Global attractor for a semilinear strongly degenerate parabolic equation on $\mathbb{R}^N$

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Abstract. The aim of this paper is to prove the existence of the global attractor for a semilinear strongly degenerate parabolic equation on $\mathbb{R}^N$ with the locally Lipschitz nonlinearity satisfying a subcritical growth condition.

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1. Introduction

The understanding of asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for a dissipative dynamical system is to consider its global attractor. The existence of global attractors has been proved for a large class of nondegenerate partial differential equations (see e.g. [5, 20, 23] and references therein). In the last few years, a number of papers are devoted to the study of long-time behavior of solutions to degenerate parabolic equations.

One of the classes of degenerate equations that has been studied widely in recent years is the class of equations involving an operator of Grushin type

$$G_\alpha u = \Delta_x u + |x|^{2\alpha} \Delta_y u, \quad \alpha \geq 0.$$

This operator was studied by Grushin in [9] when α is an integer, and by Franchi & Lanconelli in [6–8] when α is not an integer. Note that $G_0 = \Delta$ is the Laplacian operator, and G_α , when $\alpha > 0$, is not elliptic in domains intersecting the surface $x = 0$. The long-time behavior of solutions to semilinear parabolic equations involving this operator has been studied recently in [1, 2]. We also refer the reader to some recent results on the generalized Grushin operators [10, 13, 15–17]. Recently, Thuy and Tri [21] considered a strongly degenerate operator

$$P_{\alpha,\beta}u = \Delta_x u + \Delta_y u + |x|^{2\alpha}|y|^{2\beta}\Delta_z u, \quad \alpha, \beta \geq 0,$$

which is degenerate on two intersecting surfaces $x = 0$ and $y = 0$, and established some compact embedding theorems for weighted Sobolev spaces associated to this operator in bounded domains. This operator falls into the class of Δ_λ operators [12], or more general, the class of X -elliptic operators [14].

In this paper we study the existence and long-time behavior of solutions to the following semilinear strongly degenerate parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - P_{\alpha,\beta}u + \lambda u + f(X, u) = g(X), & X \in \mathbb{R}^N, t > 0, \\ u(X, 0) = u_0(X), & X \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $X = (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3} = \mathbb{R}^N$, $\lambda > 0$, $u_0 \in \mathcal{S}^1(\mathbb{R}^N)$ (see the definition of this function space below) are given, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying

$$f(X, 0) = 0, \quad (1.2)$$

$$|f'_u(X, u)| \leq C \left(a(X) + |u|^\rho \right), \quad 0 \leq \rho < \frac{4}{N_{\alpha,\beta} - 2}, \quad (1.3)$$

$$f'_u(X, u) \geq -\ell \quad (1.4)$$

$$f(X, u)u \geq -\mu u^2 - C_1(X), \quad (1.5)$$

$$F(X, u) \geq -\frac{\mu}{2}u^2 - C_2(X), \quad (1.6)$$

where $N_{\alpha,\beta} = N_1 + N_2 + (\alpha + \beta + 1)N_3$, C, ℓ are positive constants, $a \in L^{N_{\alpha,\beta}}(\mathbb{R}^N) \cap L^{\frac{N_{\alpha,\beta}}{2}}(\mathbb{R}^N)$ is a nonnegative function, $F(X, s) = \int_0^s f(X, \tau) d\tau$, $\mu < \lambda$, $C_1, C_2 \in L^1(\mathbb{R}^N)$ are two nonnegative functions;

(G) $g \in L^2(\mathbb{R}^N)$.

Under similar above conditions, following the approach used in [1], Thuy and Tri [22] recently proved the existence and uniqueness of a global mild solution to problem (1.1) in a bounded domain Ω with the homogeneous Dirichlet boundary condition, and they also proved the existence of a compact global attractor for the semigroup generated by this problem (see also [13] for a more general situation). The aim of the present paper is to extend these results to the case of unbounded domains, the more complicated case due to the lack of compactness of the embedding theorems. For other results related to problem (1.1) with the nonlinearity of polynomial type, we refer the reader to some very recent works [3, 4].

In order to study problem (1.1), we use some weighted Sobolev spaces. We use the space $\mathcal{S}^1(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{\mathcal{S}^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \left(|u|^2 + |\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2 \right) dX.$$

Then $\mathcal{S}^1(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$(u, v)_{\mathcal{S}^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \left(uv + \nabla_x u \nabla_x v + \nabla_y u \nabla_y v + |x|^{2\alpha} |y|^{2\beta} \nabla_z u \nabla_z v \right) dX.$$

We also use the space $\mathcal{S}^2(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{\mathcal{S}^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \left(|u|^2 + |P_{\alpha, \beta} u|^2 \right) dX.$$

In Lemma 2.1 below, we will establish some embedding results related to these function spaces.

Denote $A = -P_{\alpha, \beta} + \lambda u$, the positive and self-adjoint unbounded linear operator with domain of the definition

$$D(A) = \{u \in \mathcal{S}^1(\mathbb{R}^N) : Au \in L^2(\mathbb{R}^N)\} = \mathcal{S}^2(\mathbb{R}^N),$$

and define the corresponding Nemytski map f by

$$\hat{f}(u)(X) = f(X, u(X)), \quad u \in \mathcal{S}^1(\mathbb{R}^N).$$

Then, (1.1) can be formulated as an abstract evolutionary equation

$$\begin{cases} \frac{du}{dt} + Au + \hat{f}(u) = g, \\ u(0) = u_0. \end{cases}$$

The aim of this paper is to study the existence of a global mild solution and of a global attractor for the semigroup generated by problem (1.1).

