosing, e.g., $\rho(x, a) \doteq \mathbf{d}^r(x)$ for some integer $r \geq 1$, then for any $l \geq 0$, $l(x, a) + \rho(x, a)$ satisfies (SC2) and, as in the following example, we can have uniqueness of the solution for the perturbed problems. Let us remark that in the next example the trivial choice $\rho \equiv 1$ does not give an approximation of $\mathcal{V}$ (see also Proposition 3.2).

Example 3.1. Let $\mathcal{T} \doteq \{0\}$, consider the scalar control system of Example 2.1,

$$y'(t) = -y(t)\alpha(t) \quad \forall t > 0, \quad y(0) = x, \quad \alpha(t) \in [0, 1],$$

and define

$$\mathcal{J}(t, x, \alpha) \doteq \int_0^t |y_x^2(s, \alpha) - y_x(s, \alpha)| ds \quad \forall t > 0.$$

⁵ The last hypothesis can be replaced by the weaker explicit sufficient conditions in Theorem 4.2 below, implying the continuity of the limit function $\mathcal{U}^\rho$ on $\partial\mathcal{T}$.

Notice that $\mathcal{V}^m \neq \mathcal{V}$. Indeed, implementing the control $\alpha \equiv 0$, one gets e.g. $\mathcal{V}^m(1) = 0$, while $\mathcal{V}(1) = 1/2$, being

$$\mathcal{V}(x) = \left| x - \frac{x^2}{2} \right| \quad \forall x \in \mathbb{R}, \quad (25)$$

as we show below. Let us introduce for every $\varepsilon > 0$ the ε -penalized value function

$$\mathcal{V}_\varepsilon^\rho(x) \doteq \inf_{\alpha \in \mathcal{A}(x)} \int_0^{t_x(\alpha)} (|y_x^2(t, \alpha) - y_x(t, \alpha)| + \varepsilon |y_x(t, \alpha)|) dt.$$

Setting $f(x, a) \doteq -xa$, $l(x, a) \doteq |x^2 - x|$ and $\rho(x, a) \doteq |x|$, in view of Theorem 3.1, for any $\varepsilon > 0$ the Kruzkov transform $W_\varepsilon \doteq 1 - e^{-\mathcal{V}_\varepsilon^\rho}$ is the unique nonnegative solution to (24) among the continuous functions, since (CV) is not verified. Moreover, as $\varepsilon \rightarrow 0^+$ the W_ε converge to $W(x) = 1 - e^{|x - \frac{x^2}{2}|}$ for all $x \in \mathbb{R}$. Notice that the convergence is locally uniform in view of the continuity of W . Finally, $\mathcal{V} \equiv -\log(1 - W)$. Indeed, straightforward calculations show that the function

$$U(x) \doteq \left| x - \frac{x^2}{2} \right| + |x| \quad \forall x \in \mathbb{R}$$

is a MRF function for $l + \rho$, that verifies (19) for any $0 < k < 1$ and the lagrangian $|x^2 - x| + |x|$ obviously verifies (SC2). At this point it is easy to show that

$$\mathcal{V}_\varepsilon^\rho(x) = \left| x - \frac{x^2}{2} \right| + \varepsilon |x| \quad \forall x \in \mathbb{R}.$$

Finally, $\mathcal{V}$ is given by (25) and can be characterized as the locally uniform limit of $\mathcal{V}_\varepsilon^\rho$.

In Theorem 3.3 below we shall give a representation formula for the limit of the penalized problems in terms of the value function $\mathcal{U}^\rho$ defined in (22). Since the zero level set of l is arbitrary such a limit does not coincide in general with $\mathcal{V}$. In this case, problem (8) is sometimes said to exhibit the *Lavrentiev phenomenon* or, more precisely, to have a $\mathcal{A}(x) - \mathcal{A}_\rho(x)$ *Lavrentiev gap*. As a first result, the next theorem shows that, on the one hand, such a phenomenon cannot occur whenever $\mathcal{V} \equiv \mathcal{V}^f$. On the other hand, it provides a sufficient condition to avoid the Lavrentiev gap in case $\mathcal{V} \neq \mathcal{V}^f$. Theorem 3.2 includes a result in this sense due to Guerra and Sarychev [9], concerning the special case of an affine system and $l(x, a) \equiv x'Px$, P a symmetric and positive matrix, under a local stabilizability assumption, involving bounded controls.

Theorem 3.2. (i) Let $x \in \mathcal{T}^c$. If $\mathcal{V}(x) = \mathcal{V}^f(x)$, then $\mathcal{U}^\rho(x) = \mathcal{V}(x)$ for any continuous function ρ .

(ii) If, for some ρ , $\mathcal{U}^\rho$ is continuous on $\partial\mathcal{T}$ then $\mathcal{U}^\rho \equiv \mathcal{V}$. Therefore, in particular, if $\mathcal{V}^f$ is continuous on $\partial\mathcal{T}$, then $\mathcal{V}^f \equiv \mathcal{U}^\rho \equiv \mathcal{V}$ for any continuous, nonnegative function ρ .

Proof. The inequality $\mathcal{V}(x) \leq \mathcal{U}^\rho(x)$ is obvious and, if $\mathcal{V}(x) = +\infty$, it implies immediately $\mathcal{V}(x) = \mathcal{U}^\rho(x)$. Let $x \in \mathcal{T}^c$ with $\mathcal{V}(x) < +\infty$ and let $\eta > 0$. By the definition of $\mathcal{V}$, there exists a control $\tilde{\alpha} \in \mathcal{A}(x)$ such that

$$\int_0^{t_x(\tilde{\alpha})} l(y_x(t, \tilde{\alpha}), \tilde{\alpha}(t)) dt < \mathcal{V}(x) + \eta. \quad (26)$$

Case 1: $\mathcal{V}(x) = \mathcal{V}^f(x)$, so that for any $\eta > 0$ we can assume $t_x(\tilde{\alpha}) < +\infty$. By standard estimates there exists some $R > 0$ such that $|y_x(t, \tilde{\alpha})| \leq R$ for all $t \in [0, t_x(\tilde{\alpha})]$. Hence, by continuity there is some $\bar{M}_R > 0$ such that $\sup_{t \in [0, t_x(\tilde{\alpha})]} \rho(y_x(t, \tilde{\alpha}), \tilde{\alpha}(t)) \leq \bar{M}_R$ and

$$\int_0^{t_x(\tilde{\alpha})} \rho(y_x(t, \tilde{\alpha}), \tilde{\alpha}(t)) dt \leq \bar{M}_R t_x(\tilde{\alpha}) < +\infty,$$

Therefore $\tilde{\alpha} \in \mathcal{A}_\rho(x)$ and by (26) we get that $\mathcal{U}^\rho(x) \leq \mathcal{V}(x) + \eta$. By the arbitrariness of $\eta > 0$, this concludes the proof of statement (i).

