On the solutions of a singular elliptic equation concentrating on two orthogonal spheres

B. B. Manna and P. N. Srikanth

Abstract. Let $A = \{x \in \mathbb{R}^{2m} : 0 < a < |x| < b\}$ be an annulus. We consider the following singularly perturbed elliptic problem on A

$$\begin{cases} -\varepsilon^2 \Delta u + |x|^\eta u = |x|^\eta u^p, & \text{in } A, \\ u > 0, & \text{in } A, \\ u = 0, & \text{on } \partial A, \end{cases}$$

where $1 < p < \frac{m+3}{m-1}$. We shall prove the existence of a positive solution u_ε which concentrates on two different orthogonal spheres of dimension $(m-1)$ as $\varepsilon \rightarrow 0$. We achieve this by studying a reduced problem on an annular domain in $\mathbb{R}^{m+1}$ and analysing the profile of a two point concentrating solution in this domain.

Mathematics Subject Classification. 35J60, 35J91, 35J20, 35B09.

Keywords. Orthogonal sphere concentration, Morse index.

1. Introduction

Consider the following singularly perturbed elliptic equation with super linear non linearity in an annulus in $\mathbb{R}^{2m}$.

$$\begin{cases} -\varepsilon^2 \Delta u + |x|^\eta u = |x|^\eta u^p, & \text{in } A, \\ u > 0, & \text{in } A, \\ u = 0, & \text{on } \partial A, \end{cases} \quad (1.1)$$

where $1 < p < \frac{m+3}{m-1}$ and $\varepsilon > 0$, is a singular perturbation parameter.

$$A = \{x \in \mathbb{R}^{2m} : 0 < a < |x| < b\}.$$

We consider A under the co-ordinate system

$$A = I \times (J \times S^{m-1} \times S^{m-1}),$$

where $I = (a, b)$, $J = [0, \pi/2)$ and S^{m-1} has the standard polar co-ordinate representation. For $x \in A$, we can write $x \equiv x(r, \theta, \theta_1^1, \theta_2^1, \dots, \theta_{m-1}^1, \theta_1^2, \theta_2^2, \dots,$

θ_{m-1}^2), where $r \in I$, $\theta \in J$, $\theta_1^i \in [0, 2\pi)$ for $i = 1, 2$, $\theta_j^i \in [0, \pi)$ for $i = 1, 2$ and $j = 2, \dots, m-1$. In the framework of this co-ordinate system we shall prove the following main theorem:

Theorem 1.1. *The equation (1.1) has a solution u_ε which concentrates on two orthogonal spheres as $\varepsilon \rightarrow 0$. The spheres are characterized by the angle $\theta = 0$ and $\theta = \pi/2$ and by the radial co-ordinate*

- (i) $r = a$ if $\eta < 2$ (inner boundary concentration),
- (ii) $r = (a+b)/2$ if $\eta = 2$ (concentration at maximum distance from boundary),
- (iii) $r = b$ if $\eta > 2$ (outer boundary concentration).

This is the first result of its kind where one shows the concentration along two spheres which are orthogonal to each other. However our result, even it is specific, gives precise information of the concentrating solution and their profile. Note that the solutions we obtain can not be minimal energy solution in $H_0^1(\Omega)$ but is a minimal energy solution in an appropriate subspace of $H_0^1(\Omega)$.

The existence of least energy solutions of such singularly perturbed elliptic problem has been well studied (see [1–4]). It is well known that such a solution concentrates at a single point and the point of concentration converges to a point with the maximum distance from the boundary of the domain. The existence of multi peak solution, for domains with holes, have been shown by Dancer and Yan in [5]. See the work of Monica Clapp et al. in [6] for similar type of result.

Starting with the work of Ambrosetti et al., [7], there has been considerable interest in the question of solutions concentrating on m -spheres. An important contribution in this direction is by Ruf and Srikanth, [8], where the existence of S^1 concentrating solutions has been proved in dimension 4. Pacella and Srikanth in [9], have considered the problem in $2m$ -dimension and have shown the existence of solutions concentrating on $(m-1)$ dimensional sphere.

In proving our main theorem, the essential idea is to reduce the problem on an annulus in $\mathbb{R}^{m+1}$ and look for least energy solution in an appropriate space. There we shall show the existence of a solution which concentrates exactly at two points.

2. The reduction and the energy

We can express a point $z \in A$ as $z = (z_1, z_2)$, $z_i \in \mathbb{R}^m$ for $i = 1, 2$. Let $\rho_1 = r \cos\theta$ and $\rho_2 = r \sin\theta$. Then we can write z_1, z_2 as $z_1 \equiv z_1(\rho_1, \sigma_1)$ and $z_2 \equiv z_2(\rho_2, \sigma_2)$, $\sigma_i \in S^{m-1} \subset \mathbb{R}^m$ for $i = 1, 2$. Thus finally we can express z precisely as

$$z \equiv z(r, \sigma_1, \sigma_2, \theta). \quad (2.1)$$

The expression of the Laplacian in these co-ordinates is given as

$$\begin{aligned} \Delta_{\mathbb{R}^{2m}} u &= u_{rr} + \frac{2m-1}{r} u_r + \frac{m-1}{r^2} u_{\theta\theta} \left[\frac{2\cos 2\theta}{\sin 2\theta} \right] + \frac{u_{\theta\theta}}{r^2} \\ &+ \frac{1}{r^2 \cos^2 \theta} \Delta_{S^{m-1}}^{\sigma_1} u + \frac{1}{r^2 \sin^2 \theta} \Delta_{S^{m-1}}^{\sigma_2} u, \end{aligned} \quad (2.2)$$

where $\Delta_{S^{m-1}}^{\sigma_i}$ is the Laplace–Beltrami operator on S^{m-1} in the variables in σ_i . Define

$$X = \{u \in H_0^1(\bar{A}) : u \text{ is independent of } \sigma_1, \sigma_2\}. \quad (2.3)$$