One basic feature of (1.1) is that it admits the following (strict) natural Lyapunov function

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2 \right) dX + \lambda \int_{\mathbb{R}^N} u^2 dX \\ &\quad + \int_{\mathbb{R}^N} (F(X, u) - gu) dX. \end{aligned} \quad (1.7)$$

One can check that

$$\frac{d}{dt} \Phi(u(t)) = -\|u_t\|_{L^2(\mathbb{R}^N)}^2$$

for any solution u of problem (1.1). This Lyapunov function plays an essential role in proving the global existence of solutions and it implies the structure of the global attractor obtained later.

The structure of the paper is as follows. In Sect. 2, we prove the existence of a global mild solution to problem (1.1) by using the existence theorem for abstract parabolic equations and the Lyapunov function Φ . Then we can construct the semigroup $S(t)$ generated by problem (1.1). In Sect. 3, we prove the existence of a compact global attractor for $S(t)$ by showing the existence of a bounded absorbing set and the asymptotic compactness of the semigroup $S(t)$. The existence of bounded absorbing sets can be quite easily deduced by using a priori estimates of solutions, while the proof of the asymptotic compactness is much more involved. To prove the asymptotic compactness

of $S(t)$ in $L^2(\mathbb{R}^N)$ we exploit the tail-estimates method introduced by Wang [24] to overcome the difficulty arising due to the lack of compactness of the embedding theorems, while the asymptotic compactness of $S(t)$ in $S^1(\mathbb{R}^N)$ is proved by combining the asymptotic compactness in $L^2(\mathbb{R}^N)$, the singular Gronwall inequality and the arguments introduced by Prizzi and Rybokowski in [18, 19]. It is noticed that the results obtained in the paper are also true for problem (1.1) in an arbitrary (bounded or unbounded) domain Ω in $\mathbb{R}^N$, not necessary the whole space $\mathbb{R}^N$, with the homogeneous Dirichlet boundary condition. Then, instead of $\mathcal{S}^1(\mathbb{R}^N)$ and $\mathcal{S}^2(\mathbb{R}^N)$, we use the spaces $\mathcal{S}_0^1(\Omega)$ and $\mathcal{S}_0^2(\Omega)$, defined as the completions of $C_0^\infty(\Omega)$ in the corresponding norms.

2. Existence and uniqueness of a global mild solution

First, we prove the following embedding results.

Lemma 2.1. *The following embeddings are continuous:*

- (i) $\mathcal{S}^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for all $2 \leq p \leq 2_{\alpha, \beta}^* := \frac{2N_{\alpha, \beta}}{N_{\alpha, \beta} - 2}$, where $N_{\alpha, \beta} = N_1 + N_2 + (\alpha + \beta + 1)N_3$;
- (ii) $\mathcal{S}^2(\mathbb{R}^N) \hookrightarrow \mathcal{S}^1(\mathbb{R}^N)$.

Proof. (i) We follow the ideas in the case of bounded domains (see the proofs of Theorem 3.3 and Proposition 3.2 in [12]). More precisely, we first embed $\mathcal{S}^1(\mathbb{R}^N)$ into an anisotropic Sobolev-type space, and then use an embedding theorem for classical anisotropic Sobolev-type spaces of fractional orders. Because the proof is very similar to the case of bounded domains [12], so we omit it here.

(ii) For all $u \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} \|u\|_{\mathcal{S}^1(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} (|u|^2 + |\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX \\ &= \int_{\mathbb{R}^N} u(u - P_{\alpha, \beta} u) dX \leq C \left(\int_{\mathbb{R}^N} |u|^2 dX \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}^N} (|u|^2 + |P_{\alpha, \beta} u|^2) dX \right)^{1/2} \\ &= C \|u\|_{L^2(\mathbb{R}^N)} \|u\|_{\mathcal{S}^2(\mathbb{R}^N)}. \end{aligned}$$

Noting that $\|u\|_{L^2(\mathbb{R}^N)} \leq \|u\|_{\mathcal{S}^1(\mathbb{R}^N)}$, we get $\|u\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq C \|u\|_{\mathcal{S}^2(\mathbb{R}^N)}$, that is, the embedding $\mathcal{S}^2(\mathbb{R}^N) \hookrightarrow \mathcal{S}^1(\mathbb{R}^N)$ is continuous. $\square$

Next, we discuss on the operator $A = -P_{\alpha, \beta} + \lambda u$. First, one can check that A is a positive definite sectorial operator on the Hilbert space $X = L^2(\mathbb{R}^N)$. Thus, we can define fractional powers of A on $X = L^2(\mathbb{R}^N)$ as follows (see [11, Chapter 1]):

For $\theta > 0$, we define, as usual, the operator $A^{-\theta} : X \rightarrow X$ as

$$A^{-\theta} u = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-At} u dt, \quad u \in X.$$

Then $A^{-\theta}$ is injective and we define X^θ to be the range of $A^{-\theta}$, and $A^\theta : X^\theta \rightarrow X$ to be the inverse of $A^{-\theta}$. We also set $X^0 = X$ and $A^0 = Id_X$.

We know that $X^\theta = D(A^\theta)$ is a separable Hilbert space endowed with the inner scalar product

$$(u, v)_{X^\theta} = (A^\theta u, A^\theta v), \quad \|u\|_{X^\theta} = \|A^\theta u\|_X.$$

For any $\theta, \eta > 0$ and $\theta \geq \eta$, X^θ is continuously embedded into X^η .

From the above definition, one can see that

$$\begin{aligned} X^1 &= \{u \in \mathcal{S}^1(\mathbb{R}^N) : Au \in L^2(\mathbb{R}^N)\} = \mathcal{S}^2(\mathbb{R}^N), \\ X^{1/2} &= \mathcal{S}^1(\mathbb{R}^N). \end{aligned}$$

We have the following basic estimate (see e.g. [11, Theorem 1.4.3, p. 26]):

$$\|A^\theta e^{-At}\| \leq C_\theta t^{-\theta} e^{-\delta t} \quad \text{for all } t > 0, \quad (2.1)$$

Now we deal with the nonlinear term $f(u)$.