Case 2: $\mathcal{V}(x) < \mathcal{V}^f(x)$, so that $t_x(\tilde{\alpha}) = +\infty$ if $\eta < \mathcal{V}^f(x) - \mathcal{V}(x)$. By the continuity of $\mathcal{U}^\rho$ on the compact set $\partial\mathcal{T}$, there is some $\delta > 0$ such that

$$\mathcal{U}^\rho(\bar{x}) < \eta \quad \forall \bar{x} \in \mathcal{T}^c \quad \text{with} \quad \mathbf{d}(\bar{x}) < \delta. \quad (27)$$

Moreover for $\tilde{\alpha} \in \mathcal{A}(x)$ satisfying (26), there is some $\bar{t} < t_x(\tilde{\alpha})$ such that

$$\mathbf{d}(y_x(\bar{t}, \tilde{\alpha})) < \delta/2. \quad (28)$$

At this point, we get that $\int_0^{\bar{t}} \rho(y_x(t, \tilde{\alpha}), \tilde{\alpha}(t)) dt < +\infty$, arguing as in Case 1. Let $\tilde{x} \doteq y_x(\bar{t}, \tilde{\alpha})$ and let $\bar{\alpha} \in \mathcal{A}_\rho(\tilde{x})$ be a control such that

$$\int_0^{t_{\tilde{x}}(\bar{\alpha})} l(y_{\tilde{x}}(t, \bar{\alpha}), \bar{\alpha}(t)) dt < \eta,$$

which exists in view of (27). Then, the control $\alpha(t) = \tilde{\alpha}(t)\chi_{[0, \bar{t}]}(t) + \bar{\alpha}(t - \bar{t})\chi_{[\bar{t}, +\infty]}(t)$ belongs to $\mathcal{A}_\rho(x)$ and

$$\int_0^{t_x(\alpha)} l(y_x(t, \alpha), \alpha(t)) dt < \mathcal{V}(x) + 3\eta, \quad (29)$$

so that $\mathcal{U}^\rho(x) < \mathcal{V}(x) + 3\eta$. Statement (ii) is thus proved, by the arbitrariness of $\eta > 0$. $\square$

For the proof of the uniform convergence result in Theorem 3.3 below, we need the following proposition, whose quite easy proof is omitted for the sake of brevity.

Proposition 3.1. *Let $\rho : \mathbb{R}^n \times A \rightarrow \mathbb{R}_+$ be a continuous function and let $\mathcal{V}_\varepsilon^\rho$ be continuous on $\partial\mathcal{T}$ for some $\varepsilon > 0$. Then $\mathcal{U}^\rho$ is continuous on $\partial\mathcal{T}$; for any $\varepsilon > 0$, $\mathcal{V}_\varepsilon^\rho$ is continuous on $\partial\mathcal{T}$; $\text{Dom}(\mathcal{V}_\varepsilon^\rho) = \text{Dom}(\mathcal{U}^\rho)$ and it is an open set; $\mathcal{V}_\varepsilon^\rho$ is locally bounded and upper semicontinuous in $\text{Dom}(\mathcal{U}^\rho)$.*

Theorem 3.3. *Let $\rho : \mathbb{R}^n \times A \rightarrow \mathbb{R}_+$ be a continuous function.*

(i) One has

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{V}_\varepsilon^\rho(x) = \mathcal{U}^\rho(x) \quad \forall x \in \mathcal{T}^c \quad (30)$$

and, by Theorem 3.2, $\mathcal{U}^\rho \equiv \mathcal{V}$ if either $\mathcal{V}^f = \mathcal{V}$ or $\mathcal{U}^\rho$ is continuous on $\partial\mathcal{T}$.

(ii) If there is some $\bar{\varepsilon} > 0$ such that $\mathcal{V}_{\bar{\varepsilon}}^\rho$ is continuous on $\partial\mathcal{T}$ and $\mathcal{V}$ is continuous on its whole domain, then $\mathcal{U}^\rho \equiv \mathcal{V}$ and the limit (30) is uniform on any compact set $Q \subset \text{Dom}(\mathcal{V})$.

Proof. Let $\varepsilon > 0$ and fix $x \in \mathcal{T}^c$. Since for any $\alpha \in \mathcal{A}(x)$, $\mathcal{J}_\varepsilon(t_x(\alpha), x, \alpha) < +\infty$ implies $\alpha \in \mathcal{A}_\rho(x)$, then $\mathcal{U}^\rho(x) \leq \mathcal{V}_\varepsilon^\rho(x)$ and in order to prove (i) it remains only to show that the strict inequality $\mathcal{U}^\rho(x) < \inf_{\varepsilon > 0} \mathcal{V}_\varepsilon^\rho(x)$ cannot hold. If $\mathcal{U}^\rho(x) = +\infty$, the equality is obvious. If instead $\mathcal{U}^\rho(x) < +\infty$, let us assume by contradiction that, for some $\eta > 0$,

$$\mathcal{U}^\rho(x) < \mathcal{V}_\varepsilon^\rho(x) - 3\eta$$

for any $\varepsilon > 0$. By the definition of $\mathcal{U}^\rho$, there is some $\alpha \in \mathcal{A}_\rho(x)$ such that

$$\int_0^{t_x(\alpha)} l(y_x(t, \alpha), \alpha(t)) dt < \mathcal{U}^\rho(x) + \eta$$

and $\bar{\rho} \doteq \int_0^{t_x(\alpha)} \rho(y_x(t, \alpha), \alpha(t)) dt < +\infty$. Then one has

$$\mathcal{V}_\varepsilon^\rho(x) < \int_0^{t_x(\alpha)} l(y_x(t, \alpha), \alpha(t)) dt + \eta \quad (31)$$

for any $\varepsilon \leq \eta/\bar{\rho}$. The proof of (30) is thus concluded, since we immediately obtain the contradiction $\mathcal{U}^\rho(x) < \mathcal{U}^\rho(x) - \eta$.