Then $u \in X$ can be written as $u(r, \theta, \sigma_1, \sigma_2) \equiv u(r, \theta)$ and we have

$$\Delta_{\mathbb{R}^{2m}} u = u_{rr} + \frac{2m-1}{r} u_r + \frac{m-1}{r^2} \left[\frac{2\cos 2\theta}{\sin 2\theta} \right] u_\theta + \frac{u_{\theta\theta}}{r^2}. \quad (2.4)$$

Consider the change of variables $s(r) = \frac{1}{2}r^2$, $\varphi = 2\theta$. For $u \in X$, we can define $v(s, \varphi) := u(r, \theta)$. Then we see that

$$\begin{aligned} u_r &= r v_s = \sqrt{2s} v_s, & u_{rr} &= 2s v_{ss} + v_s, \\ u_\theta &= 2v_\varphi, & u_{\theta\theta} &= 4v_{\varphi\varphi}. \end{aligned}$$

Then for $u \in X$ solving (1.1), we see that v satisfies (redefining $\varepsilon^2 = 2^{1-(\eta/2)}\varepsilon^2$)

$$-\varepsilon^2 \left[v_{ss} + \frac{m}{s} v_s + \frac{m-1}{s^2} v_\varphi \left[\frac{\cos \varphi}{\sin \varphi} \right] + \frac{v_{\varphi\varphi}}{s^2} \right] + \frac{v}{s^{1-\frac{\eta}{2}}} = \frac{v^p}{s^{1-\frac{\eta}{2}}}. \quad (2.5)$$

So we have the reduced equation for v as

$$\begin{cases} -\varepsilon^2 \Delta v + \frac{v}{|x|^\alpha} = \frac{v^p}{|x|^\alpha}, & \text{in } \Omega, \\ v > 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.6)$$

where $\alpha = 1 - \frac{\eta}{2}$, $\Omega = I' \times S^m$, $I' = \left(\frac{a^2}{2}, \frac{b^2}{2} \right)$, $\varphi \in [0, \pi)$. Note that the function v is axially symmetric. Our aim is to prove the existence of a two-peak solution of (2.6) and to look for the location of the peaks as $\varepsilon \rightarrow 0$.

Define $H_\#(\Omega) \subset H_0^1(\Omega)$ as

$$H_\#(\Omega) = \left\{ u \in H_0^1(\Omega) : u(x_1, \dots, x_m, x_{m+1}) \equiv u \left(\sqrt{x_1^2 + \dots + x_m^2}, |x_{m+1}| \right) \right\}.$$

Note that any non-constant solution in $H_\#(\Omega)$ is axially symmetric and shall have at least two local maximums or minimums (antipodal points). We shall show that there exists a solution with exactly two local maximums for $\varepsilon \ll 1$.

Lemma 2.1. $H_\#(\Omega)$ is a closed subspace of $H_0^1(\Omega)$ and $H_{0,rad}^1(\Omega) \subset H_\#(\Omega)$.

The energy functional $J_\varepsilon(u)$ is defined as

$$J_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + \frac{u^2}{2|x|^\alpha} \right) dx - \frac{1}{p+1} \int_\Omega \frac{u^{p+1}}{|x|^\alpha} dx. \quad (2.7)$$

Lemma 2.2. The Morse index of any non-zero radial solution of (2.6) is ≥ 2 in $H_\#(\Omega)$.

Proof. Let $u \in H_{0,rad}^1(\Omega)$ satisfies (2.6). Then

$$D^2 J_\varepsilon(u)[u, u] = \int_\Omega \left(\varepsilon^2 |\nabla u|^2 + \frac{u^2}{|x|^\alpha} \right) dx - p \int_\Omega \frac{u^{p+1}}{|x|^\alpha} dx < 0,$$

as $p > 1$ and $u \neq 0$.

Next let us take $v = u(c + \cos(2\varphi))$, where the co-ordinate system of Ω is taken as standard polar co-ordinate system. c is chosen so that u and v are orthogonal, that is

$$\int_{a^2/2}^{b^2/2} \int_0^\pi u^2(c + \cos 2\varphi) r^m \sin^{m-1} \varphi dr d\varphi = 0.$$

Note that u being radial we have $v \equiv v(r, \varphi) \in H_\#(\Omega)$. Then Δv in standard polar co-ordinate system takes the form

$$\Delta v = v_{rr} + \frac{m}{r} v_r + \frac{1}{r^2} v_{\varphi\varphi} + \frac{m-1}{r^2} \frac{\cos \varphi}{\sin \varphi} v_\varphi.$$

So, we have

$$\begin{aligned} \Delta v &= \left(u_{rr} + \frac{m}{r} u_r \right) (c + \cos 2\varphi) - \frac{4}{r^2} u \cos 2\varphi - \frac{2(m-1)}{r^2} \frac{\cos \varphi}{\sin \varphi} \sin 2\varphi u \\ -\varepsilon^2 \Delta v &= (-\varepsilon^2 \Delta u)(c + \cos 2\varphi) + \frac{4\varepsilon^2}{r^2} u \cos 2\varphi + \frac{4(m-1)\varepsilon^2}{r^2} \cos^2 \varphi u \\ &= \frac{1}{|x|^\alpha} (-u + u^p)(c + \cos 2\varphi) + \frac{4\varepsilon^2}{r^2} u \cos 2\varphi + \frac{4(m-1)\varepsilon^2}{r^2} \cos^2 \varphi u. \end{aligned}$$

Then

$$\begin{aligned} & -\varepsilon^2 \Delta v + \frac{1}{|x|^\alpha} (1 - pu^{p-1})v \\ &= \frac{1}{|x|^\alpha} (1 - p)u^p (c + \cos 2\varphi) + \frac{4\varepsilon^2}{r^2} u \cos 2\varphi + \frac{4(m-1)\varepsilon^2}{r^2} \cos^2 \varphi u. \end{aligned}$$

