Symmetry results for nonlinear elliptic operators with unbounded drift

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Abstract. We prove the one-dimensional symmetry of solutions to elliptic equations of the form $-\operatorname{div}(e^{G(x)}a(|\nabla u|)\nabla u) = f(u)e^{G(x)}$, under suitable energy conditions. Our results holds without any restriction on the dimension of the ambient space.

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1. Introduction

In this paper we study the one-dimensional symmetry of solutions to nonlinear equations of the following type:

$$\operatorname{div}(a(|\nabla u|)\nabla u) + a(|\nabla u|) \langle \nabla G(x), \nabla u \rangle + f(u) = 0, \quad (1)$$

or in a more compact form

$$-\operatorname{div}(e^{G(x)}a(|\nabla u|)\nabla u) = f(u)e^{G(x)}, \quad (2)$$

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where $f \in C^1(\mathbb{R})^1$, $G \in C^2(\mathbb{R}^n)$ and $a \in C_{loc}^{1,1}((0, +\infty))$. We also require that the function a satisfies the following structural conditions:

$$a(t) > 0 \quad \text{for any } t \in (0, +\infty), \quad (3)$$

$$a(t) + a'(t)t > 0 \quad \text{for any } t \in (0, +\infty). \quad (4)$$

Observe that the general form of (2) encompasses, as very special cases, many elliptic singular and degenerate equations. Indeed, if $G \equiv 0$ and $a(t) = t^{p-2}$, $1 < p < +\infty$, or $a(t) = 1/\sqrt{1+t^2}$ then we obtain the p -Laplacian and the mean curvature equations respectively. Moreover, if $a(t) \equiv 1$ and $G(x) = -|x|^2/2$ Eq. (1) boils down to the classical Ornstein-Uhlenbeck operator for which we refer to [1] and the references therein. To prove the one-dimensional symmetry of solutions we follow the approach introduced in [5] and further developed in [9]. Following [3, 5, 9], we define $A : \mathbb{R}^n \rightarrow \text{Mat}(n \times n)$, $\lambda_1 \in C^0((0, +\infty))$, $\lambda_G \in C^0(\mathbb{R}^{2n})$ as follow

$$A_{hk}(\xi) := \frac{a'(|\xi|)}{|\xi|} \xi_h \xi_k + a(|\xi|) \delta_{hk} \quad \text{for any } 1 \leq h, k \leq n, \quad (5)$$

$$\lambda_1(t) := a(t) + a'(t)t \quad \text{for any } t > 0 \quad (6)$$

and

$$\lambda_G(x) := \text{maximal eigenvalue of } \nabla^2 G(x). \quad (7)$$

Definition 1.1. We say that u is a weak solution to (1) if $u \in C^1(\mathbb{R}^n)$ and denoted by $d\mu = e^{G(x)} dx$

$$\int_{\mathbb{R}^n} \langle a(|\nabla u|) \nabla u, \nabla \varphi \rangle - f(u) \varphi \, d\mu = 0 \quad \forall \varphi \in C_c^1(\mathbb{R}^n) \quad (8)$$

and either (A1) or (A2) is satisfied, where:

$$(A1) \quad \{\nabla u = 0\} = \emptyset.$$

$$(A2) \quad a \in C^0([0, +\infty)) \text{ and}$$

$$\text{the map } t \rightarrow ta(t) \text{ belongs to } C^1([0, +\infty)).$$

Notice that (8) is well-defined, thanks to (A1) or (A2).

Notice also that weak solutions to (1) are critical points of the functional

$$I(u) := \int_{\mathbb{R}^n} \left(\Lambda(|\nabla u|) + F(u) \right) d\mu \quad (9)$$

where $F'(t) = -f(t)$, $d\mu = e^{G(x)} dx$ and

$$\Lambda(t) := \int_0^t a(|\tau|) \tau d\tau.$$

The regularity assumption $u \in C^1(\mathbb{R}^n)$ is always fulfilled in many important cases, like those involving the p -Laplacian operator or the mean curvature operator. For instance, when $a(t) = t^{p-2}$, $1 < p < +\infty$, any distribution solution $u \in W_{loc}^{1,p}(\mathbb{R}^n) \cap L_{loc}^\infty(\mathbb{R}^n)$ is of class C^1 , by the well-known results in

¹ One could consider functions f which are only locally lipschitz continuous, as in [9]. To avoid inessential technicalities, we do not treat this case here.

[16, 22]). In light of this, and in view of the great generality of the function a , it is natural to work in the above setting.