Let us now prove (ii). In view of Proposition 3.1, for any $\varepsilon > 0$, $\mathcal{V}_\varepsilon^\rho$ is locally bounded and upper semicontinuous in its domain, which is an open set and coincides with $\text{Dom}(\mathcal{U}^\rho)$. Moreover, $\mathcal{U}^\rho \equiv \mathcal{V}$ in view of Theorem 3.2. Therefore, $\{\mathcal{V}_\varepsilon^\rho - \mathcal{V}\}_{\varepsilon > 0}$ is a decreasing sequence of nonnegative and upper semicontinuous functions defined in the open set $\text{Dom}(\mathcal{V})$ and converging to the null function as $\varepsilon \rightarrow 0^+$. Now, the uniform convergence of the $\mathcal{V}_\varepsilon^\rho$ to $\mathcal{V}$ on any compact subset $Q \subset \text{Dom}(\mathcal{V})$ follows from an easy adaptation of Dini's Theorem for continuous functions to the upper semicontinuous case. $\square$

We conclude this subsection pointing out that in the special case $\rho \equiv 1$ the limit function $\mathcal{U}^\rho$ of ε -penalized problems coincides with the limit of the so-called *T-finite time value functions*, defined for any $T > 0$, as follows:

$$\mathcal{V}_T(x) = \inf_{\{\alpha \in \mathcal{A}(x): t_x(\alpha) \leq T\}} \mathcal{J}(t_x(\alpha), x, \alpha) \quad \forall x \in \mathcal{T}^c.$$

Proposition 3.2. *For every $x \in \mathcal{T}^c$,*

$$\mathcal{V}^f(x) = \lim_{T \rightarrow +\infty} \mathcal{V}_T(x).$$

If $\mathcal{V}^f$ is continuous on $\partial\mathcal{T}$, then the above limit coincides with $\mathcal{V}$. When in addition $\mathcal{V}_T$ for some $T > 0$ is continuous on $\partial\mathcal{T}$ and $\mathcal{V}$ is continuous in its domain, then the above convergence is locally uniform.

Proof. Fix $x \in \mathcal{T}^c$. Clearly, $\mathcal{V}^f(x) \leq \mathcal{V}_T(x)$ for any $T > 0$ and we have just to prove that $\mathcal{V}^f(x) < \inf_{T>0} \mathcal{V}_T(x)$ cannot hold. If $\mathcal{V}^f(x) = +\infty$, the equality is trivial. If instead $\mathcal{V}^f(x) < +\infty$, for any $\eta > 0$ there is some $\alpha \in \mathcal{A}(x)$ such that $T_\eta \doteq t_x(\alpha) < +\infty$ and

$$\mathcal{V}_{T_\eta}(x) \leq \int_0^{t_x(\alpha)} l(y_x(t, \alpha), \alpha(t)) dt \leq \mathcal{V}^f(x) + \eta.$$

This shows that the limit holds. The remaining result is a consequence of Theorem 3.3. $\square$

3.2. Target approximations

One could also consider an approximation *from below* of $\mathcal{V}$, by fattening the target. For every $\delta > 0$ and $x \in \mathcal{T}_\delta^c$, let us define

$$t_x^\delta(\alpha) \doteq \inf\{t \geq 0 : y_x(t, \alpha) \in \mathcal{T}_\delta\} \quad (\leq +\infty)$$

and the $\mathcal{T}_\delta$ -problem

$$\mathcal{V}_{\mathcal{T}_\delta}(x) \doteq \inf_{\alpha \in \mathcal{A}: t_x^\delta(\alpha) < +\infty} \mathcal{J}(t_x^\delta(\alpha), x, \alpha) \quad (\leq +\infty),$$

where only trajectories reaching the target $\mathcal{T}_\delta$ in finite time are allowed.

In the next proposition we show that $\mathcal{V}$ is the natural limit of the $\mathcal{V}_{\mathcal{T}_\delta}$ as $\delta \rightarrow 0^+$, as soon as it is continuous on $\partial\mathcal{T}$. By Proposition 2.4, a sufficient condition for such a continuity is the (LACL) for l , which easily yields the STLC for $\mathcal{T}_\delta \forall \delta > 0$ small enough, but *not* in general for $\mathcal{T}$.

Proposition 3.3. *For every $x \in \mathcal{T}^c$, assuming $\mathcal{V}$ continuous on $\partial\mathcal{T}$, we have*

$$\mathcal{V}(x) = \lim_{\delta \rightarrow 0} \mathcal{V}_{\mathcal{T}_\delta}(x)$$

and this convergence is uniform in $\text{Dom}(\mathcal{V})$.

Proof. Fix $\eta > 0$. The continuity of $\mathcal{V}$ on the compact set $\partial\mathcal{T}$ implies that there is some $\bar{\delta} > 0$ such that $\forall \bar{x} \in \mathcal{T}_{\bar{\delta}} \setminus \mathcal{T}$ one has $\int_0^{t_{\bar{x}}(\bar{\alpha})} l(y_{\bar{x}}(t, \bar{\alpha}), \bar{\alpha}(t)) dt \leq \eta/2$ for some $\bar{\alpha} \in \mathcal{A}(\bar{x})$. Let $S(x) \doteq \sup_{\delta > 0} \mathcal{V}_{\mathcal{T}_\delta}(x)$. Obviously $S(x) \leq \mathcal{V}(x)$, for all $x \in \mathcal{T}^c$, and if $S(x) = +\infty$ then $S(x) = \mathcal{V}(x)$. Therefore let $S(x) < +\infty$ and for every $\delta \in]0, \bar{\delta}[$, choose $\alpha_\delta \in \mathcal{A}$ satisfying

$$\int_0^{t_x^\delta(\alpha_\delta)} l(y_x(t, \alpha_\delta), \alpha_\delta(t)) dt \leq \mathcal{V}_{\mathcal{T}_\delta}(x) + \eta/2. \quad (32)$$