Lemma 2.2. *Suppose (F) - (G) hold. Then, there exists $0 \leq \gamma < \frac{1}{2}$ such that the map $\hat{f} : X^{1/2} = \mathcal{S}^1(\mathbb{R}^N) \rightarrow X^{-\gamma}$ is Lipschitzian continuous on every bounded subset of $\mathcal{S}^1(\mathbb{R}^N)$.*

Proof. We consider two cases:

Case 1: $0 < \rho \leq \frac{2}{N_{\alpha, \beta} - 2}$. In this case it is easy to check that the map $\hat{f} : \mathcal{S}^1(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is Lipschitzian continuous on every bounded subset of $\mathcal{S}^1(\mathbb{R}^N)$ by using Hölder inequality and the embedding $\mathcal{S}^1(\mathbb{R}^N) \hookrightarrow L^{2(\rho+1)}(\mathbb{R}^N)$.

Case 2: $\frac{2}{N_{\alpha, \beta} - 2} < \rho < \frac{4}{N_{\alpha, \beta} - 2}$. We will show that $\hat{f} : \mathcal{S}^1(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N)$, where $q := \frac{2\alpha_\beta}{\rho+1}$, is Lipschitzian continuous on every bounded subset of $\mathcal{S}^1(\mathbb{R}^N)$.

By (1.3) we have $|f(X, u)| \leq C(|a||u| + |u|^{\rho+1})$. Hence for $u \in \mathcal{S}^1(\mathbb{R}^N)$, by Hölder's inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |f(X, u)|^q dX &\leq C \int_{\mathbb{R}^N} (|a|^q |u|^q + |u|^{q(\rho+1)}) dX \\ &\leq C \left(\|a\|_{L^{2^*_{\alpha, \beta/\rho}}(\mathbb{R}^N)}^q \|u\|_{L^{2^*_{\alpha, \beta}}(\mathbb{R}^N)}^q + \|u\|_{L^{2^*_{\alpha, \beta}}(\mathbb{R}^N)}^{2^*_{\alpha, \beta}} \right) < +\infty \end{aligned}$$

since $\mathcal{S}^1(\mathbb{R}^N)$ is continuously embedded into $L^{2^*_{\alpha, \beta}}(\mathbb{R}^N)$ and the assumption of a . This shows that $\hat{f}$ is a map from $\mathcal{S}^1(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$.

Let $u, v \in \mathcal{S}^1(\mathbb{R}^N)$ and $\|u\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq R$, $\|v\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq R$, we have from (1.3)

$$\begin{aligned} \int_{\mathbb{R}^N} |f(X, u) - f(X, v)|^q dX &\leq C \int_{\mathbb{R}^N} |u - v|^q (|a|^q + |u|^{q\rho} + |v|^{q\rho}) dX \\ &\leq C \int_{\mathbb{R}^N} |a|^q |u - v|^q dX + C \int_{\mathbb{R}^N} |u|^{q\rho} |u - v|^q dX + C \int_{\mathbb{R}^N} |v|^{q\rho} |u - v|^q dX. \end{aligned}$$

Applying Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |a|^q |u - v|^q dX &\leq \|a\|_{L^{2^*_{\alpha,\beta}/\rho}(\mathbb{R}^N)}^{q\rho} \|u - v\|_{L^{2^*_{\alpha,\beta}}(\mathbb{R}^N)}^q, \\ \int_{\mathbb{R}^N} |u|^{q\rho} |u - v|^q dX &\leq \|u\|_{L^{2^*_{\alpha,\beta}}(\mathbb{R}^N)}^{q\rho} \|u - v\|_{L^{2^*_{\alpha,\beta}}(\mathbb{R}^N)}^q, \\ \int_{\mathbb{R}^N} |v|^{q\rho} |u - v|^q dX &\leq \|v\|_{L^{2^*_{\alpha,\beta}}(\mathbb{R}^N)}^{q\rho} \|u - v\|_{L^{2^*_{\alpha,\beta}}(\mathbb{R}^N)}^q.\end{aligned}$$

Since $\mathcal{S}^1(\mathbb{R}^N)$ is continuously embedded into $L^{2^*_{\alpha,\beta}}(\mathbb{R}^N)$ and $1 < q < 2^*_{\alpha,\beta}$, there exists a positive number $M(R)$ such that

$$\|\hat{f}(u) - \hat{f}(v)\|_{L^q(\mathbb{R}^N)} \leq M(R) \|u - v\|_{\mathcal{S}^1(\mathbb{R}^N)}.$$

Since $\frac{2}{N_{\alpha,\beta}-2} < \rho < \frac{4}{N_{\alpha,\beta}-2}$, we have $1 < q < 2$. Thus,

$$p = \frac{q}{q-1} = \frac{2^*_{\alpha,\beta}}{2^*_{\alpha,\beta}-1} \in (2, 2^*_{\alpha,\beta}).$$

Repeating the arguments used in the proof of Lemma 3.1 in [1], just replacing $N(k)$ by $N_{\alpha,\beta}$, one can show that there exists $\gamma \in (0, 1/2)$ such that X^γ is continuously embedded in $L^p(\mathbb{R}^N)$. Hence $L^q(\mathbb{R}^N) = (L^p(\mathbb{R}^N))'$ is continuously embedded in $X^{-\gamma}$.

Since both $L^2(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ are continuously embedded in $X^{-\gamma}$, we get the desired result. $\square$

We are now ready to prove the main result in this section.