Definition 1.2. Let $h \in L^1_{loc}(\mathbb{R}^n)$ and let u be a weak solution to (1). We say that u is h -stable if

$$\int_{\mathbb{R}^n} \langle A(\nabla u) \nabla \varphi, \nabla \varphi \rangle - f'(u) \varphi^2 \, d\mu \geq \int_{\mathbb{R}^n} a(|\nabla u|) h \varphi^2 \, d\mu \quad \forall \varphi \in C_c^1(\mathbb{R}^n). \quad (10)$$

Remark 1.3. When $a(t) \equiv 1$, Definition 1.2 boils down to the h -stability condition introduced in [2, 3]. When $h \equiv 0$, then u satisfies the classical stability condition [5, 9–11], and we simply say that u is stable. In particular, every minimum point of the functional (9) is a stable solution to (1). Let us also point out that, in view of (A1) or (A2), the integral

$$\int_{\mathbb{R}^n} \langle A(\nabla u) \nabla \varphi, \nabla \varphi \rangle - f'(u) \varphi^2 - a(|\nabla u|) h \varphi^2 \, d\mu \quad (11)$$

is well defined.² In particular, under the condition (A2) the function A can be extended by continuity at the origin, by setting $A_{hk}(0) := a(0)\delta_{hk}$.

We can now state our main symmetry results:

Theorem 1. Assume $G \in C^2(\mathbb{R}^n)$ and $h \in L^1_{loc}(\mathbb{R}^n)$ with $h \geq \lambda_G$. Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H^1_{loc}(\mathbb{R}^n)$ be a h -stable weak solution to (1). Assume that there exists $C > 0$ such that

$$\lambda_1(t) \leq C a(t) \quad \forall t > 0, \quad (12)$$

and one of the following conditions holds

- (a) there exists $C_0 \geq 1$ such that $\int_{B_R} a(|\nabla u|) |\nabla u|^2 \, d\mu \leq C_0 R^2$ for any $R \geq C_0$,
- (b) $n = 2$ and u satisfies $a(|\nabla u|) |\nabla u|^2 e^G \in L^\infty(\mathbb{R}^2)$.

Then u is one-dimensional, i.e. there exists $\omega \in \mathbb{S}^{n-1}$ and $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$u(x) = u_0(\langle \omega, x \rangle) \quad \forall x \in \mathbb{R}^n. \quad (13)$$

Moreover,

$$\langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle = 0 \quad \forall x \in \mathbb{R}^n. \quad (14)$$

In particular, if u_0 is not constant, there are C and g of class C^2 such that

$$G(x) = C(\langle \omega, x \rangle) + g(x'), \quad (15)$$

where $x' := x - \langle \omega, x \rangle \omega$ and $\lambda_G(x) = h(x) = C''(\langle \omega, x \rangle)$ for all $x \in \mathbb{R}^n$.

Remark 1.4. Paradigmatic examples satisfying the assumption (12) are the p -Laplacian operator, for any $p \in (1, +\infty)$, and the generalized mean curvature operator obtained by setting $a(t) := (1 + t^q)^{-\frac{1}{q}}$, with $q > 1$.

² cfr. also [9, footnote 1 at p. 742 and footnote 2 at page 743].

Theorem 2. Let $G(x) := -|x|^2/2$, $a(t) := t^{p-2}$ with $p > 1$ and let $u \in C^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ be a monotone weak solution to (1), i.e., such that

$$\partial_i u(x) > 0 \quad \forall x \in \mathbb{R}^n, \quad (16)$$

for some $i \in \{1, \dots, n\}$. Suppose that u satisfies either (a) or (b) in Theorem 1. Then u is one-dimensional. Moreover, if either $p = 2$ or $a(t) := (1 + t^q)^{-\frac{1}{q}}$ with $q > 1$, then the same conclusion holds for every monotone weak solution $u \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Theorem 3. Let u be a bounded weak solution to

$$\Delta u - \langle x, \nabla u \rangle + f(u) = 0 \quad (17)$$

with Morse index k . Then,

- (i) if $k \leq 2$ then u is one-dimensional;
- (ii) if $3 \leq k \leq n$ then u is a function of at most $k - 1$ variables, i.e. there exists $C \in \text{Mat}((k - 1) \times n)$ and $u_0 : \mathbb{R}^{k-1} \rightarrow \mathbb{R}$ such that

$$u(x) = u_0(Cx) \quad \forall x \in \mathbb{R}^n. \quad (18)$$

The result in Theorem 3 should be compared with the analysis in [14], where the author shows that a minimal surface in the Gauss space, with Morse index $\leq n$, is necessarily a hyperplane through the origin. These minimal surfaces are important geometric objects as they correspond to self-shrinkers for the mean curvature flow, which are the model of generic singularities. Since the minimal surface equation in the Gauss space arises as singular limit, as $\epsilon \rightarrow 0$, of the equations

$$\Delta u - \langle x, \nabla u \rangle - \frac{W'(u)}{\epsilon} = 0,$$

where W is a double-well potential (see for instance [23]), it is natural to ask if there exist bounded solutions to (17), with Morse index less than or equal to n , which are not one-dimensional.

2. A geometric Poincaré inequality

We recall the following result which has been proved in [9].