Set $\bar{t} \doteq t_x^\delta(\alpha_\delta)$, $\bar{x} \doteq y_x(\bar{t}, \alpha_\delta) \in \mathcal{T}_\delta$ and $\alpha(t) \doteq \alpha_\delta(t)\chi_{[0, \bar{t}]} + \bar{\alpha}(t - \bar{t})\chi_{[\bar{t}, +\infty[}$. Then $\alpha \in \mathcal{A}(x)$ and

$$\begin{aligned} \mathcal{V}(x) &\leq \int_0^{t_x(\alpha)} l(y_x(t, \alpha), \alpha(t)) dt \leq \int_0^{\bar{t}} l(y_x(t, \alpha_\delta), \alpha_\delta(t)) dt + \eta/2 \\ &\leq \mathcal{V}_{\mathcal{T}_\delta}(x) + \eta \leq S(x) + \eta \end{aligned} \quad (33)$$

and, being $\eta > 0$ arbitrary, the equality is proved. As we see from the proof, this convergence is uniform in $\text{Dom}(\mathcal{V})$ with no further assumptions. $\square$

It is easy to see that sufficient conditions in order to have uniqueness to (BVPK) (see e.g. Corollary 2.1) yield the uniqueness to the boundary value problem naturally associated to the Kruzkov transform of $\mathcal{V}_{\mathcal{T}_\delta}$, for any $\delta > 0$ sufficiently small.

4. Continuity results

In the present section, we give sufficient condition for the continuity of $\mathcal{V}$ in its whole domain. Furthermore, exploiting some results of Motta and Rampazzo [14], we give some sufficient conditions in order to have the continuity of $\mathcal{U}^\rho$ on $\partial\mathcal{T}$ and we particularize the results for the value function $\mathcal{V}^f$.

4.1. Global continuity

The continuity of the function $\mathcal{V}$ on the target does not propagate in general to the whole domain, but, as stated in Proposition 2.5, it implies just the upper semicontinuity of $\mathcal{V}$. In Theorem 4.1 we prove that $\mathcal{V}$ is globally continuous assuming its continuity on the target and the *turnpike-type condition* (TPC) below. Turnpike conditions have their roots in economic growth theory. More recently their use has been extended to a wide range of variational and optimal control problems. The novelty here is to introduce such a kind of notion in order to study the continuity of the value function $\mathcal{V}$. We refer to the book [20] and to [19] for interesting surveys on the subject.

Loosely speaking (TPC) says that for any fixed δ -neighborhood of the target, for every $x \in \text{Dom}(\mathcal{V})$ one can select a nearly optimal asymptotic trajectory reaching $\mathcal{T}_\delta$ in finite time T , depending just on $\mathcal{V}(x)$ and δ (that is, uniformly w.r.t. x and α).

(TPC) $\forall R, \eta, \delta > 0$, there exists some increasing function $T(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for every $x \in \mathcal{T}_\delta^c \cap \text{Dom}(\mathcal{V})$, $\mathbf{d}(x) \leq R$, there is a control $\alpha \in \mathcal{A}(x)$ verifying

$$\int_0^{t_x(\alpha)} l(y_x(t, \alpha), \alpha(t)) dt \leq \mathcal{V}(x) + \eta, \\ t_x^\delta(\alpha) \doteq \inf \{t > 0 : y_x(t, \alpha) \in \mathcal{T}_\delta\} \leq T(\mathcal{V}(x)). \quad (34)$$

Theorem 4.1. Assume (TPC). If $\mathcal{V}$ is continuous on $\partial\mathcal{T}$, then $\mathcal{V}$ is continuous in its domain and $\lim_{x \rightarrow \bar{x}} \mathcal{V}(x) = +\infty$ for every $\bar{x} \in \partial\text{Dom}(\mathcal{V})$.

Proof. Owing to Proposition 2.5, $\text{Dom}(\mathcal{V})$ is an open subset of $\mathcal{T}^c$ where $\mathcal{V}$ is locally bounded and upper semicontinuous. Let $x_0 \in \text{Dom}(\mathcal{V})$ and let $\nu > 0$ be such that $B(x_0, \nu) \subset \text{Dom}(\mathcal{V})$. Let $M \doteq \sup\{\mathcal{V}(x) : x \in B(x_0, \nu)\} + 3$ and set $R \doteq \mathbf{d}(x_0) + \nu$. Fix $\eta \in]0, 1[$. By the continuity of $\mathcal{V}$ on the compact set $\partial\mathcal{T}$, there is some $\delta \equiv \delta_\eta > 0$ such that for any $x \in \mathcal{T}_{2\delta}$ there is a control $\tilde{\alpha}_x \in \mathcal{A}(x)$ verifying

$$\int_0^{t_x(\tilde{\alpha}_x)} l(y_x(t, \tilde{\alpha}_x), \tilde{\alpha}_x(t)) dt \leq \eta. \quad (35)$$

Fix $x_1, x_2 \in B(x_0, \nu)$ and assume e.g. $\mathcal{V}(x_2) \geq \mathcal{V}(x_1)$. By (TPC), in correspondence of R, η and δ introduced above, in view of the definition of M , there exist some $T \doteq T(M)$ and $\alpha_1 \in \mathcal{A}(x_1)$ such that

$$\int_0^{t_{x_1}(\alpha_1)} l(y_{x_1}(t, \alpha_1), \alpha_1(t)) dt \leq \mathcal{V}(x_1) + \eta < M$$

and

$$\mathbf{d}(y_{x_1}(\bar{t}, \alpha_1)) < \delta$$

for some $\bar{t} \leq T \wedge t_{x_1}(\alpha_1)$.