Theorem 2.1. *Suppose (F) – (G) hold. Then for any $u_0 \in \mathcal{S}^1(\mathbb{R}^N)$ given, problem (1.1) has a unique global mild solution $u \in C([0, \infty); \mathcal{S}^1(\mathbb{R}^N))$.*

Proof. Because $\hat{f} : X^{1/2} \rightarrow X^{-\gamma}$ is Lipschitzian continuous on every bounded subset of $X^{1/2}$, by Theorem 3.3.3 in [11, p. 54], we get the local existence of a mild solution u .

Suppose that the solution u is defined on the maximal interval $[0, t_{\max})$. We now show that $t_{\max} = +\infty$ by using the Lyapunov function (1.7) and the dissipativeness condition (1.6). Using (1.6) and the Cauchy inequality we get

$$\begin{aligned}\Phi(u(t)) &\geq \frac{1}{2} \int_{\mathbb{R}^N} \left(|u(t)|^2 + |\nabla_x u(t)|^2 + |\nabla_y u(t)|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u(t)|^2 \right) dX \\ &\quad + \frac{\lambda}{2} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{2} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 - \|C_2\|_{L^1(\mathbb{R}^N)} - \varepsilon \|u(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{4\varepsilon} \|g\|_{L^2(\mathbb{R}^N)}^2.\end{aligned}$$

Choosing ε small enough such that $\mu + 2\varepsilon < \lambda$ we obtain

$$\begin{aligned}\Phi(u(0)) &\geq \Phi(u(t)) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} \left(|u(t)|^2 + |\nabla_x u(t)|^2 + |\nabla_y u(t)|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u(t)|^2 \right) dX \\ &\quad + \frac{\lambda - \mu - 2\varepsilon}{2} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{4\varepsilon} \|g\|_{L^2(\mathbb{R}^N)}^2 - \|C_2\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$

Hence

$$\|u(t)\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq M \quad \forall t \in [0, t_{\max}).$$

This implies that $t_{\max} = +\infty$. Indeed, let $t_{\max} < \infty$ and $\limsup_{t \rightarrow t_{\max}^-} \|u(t)\|_{\mathcal{S}^1(\mathbb{R}^N)} < +\infty$. Then there exist a sequence $(t_n)_{n \geq 1}$ and a constant K such that $t_n \rightarrow t_{\max}^-$ as $n \rightarrow +\infty$ and $\|u(t_n)\|_{\mathcal{S}^1(\mathbb{R}^N)} < K, n = 1, 2, \dots$. As we have already shown above, for each $n \in \mathbb{N}$ there exists a unique mild solution of problem (1.1) with the initial datum $u(t_n)$ on $[t_n, t_n + T^*]$, where $T^* > 0$ depending on K and independent of $n \in \mathbb{N}$. Thus, we can get $t_{\max} < t_n + T^*$, for $n \in \mathbb{N}$ large enough. This contradicts the maximality of $t_{\max}$. $\square$

3. Existence of a global attractor in $\mathcal{S}^1(\mathbb{R}^N)$

By Theorem 2.1, we can define a continuous semigroup $S(t) : \mathcal{S}^1(\mathbb{R}^N) \rightarrow \mathcal{S}^1(\mathbb{R}^N)$ as follows

$$S(t)u_0 := u(t),$$

where $u(t)$ is the unique global mild solution of (1.1) subject to u_0 as initial datum.

3.1. Existence of bounded absorbing sets

For the sake of brevity, in the following lemmas, we give some formal calculations, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [20].

We first prove the existence of a bounded absorbing set for $S(t)$ in $\mathcal{S}^1(\mathbb{R}^N)$.

Lemma 3.1. *Suppose (F) – (G) hold. Then the semigroup $S(t)$ generated by (1.1) has a bounded absorbing set in $\mathcal{S}^1(\mathbb{R}^N)$, that is, there exists a positive constant ρ , such that for every bounded subset B in $\mathcal{S}^1(\mathbb{R}^N)$, there is a number $T = T(B) > 0$, such that for all $t \geq T$, $u_0 \in B$, we have*

$$\|u(t)\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq \rho.$$

Proof. Taking the inner product of (1.1) with u in $L^2(\mathbb{R}^N)$ we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX \\ + \lambda \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} f(X, u) u dX = (g, u)_{L^2(\mathbb{R})}. \end{aligned} \quad (3.1)$$

Using (1.5), we have

$$\int_{\mathbb{R}^N} f(X, u) u dX \geq -\mu \int_{\mathbb{R}^N} |u|^2 dX - \int_{\mathbb{R}^N} C_1(X) dX. \quad (3.2)$$

By the Cauchy inequality, the right-hand side of (3.1) is estimated as follows

$$|(g, u)_{L^2(\mathbb{R})}| \leq \|g\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R}^N)} \leq \frac{\lambda - \mu}{2} \|u\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2(\lambda - \mu)} \|g\|_{L^2(\mathbb{R})}^2. \quad (3.3)$$

It follows from (3.1)–(3.3) that

$$\begin{aligned} \frac{d}{dt} \|u\|_{L^2(\mathbb{R}^N)}^2 + 2 \int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX + (\lambda - \mu) \|u\|_{L^2(\mathbb{R}^N)}^2 \\ \leq 2 \|C_1\|_{L^1(\mathbb{R}^N)} + \frac{1}{\lambda - \mu} \|g\|_{L^2(\mathbb{R}^N)}^2. \end{aligned} \quad (3.4)$$

Hence, in particular, we have

$$\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^N)}^2 + (\lambda - \mu) \|u(t)\|^2 \leq 2 \|C_1\|_{L^1(\mathbb{R}^N)} + \frac{1}{\lambda - \mu} \|g\|_{L^2(\mathbb{R}^N)}^2.$$