Lemma 2.1. For any $\xi \in \mathbb{R}^n \setminus \{0\}$, the matrix $A(\xi)$ is symmetric and positive definite and its eigenvalues are $\lambda_1(|\xi|), \dots, \lambda_n(|\xi|)$, where λ_1 is as in (6) and $\lambda_i(t) = a(t)$ for every $i = 2, \dots, n$. Moreover,

$$\langle A(\xi)\xi, \xi \rangle = |\xi|^2 \lambda_1(|\xi|), \quad (19)$$

and

$$0 \leq \langle A(\xi)(V - W), (V - W) \rangle = \langle A(\xi)V, V \rangle + \langle A(\xi)W, W \rangle - 2 \langle A(\xi)V, W \rangle, \quad (20)$$

for any $V, W \in \mathbb{R}^n$ and any $\xi \in \mathbb{R}^n \setminus \{0\}$.

Lemma 2.2. *Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H_{loc}^1(\mathbb{R}^n)$ be a weak solution to (1). Then for any $i = 1, \dots, n$, and any $\varphi \in C_c^1(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \langle A(\nabla u) \nabla u_i, \nabla \varphi \rangle - a(|\nabla u|) \langle \nabla u, \nabla(G_i) \rangle \varphi - f'(u) u_i \varphi \, d\mu = 0. \quad (21)$$

Proof. By Lemma 2.2 in [9] we have

$$\text{the map } x \rightarrow W(x) := a(|\nabla u(x)|) \nabla u(x) \text{ belongs to } W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n), \quad (22)$$

therefore, since $e^{G(x)} \in C^2(\mathbb{R}^n)$ we get

$$We^G \in W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n). \quad (23)$$

By Stampacchia's Theorem (see, e.g. [18, Theorem 6:19]), we get $\partial_i(We^G) = 0$ for almost any $x \in \{We^G = 0\} = \{W = 0\}$, that is

$$\partial_i(We^G) = 0$$

for almost any $x \in \{\nabla u = 0\}$. In the same way, by Stampacchia's Theorem and (A2), it can be proven that $\nabla u_i(x) = 0$, and hence $A(\nabla u(x)) \nabla u_i(x) = 0$, for almost any $x \in \{\nabla u = 0\}$. Moreover, the following relation holds (see [9] for the proof)

$$\partial_i(We^G) = (A(\nabla u) \nabla u_i + a(|\nabla u|) \nabla u G_i) e^G \quad \text{on } \{\nabla u \neq 0\}, \quad (24)$$

and thanks to the previous observations

$$\partial_i(We^G) = (A(\nabla u) \nabla u_i + a(|\nabla u|) \nabla u G_i) e^G \quad \text{a.e. in } \mathbb{R}^n. \quad (25)$$

Applying (8) with φ replaced by φ_i and making use of (23) and (25), we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} a(|\nabla u|) \langle \nabla u, \nabla \varphi_i \rangle - f(u) \varphi_i \, d\mu \\ &= \int_{\mathbb{R}^n} - \langle A(\nabla u) \nabla u_i, \nabla \varphi \rangle - a(|\nabla u|) \langle \nabla u, \nabla \varphi \rangle G_i \, d\mu \\ &\quad + \int_{\mathbb{R}^n} f'(u) u_i \varphi + f(u) \varphi G_i \, d\mu \\ &= \int_{\mathbb{R}^n} - \langle A(\nabla u) \nabla u_i, \nabla \varphi \rangle - a(|\nabla u|) \langle \nabla u, \nabla(\varphi G_i) \rangle \, d\mu \\ &\quad + \int_{\mathbb{R}^n} a(|\nabla u|) \langle \nabla u, \nabla G_i \rangle \varphi + f'(u) u_i \varphi + f(u) \varphi G_i \, d\mu. \end{aligned}$$

Recalling (8), applied with φ replaced by φG_i , we obtain the thesis. $\square$

From now on, we use A and a , as a short-hand notation for $A(\nabla u)$ and $a := a(|\nabla u|)$ respectively. In the following result we prove that every monotone solution to (1) is indeed h -stable.

Lemma 2.3. *Assume that u is a weak solution to (1) and that there exists $i \in \{1, \dots, n\}$ such that*

$$u_i := \partial_i u(x) > 0 \quad \forall x \in \mathbb{R}^n \quad (26)$$

then u is h -stable, with

$$h(x) := \frac{\langle \nabla u(x), \nabla G_i(x) \rangle}{u_i(x)}.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi := \varphi^2/u_i$. We use (20) with $V := \varphi \nabla u_i/u_i$ and $W := \nabla \varphi$ to obtain that

$$\frac{2\varphi}{u_i} \langle A \nabla u_i, \nabla \varphi \rangle - \frac{\varphi^2}{u_i^2} \langle A \nabla u_i, \nabla u_i \rangle \leq \langle A \nabla \varphi, \nabla \varphi \rangle.$$