By standard estimates, all trajectories starting from points $x \in B(x_0, \nu)$ and corresponding to a control α , verify $|y_x(t, \alpha)| \leq R' \quad \forall t \in [0, T]$ for some $R' > 0$. Moreover,

$$|y_{x_2}(t, \alpha) - y_{x_1}(t, \alpha)| \leq L|x_2 - x_1| \quad \forall t \in [0, T], \quad \forall x_1, x_2 \in B(x_0, \nu),$$

for a suitable $L > 0$. Therefore, choosing $0 < \nu' < \nu$ small enough in order to have $\mathbf{d}(y_{x_2}(\bar{t}, \alpha_1)) < 2\delta$ and denoting by $\omega(\cdot, R')$ the modulus of l in $B(0, R')$, for $x_1, x_2 \in B(x_0, \nu')$, by considering the control $\alpha_2 \in \mathcal{A}(x_2)$ given by

$$\alpha_2(t) = \alpha_1(t)\chi_{[0, \bar{t}]} + \tilde{\alpha}_{y_{x_2}(\bar{t}, \alpha_1)}(t - \bar{t})\chi_{[\bar{t}, +\infty[},$$

we get

$$\begin{aligned} 0 \leq \mathcal{V}(x_2) - \mathcal{V}(x_1) &\leq \int_0^{\bar{t}} |l(y_{x_2}(t, \alpha_1), \alpha_1(t)) - l(y_{x_1}(t, \alpha_1), \alpha_1(t))| dt + 3\eta \\ &\leq T\omega(L|x_2 - x_1|, R') + 3\eta < 4\eta, \end{aligned}$$

which implies the continuity of $\mathcal{V}$ by the arbitrariness of $\eta > 0$.

To prove that $\lim_{x \rightarrow \bar{x} \in \partial \text{Dom}(\mathcal{V})} \mathcal{V}(x) = +\infty$, assume by contradiction that there are some $\bar{x} \in \partial \text{Dom}(\mathcal{V})$, $M > 0$ and $x_n \in \text{Dom}(\mathcal{V})$ such that $|x_n - \bar{x}| \leq 1/n$ and $\mathcal{V}(x_n) \leq M$ for any $n \geq 1$. Hence, arguing as in the previous step it not difficult to show that, thanks to the continuity of $\mathcal{V}$ on the compact set $\partial \mathcal{T}$ and using the (TPC), starting from some nearly optimal control $\alpha_n \in \mathcal{A}(x_n)$ for n large enough one can construct an admissible control $\alpha \in \mathcal{A}(x)$. Therefore $\bar{x}$ does not belong to $\partial \text{Dom}(\mathcal{V})$, being $\text{Dom}(\mathcal{V})$ an open set. $\square$

Hypotheses (SC1), (SC2) introduced in Sect. 2 guarantee the propagation of the continuity of $\mathcal{V}$ from $\partial \mathcal{T}$.

Proposition 4.1. *Assume $\mathcal{V}$ continuous on $\partial \mathcal{T}$ and either (SC1) or (SC2). Then $\mathcal{V}$ is continuous in its domain and $\lim_{x \rightarrow \bar{x}} \mathcal{V}(x) = +\infty$ for every $\bar{x} \in \partial \text{Dom}(\mathcal{V})$.*

Proof. Fix R, η and $\delta > 0$. Assume first (SC1). The UGAS property w.r.t. an invariant set $\mathcal{T}$ yields the existence of a function $\beta \in \mathcal{KL}$ such that for all $x \in \mathcal{T}^c$ such that $\mathbf{d}(x \leq R$ and for all $\alpha \in \mathcal{A}$, one has

$$\mathbf{d}(y_x(t, \alpha)) \leq \beta(\mathbf{d}(x), t) \leq \beta(R, t) \quad \forall t > 0$$

(see [2]). Hence for all (not necessarily nearly optimal) controls $\alpha \in \mathcal{A}$, the exit time $t_x^\delta(\alpha)$ defined as in (TPC) is not greater than $T \doteq \inf\{t > 0 : \beta(R, t) \leq \delta\} < +\infty$. Therefore, also in view of Proposition 2.5, (SC1) together with the continuity of $\mathcal{V}$ on $\partial \mathcal{T}$ implies (TPC).

If condition (SC2) is assumed, hypothesis (TPC) is easily fulfilled. Indeed, for all $x \in \mathcal{T}^c$ and $\alpha \in \mathcal{A}$ with finite cost $M > 0$, one has $M = \int_0^{t_x^\delta(\alpha)} l(y_x(t, \alpha), \alpha(t)) dt \geq c(\delta) t_x^\delta(\alpha)$, so that (TPC) is satisfied by choosing $T(r) \doteq (r + \eta)/c(\delta)$ for any $r \geq 0$.

At this point, the statement follows in both cases from Theorem 4.1. $\square$

4.2. Continuity on the target

Analogously to Proposition 2.4, by Theorem 1.1 in [14] it follows that the (LACL) for $l + \rho$ yields the continuity on $\partial\mathcal{T}$ of $\mathcal{V}_\varepsilon^\rho$ for any $\varepsilon \in]0, 1]$ and of $\mathcal{U}^\rho$.

In this subsection we provide weaker conditions sufficient for the continuity of $\mathcal{U}^\rho$ on $\partial\mathcal{T}$, involving just a MRF U for l . To this aim let us first recall from [14] that inequality (19) is equivalent to (37) below, involving a Lagrangian g which, differently from l , is always strictly positive outside the target.

Proposition 4.2. [14] *Let U be a local MRF for h . Then $\forall \sigma \in]0, U_0[$ there exists a continuous, strictly increasing function $m :]0, \sigma] \rightarrow]0, +\infty[$ such that, setting*

$$g(x, a) \doteq k h(x, a) + m(U(x)) \quad \forall (x, a) \in U^{-1}(]0, \sigma]) \times A, \quad (36)$$

(k the same as in (19)), one has

$$\min_{a \in A} \left\{ \langle p, f(x, a) \rangle + g(x, a) \right\} \leq 0 \quad \forall p \in D^*U(x). \quad (37)$$

Following the notations introduced above, for any $x \in U^{-1}(]0, \sigma])$ let us define for $t \in [0, t_x(\alpha)[$,

$$c(t) \doteq \int_0^t g(y_x(\tau, \alpha), \alpha(\tau)) d\tau \quad \text{and} \quad \bar{c} \doteq \int_0^{t_x(\alpha)} g(y_x(\tau, \alpha), \alpha(\tau)) d\tau.$$

This function is invertible and its inverse, $t(c)$, is a continuous time-change such that

$$t(c) = \int_0^c \frac{dc'}{g(y_x(t(c'), \alpha), \alpha(t(c')))}, \quad c \in [0, \bar{c}[.$$

Therefore, $t_x(\alpha) < +\infty$ if and only if

$$\int_0^{\bar{c}} \frac{dc'}{g(y_x(t(c'), \alpha), \alpha(t(c')))} < +\infty \quad (38)$$

and $\int_0^{t_x(\alpha)} \rho(y_x(t, \alpha), \alpha(t)) dt < +\infty$ if and only if

$$\int_0^{\bar{c}} \frac{\rho(y_x(t(c'), \alpha), \alpha(t(c')))}{g(y_x(t(c'), \alpha), \alpha(t(c')))} dc' < +\infty. \quad (39)$$