Hence

$$\begin{aligned} & \langle -\varepsilon^2 \Delta v + \frac{1}{|x|^\alpha} (1 - pu^{p-1})v, v \rangle \\ &= (1-p) \int_\Omega \frac{u^{p+1}}{r^\alpha} (c + \cos 2\varphi)^2 dx + \int_\Omega \frac{4\varepsilon^2}{r^2} u^2 \cos 2\varphi (c + \cos 2\varphi) dx \\ & \quad + \int_\Omega \frac{4(m-1)\varepsilon^2}{r^2} \cos^2 \varphi (c + \cos 2\varphi) u^2 dx \\ &= 2\pi \Pi_{i=1}^{m-2} \int_0^\pi \sin^i \varphi d\varphi \left[(1-p) \int_{a^2/2}^{b^2/2} r^{m-\alpha} u^{p+1} dr \int_0^\pi (c + \cos \varphi)^2 \sin^{m-1} \varphi d\varphi \right. \\ & \quad + 4\varepsilon^2 \int_{a^2/2}^{b^2/2} r^{m-2} u^2 dr \int_0^\pi \cos 2\varphi (c + \cos \varphi) \sin^{m-1} \varphi d\varphi \\ & \quad \left. + 4(m-1)\varepsilon^2 \int_{a^2/2}^{b^2/2} r^{m-2} u^2 dr \int_0^\pi \cos^2 \varphi (c + \cos \varphi) \sin^{m-1} \varphi d\varphi \right] \\ &= 2\pi \Pi_{i=1}^m \int_0^\pi \sin^i \varphi d\varphi \left[C_1 (1-p) \int_{a^2/2}^{b^2/2} r^{m-\alpha} u^{p+1} dr \right. \\ & \quad \left. + 4\varepsilon^2 C_2 \int_{a^2/2}^{b^2/2} r^{m-2} u^2 dr + 4(m-1)\varepsilon^2 C_3 \int_{a^2/2}^{b^2/2} r^{m-2} u^2 dr \right], \end{aligned}$$

for some $C_1 > 0$, C_2, C_3 , which are independent of ε . Using $a^2/2 < r < b^2/2$, we get, for some positive constant (generic) C ,

$$\langle -\varepsilon^2 \Delta v + \frac{1}{|x|^\alpha} (1 - pu^{p-1})v, v \rangle \leq C \int_{a^2/2}^{b^2/2} \left[(1-p)u^{p+1} + C_4 \varepsilon^2 u^2 \right] r^m dr.$$

Choose ε small enough, so that $C_4 \varepsilon^2 < p-1$. Then we have a $C > 0$ such that

$$\begin{aligned} & \langle -\varepsilon^2 \Delta v + \frac{1}{|x|^\alpha} (1 - pu^{p-1})v, v \rangle \\ & \leq C(1-p) \int_{\Omega} (u^{p+1} - u^2) dx = C(1-p) \int_{\Omega} |\nabla u|^2 dx < 0. \end{aligned}$$

So u and v being orthogonal we see that the Morse index of any radial solution u is greater or equal to 2. $\square$

Remark 2.3. Note that $H_{\sharp}$ is closed in $H_0^1(\Omega)$ (hence compactly embedded in $L^2(\Omega)$). It can be easily shown that J_ε satisfies $(PS)_c$ condition for any critical value c . Also it has MP geometry near the origin. Next we shall construct a test function, from which we shall get a MP critical level c_ε . Also using the test function we shall get an upper bound of c_ε which shall help us analysing the behaviour of a sequence of solutions as $\varepsilon \rightarrow 0$.

Consider the following equation

$$\begin{cases} -\Delta U + U = U^p, & \text{in } \mathbb{R}^{m+1}, \\ U > 0, & \text{in } \mathbb{R}^{m+1}, \\ \lim_{|x| \rightarrow \infty} U(x) = 0, & U(0) = \max_{x \in \mathbb{R}^{m+1}} U(x). \end{cases} \quad (2.8)$$

The energy functional $J(U)$ of (2.8) is given by

$$J(U) = \frac{1}{2} \int_{\mathbb{R}^{m+1}} (|\nabla U|^2 + U^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^{m+1}} U^{p+1} dx, \quad U \in H^1(\mathbb{R}^{m+1}). \quad (2.9)$$

It is well known that there exists a least positive critical value c^* , which is characterized by

$$c^* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \quad (2.10)$$

where $\Gamma = \{\gamma(t) \in C([0,1], H^1(\mathbb{R}^{m+1})); \gamma(0) = 0, I(\gamma(1)) < 0\}$ (see [10]). And there is a least energy solution U of (2.8) such that $U(x) = U(|x|)$ and

$$|D^\alpha U(x)| \leq C \exp(-\delta|x|), \quad (2.11)$$

for some $c, \delta > 0$ and for all $|\alpha| \leq 2$. Furthermore, U satisfies the Pohozaev identity

$$\frac{m-1}{2} \int_{\mathbb{R}^{m+1}} |\nabla U|^2 dx + \frac{m+1}{2} \int_{\mathbb{R}^{m+1}} U^2 dx - \frac{m+1}{p+1} \int_{\mathbb{R}^{m+1}} U^{p+1} dx = 0. \quad (2.12)$$

For $a^2/2 < d < b^2/2$, let $W(x) = U\left(\frac{x}{d^{\alpha/2}}\right)$. Then W will satisfy

$$\begin{cases} -\Delta W + \frac{W}{d^\alpha} = \frac{W^p}{d^\alpha}, & \text{in } \mathbb{R}^{m+1}, \\ W > 0, & \text{in } \mathbb{R}^{m+1}, \\ \lim_{|x| \rightarrow \infty} W(x) = 0, & W(0) = \max_{x \in \mathbb{R}^{m+1}} W(x). \end{cases} \quad (2.13)$$