From this and Lemma 2.2 we get

$$\begin{aligned} 0 &= \int \langle A \nabla u_i, \nabla \psi \rangle - a \langle \nabla u, \nabla G_i \rangle \psi - f'(u) u_i \psi \, d\mu \\ &= \int 2 \frac{\varphi}{u_i} \langle A \nabla u_i, \nabla \varphi \rangle - \frac{\varphi^2}{u_i^2} \langle A \nabla u_i, \nabla u_i \rangle - a \frac{\varphi^2}{u_i} \langle \nabla u, \nabla G_i \rangle - f'(u) \varphi^2 \, d\mu \\ &\leq \int \langle A \nabla \varphi, \nabla \varphi \rangle - a \frac{\varphi^2}{u_i} \langle \nabla u, \nabla G_i \rangle - f'(u) \varphi^2 \, d\mu. \end{aligned} \quad (27)$$

Notice that we can apply Lemma 2.2 since, in view of (26), u has no critical points and thus it is of class C^2 , by the classical regularity results. $\square$

The following Lemma can be proved using the same techniques implemented in [9, Lemma 2.4],

Lemma 2.4. *Let $h \in L_{loc}^1(\mathbb{R}^n)$. Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H_{loc}^1(\mathbb{R}^n)$ be a h -stable weak solution to (1). Then, (10) holds for any $\varphi \in H_0^1(B)$ and for any ball $B \subset \mathbb{R}^n$. Moreover, under the assumptions of Lemma 2.2,*

$$\int_{\mathbb{R}^n} \langle A(\nabla u) \nabla u_i, \nabla \varphi \rangle - a(|\nabla u|) \langle \nabla u, \nabla(G_i) \rangle \varphi - f'(u) u_i \varphi \, d\mu = 0, \quad (28)$$

for any $i = 1, \dots, n$, any $\varphi \in H_0^1(B)$ and any ball $B \subset \mathbb{R}^n$.

Proposition 2.5. *Let $h \in L_{loc}^1(\mathbb{R}^n)$ and $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H_{loc}^1(\mathbb{R}^n)$ be a h -stable weak solution to (1). Then, for every $\varphi \in C_c^1(\mathbb{R}^n)$ it holds*

$$\begin{aligned} &\int_{\mathbb{R}^n} a(|\nabla u|) h(x) |\nabla u|^2 \varphi^2 \, d\mu \\ &\leq \int_{\mathbb{R}^n} |\nabla u|^2 \langle A \nabla \varphi, \nabla \varphi \rangle + a(|\nabla u|) \langle \nabla^2 G \nabla u, \nabla u \rangle \varphi^2 \\ &\quad + \varphi^2 \left[\langle A \nabla |\nabla u|, \nabla |\nabla u| \rangle - \sum_{i=1}^n \langle A(\nabla u) \nabla u_i, \nabla u_i \rangle \right] d\mu. \end{aligned} \quad (29)$$

Proof. We start observing that by Stampacchia's Theorem, since $\mu \ll \mathcal{L}^n$, we get

$$\nabla |\nabla u|(x) = 0 \quad \mu - \text{a.e. } x \in \{|\nabla u| = 0\}, \quad (30)$$

$$\nabla u_j(x) = 0 \quad \mu - \text{a.e. } x \in \{|\nabla u| = 0\} \subseteq \{u_j = 0\}, \quad (31)$$

for any $j = 1, \dots, n$. Let $\varphi \in C_c^1(\mathbb{R}^n)$ and $i = 1, \dots, n$. Using (21) with test function $u_i \varphi^2$ and summing over $i = 1, \dots, n$ we get

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{i=1}^n \langle A(\nabla u) \nabla u_i, \nabla(u_i \varphi^2) \rangle - f'(u) |\nabla u|^2 \varphi^2 \, d\mu \\ = \int_{\mathbb{R}^n} a(|\nabla u|) \langle \nabla^2 G \nabla u, \nabla u \rangle \varphi^2 \, d\mu \end{aligned} \quad (32)$$

Using (10) with test function $|\nabla u| \varphi$ (note that this choice is possible thanks to Lemma 2.4) we then get

$$\begin{aligned} \int_{\mathbb{R}^n} a(|\nabla u|) h(x) |\nabla u|^2 \varphi^2 \, d\mu &\leq \int_{\mathbb{R}^n} \left\langle \left(A(\nabla u(x)) \nabla(|\nabla u| \varphi) \right), \nabla(|\nabla u| \varphi) \right\rangle \\ &\quad - f'(u) |\nabla u|^2 \varphi^2 \, d\mu \\ &= \int_{\mathbb{R}^n} |\nabla u|^2 \langle A \nabla \varphi, \nabla \varphi \rangle \, d\mu \\ &\quad + \int_{\{|\nabla u| \neq 0\}} \varphi^2 \langle A \nabla |\nabla u|, \nabla |\nabla u| \rangle \\ &\quad + 2\varphi |\nabla u| \langle A \nabla \varphi, \nabla |\nabla u| \rangle - f'(u) |\nabla u|^2 \varphi^2 \, d\mu \end{aligned} \quad (33)$$