In order to state, in Theorem 4.2 below, explicit sufficient conditions implying either (38) or (39), we need the following definitions. Owing to Proposition 4.2, fixed a selection $p(x) \in D^*U(x)$ there is some $a(x) \in A$ such that

$$\langle p(x), f(x, a(x)) \rangle + g(x, a(x)) \leq 0, \quad (40)$$

where $g(x, a) = k l(x, a) + m(U(x))$. Let $A(x)$ denote the set of all $a(x)$ verifying (40). For any $s \in]0, \sigma]$, let us define

$$\hat{g}(s) \doteq \inf\{g(x, a) : x \in U^{-1}([s, \sigma]), a \in A(U^{-1}([s, \sigma]))\}, \quad (41)$$

and

$$\hat{\rho}(s) \doteq \sup\{\rho(x, a) : x \in U^{-1}(]0, s]), a \in A(U^{-1}(]0, s]))\}. \quad (42)$$

Notice that $\hat{g}(s) \geq m(s) > 0$ and $\hat{\rho}(s) \leq \bar{M} \doteq \max\{\rho(x, a) : x \in U^{-1}(]0, \sigma]), a \in A\} < +\infty$ for all $s \in]0, \sigma]$. As it is not restrictive, let us assume $\hat{g}$ and $\hat{\rho}$ continuous. In fact, it is easy to construct continuous approximations of $\hat{g}$ and $\hat{\rho}$ from below and from above, respectively, verifying all the properties described above.

Theorem 4.2. *Let ρ be a continuous function. Assume that there exists a local MRF U for l , defined in some open set Ω .*

(i) *If $\hat{g}$ and $\hat{\rho}$, defined as in (41), (42), respectively, verify*

$$\int_0^\sigma \frac{ds}{\hat{g}(s)} = +\infty, \quad \int_0^\sigma \frac{\hat{\rho}(s)}{\hat{g}(s)} ds < +\infty, \quad (43)$$

then $\mathcal{U}^\rho(x) \leq k^{-1} U(x) \quad \forall x \in \Omega$, so that $\mathcal{U}^\rho$ is continuous on $\partial\mathcal{T}$. Moreover, $\mathcal{U}^\rho \equiv \mathcal{V}$.

(ii) *If $\hat{g}$ defined as in (41) verifies*

$$\int_0^\sigma \frac{ds}{\hat{g}(s)} < +\infty, \quad (44)$$

then $\mathcal{V}^f(x) \leq k^{-1} U(x) \quad \forall x \in \Omega$, so that $\mathcal{V}^f$ is continuous on $\partial\mathcal{T}$. Moreover, $\mathcal{V}^f \equiv \mathcal{U}^\rho \equiv \mathcal{V}$ for every ρ .

Proof. The results in (i) and (ii) are straightforward consequences of Theorem 3.2 and of the following Lemma.

In order to state the lemma, we recall by Section 3 in [14] that the existence of a local MRF U for l implies the existence of a $\mathcal{KL}$ function β such that for any $\eta > 0$ there is some $\alpha \in \mathcal{A}$ verifying

$$\mathbf{d}(y_x(t, \alpha)) \leq \beta(\mathbf{d}(x), t) \quad \forall t \in \mathbb{R}_+, \quad (45)$$

and the estimate

$$\int_0^{t_x(\alpha)} g(y_x(t, \alpha), \alpha(t)) dt \leq (1 + \eta) U(x). \quad (46)$$

□

Lemma 4.1. *Under the same assumptions and notations of Theorem 4.2, if for some ρ condition (43) holds, then $\forall \eta > 0, \forall x \in U^{-1}(]0, \sigma])$, there exists $\alpha \in \mathcal{A}(x)$ verifying (45), (46) and also*

$$\int_0^{t_x(\alpha)} \rho(y_x(t, \alpha), \alpha(t)) dt \leq (1 + \eta) \hat{R}(U(x)), \quad (47)$$

where $\hat{R}(s) \doteq \int_0^s \frac{\hat{\rho}(s')}{\hat{g}(s')} ds'$ for $s \in]0, \sigma]$.

If the stronger condition (44) is satisfied, then, $\forall \eta > 0$, $\forall x \in U^{-1}(]0, \sigma[)$ there exists $\alpha \in \mathcal{A}(x)$ verifying (45), (46), and

$$t_x(\alpha) \leq (1 + \eta) \hat{G}(U(x)), \quad (48)$$

where $\hat{G}(s) \doteq \int_0^s \frac{ds'}{\hat{g}(s')}$ for $s \in]0, \sigma]$.

Proof. Fix $\eta > 0$. Let $(\nu_k)_k \subset]0, 1]$ be any infinitesimal, strictly decreasing sequence such that $\nu_0 = 1$. Using the same notations of Proposition 4.2, for any $x \in U^{-1}(]0, \sigma[)$, set $\mu_k \doteq \nu_k U(x)$. In view of the proof of Theorem 1.1 in Section 3 of [14], there exist a trajectory-control pair $(y, \alpha) : [0, \bar{t}] \rightarrow \mathcal{T}^c \times A'$ and a sequence $t_0 \doteq 0 < t_1 < \dots < t_k < t_{k+1} < \dots$, $\bar{t} = \lim_{k \rightarrow +\infty} t_k$ (possibly, $\bar{t} = +\infty$), such that, for each $k \geq 0$,

$$\begin{aligned} U(y(t_k)) &= \mu_k, \quad U(y(t_{k+1})) < U(y(t)) \leq U(y(t_k)) \quad \forall t \in [t_k, t_{k+1}[\\ \alpha(t) &\in A'(U^{-1}(]0, \mu_{k+1}]) \quad \forall t \in [t_k, t_{k+1}[\end{aligned}$$

and

$$\int_{t_k}^{t_{k+1}} g(y(t), \alpha(t)) dt \leq (1 + \eta) [U(y(t_k)) - U(y(t_{k+1}))]. \quad (49)$$