Define

$$Z_{\varepsilon,t}^\gamma = \Phi_\gamma\left(\frac{x-Q}{t}\right) W\left(\frac{x-Q}{\varepsilon t}\right) + \Phi_\gamma\left(\frac{x+Q}{t}\right) W\left(\frac{x+Q}{\varepsilon t}\right), \quad (2.14)$$

where $Q = (0, \dots, 0, d) \in \Omega$ and $\Phi_\gamma(x) = \phi_\gamma\left(\frac{x}{d^{\alpha/2}}\right)$, where ϕ_γ is a non-negative smooth radial function supported in $B(0, 2\gamma)$ satisfying $\nabla \phi_\gamma < 2/\gamma$ and

$$\phi_\gamma(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq \gamma \\ 0 & \text{for } r \geq 2\gamma. \end{cases}$$

Also γ is chosen so that $B(0, 2t\gamma d^{\alpha/2}) \subset B(0, \text{dist}(Q, \partial\Omega))$, for all $0 < t < t_0$, where t_0 is a fixed positive number which will be chosen later. Clearly $Z_{\varepsilon,t}^\gamma \in H_\#(\Omega)$.

The corresponding energy functional is

$$J_\varepsilon(Z_{\varepsilon,t}^\gamma) = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla Z_{\varepsilon,t}^\gamma|^2 + \frac{|Z_{\varepsilon,t}^\gamma|^2}{2|x|^\alpha} \right) dx - \frac{1}{p+1} \int_\Omega \frac{|Z_{\varepsilon,t}^\gamma|^{p+1}}{|x|^\alpha} dx. \quad (2.15)$$

First note that the supports of $\Phi_\gamma\left(\frac{x-Q}{t}\right) W\left(\frac{x-Q}{\varepsilon t}\right)$ and $\Phi_\gamma\left(\frac{x+Q}{t}\right) W\left(\frac{x+Q}{\varepsilon t}\right)$ are disjoint. To get the mountain-pass solution, we shall calculate the above energy explicitly. Denote $\Phi_\gamma^\pm \equiv \Phi_\gamma(\frac{1}{t}(x \pm Q))$ and $W^\pm \equiv W(\frac{1}{\varepsilon t}(x \pm Q))$. Then we have

$$\begin{aligned} & \int_\Omega \frac{\varepsilon^2}{2} |\nabla Z_{\varepsilon,t}^\gamma|^2 dx \\ &= \int_\Omega \frac{\varepsilon^2}{2} \left| \left((\nabla_x \Phi_\gamma^-) W^- + \Phi_\gamma^- (\nabla_x W^-) \right) + \left((\nabla_x \Phi_\gamma^+) W^+ + \Phi_\gamma^+ (\nabla_x W^+) \right) \right|^2 dx \\ &= \int_\Omega \frac{\varepsilon^2}{2} \left| (\nabla_x \Phi_\gamma^-) W^- + \Phi_\gamma^- (\nabla_x W^-) \right|^2 dx + \int_\Omega \frac{\varepsilon^2}{2} \left| (\nabla_x \Phi_\gamma^+) W^+ + \Phi_\gamma^+ (\nabla_x W^+) \right|^2 dx \\ &= I_1 + I_2 \text{ (say) }. \end{aligned}$$

Make the change of variable $d^{\alpha/2}y = \frac{1}{\varepsilon t}(x-Q)$ in I_1 and $d^{\alpha/2}y = \frac{1}{\varepsilon t}(x+Q)$ in I_2 and using the decay estimates of the solution U and ∇U of (2.6), we get

$$\begin{aligned} I_1 &= \int_\Omega \frac{\varepsilon^2}{2} \left| (\nabla_x \Phi_\gamma^-) W^- + \Phi_\gamma^- (\nabla_x W^-) \right|^2 dx \quad (2.16) \\ &= \frac{1}{2} d^{(m-1)\frac{\alpha}{2}} \varepsilon^{m+1} t^{m-1} \int_{B(0, 2\gamma/\varepsilon)} \left[\nabla \phi_\gamma(\varepsilon y) U(y) + \phi_\gamma(\varepsilon y) \nabla U(y) \right]^2 dy \\ &= \frac{1}{2} d^{(m-1)\frac{\alpha}{2}} \varepsilon^{m+1} t^{m-1} \int_{B(0, 2\gamma/\varepsilon)} \left[|\nabla U|^2 \phi_\gamma^2(\varepsilon y) \right. \\ &\quad \left. + 2\phi_\gamma(\varepsilon y) U(y) \nabla \phi_\gamma(\varepsilon y) \nabla U(y) + |\nabla \phi_\gamma(\varepsilon y)|^2 U^2(y) \right] dy \\ &= \frac{1}{2} d^{(m-1)\frac{\alpha}{2}} \varepsilon^{m+1} t^{m-1} \left[\int_{B(0, \gamma/\varepsilon)} |\nabla U|^2 dy + \int_{B(0, 2\gamma/\varepsilon) \setminus B(0, \gamma/\varepsilon)} |\nabla U|^2 \phi_\gamma^2(\varepsilon y) dy \right] \\ &\quad + \frac{1}{2} \varepsilon^{m+1} t^{m-1} \int_{B(0, 2\gamma/\varepsilon) \setminus B(0, \gamma/\varepsilon)} \left[2\phi_\gamma(\varepsilon y) U(y) \nabla \phi_\gamma(\varepsilon y) \nabla U(y) + |\nabla \phi_\gamma(\varepsilon y)|^2 U^2(y) \right] dy \\ &= \frac{1}{2} d^{(m-1)\frac{\alpha}{2}} \varepsilon^{m+1} t^{m-1} \left[\int_{B(0, \gamma/\varepsilon)} |\nabla U|^2 dy + O(\varepsilon) \right] \\ &= \frac{1}{2} d^{(m-1)\frac{\alpha}{2}} \varepsilon^{m+1} t^{m-1} \left[\int_{\mathbb{R}^{m+1}} |\nabla U|^2 dy + O(\varepsilon) \right]. \end{aligned}$$