Using the Gronwall inequality, we obtain

$$\|u(t)\|_{L^2(\mathbb{R}^N)}^2 \leq e^{-(\lambda - \mu)t} \|u_0\|_{L^2(\mathbb{R}^N)}^2 + \left(\frac{2 \|C_1\|_{L^1(\mathbb{R}^N)}}{\lambda - \mu} + \frac{1}{(\lambda - \mu)^2} \|g\|_{L^2(\mathbb{R}^N)}^2 \right).$$

Hence we deduce the existence of a bounded absorbing set in $L^2(\mathbb{R}^N)$: There are a constant R and a time $t_0(\|u_0\|_{L^2(\mathbb{R}^N)})$ such that for the solution $u(t) = S(t)u_0$,

$$\|u(t)\|_{L^2(\mathbb{R}^N)} \leq R \quad \text{for all } t \geq t_0(\|u_0\|_{L^2(\mathbb{R}^N)}). \quad (3.5)$$

Integrating (3.4) on $(t, t+1)$, $t \geq t_0(\|u_0\|_{L^2(\mathbb{R}^N)})$, and using (1.6), we find that

$$\begin{aligned} \int_t^{t+1} \left(\int_{\mathbb{R}^N} (|\nabla_x u(s)|^2 + |\nabla_y u(s)|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u(s)|^2) dX \right) ds \\ + (\lambda - \mu) \int_t^{t+1} \|u(s)\|_{L^2(\mathbb{R}^N)}^2 ds \leq C \left(\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + 1 + \|g\|_{L^2(\mathbb{R}^N)}^2 \right) \\ \leq C \left(R^2 + 1 + \|g\|_{L^2(\mathbb{R}^N)}^2 \right). \end{aligned} \quad (3.6)$$

On the other hand, multiplying (1.1) by $-P_{\alpha,\beta}u$ and integrating over $\mathbb{R}^N$, then integrating by parts and using (1.4), we obtain

$$\begin{aligned} \frac{d}{dt} \left(\int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX \right) + \int_{\mathbb{R}^N} |P_{\alpha,\beta}u|^2 dX \\ + \lambda \int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX \\ = \int_{\mathbb{R}^N} f(X, u) P_{\alpha,\beta}u dX - \int_{\mathbb{R}^N} g P_{\alpha,\beta}u dX \\ = - \int_{\mathbb{R}^N} f'_u(X, u) (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX - \int_{\mathbb{R}^N} g P_{\alpha,\beta}u dX \\ \leq \ell \int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX + \int_{\mathbb{R}^N} |P_{\alpha,\beta}u|^2 dX \\ + \frac{1}{4} \int_{\mathbb{R}^N} |g|^2 dX. \end{aligned} \quad (3.7)$$

Hence

$$\begin{aligned} & \frac{d}{ds} \left(\int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX \right) \\ & \leq \ell \int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX + \frac{1}{4} \|g\|_{L^2(\mathbb{R}^N)}^2. \end{aligned} \quad (3.8)$$

Combining (3.6), (3.8), and applying the uniform Gronwall inequality, we have

$$\int_{\mathbb{R}^N} (|\nabla_x u(t)|^2 + |\nabla_y u(t)|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u(t)|^2) dX \leq C \left(R^2 + 1 + \|g\|_{L^2(\mathbb{R}^N)}^2 \right) \quad (3.9)$$

for all $t \geq t_0$. Using (3.5), we finish the proof. $\square$

We now derive uniform estimates of the derivative of solutions in time.

Lemma 3.2. *Suppose (F) – (G) hold. Then for every bounded subset B in $S^1(\mathbb{R}^N)$, there exists a constant $T = T(B) > 0$ such that*

$$\|u_t(s)\|_{L^2(\mathbb{R}^N)}^2 \leq \rho_1 \text{ for all } u_0 \in B, \text{ and } s \geq T,$$

where $u_t(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$ and ρ_1 is a positive constant independent of B .

Proof. By differentiating (1.1) in time and denoting $v = u_t$, we get

$$\frac{\partial v}{\partial t} - P_{\alpha,\beta} v + \lambda v + \frac{\partial f}{\partial u}(X, u)v = 0.$$

Taking the inner product of the above equality with v in $L^2(\mathbb{R}^N)$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} (|\nabla_x v|^2 + |\nabla_y v|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z v|^2) dX \\ & + \lambda \|v\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \frac{\partial f}{\partial u}(X, u) |v|^2 dX = 0. \end{aligned}$$

By (1.4), it follows that

$$\frac{d}{dt} \|v\|_{L^2(\mathbb{R}^N)}^2 \leq 2\ell \|v\|_{L^2(\mathbb{R}^N)}^2. \quad (3.10)$$

On the other hand, integrating (3.7) from t to $t+1$ and using (3.9), we obtain

$$\int_t^{t+1} \|u_t(s)\|_{L^2(\mathbb{R}^N)}^2 ds \leq C(\rho, \|g\|_{L^2(\mathbb{R}^N)}^2) \quad (3.11)$$

as t large enough. Combining (3.10) with (3.11), and using the uniform Gronwall inequality, we have

$$\|u_t(s)\|_{L^2(\mathbb{R}^N)}^2 \leq C(\rho, \|g\|_{L^2(\mathbb{R}^N)}^2).$$

The proof is complete. $\square$

We now show the existence of a bounded absorbing set in $\mathcal{S}^2(\mathbb{R}^N)$.