By the definition (41) of $\hat{g}$,

$$\int_{t_k}^{t_{k+1}} \hat{g}(U(y(t_{k+1}))) dt \leq \int_{t_k}^{t_{k+1}} g(y(t), \alpha(t)) dt \leq (1 + \eta) [\mu_k - \mu_{k+1}]$$

since $U(y(t_k)) - U(y(t_{k+1})) = \mu_k - \mu_{k+1}$. Therefore,

$$\bar{t} = \sum_{k=0}^{+\infty} [t_{k+1} - t_k] \leq (1 + \eta) \sum_{k=0}^{+\infty} \frac{\mu_k - \mu_{k+1}}{\hat{g}(\mu_{k+1})}.$$

Assume (44). Then, $\forall \varepsilon > 0$, since $(\nu_k)_k$ is arbitrary, we can choose $(\mu_k)_k$ so that

$$\left| \sum_{k=0}^{+\infty} \frac{\mu_k - \mu_{k+1}}{\hat{g}(\mu_{k+1})} - \int_0^{U(x)} \frac{ds}{\hat{g}(s)} \right| = \left| \sum_{k=0}^{+\infty} \frac{\mu_k - \mu_{k+1}}{\hat{g}(\mu_{k+1})} - \hat{G}(U(x)) \right| \leq \varepsilon$$

and by the arbitrariness of ε , $t_x(\alpha) \leq \bar{t} \leq (1 + \eta) \hat{G}(U(x))$. The second part of the lemma is thus proved, since the upper bound on the cost in (46) follows straightforwardly from (49).

Let now hypothesis (43) hold. Then

$$\int_0^{\bar{t}} |\alpha(t)|^r dt = \sum_{k=0}^{+\infty} \int_{t_k}^{t_{k+1}} |\alpha(t)|^r dt \leq \sum_{k=0}^{+\infty} \hat{a}^r(U(y(t_k))) [t_{k+1} - t_k]$$

and by the previous estimates we get

$$\int_0^{\bar{t}} |\alpha(t)|^r dt \leq \sum_{k=0}^{+\infty} \frac{\hat{a}^r(\mu_k)}{\hat{g}(\mu_{k+1})} [\mu_k - \mu_{k+1}].$$

Since, as it is not restrictive by the arbitrariness of the sequence $(\nu_k)_k$, we can assume that $\hat{a}(\mu_k) \leq C \hat{a}(\mu_{k+1})$ for some $C > 0$ for all k , the proof can be easily concluded arguing as above. $\square$

Example 4.1. Let us consider a minimization problem where $U(x) = \mathbf{d}^\gamma(x)$ is a local MRF in $\mathcal{T}_\sigma$ for some $\sigma, \gamma > 0$. Let us also assume that, for any $s \in]0, \sigma[$,

$$\begin{aligned}\tilde{g}(s) &\doteq \inf\{g(x, a) : s \leq \mathbf{d}(x) \leq \sigma, a \in A'(\mathbf{d}^{-1}([s, \sigma]))\} = \bar{C}_1 s^{\beta_1}, \\ \tilde{a}(s) &\doteq \sup\{|a| : a \in A'(\mathbf{d}^{-1}([0, s]))\} = \bar{C}_2 s^{\beta_2},\end{aligned}$$

for some $\bar{C}_1, \bar{C}_2 > 0$, and $\beta_1, \beta_2 \geq 0$. Let us notice that, when $U = \mathbf{d}^\gamma$, using the notations of Proposition 4.2, one has

$$\hat{g}(s) = \tilde{g}(s^{1/\gamma}); \quad \hat{a}(s) = \tilde{a}(s^{1/\gamma}).$$

Then condition (44) becomes

$$\int_0^\sigma \frac{ds}{s^{\beta_1/\gamma}} < +\infty,$$

and it turns out to be satisfied iff $\beta_1 < \gamma$ (without restrictions on the control size, since we can set $\beta_2 = 0$). In case $\beta_1 \geq \gamma$, instead, for any $r \geq 1$ the weaker assumption (43), becoming

$$\int_0^\sigma \frac{ds}{s^{(\beta_1 - r\beta_2)/\gamma}} < +\infty,$$

is verified iff $r\beta_2 > \beta_1 - \gamma$.

Let for instance $\gamma = 1$. Then the stronger condition $\beta_1 < 1$ cannot be satisfied if, for instance, g has polynomial growth around the target (w.r.t. the distance function) of degree $\beta_1 \geq 1$. Notice that g could have $\beta_1 < 1$ even in cases in which the original Lagrangian l grows as $\mathbf{d}^\beta(x)$ with $\beta \geq 1$.

5. Infinite horizon problem

Let us introduce the *infinite horizon value function*, defined for any $x \in \mathbb{R}^n$ as

$$\mathcal{V}^\infty(x) = \inf_{\alpha \in \mathcal{A}} \mathcal{J}(+\infty, x, \alpha). \quad (50)$$

Clearly, for any target $\mathcal{T}$ one has $\mathcal{V}^m \leq \mathcal{V}^\infty$ but, choosing $\mathcal{T}$ in a suitable way, $\mathcal{V}^\infty$ does actually coincide with $\mathcal{V}^m$. As a consequence, in this case all the results obtained for $\mathcal{V}$ yield analogous results for $\mathcal{V}^\infty$ as soon as we have $\mathcal{V} \equiv \mathcal{V}^m$. We recall by Proposition 2.3 that either hypothesis (SC1) or (SC2) implies such an equality. For instance, we have

Proposition 5.1. *If $\mathcal{T} \times \{0\}$ is a viability set⁶ for (f, l) , then $\mathcal{V}^\infty \equiv \mathcal{V}^m$.*

As an immediate consequence we get the following

Corollary 5.1. *If $\mathcal{T} \times \{0\}$ is a viability set for (f, l) and either (SC1) or (SC2) holds true for $\mathcal{T}$, then $\mathcal{V}^\infty \equiv \mathcal{V}^m \equiv \mathcal{V}$.*

⁶ Let $F(x) \doteq \{(f(x, a), l(x, a)) : a \in A\}$. Any closed subset $K \subset \mathbb{R}^n \times \mathbb{R}$ will be called a *viability set* for (f, l) if for any $(x_0, \lambda_0) \in K$ there is a solution (y, λ) of the differential inclusion

$$(\dot{y}(t), \dot{\lambda}(t)) \in F(y(t)) \quad t \geq 0$$

such that $(y(0), \lambda(0)) = (x_0, \lambda_0)$ and $(y(t), \lambda(t)) \in K \quad \forall t > 0$ (see e.g. [1]).