Similar estimate will show that,

$$I_2 = \frac{1}{2}d^{(m-1)\frac{\alpha}{2}}\varepsilon^{m+1}t^{m-1} \int_{\mathbb{R}^{m+1}} |\nabla U|^2 dy + O(\varepsilon).$$

Adding I_1 and I_2 , we get

$$\int_{\Omega} \frac{\varepsilon^2}{2} |\nabla Z_{\varepsilon,t}^{\gamma}|^2 dx = d^{(m-1)\frac{\alpha}{2}}\varepsilon^{m+1}t^{m-1} \int_{\mathbb{R}^{m+1}} |\nabla U|^2 dy + O(\varepsilon). \quad (2.17)$$

The second term of the energy (2.15), is

$$\begin{aligned} & \int_{\Omega} \frac{1}{2|x|^{\alpha}} |Z_{\varepsilon,t}^{\gamma}|^2 dx \\ &= \int_{\Omega} \frac{1}{2|x|^{\alpha}} \left| \Phi_{\gamma}^{-} W^{-} + \Phi_{\gamma}^{+} W^{+} \right|^2 dx \\ &= \frac{1}{2}d^{(m+1)\frac{\alpha}{2}}\varepsilon^{m+1}t^{m+1} \int_{B(0,2\gamma/\varepsilon)} \left[\frac{\phi_{\gamma}^2(\varepsilon y)U^2(y)}{|Q + \varepsilon t d^{\alpha/2}y|^{\alpha}} + \frac{\phi_{\gamma}^2(\varepsilon y)U^2(y)}{|Q - \varepsilon t d^{\alpha/2}y|^{\alpha}} \right] dy \\ &= d^{(m+1)\frac{\alpha}{2}}\varepsilon^{m+1}t^{m+1} \int_{B(0,2\gamma/\varepsilon)} \phi_{\gamma}^2(\varepsilon y)U^2(y)(d^{-\alpha} + O(\varepsilon)) dy \\ &= d^{(m-1)\frac{\alpha}{2}}\varepsilon^{m+1}t^{m+1} \left[\int_{\mathbb{R}^{m+1}} U^2(y) dy + O(\varepsilon) \right]. \end{aligned} \quad (2.18)$$

Similarly,

$$\frac{1}{p+1} \int_{\Omega} |Z_{\varepsilon,t}^{\gamma}|^{p+1} = \frac{2}{p+1}d^{(m-1)\frac{\alpha}{2}}\varepsilon^{m+1}t^{m+1} \left[\int_{\mathbb{R}^{m+1}} U^{p+1}(y) dy + O(\varepsilon) \right]. \quad (2.19)$$

Combining all these terms in (2.17), (2.18), (2.19), we have

$$\begin{aligned} \varepsilon^{-(m+1)}J_{\varepsilon}(Z_{\varepsilon,t}^{\gamma}) &= 2d^{(m-1)\frac{\alpha}{2}} \left[\frac{t^{m-1}}{2} \int_{\mathbb{R}^{m+1}} |\nabla U|^2 dx + \right. \\ &\quad \left. \frac{t^{m+1}}{2} \int_{\mathbb{R}^{m+1}} U^2(x) dx - \frac{t^{m+1}}{p+1} \int_{\mathbb{R}^{m+1}} U^{p+1}(x) dx + O(\varepsilon) \right] \end{aligned} \quad (2.20)$$

Then from (2.12), we have

$$\varepsilon^{-(m+1)}J_{\varepsilon}(Z_{\varepsilon,t}^{\gamma}) = d^{(m-1)\frac{\alpha}{2}}(t^{m-1} - (\frac{m-1}{m+1})t^{m+1}) \int_{\mathbb{R}^{m+1}} |\nabla U|^2 dx + O(\varepsilon). \quad (2.21)$$

We choose t_0 such that

$$d^{(m-1)\frac{\alpha}{2}}(t_0^{m-1} - (\frac{m-1}{m+1})t_0^{m+1}) \int_{\mathbb{R}^{m+1}} |\nabla U|^2 dx + O(\varepsilon) < -1.$$

Now we can choose γ such that $B(0, 2t_0\gamma d^{\alpha/2}) \subset B(0, \text{dist}(Q, \partial\Omega))$. Then there exists ε_0 such that

$$J_{\varepsilon}(Z_{\varepsilon,t_0}^{\gamma}) < 0 \quad \text{for all} \quad \varepsilon < \varepsilon_0. \quad (2.22)$$

We define

$$c_\varepsilon = \inf_{\beta \in \mathcal{P}} \max_{t \in [0,1]} J_\varepsilon(\beta(t)), \quad (2.23)$$

where $\mathcal{P} = \{\beta \in C([0,1], H_\sharp) : \beta(0) = 0, \beta(1) = Z_{\varepsilon, t_0}^\gamma\}$. From MP-Lemma we have c_ε as a critical value of J_ε .