and by (32) we conclude that

$$\begin{aligned} \int_{\mathbb{R}^n} a(|\nabla u|) h(x) |\nabla u|^2 \varphi^2 \, d\mu &\leq \int_{\mathbb{R}^n} |\nabla u|^2 \langle A \nabla \varphi, \nabla \varphi \rangle \, d\mu \\ &\quad + \int_{\{|\nabla u| \neq 0\}} a(|\nabla u|) \langle \nabla^2 G \nabla u, \nabla u \rangle \varphi^2 \, d\mu \\ &\quad + \int_{\{|\nabla u| \neq 0\}} \varphi^2 \left[\langle A \nabla |\nabla u|, \nabla |\nabla u| \rangle \right. \\ &\quad \left. - \sum_{i=1}^n \langle A(\nabla u) \nabla u_i, \nabla u_i \rangle \right] \, d\mu. \end{aligned} \quad (34)$$

which is the thesis. $\square$

Remark 2.6. Letting

$$L_{u,x} := \{y \in \mathbb{R}^n \mid u(y) = u(x)\},$$

we denote by $\nabla_T u$ the tangential gradient of u along $L_{u,x} \cap \{|\nabla u| \neq 0\}$, and by $k_1, \dots, k_{n-1}$ the principal curvatures of $L_{u,x} \cap \{|\nabla u| \neq 0\}$.

$$\begin{aligned} \langle A \nabla |\nabla u|, \nabla |\nabla u| \rangle - \sum_{i=1}^n \langle A(\nabla u) \nabla u_i, \nabla u_i \rangle &= a \left[|\nabla |\nabla u||^2 - \sum_{i=1}^n |\nabla u_i|^2 \right] \\ &\quad - a' |\nabla u| |\nabla_T |\nabla u||^2 \end{aligned} \quad (35)$$

and using (6) we get

$$\begin{aligned} & \langle A \nabla |\nabla u|, \nabla |\nabla u| \rangle - \sum_{i=1}^n \langle A(\nabla u) \nabla u_i, \nabla u_i \rangle \\ &= -\lambda_1 |\nabla_T |\nabla u||^2 - a(|\nabla u|) \left(\sum_{i=1}^n |\nabla u_i|^2 - |\nabla_T |\nabla u||^2 - |\nabla |\nabla u||^2 \right). \end{aligned} \quad (36)$$

Notice that the quantity

$$\sum_{i=1}^n |\nabla u_i|^2 - |\nabla |\nabla u||^2 - |\nabla_T |\nabla u||^2$$

has a geometric interpretation, in the sense that it can be expressed in terms of the principal curvatures of level sets of u .

More precisely, the following formula holds (see [9, 20, 21])

$$\sum_{i=1}^n |\nabla u_i|^2 - |\nabla |\nabla u||^2 - |\nabla_T |\nabla u||^2 = |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \quad \text{on } L_{u,x} \cap \{\nabla u \neq 0\}, \quad (37)$$

so that (34) becomes

$$\begin{aligned} & \int_{\{\nabla u \neq 0\}} a(|\nabla u|) h(x) |\nabla u|^2 \varphi^2 + \left[\lambda_1 |\nabla_T |\nabla u||^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \right] \varphi^2 \\ & - a(|\nabla u|) \langle \nabla^2 G \nabla u, \nabla u \rangle \varphi^2 \, d\mu \leq \int_{\mathbb{R}^n} \langle A \nabla \varphi, \nabla \varphi \rangle |\nabla u|^2 d\mu. \end{aligned}$$

Rearranging the terms, we obtain

$$\begin{aligned} & \int_{\{\nabla u \neq 0\}} a(|\nabla u|) \langle (h(x)I - \nabla^2 G) \nabla u, \nabla u \rangle \varphi^2 \\ & + \left[\lambda_1 |\nabla_T |\nabla u||^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \right] \varphi^2 \, d\mu \\ & \leq \int_{\mathbb{R}^n} \langle A \nabla \varphi, \nabla \varphi \rangle |\nabla u|^2 d\mu, \end{aligned} \quad (38)$$

where $I \in \text{Mat}(n \times n)$ denotes the identity matrix.

Notice that from (38) we also obtain

$$\int_{\{\nabla u \neq 0\}} a(|\nabla u|) \langle (h(x)I - \nabla^2 G) \nabla u, \nabla u \rangle \varphi^2 d\mu \leq \int_{\mathbb{R}^n} \langle A \nabla \varphi, \nabla \varphi \rangle |\nabla u|^2 d\mu \quad (39)$$

3. One-dimensional symmetry of solutions

In this section we will use (38) to prove several one-dimensional results for solutions to (1), following the approach introduced in [5] and then developed in [9]. Notice that, more recently, a similar approach has also been used to handle semilinear equations in Riemannian and subriemannian spaces (see

[6–8, 12, 13, 19]) and also to study problems involving the Ornstein-Uhlenbeck operator [2], as well as semilinear equations with unbounded drift [3].

The following Lemma is proved in [9, 13].