Proof of Proposition 5.1. Let $x \in \mathbb{R}^n$ be such that $\mathcal{V}^m(x) < +\infty$ (otherwise, the equality $\mathcal{V}^\infty(x) = \mathcal{V}^m(x) = +\infty$ is trivial). For any $\eta > 0$ there is some $\hat{\alpha} \in \mathcal{A}$ such that

$$\int_0^{t_x(\hat{\alpha})} l(y_x(t, \hat{\alpha}), \hat{\alpha}(t)) dt \leq \mathcal{V}^m(x) + \eta. \quad (51)$$

If $t_x(\hat{\alpha}) = +\infty$, (51) implies that $\mathcal{V}^\infty(x) \leq \mathcal{V}^m(x) + \eta$. If $t_x(\hat{\alpha}) < +\infty$, set $\bar{x} \doteq y_x(t_x(\hat{\alpha}), \hat{\alpha})$. Since $\mathcal{T} \times \{0\}$ is viable for (f, l) and $\bar{x} \in \mathcal{T}$, there exists a control $\bar{\alpha} \in \mathcal{A}$ such that

$$y_{\bar{x}}(t, \bar{\alpha}) \in \mathcal{T} \quad \text{and} \quad \int_0^t l(y_{\bar{x}}(t, \bar{\alpha}), \bar{\alpha}(t)) dt = 0 \quad \forall t > 0$$

(incidentally, this argument yields that $\mathcal{V}^\infty \equiv 0$ in $\mathcal{T}$). Therefore considering the control $\alpha(t) \doteq \hat{\alpha}\chi_{[0, t_x(\hat{\alpha})]}(t) + \bar{\alpha}(t - t_x(\hat{\alpha}))\chi_{[t_x(\hat{\alpha}), +\infty)}(t)$ for all $t \geq 0$, we get that $\int_0^{+\infty} l(y_x(t, \alpha), \alpha(t)) dt = \int_0^{t_x(\hat{\alpha})} l(y_x(t, \hat{\alpha}), \hat{\alpha}(t)) dt$. Hence (51) implies again that $\mathcal{V}^\infty(x) \leq \mathcal{V}^m(x) + \eta$. By the arbitrariness of $\eta > 0$, this yields the equality $\mathcal{V}^\infty \equiv \mathcal{V}^m$ and the proof is concluded.

Remark 5.1. Viability sufficient conditions can be found e.g. in [1]. In particular, we recall that, if $\mathcal{T}$ and the sets $F(x) \doteq \{(f(x, a), l(x, a)) : a \in A\} \forall x \in \mathcal{T}_\sigma$ are convex and the (LACL) for l holds, the viability hypothesis in Proposition 5.1 is satisfied by well known results of convex analysis. Actually in this case there exists an equilibrium point $\bar{x} \in \mathcal{T}$ for system (3). For nonconvex sets, the implication is in general false.

Remark 5.2. The viability hypothesis in Proposition 5.1 is a sufficient condition in order to have $\mathcal{V}^\infty \equiv 0$ on $\mathcal{T}$. Actually, by the proof above one easily deduces that, if $\mathcal{Z}_\infty \doteq \{x \in \mathbb{R}^n : \mathcal{V}^\infty(x) = 0\} \neq \emptyset$ then

$$\mathcal{V}^\infty \equiv \mathcal{V}^m \text{ for any target } \mathcal{T} \subset \mathcal{Z}_\infty.$$

Under the hypotheses of Corollary 5.1, all the results about uniqueness, stability and continuity obtained in the previous sections for $\mathcal{V}$ give rise to analogous results for the infinite horizon value function, $\mathcal{V}^\infty$, which we omit for the sake of brevity. For instance, in view of Remark 5.1, Corollary 2.1 implies the following.

Corollary 5.2. *Let either $\mathcal{T} \times \{0\}$ be a viability set for (f, l) or $\mathcal{T}$ and the sets $F(x) \doteq \{(f(x, a), l(x, a)) : a \in A\} \forall x \in \mathcal{T}_\sigma$ be convex. Assume (CV), (LACL) for l and either (SC1) or (SC2). Then*

- (i) *there is a unique nonnegative viscosity solution W to (BVPC), which turns out to be continuous. Moreover, $\mathcal{V}^\infty \equiv \Psi^{-1}(W) = -\log(1 - W)$ and $\text{Dom}(\mathcal{V}^\infty) = \{x : W(x) < 1\}$;*
- (ii) *$(\mathcal{V}^\infty, \text{Dom}(\mathcal{V}^\infty))$ is the unique nonnegative viscosity solution to (BVP) among the pairs (u, Ω) , where u satisfies (13). Moreover, $\mathcal{V}^\infty$ is continuous.*

If (CV) is not satisfied, the above uniqueness results hold just among the continuous functions.

We conclude by observing that Examples 2.1 and 2.2 are in fact examples of infinite horizon problems. The value function $\mathcal{V}$ coincides with $\mathcal{V}^\infty$ and it is the unique nonnegative solution of the (BVP) with $\mathcal{T} = \{0\}$.

Remark 5.3. We can cover affine-quadratic problems, where

$$l(x, a) = x^T Q x + a^T R a, \quad f(x, a) = A(x) + \langle B(x), a \rangle, \quad x \in \mathbb{R}^n, a \in A \subset \mathbb{R}^m$$

with Q and R symmetric and *positive semi-definite* matrices and the control set A convex. In the literature, Q is usually assumed to be positive definite. In this case, (SC2) for $\mathcal{T} = \{0\}$ is trivially satisfied, so that one has $\mathcal{V} \equiv \mathcal{V}^m$.

For such problems we have uniqueness among the nonnegative solutions of (BVP) for $\mathcal{T} = \{0\}$ assuming the (LACL) and either Q is positive definite or f has the UGAS property w.r.t. $\{0\}$, namely (SC1) holds.

Such kind of problems are widely studied, both for unbounded and bounded controls. In the last case they are also known in the literature as constrained or saturated problems (see e.g. [8]).

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Received: 1 July 2013.

Accepted: 31 March 2014.