Remark 2.4. *It can be easily shown that J_ε satisfies all the hypothesis of [11, Thm 10.2] (also look at [12, p.222 and Thm 5.1] and [13, p.1598]). Hence at the level c_ε , there is a solution u_ε of (2.6) with Morse index less or equal to one. Then we can readily see from Lemma 2.2 that there is a non radial MP solution corresponding to the MP critical value c_ε . Also note that the solution is not the least energy solution in the space $H_0^1(\Omega)$.*

3. Profile of the solution

In this section, we follow the line of proof in Beyon and Park [14]. Let $\beta(t) = Z_{\varepsilon, tt_0}^\gamma$. Then it follows that $\lim_{t \rightarrow 0} \beta(t) = 0$ and $\beta(1) = Z_{\varepsilon, t_0}^\gamma$. Note, $Z_{\varepsilon, tt_0}^\gamma \in H_\sharp$ for all $t \in [0, 1]$. Moreover, from (2.20) we have

$$\begin{aligned} J_\varepsilon(Z_{\varepsilon, t}^\gamma) = & 2d^{(m-1)\frac{\alpha}{2}} \varepsilon^{m+1} \left[\frac{1}{2} (tt_0)^{m-1} \int_{\mathbb{R}^{m+1}} |\nabla U|^2 dx + \frac{1}{2} (tt_0)^{m+1} \int_{\mathbb{R}^{m+1}} U^2 dx \right. \\ & \left. - \frac{1}{p+1} (tt_0)^{m+1} \int_{\mathbb{R}^{m+1}} U^{p+1}(x) dx + O(\varepsilon) \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon^{-(m+1)} c_\varepsilon \\ & \leq 2d^{(m-1)\frac{\alpha}{2}} \max_{t \in (0, t_0)} \left[\frac{t^{m-1}}{2} \int_{\mathbb{R}^{m+1}} |\nabla U|^2 dx + \frac{t^{m+1}}{2} \int_{\mathbb{R}^{m+1}} U^2 dx \right. \\ & \quad \left. - \frac{t^{m+1}}{p+1} \int_{\mathbb{R}^{m+1}} U^{p+1}(x) dx \right]. \end{aligned}$$

Again, using (2.12) we can easily show that the maximum occurs at $t = 1$ and we get

$$\overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon^{-(m+1)} c_\varepsilon \leq 2d^{(m-1)\frac{\alpha}{2}} J(U). \quad (3.1)$$

By Sobolev embedding and the bootstrap argument we can find $\{\|u_\varepsilon\|_\infty\}$ is bounded and by maximum principle we see that it is bounded away from 0. Also from Pohozaev identity (2.12) we can have $\{\varepsilon^{-(m+1)} \|u_\varepsilon\|_\varepsilon^2\}$ is bounded where

$$\|u_\varepsilon\|_\varepsilon^2 = \int_{\Omega} (\varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2) dx. \quad (3.2)$$

Let $q_\varepsilon \in A$ such that $\liminf_{\varepsilon \rightarrow 0} u_\varepsilon(q_\varepsilon) > 0$. Consider

$$\omega_\varepsilon = u_\varepsilon(\varepsilon x - q_\varepsilon). \quad (3.3)$$

Lemma 3.1. $\frac{1}{\varepsilon} \text{dist}(q_\varepsilon, \partial A) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Proof. Similar proof as given in Proposition 4 of [8] □

Then it can be easily shown that, up to a subsequence, ω_ε converges (locally in C^2) to a finite energy solution $\widetilde{W}$ of (2.8). Note that $\widetilde{W}$ is non-negative and hence by uniqueness $W = \widetilde{W}$.

Also from (3.1), we can say that there can be at most finitely many $q_\varepsilon^1, \dots, q_\varepsilon^k \in A$ satisfying

- (i) $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \inf_{1 \leq i \neq j \leq k} |q_\varepsilon^i - q_\varepsilon^j| = \infty$ for $i \neq j$ and
- (ii) $\liminf_{\varepsilon \rightarrow 0} u_\varepsilon(q_\varepsilon^i) > 0$ for $j = 1, 2, \dots, k$.

Then from comparison principle we get c and $C > 0$ such that

$$u_\varepsilon(x) + |\nabla u_\varepsilon(x)| \leq C \exp\left(-\frac{c}{\varepsilon} \text{dist}(x, q_\varepsilon^1, \dots, q_\varepsilon^k)\right). \quad (3.4)$$

And this implies

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-(m+1)} J_\varepsilon(u_\varepsilon) = k d^{(m-1)\frac{\alpha}{2}} J(U). \quad (3.5)$$

Then from (3.1) we have $k \leq 2$. Since $u_\varepsilon \in H_\sharp$, for any point of maximum q_ε of u_ε , the antipodal point $-q_\varepsilon$ is also a max of u_ε and q_ε must lie on x_{m+1} axis.

4. Location of the spike layers when $\eta = 2$

We shall determine the location of the peaks by estimating the upper and lower bounds of the energy. Note that for $\eta = 2$, $\alpha = 1 - \eta/2 = 0$. So we have our reduced equation as

$$\begin{cases} -\varepsilon^2 \Delta v + v = v^p, & \text{in } \Omega, \\ v > 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

Let q_ε be one of the maximum points of the solution u_ε . And it has been proved that $\frac{1}{\varepsilon} \text{dist}(q_\varepsilon, \partial A) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Let $q_\varepsilon \rightarrow \bar{q}$ as $\varepsilon \rightarrow 0$. Then another point of concentration is $-\bar{q}$.

Now consider the following problem in a ball $B_\rho = \{x \in \mathbb{R}^{m+1} : |x| \leq \rho\}$, for some $\rho > 0$.

$$\begin{cases} -\Delta u + u = u^p, & \text{in } B_\rho, \\ u > 0, & \text{in } B_\rho, \\ u = 0, & \text{on } \partial B_\rho. \end{cases} \quad (4.2)$$

This functional has a least positive critical value, denoted by c_ρ , which can be characterized similarly to (2.23). Using Schwarz symmetrization, we can have a radially symmetric least energy solution of (4.2).

Lemma 4.1.

$$c_\rho = c^* + \exp[-2\rho(1 + o(1))],$$

where c^* is given in (2.10).