Lemma 3.1. *Let $g \in L_{loc}^\infty(\mathbb{R}^n, [0, +\infty))$ and let $q > 0$. Let also, for any $\tau > 0$,*

$$\eta(\tau) := \int_{B_\tau} g(x) dx. \quad (40)$$

Then, for any $0 < r < R$,

$$\int_{B_R \setminus B_r} \frac{g(x)}{|x|^q} dx \leq q \int_r^R \frac{\eta(\tau)}{|\tau|^{q+1}} d\tau + \frac{1}{R^q} \eta(R). \quad (41)$$

Proof of Theorem 1.

Let us fix $R > 0$ (to be taken appropriately large in what follows) and $x \in \mathbb{R}^n$ and let us define

$$\varphi(x) := \begin{cases} 1 & \text{if } x \in B_{\sqrt{R}} \\ 2 \frac{\log(R/|x|)}{\log(R)} & \text{if } x \in B_R \setminus B_{\sqrt{R}} \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_R, \end{cases} \quad (42)$$

where $B_R := \{y \in \mathbb{R}^n \mid |y| < R\}$. Obviously $\varphi \in Lip(\mathbb{R}^n)$ and

$$|\nabla \varphi(x)| \leq C_2 \frac{\chi_{\sqrt{R}, R}(x)}{\log(R)|x|}$$

for suitable $C_2 > 0$. Hence for every $R > e$, (38) together with $h \geq \lambda_G$ yields

$$\begin{aligned} & \int_{\{\nabla u \neq 0\} \cap \overline{B_R}} \left[\lambda_1 |\nabla_T |\nabla u||^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \right] \varphi^2 d\mu \\ & \leq \int_{\mathbb{R}^n} \langle A(\nabla u) \nabla \varphi, \nabla \varphi \rangle |\nabla u|^2 d\mu \end{aligned} \quad (43)$$

therefore, by (12)

$$\begin{aligned} & \int_{\{\nabla u \neq 0\} \cap \overline{B_R}} \left[\lambda_1 |\nabla_T |\nabla u||^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \right] \varphi^2 d\mu \\ & \leq (1 + C) \int_{\mathbb{R}^n} a(|\nabla u|) |\nabla \varphi|^2 |\nabla u|^2 d\mu \\ & \leq \frac{(1 + C)C_2^2}{\log(R)^2} \int_{B_R \setminus B_{\sqrt{R}}} \frac{a(|\nabla u|) |\nabla u|^2}{|x|^2} d\mu. \end{aligned} \quad (44)$$

Applying Lemma 3.1 with $g = a(|\nabla u|) |\nabla u|^2 e^G$ and $q = 2$, and recalling that

$$\int_{B_R} a(|\nabla u|) |\nabla u|^2 d\mu \leq C_0 R^2$$

for R large, we obtain

$$\begin{aligned} & \int_{\{\nabla u \neq 0\} \cap \overline{B_R}} \left[\lambda_1 |\nabla_T |\nabla u||^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \right] \varphi^2 \, d\mu \\ & \leq \frac{(1+C)C_0 C_2^2}{\log(R)^2} \left[2 \int_{\sqrt{R}}^R \frac{1}{|\tau|} d\tau + 1 \right] \leq 2 \frac{(1+C)C_0 C_2^2}{\log(R)}. \end{aligned} \quad (45)$$

Therefore, sending $R \rightarrow +\infty$ in (45) we get

$$k_j(x) = 0 \quad \text{and} \quad |\nabla_T |\nabla u|| (x) = 0 \quad (46)$$

for every $j = 1, \dots, n-1$ and every $x \in \{\nabla u \neq 0\}$. From this and Lemma 2.11 in [9] we get the one-dimensional symmetry of u .

Let us now suppose $n = 2$ and $a(|\nabla u|) |\nabla u|^2 e^G \in L^\infty(\mathbb{R}^2)$. Taking in (38) the following test function

$$\phi(x) = \max \left[0, \min \left(1, \frac{\ln R^2 - \ln |x|}{\ln R} \right) \right], \quad (47)$$

recalling that $h \geq \lambda_G$ and following [9, Cor. 2.6], we then obtain

$$\begin{aligned} & \int_{\{\nabla u \neq 0\} \cap \overline{B_R}} \left[\lambda_1 |\nabla_T |\nabla u||^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \right] \varphi^2 \, d\mu \\ & \leq C' \int_{B_{R^2} \setminus B_R} \frac{a(|\nabla u|(x))}{|x|^2 (\ln R)^2} |\nabla u|^2 e^{G(x)} \, dx \end{aligned}$$

for some constant $C' > 0$. When $R \rightarrow +\infty$, since $a(|\nabla u|) |\nabla u|^2 e^{G(x)}$ is bounded, the r.h.s. term of the previous inequality goes to zero, and we conclude again that u is one-dimensional.