Lemma 3.3. *Let assumptions (F) – (G) hold. Then there exists a positive number R such that for any solution u of problem (1.1) we have*

$$\|P_{\alpha,\beta} u\|_{L^2(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2 \leq R^2. \quad (3.12)$$

Proof. Taking the L^2 -inner product of (1.1) with $-P_{\alpha,\beta}u + u$, we have

$$\begin{aligned} & \|P_{\alpha,\beta}u\|_{L^2(\mathbb{R}^N)}^2 + \lambda \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} f(X, u)u dX \\ & \leq - \int_{\mathbb{R}^N} u_t (-P_{\alpha,\beta}u + u) dX + (\lambda + 1) \int_{\mathbb{R}^N} u P_{\alpha,\beta}u dX \\ & \quad - \int_{\mathbb{R}^N} f(X, u) P_{\alpha,\beta}u dX + \int_{\mathbb{R}^N} g(-P_{\alpha,\beta}u + u) dX. \end{aligned}$$

Using (1.5) and integrating by parts, we get

$$\begin{aligned} & \|P_{\alpha,\beta}u\|_{L^2(\mathbb{R}^N)}^2 + (\lambda - \mu) \|u\|_{L^2(\mathbb{R}^N)}^2 \\ & \leq - \int_{\mathbb{R}^N} u_t (-P_{\alpha,\beta}u + u) dX - (\lambda + 1) \int_{\mathbb{R}^N} (|\nabla_x u|^2 + |\nabla_y u|^2 \\ & \quad + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX \\ & \quad - \int_{\mathbb{R}^N} f'_u(X, u) (|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha} |y|^{2\beta} |\nabla_z u|^2) dX \\ & \quad + \int_{\mathbb{R}^N} g(-P_{\alpha,\beta}u + u) dX + \|C_1\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

Using (1.4) and the Cauchy inequality, we have

$$\begin{aligned} & \|P_{\alpha,\beta}u\|_{L^2(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2 \\ & \leq C \left(\|u_t\|_{L^2(\mathbb{R}^N)}^2 + \|u\|_{S^1(\mathbb{R}^N)}^2 + \|g\|_{L^2(\mathbb{R}^N)}^2 + \|C_1\|_{L^1(\mathbb{R}^N)} \right). \end{aligned}$$

Hence, by Lemmas 3.1 and 3.2, we get (3.12). $\square$

3.2. Asymptotic compactness of the semigroup $S(t)$

In order to prove the existence of a global attractor, we mainly have to prove the asymptotic compactness of $S(t)$. Firstly, in Lemma 3.4 below, we prove a tail-estimate of solutions. Then, using this estimate, we prove the asymptotic compactness of $S(t)$ in $L^2(\mathbb{R}^N)$. Finally, using the singular Gronwall inequality and properties of analytic semigroup generated by sectorial operators, we obtain the asymptotic compactness of the semigroup $S(t)$ in $\mathcal{S}^1(\mathbb{R}^N)$.

Lemma 3.4. *Let $\bar{\vartheta} \in C^1([0, \infty))$ be a function such that $\bar{\vartheta} \in [0, 1]$ and*

$$\bar{\vartheta}(s) = 0 \text{ for } s \in [0, 1] \text{ and } \bar{\vartheta}(s) = 1 \text{ for } s \in [2, \infty).$$

Setting $\vartheta = \bar{\vartheta}^2$ and the functions $\bar{\vartheta}_k : \mathbb{R}^N \rightarrow \mathbb{R}$ and $\vartheta_k : \mathbb{R}^N \rightarrow \mathbb{R}$ be defined by

$$\bar{\vartheta}_k(X) = \bar{\vartheta} \left(\frac{|X|^2}{k^2} \right) \text{ and } \vartheta_k(X) = \vartheta \left(\frac{|X|^2}{k^2} \right) \text{ for any } k \in \mathbb{N}.$$

Then whenever $u : [0, \infty) \rightarrow \mathcal{S}^1(\mathbb{R}^N)$ is a solution of (1.1), for every $\eta > 0$ there exist two constants $K_0 > 0$ and $T_0 > 0$ we have

$$\int_{\mathbb{R}^N} \vartheta_k(X) |u(t, X)|^2 dX < \eta \text{ for all } t \geq T_0, k \geq K_0. \quad (3.13)$$

Proof. We follow the arguments in the proof of Theorem 6.5 in [19]. Since $\nabla \vartheta_k(X) = \frac{2}{k^2} \vartheta' \left(\frac{|X|^2}{k^2} \right) X$ and $\nabla \bar{\vartheta}_k(X) = \frac{2}{k^2} \bar{\vartheta}' \left(\frac{|X|^2}{k^2} \right) X$, we have

$$\sup_{X \in \mathbb{R}^N} |\nabla \vartheta_k(X)| \leq \frac{C_{\vartheta}}{k} \quad \text{and} \quad \sup_{X \in \mathbb{R}^N} |\nabla \bar{\vartheta}_k(X)| \leq \frac{C_{\bar{\vartheta}}}{k}.$$

Let $V_k : \mathcal{S}^1(\mathbb{R}^N) \longrightarrow \mathbb{R}^+$ be a function defined by

$$V_k(u) = \frac{1}{2} \int_{\mathbb{R}^N} \vartheta_k(X) |u(X)|^2 dX, \quad u \in \mathcal{S}^1(\mathbb{R}^N).$$

It is easy to check that V_k is Fréchet differentiable and

$$(V_k(u))'(v) = \int_{\mathbb{R}^N} \vartheta_k(X) u(X) v(X) dX = (\vartheta_k u, v)_{L^2(\mathbb{R}^N)}, \quad u, v \in \mathcal{S}^1(\mathbb{R}^N).$$

Therefore, if $u : [0, \infty) \longrightarrow \mathcal{S}^1(\mathbb{R}^N)$ is a positive semi-trajectory of the semi-group $S(t)$, we have

$$(V_k \circ u)'(t) = \int_{\mathbb{R}^N} \vartheta_k(X) u(t) u_t(t) dX = (u_t(t), \vartheta_k u(t))_{L^2(\mathbb{R}^N)}.$$