Proof. The proof can be found in Lemma 2.1 in [15]. $\square$

We shall now use Lemma 4.1 to give the upper and lower bound for c_ε . Let U_ρ be a corresponding radial least energy solution to (4.2). Define

$$U_\rho^\varepsilon = U_\rho(\varepsilon x - \bar{p}) + U_\rho(\varepsilon x + \bar{p}),$$

for some $\bar{p} \in \Omega$. Then we see that for any $\rho > 0$, there is an ε_ρ such that $U_\rho^\varepsilon \in H_\#$ for all $\varepsilon < \varepsilon_\rho$ and such that the support of $U_\rho(\varepsilon x - \bar{p})$ and $U_\rho(\varepsilon x + \bar{p})$ are disjoint. Estimating as before we can have the energy at U_ρ^ε as

$$J_\varepsilon(U_\rho^\varepsilon) = 2\varepsilon^{m+1} \int_{B_\rho(0)} \left(\frac{1}{2}(|\nabla U_\rho|^2 + |U_\rho|^2) - \frac{1}{p+1}|U_\rho|^{p+1} \right) = 2\varepsilon^{m+1}c_\rho. \quad (4.3)$$

Hence we have (following the line of proof of Lemma 6 in [8])

$$c_\varepsilon \leq 2\varepsilon^{m+1}c_\rho = 2\varepsilon^{m+1}(c^* + \exp[-2\rho(1 + o(1))]). \quad (4.4)$$

We can take $\rho = \frac{1}{\varepsilon} \text{dist}(\bar{p}, \partial\Omega)$ and $\bar{p}$ is at maximum distance from $\partial\Omega$. Finally we have

$$c_\varepsilon \leq 2\varepsilon^{m+1}c_\rho = 2\varepsilon^{m+1} \left(c^* + \exp \left[-\frac{2}{\varepsilon} \text{dist}(\bar{p}, \partial\Omega)(1 + o(1)) \right] \right). \quad (4.5)$$

Next we shall estimate c_ε from below. Let

$$d_\varepsilon = \text{dist}(q_\varepsilon, \partial\Omega) = \text{dist}(-q_\varepsilon, \partial\Omega) \rightarrow d_0 = \text{dist}(\bar{q}, \partial\Omega) \quad \text{as } \varepsilon \rightarrow 0.$$

Given δ , choose a number $d'_0 > 0$ such that

$$\text{vol}(B(\bar{q}, d'_0)) = \text{vol}(\Omega \cap B(\bar{q}, d_0 + \delta)).$$

Now take $\delta' > 0$ slightly smaller than δ with $d'_0 < d_0 + \delta'$. Let us consider a C^∞ cut-off function $\eta_\varepsilon(s)$ such that $\eta_\varepsilon(s) = 1$ for $0 \leq s \leq d_\varepsilon + \delta'$ and $\eta_\varepsilon(s) = 0$ for $s \geq d_\varepsilon + \delta$, with $0 \leq \eta_\varepsilon \leq 1$ and with uniformly bounded derivative. Let us take $\tilde{u}_\varepsilon = u_\varepsilon \eta_\varepsilon(x - \bar{q}) + u_\varepsilon \eta_\varepsilon(x + \bar{q})$. Note that the support of $u_\varepsilon \eta_\varepsilon(x - \bar{q})$ and $u_\varepsilon \eta_\varepsilon(x + \bar{q})$ are disjoint and we find that

$$c_\varepsilon \geq J_\varepsilon(tu_\varepsilon) \geq J_\varepsilon(t\tilde{u}_\varepsilon) - 2\exp \left[-\frac{2}{\varepsilon}(d_\varepsilon + \delta') \right], \quad (4.6)$$

for ε sufficiently small. Here we have used the estimates in (3.4). Also note that

$$J_\varepsilon(t\tilde{u}_\varepsilon) = J_\varepsilon(tu_\varepsilon \eta_\varepsilon(x - \bar{q})) + J_\varepsilon(tu_\varepsilon \eta_\varepsilon(x + \bar{q})).$$

So using the same analysis as in [15] we can get

$$c_\varepsilon \geq 2\varepsilon^{m+1} \left\{ c^* + \exp \left[-\frac{2}{\varepsilon}(d_\varepsilon + \delta') \right] \right\}. \quad (4.7)$$

Taking $\bar{p}$ at the maximum distance from the boundary and comparing (4.5) and (4.7) we have $\text{dist}(q_\varepsilon, \partial\Omega) \rightarrow \max_{x \in \Omega} \text{dist}(x, \partial\Omega)$. That is, q_ε concentrates at $(0, 0, \frac{a+b}{2})$.

Proof of the Theorem 1.1 (for $\eta = 2$)

Proof. Let v_ε be a non radial MP solution of (2.6) at the critical level c_ε with Morse index less or equal to 1 (actually it is 1 as u_ε being positive radial solution). Now from the above discussion we see that the sequence $\{v_\varepsilon\}_\varepsilon$ concentrates exactly at two points in Ω . Also we have got the co-ordinates of those concentration points as $(\frac{a+b}{2}, 0, 0, \dots, 0)$ and $(\frac{a+b}{2}, \pi, 0, \dots, 0)$. Now corresponding to v_ε there is a solution u_ε of Eq. (1.1). And we get that $\{u_\varepsilon\}_\varepsilon$ concentrates exactly at two $m-1$ dimensional spheres in A , placed at the angle $\theta = 0$ and $\theta = \pi/2$ with the radial co-ordinate $(a+b)/2$. $\square$

5. The case $\eta \neq 2$ and proof of Theorem 1.1

Let us first assume that $\eta > 2$ and we have $\alpha < 0$. The other case will follow exactly in the same way as done in this case. First note that for $\eta \neq 2$, we have $\alpha \neq 0$ and v satisfies (2.6). Our aim is to show the existence of a two-peak solution of (2.6) and to prove the peaks goes to outer boundary as $\varepsilon \rightarrow 0$, for $\eta > 2$.

Let $\delta > 0$ be any positive number. Take p_0 near to the outer boundary such that $\text{dist}(p_0, \partial\Omega) < \delta/2$ and p_0 is on $(m+1)^{\text{th}}$ axis. Define the test function $Z_{\varepsilon,t}^\gamma$ as in (2.14) with $Q = p_0$.