Assume now that u is not constant. If we take in (39) the same test functions as above, we get

$$\int_{\mathbb{R}^n} a(|\nabla u|) \langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle \, d\mu(x) = 0.$$

Using the fact that $u(x) = u_0(\langle \omega, x \rangle)$ and $a(t) > 0$ we obtain that $\langle (h(x)I_n - \nabla^2 G(x)) \omega, \omega \rangle = 0$ for all x such that $u'_0(\langle \omega, x \rangle) \neq 0$. Since u is not constant and is a solution to the elliptic Eq. (1), the set of points such that $u'_0(\langle \omega, x \rangle) = 0$ has zero measure, so, by the regularity of G we conclude that

$$\langle (h(x)I_n - \nabla^2 G(x)) \omega, \omega \rangle = 0 \quad \forall x \in \mathbb{R}^n,$$

which gives (14) and (15).

As pointed out in [3], a Liouville type result follows from Theorem 1.

Corollary 3.2. *Let G, h, u satisfy the assumptions in Theorem 1. Assume further that $h \in C^0(\mathbb{R}^n)$ and $h(x) > \lambda_G(x)$ for some $x \in \mathbb{R}^n$. Then u is constant. In particular, if u is a stable solution, that is $h \equiv 0$, and $\lambda_G(x) < 0$ for some $x \in \mathbb{R}^n$, then u is constant.*

In the following lemma we give a sufficient condition for a solution u to satisfy condition (a) in Theorem 1.

Lemma 3.3. *Let u be a weak solution to (1). Then, for each $\varphi \in C_c^1(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} a(|\nabla u|)|\nabla u|^2 \varphi d\mu = - \int_{\mathbb{R}^n} a(|\nabla u|) \langle \nabla u, \nabla \varphi \rangle u d\mu + \int_{\mathbb{R}^n} f(u) u \varphi d\mu. \quad (48)$$

In particular, if $t \rightarrow ta(t) \in L^\infty((0, +\infty))$, $u \in L^\infty(\mathbb{R}^n)$ and $\mu(\mathbb{R}^n) < +\infty$ then there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} a(|\nabla u|)|\nabla u|^2 d\mu \leq C. \quad (49)$$

Proof. Clearly (48) follows by taking $u\varphi$ as test function in (8). Let us show (49). For every $R > 1$ let $\Phi_R \in C^\infty(\mathbb{R})$ be such that $\Phi_R(t) = 1$ if $t \leq R$, $\Phi_R(t) = 0$ if $t \geq R+1$ and $\Phi'_R(t) \leq 3$ for $t \in [R, R+1]$, and define $\varphi(x) := \Phi_R(|x|)$. Then $|\nabla \varphi(x)| \leq |\Phi'_R(|x|)| \leq 3$, and (48) yields

$$\int_{B_R} a(|\nabla u|)|\nabla u|^2 d\mu \leq 3 \int_{B_{R+1} \setminus B_R} a(|\nabla u|)|\nabla u||u| d\mu + \int_{B_{R+1}} |f(u)||u| d\mu \leq C,$$

which gives (49) by letting $R \rightarrow +\infty$. $\square$

In the rest of the section we fix $G(x) = -|x|^2/2$. We start with a result which follows directly from Lemma 2.3.

Lemma 3.4. *Let $G(x) := -|x|^2/2$ and assume that u is a monotone weak solution to (1), i.e. there exists $i \in \{1, \dots, n\}$ such that*

$$\partial_i u(x) > 0 \quad \forall x \in \mathbb{R}^n, \quad (50)$$

then $u \in C^2(\mathbb{R}^n)$ and u is (-1) -stable.

Proof of Theorem 2. We start observing that u is (-1) -stable by Lemma 2.3. Since $\nabla^2 G(x) = -Id$ we have

$$-1 = h(x) = \lambda_G(x) = -1. \quad (51)$$

If $a(t) = t^{p-2}$ for some $p > 1$ then

$$\lambda_1(t) = (p-1)t^{p-2} = (p-1)a(t) \quad \forall t > 0 \quad (52)$$

and the conclusion follows by Theorem 1. If $a(t) = (1+t^q)^{-\frac{1}{q}}$ with $q > 1$ then

$$\lambda_1(t) = (1+t^q)^{-\frac{1}{q}} - (1+t^q)^{-\frac{q+1}{q}} t^q \leq a(t) \quad \forall t > 0, \quad (53)$$

$$ta(t) \leq 1 \quad \forall t > 0. \quad (54)$$

By Lemma 3.3 and (54) there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} a(|\nabla u|)|\nabla u|^2 d\mu \leq C. \quad (55)$$

Notice that, if $a(t) = 1$ for every $t > 0$, by Theorem [17, Theorem 4.1] we have $u \in H^2(\mathbb{R}^n, \mu)$, so that (55) holds in this case, too. The conclusion follows by (53), (55) and Theorem 1. $\square$

4. Solutions with morse index bounded by the Euclidean dimension

In this section we will focus on the Ornstein-Uhlenbeck operator. More precisely, we will consider weak solutions $u \in H^1(\mathbb{R}^n, \mu) \cap L^\infty(\mathbb{R}^n)$ to

$$\Delta u - \langle x, \nabla u \rangle + f(u) = 0 \quad (56)$$

where $f \in C^1(\mathbb{R})$, and we will prove some new symmetry results for solutions with Morse index $k \leq n$. We recall that, by Theorem [17, Theorem 4.1], bounded weak solutions to (56) satisfy $u \in H^2(\mathbb{R}^n, \mu) \cap L^\infty(\mathbb{R}^n)$.