Let q be a positive number such that $q + \mu < \lambda$. We have

$$\begin{aligned} (V_k \circ u)'(t) + 2q(V_k \circ u)(t) &= (u_t, \vartheta_k u(t))_{L^2(\mathbb{R}^N)} + 2q(V_k \circ u)(t) \\ &= (P_{\alpha, \beta} u, \vartheta_k(X) u)_{L^2(\mathbb{R}^N)} - \lambda \int_{\mathbb{R}^N} \vartheta_k(X) |u|^2 dX - \int_{\mathbb{R}^N} f(X, u) \vartheta_k(X) u dX \\ &\quad - \int_{\mathbb{R}^N} g \vartheta_k(X) u dX + q \int_{\mathbb{R}^N} \vartheta_k(X) |u|^2 dX \\ &\leq (P_{\alpha, \beta} u, \vartheta_k u) + (q + \mu - \lambda) \int_{\mathbb{R}^N} \vartheta_k(X) |u|^2 dX \\ &\quad - \int_{\mathbb{R}^N} g(X) \vartheta_k(X) u dX + \int_{\mathbb{R}^N} C_1(X) \vartheta_k(X) dX, \end{aligned}$$

where we have used (1.5). Since

$$\begin{aligned} &(P_{\alpha, \beta} u, \vartheta_k u) \\ &\leq \int_{\mathbb{R}^N} \vartheta_k(X) (|u|^2 + |P_{\alpha, \beta} u|^2) dX \leq \int_{|X| \geq k} (|u|^2 + |P_{\alpha, \beta} u|^2) dX, \end{aligned}$$

we have

$$\begin{aligned} &(P_{\alpha, \beta} u, \vartheta_k u) + (q + \mu - \lambda) \int_{\mathbb{R}^N} \vartheta_k(X) |u|^2 dX \\ &\quad - \int_{\mathbb{R}^N} g(X) \vartheta_k(X) u dX + \int_{\mathbb{R}^N} C_1(X) \vartheta_k(X) dX \\ &\leq \int_{|X| \geq k} (|u|^2 + |P_{\alpha, \beta} u|^2) dX - (\lambda - q - \mu - \varepsilon) \|\bar{\vartheta}_k u\|_{L^2(\mathbb{R}^N)}^2 \\ &\quad + \frac{1}{4\varepsilon} \|\bar{\vartheta}_k g\|_{\mathbb{R}^N}^2 + \int_{\mathbb{R}^N} |C_1(X)| |\vartheta_k(X)| dX \\ &\leq \int_{|X| \geq k} (|u|^2 + |P_{\alpha, \beta} u|^2) dX + c_k, \end{aligned}$$

where the assumption $0 < \varepsilon < \lambda - q - \mu$ has been used, and

$$c_k = \frac{1}{4\varepsilon} \|\bar{\vartheta}_k g\|_{\mathbb{R}^N}^2 + \int_{\mathbb{R}^N} |C_1(X)| |\vartheta_k(X)| dX.$$

It follows from here that

$$(V_k \circ u)'(t) + 2q(V_k \circ u)(t) \leq \int_{|X| \geq k} (|u|^2 + |P_{\alpha, \beta} u|^2) dX + c_k, \quad \forall t \geq 0.$$

Hence, we have

$$(V_k \circ u)(t) \leq e^{-2qt} \int_0^t e^{2q\xi} \int_{|X| \geq k} (|u|^2 + |P_{\alpha, \beta} u|^2) dX d\xi + c_k + e^{-2qt} (V_k \circ u)(0).$$

For any $\eta > 0$ given, since $C_1 \in L^1(\mathbb{R}^N)$, $g \in L^2(\mathbb{R}^N)$, there exists $K_1 > 0$ such that for all $k \geq K_1$, we have

$$c_k < \frac{\eta}{3}.$$

On the other hand, by Lemma 3.3, there exist $K_2 > 0$ and $T_1 > 0$ such that for all $t \geq T_1$, $k \geq K_2$,

$$e^{-2qt} \int_0^t e^{2q\xi} \int_{|X| \geq k} (|u|^2 + |P_{\alpha, \beta} u|^2) dX d\xi < \frac{\eta}{3}.$$

Obviously, there exists $T_3 > 0$ such that for all $t \geq T_3$,

$$e^{-2qt} (V_k \circ u)(0) < \frac{\eta}{3}.$$

Choosing $T_0 = \max\{T_1, T_2, T_3\}$ and $K_0 = \max\{K_1, K_2\}$, we get the estimate (3.13). $\square$

Lemma 3.5. *The semigroup $S(t)$ is asymptotically compact in the strong $L^2(\mathbb{R}^N)$ topology.*

Proof. The proof is similar to that of Theorem 6.6 in [19], however, we recall it here for convenience of the reader.

Let (u_n) be bounded in $\mathcal{S}^1(\mathbb{R}^N)$, (t_n) be a number sequence such that $\lim_{n \rightarrow \infty} t_n = +\infty$. We have to show that there exist a strictly increase sequence (n_k) and $u \in \mathcal{S}^1(\mathbb{R}^N)$ such that $\lim_{n \rightarrow \infty} S(t_{n_k})u_{n_k} = u$ in $L^2(\mathbb{R}^N)$.