From (3.1), we have

$$\overline{\lim}_{\varepsilon \rightarrow 0} \varepsilon^{-(m+1)} c_\varepsilon \leq 2 |p_0|^{\frac{m-1}{2}\alpha} J(U) \leq 2 \left(\frac{b^2}{2} - \frac{\delta}{2} \right)^{\frac{m-1}{2}\alpha} J(U). \quad (5.1)$$

Now from (3.5), and using the fact $K = 2$, we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-(m+1)} J_\varepsilon(u_\varepsilon) = 2 |p_0|^{\frac{m-1}{2}\alpha} J(U) \geq 2 |b^2/2|^{\frac{m-1}{2}\alpha} J(U). \quad (5.2)$$

Lemma 5.1. *For $\eta > 2$, we have $|p_\varepsilon| \rightarrow \frac{b^2}{2}$ as $\varepsilon \rightarrow 0$.*

Proof. From (5.1) and (5.2) and the conclusion there after, we have

$$2 |b^2/2|^{\frac{m-1}{2}\alpha} J(U) + O(\varepsilon) \leq 2 |p_\varepsilon|^{\frac{m-1}{2}\alpha} J(U) \leq 2 \left(\frac{b^2}{2} - \frac{\delta}{2} \right)^{\frac{m-1}{2}\alpha} J(U) + O(\varepsilon). \quad (5.3)$$

Then taking $\delta \rightarrow 0$ we see that $|p_\varepsilon| \rightarrow \frac{b^2}{2}$ as $\varepsilon \rightarrow 0$. $\square$

Proof of the Theorem 1.1

Proof. In the Sects. 4 and 5 we have prove that there is a non radial least energy solution v_ε concentrating exactly at two points in Ω . Hence the corresponding solution u_ε of (1.1) will concentrates exactly at two $(m-1)$ -dimensional spheres in A , placed the angle $\theta = 0$ and $\theta = \pi/2$. The radial co-ordinates of the concentrating spheres is

- i. $\frac{a+b}{2}$, for $\eta = 2$.
- ii. b , for $\eta > 2$

Now for $\eta < 2$, $\alpha > 0$. We can take a point near to the inner boundary on the $(m + 1)$ th axis and construct the test function as in (2.14). Proceeding similarly as the previous section we can have a non radial least energy solution v_ε concentrating exactly at two points in Ω on the $(m + 1)$ th axis and the points of concentration will converge to the inner boundary as $\varepsilon \rightarrow 0$. Hence, the corresponding solution of Eq. (1.1) will concentrates exactly at two $(m - 1)$ -dimensional orthogonal spheres in A , and the radial co-ordinates of the concentrating spheres is a . $\square$

References

- [1] Ni, W.-M., Takagi, I.: On the shape of least-energy solutions to a semilinear Neumann problem. *Commun. Pure Appl. Math.* **44**(7), 819–851 (1991)
- [2] Ni, W.-M., Takagi, I.: Locating the peaks of least-energy solutions to a semilinear Neumann problem. *Duke Math. J.* **70**(2), 247–281 (1993)
- [3] Ni, W.-M., Takagi, I., Wei, J.: On the location and profile of spike-layer solutions to a singularly perturbed semilinear Dirichlet problem: intermediate solutions. *Duke Math. J.* **94**(3), 597–618 (1998)
- [4] Ni, W.-M., Wei, J.: On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems. *Commun. Pure Appl. Math.* **48**(7), 731–768 (1995)
- [5] Dancer, E.N., Yan, S.: A singularly perturbed elliptic problem in bounded domains with nontrivial topology. *Adv. Differ. Equ.* **4**(3), 347–368 (1999)
- [6] Pistoia, A., Clapp, M., Faya, J.: Nonexistence and multiplicity of solutions to elliptic problems with supercritical exponents. *arXiv:1212.5137*
- [7] Ambrosetti, A., Malchiodi, A., Ni, W.-M.: Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres II. *Indiana Univ. Math. J.* **53**(2), 297–329 (2004)
- [8] Ruf, B., Srikanth, P.N.: Singularly perturbed elliptic equations with solutions concentrating on a 1-dimensional orbit. *J. Eur. Math. Soc. (JEMS)* **12**(2), 413–427 (2010)
- [9] Pacella, F., Srikanth, P.N.: A reduction method for semilinear elliptic equations and solutions concentrating on spheres. *arXiv:1210.0782*
- [10] Jeanjean, L., Tanaka, K.: A remark on least energy solutions in $\mathbf{R}^N$. *Proc. Amer. Math. Soc.* **131**(8), 2399–2408 (2003) (electronic)
- [11] Ghoussoub, N.: Duality and perturbation methods in critical point theory. In: *Cambridge Tracts in Mathematics*, vol. 107. Cambridge University Press, Cambridge (1993) (with appendices by David Robinson)
- [12] Ekeland, I., Ghoussoub, N.: Selected new aspects of the calculus of variations in the large. *Bull. Am. Math. Soc. (NS)* **39**(2), 207–265 (2002) (electronic)

- [13] Fang, G., Ghoussoub, N.: Morse-type information on Palais–Smale sequences obtained by min–max principles. *Commun. Pure Appl. Math* **47**(12), 1595–1653 (1994)
- [14] Byeon, J., Park, J.: Singularly perturbed nonlinear elliptic problems on manifolds. *Calc. Var. Part. Differ. Equ.* **24**(4), 459–477 (2005)
- [15] Del Pino, M., Felmer, P.L.: Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting. *Indiana Univ. Math. J.* **48**(3), 883–898 (1999)

B. B. Manna and P. N. Srikanth
TIFR CAM
Bangalore
India
e-mail: srikanth@math.tifrbng.res.in

B. B. Manna
e-mail: bhakti@math.tifrbng.res.in

Received: 19 September 2013.

Accepted: 21 March 2014.