Definition 4.1. A bounded weak solution u to the Ornstein-Uhlenbeck operator has Morse index $k \in \mathbb{N}$ if k is the maximal dimension of a subspace X of $H^1(\mathbb{R}^n, \mu)$ such that

$$Q_u(\varphi) := \int_{\mathbb{R}^n} |\nabla \varphi|^2 - f'(u)\varphi^2 d\mu < 0 \quad \forall \varphi \in X \setminus \{0\}. \quad (57)$$

Remark 4.2. Let u be a bounded solution to (56) and let $L : H^2(\mathbb{R}^n, \mu) \rightarrow L^2(\mathbb{R}^n, \mu)$ be the linear operator defined as

$$L(v) := -\Delta v + \langle \nabla v, x \rangle - f'(u)v. \quad (58)$$

Notice that L is self-adjoint in $L^2(\mathbb{R}^n, \mu)$ with compact inverse, so that by the Spectral Theorem [15] there exists an orthonormal basis of $L^2(\mathbb{R}^n, \mu)$ consisting of eigenvectors of L , and each eigenvalue of L is real. Then, u has Morse index k if and only if L has exactly k strictly negative eigenvalues, repeated according to their geometric multiplicity (see for instance [17, Theorem 4.1]).

The following Proposition is proved in [2, Lemma 3.2].

Proposition 4.3. *Let u be a bounded weak solution to (56). If for some $i = 1, \dots, n$, u_i is not identically zero then it is an eigenfunction of L with eigenvalue -1 , i.e.*

$$\int_{\mathbb{R}^n} \langle \nabla u_i, \nabla \varphi \rangle + u_i \varphi - f'(u)u_i \varphi d\mu = 0, \quad \forall \varphi \in H^1(\mathbb{R}^n, \mu). \quad (59)$$

We are now in a position to prove Theorem 3.

Proof of Theorem 3. By [17, Theorem 4.1] every bounded weak solution to (56) belongs to $H^2(\mathbb{R}^n, \mu)$, hence $u_i \in H^1(\mathbb{R}^n, \mu)$ for all $i = 1, \dots, n$. Therefore, using (59) with u_i as test function we obtain

$$Q_u(u_i) = \int_{\mathbb{R}^n} |\nabla u_i|^2 - f'(u)u_i^2 d\mu = - \int_{\mathbb{R}^n} u_i^2 d\mu \leq 0, \quad \forall i = 1, \dots, n. \quad (60)$$

In particular

$$Q_u(u_i) < 0 \quad (61)$$

for every $i = 1, \dots, n$ such that u_i is not identically zero. Let L be the operator defined in (58). If $k = 0$ then u is stable, hence it is constant by Corollary 3.2.

If $k = 1$ then, by Remark 4.2 and Proposition 4.3, it follows that -1 is the smallest eigenvalue of L , that is

$$\inf_{\varphi \in H^1(\mathbb{R}^n, \mu), \|\varphi\|_{L^2(\mathbb{R}^n, \mu)}=1} \left(\int_{\mathbb{R}^n} |\nabla \varphi|^2 - f'(u) \varphi^2 \, d\mu \right) = -1. \quad (62)$$

Using (62) it follows that u is (-1) -stable and therefore, by Theorem 1, u is one-dimensional. Assume now $2 \leq k \leq n$ and define $S := \{i \in \{1, \dots, n\} \mid u_i(x) \neq 0, \text{ for some } x \in \mathbb{R}^n\}$ and $X := \text{span}_{i \in S} \{u_i\} \subset H^1(\mathbb{R}^n, \mu)$. Clearly,

$$Q_u(v) < 0 \quad \forall v \in X \setminus \{0\} \quad (63)$$

therefore, by Definition 4.1, X has dimension less or equal than k , i.e. there exists $I \subset S$ with $|I| \geq |S| - k$ such that $\{u_i\}_{i \in I}$ are linearly dependent [15]. This means that, up to an orthogonal change of variables, u depends on at most k variables. Let us assume by contradiction that u is a function of exactly k variables. We claim that -1 is the smallest eigenvalue of L , as before. Indeed, if this is not the case, then there exist $\lambda < -1$ and $v \in H^1(\mathbb{R}^n, \mu)$, with $v \not\equiv 0$, such that $L(v) = \lambda v$. Therefore, by the linear independence of eigenvectors associated to different eigenvalues, it follows that $Y := \text{span}\{u_i, v\}$ has dimension equal to $k + 1$ and $Q_u(w) < 0$ for every $w \in Y \setminus \{0\}$ which is in contradiction with the fact that u has Morse index k . This proves that u is a function of at most $(k - 1)$ variables, as claimed. $\square$

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