Let $\|u_n\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq R$, $\forall n \geq 1$. Since orbits of bounded sets are bounded, we have

$$\|S(t)u_n\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq R \quad \forall t \geq 0, \quad \forall n \geq 1.$$

Put α is the Kuratowski measure of non-compactness on $L^2(\mathbb{R}^N)$

$$\alpha(B) = \inf\{l > 0 : B \text{ admits a finite cover by sets of diameter } \leq l\},$$

where B is a bounded subset in $L^2(\mathbb{R}^N)$. Using properties of this measure, we have for every k, n_0 ,

$$\begin{aligned} \alpha\{u_n(t_n) | n \in \mathbb{N}\} &\leq \alpha\{(1 - \vartheta_k)u_n(t_n) | n \in \mathbb{N}\} + \alpha\{\vartheta_k u_n(t_n) | n \in \mathbb{N}\} \\ &= \alpha\{(1 - \vartheta_k)u_n(t_n) | n \in \mathbb{N}\} + \alpha\{\vartheta_k u_n(t_n) | n \geq n_0\}. \end{aligned}$$

Noting that $t_n \rightarrow \infty$ when $n \rightarrow \infty$, then using (3.13), we have for every $\varepsilon > 0$, $\exists k, n_0 : \|\vartheta_k u_n(t_n)\|_{L^2(\mathbb{R}^N)} < \varepsilon, \forall n \geq n_0$. Thus,

$$\alpha\{u_n(t_n) | n \in \mathbb{N}\} \leq \alpha\{(1 - \vartheta_k)u_n(t_n) | n \in \mathbb{N}\} + \varepsilon.$$

Since $1 - \vartheta_k \in C_0^1(\mathbb{R}^N)$, it is easy to check that the map $u \mapsto (1 - \vartheta_k)u$ is compact from $\mathcal{S}^1(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$ by using the compactness of the embedding $\mathcal{S}^1(\Omega) \hookrightarrow L^2(\Omega)$ when Ω is bounded (see [21]). Hence

$$\alpha\{(1 - \vartheta_k)u_n(t_n) | n \in \mathbb{N}\} = 0.$$

We therefore obtain that $\alpha\{u_n(t_n) | n \in N\} = 0$, i.e., $\{u_n(t_n)\}$ is precompact on $L^2(\mathbb{R}^N)$. This completes the proof. $\square$

Theorem 3.1. *Suppose (F) and (G) hold. Then the semigroup $S(t)$ generated by problem (1.1) has a compact global attractor $\mathcal{A}$ in the space $\mathcal{S}^1(\mathbb{R}^N)$.*

Proof. We use some ideas in Lemma 2.2 and Theorem 2.3 in [18]. By Lemma 3.1, in order to prove the existence of the global attractor in $\mathcal{S}^1(\mathbb{R}^N)$, we only need to prove that $S(t)$ is asymptotically compact in $\mathcal{S}^1(\mathbb{R}^N)$.

Let $\{u_n\}_{n \leq 1}$ be a bounded sequence in $\mathcal{S}^1(\mathbb{R}^N)$ and $t_n \rightarrow +\infty$. Fix $T > 0$, since $\{u_n\}$ is bounded and orbits of bounded sets are bounded, $\{S(t_n - T)u_n\}$ is bounded in $\mathcal{S}^1(\mathbb{R}^N)$. Since $S(t)$ is asymptotically compact in $L^2(\mathbb{R}^N)$, there is a subsequence $\{S(t_{n_k} - T)u_{n_k}\}$ and $v \in \mathcal{S}^1(\mathbb{R}^N)$ such that $v_k = S(t_{n_k} - T)u_{n_k} \rightharpoonup v$ weakly in $\mathcal{S}^1(\mathbb{R}^N)$ and strongly to $S(t)v$ in $L^2(\mathbb{R}^N)$ as $k \rightarrow \infty$. We will prove that $S(t_{n_k})u_{n_k} = S(T)v_k$ converges strongly to $S(T)v$ in $\mathcal{S}^1(\mathbb{R}^N)$, and thus $S(t)$ is asymptotically compact. Denote $v_k(t) = S(t)v_k, v(t) = S(t)v$, we have

$$\begin{aligned} v_k(t) &= e^{-At}v_k - \int_0^t e^{-A(t-s)}(\hat{f}(v_k(s)) + g)ds, \\ v(t) &= e^{-At}v - \int_0^t e^{-A(t-s)}(\hat{f}(v(s)) + g)ds. \end{aligned}$$

Hence applying the estimate (2.1), we have

$$\begin{aligned} \|v_k(t) - v(t)\|_{\mathcal{S}^1(\mathbb{R}^N)} &\leq C_4\|v_k - v\|_{L^2(\mathbb{R}^N)} \\ &\quad + C_5 \int_0^t (t-s)^{-1/2-\gamma} \|v_k(s) - v(s)\|_{\mathcal{S}^1(\mathbb{R}^N)} ds. \end{aligned}$$

By the singular Gronwall inequality [11, Chapter 7], there is a constant C such that, for $t \in (0, T]$,

$$\|v_k(t) - v(t)\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq Ct^{-1/2}\|v_k - v\|_{L^2(\mathbb{R}^N)}.$$

In particular,

$$\|v_k(T) - v(T)\|_{\mathcal{S}^1(\mathbb{R}^N)} \leq Ct^{-1/2}\|v_k - v\|_{L^2(\mathbb{R}^N)}.$$

Since $v_k \rightarrow v$ in $L^2(\mathbb{R}^N)$, $v_k(T) \rightarrow v(T)$ in $\mathcal{S}^1(\mathbb{R}^N)$ as $k \rightarrow +\infty$. This implies that $S(t)$ is asymptotically compact. Therefore, the semigroup $S(t)$ generated by (1.1) has a compact connected global attractor $\mathcal{A}$ in $\mathcal{S}^1(\mathbb{R}^N)$. $\square$

Remark 3.1. Thanks to the Lyapunov function Φ , one can show that the semigroup $S(t)$ is a gradient system. Therefore, by Theorem 10.13 in [20], we have the structure of the global attractor obtained, namely $\mathcal{A} = W^u(E)$, the unstable manifold of the set E of fixed points of the semigroup $S(t)$. In particular, we have

$$\lim_{t \rightarrow +\infty} \text{dist}(S(t)u_0, E) = 0 \text{ for every } u_0 \in \mathcal{S}^1(\mathbb{R}^N).$$

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