Principal eigenvalue of the fractional Laplacian with a large incompressible drift

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Abstract. We study the principal Dirichlet eigenvalue of the operator $L_A = \Delta^{\alpha/2} + Ab(x) \cdot \nabla$, on a bounded $C^{1,1}$ regular domain D . Here $\alpha \in (1, 2)$, $\Delta^{\alpha/2}$ is the fractional Laplacian, $A \in \mathbb{R}$, and b is a bounded d -dimensional divergence-free vector field in the Sobolev space $W^{1,2d/(d+\alpha)}(D)$. We prove that the eigenvalue remains bounded, as $A \rightarrow +\infty$, if and only if b has non-trivial first integrals in the domain of the quadratic form of $\Delta^{\alpha/2}$ for the Dirichlet condition.

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1. Introduction

This article is motivated by the following result of Berestycki et al. [4] for the Laplacian perturbed by a divergence-free drift in dimensions $d \geq 2$. Let $D \subset \mathbb{R}^d$ be a bounded C^2 regular open set and let $b(x) = (b_1(x), \dots, b_d(x)) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a bounded d -dimensional vector field such that $\operatorname{div} b = 0$ on D in the sense of distributions (distr.), i.e.

$$\int_{\mathbb{R}^d} b(x) \cdot \nabla \phi(x) dx = 0, \quad \phi \in C_c^\infty(D). \quad (1.1)$$

For $A \in \mathbb{R}$, let (ϕ_A, λ_A) be the principal eigen-pair corresponding to the Dirichlet problem for the operator $\Delta + Ab(x) \cdot \nabla$. Theorem 0.3 of [4] asserts that λ_A remains bounded as $A \rightarrow +\infty$, if and only if the equation

$$\operatorname{div}(wb) = 0 \quad (\text{distr.}) \text{ on } D \quad (1.2)$$

has a solution w (called a first integral of b), such that $w \neq 0$ and $w \in H_0^1(D)$.

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The result can be interpreted intuitively in the following way: functions w satisfying (1.2) are constant along the flow of the vector field $Ab(x)$ (see Sect. 5.2), and the existence of (non-trivial) first integrals allows for flow lines that are contained in D . On the other hand, if no such w exist, then the flow leaves D with speed proportional to A . Adding the Laplacian Δ to $b \cdot \nabla$, or equivalently the Brownian motion to the flow, results in a stochastic process whose trajectories gradually depart from the integral curves of b , but the general picture is similar: if nontrivial first integrals exist, then the trajectories may remain in D with positive probability during a finite time interval, even as $A \rightarrow +\infty$. In this case we are lead to a nontrivial limiting transition mechanism between the flow lines. The result described in the foregoing enjoys many extensions and has proved quite useful in various applications describing the influence of a fluid flow on a diffusion, see for example [5, 16, 20, 39]. In the context of a compact, connected Riemannian manifold a sufficient and necessary condition for λ_A to remain bounded, as $A \rightarrow +\infty$, expressed in terms of the eigenspaces of the advection operator $b(x) \cdot \nabla$, has been given in [21, Theorem 1].

The purpose of the present paper is to verify that a similar property of the principal eigenvalue holds when the classical Laplacian is replaced by the fractional Laplacian $\Delta^{\alpha/2}$ with $\alpha \in (1, 2)$. We consider $\mathcal{I}_0^\alpha$ defined as the set of all the nonzero first integrals in the Sobolev space $H_0^{\alpha/2}(D)$ equipped with the norm coming from the Dirichlet form $\mathcal{E}^\alpha$ of $\Delta^{\alpha/2}$ [see (2.16) below]. The Sobolev norm condition on the first integrals reflects smoothing properties of the Green function of the fractional Laplacian, while (1.2) is related to the flow defined by b .

The main difficulty in our development stems from roughness of general elements of $H_0^{\alpha/2}(D)$ and non-locality of $\Delta^{\alpha/2}$, which prevent us from a direct application of the differential calculus in the way it has been done in [4]. Instead, we use conditioning suggested by a paper of Bogdan and Dyda [9], approximation techniques for flows given by DiPerna and Lions in [17], and the properties of the Green function and heat kernel of gradient perturbations of $\Delta^{\alpha/2}$ obtained by Bogdan and Jakubowski in [11] for $\alpha \in (1, 2)$ and bounded $C^{1,1}$ -regular open sets D . These properties allow to define and study, via the classical Krein–Rutman theorem and compactness arguments, the principal eigen-pair (λ_A, ϕ_A) for $L_A = \Delta^{\alpha/2} + Ab \cdot \nabla$ and $\alpha \in (1, 2)$. Our main result can be stated as follows.

Theorem 1.1. *Suppose that $D \subset \mathbb{R}^d$ is a bounded $C^{1,1}$ regular domain. If $\alpha \in (1, 2)$, and $b \in L^\infty(D) \cap W^{1,2d/(d+\alpha)}(D)$ is of zero divergence, then*

$$\lim_{A \rightarrow +\infty} \lambda_A = \inf_{w \in \mathcal{I}_0^\alpha, \|w\|_2=1} \mathcal{E}^\alpha(w, w), \quad (1.3)$$

and the infimum is attained. Here we use the convention that $\inf \emptyset = +\infty$, hence $\lim_{A \rightarrow +\infty} \lambda_A = +\infty$ if and only if the zero function is the only first integral of b .

Equality (1.3) results from the following lower and upper bounds of λ_A ,

$$\liminf_{A \rightarrow +\infty} \lambda_A \geq \inf_{w \in \mathcal{I}^\alpha, \|w\|_2=1} \mathcal{E}^\alpha(w, w), \quad (1.4)$$

$$\sup_{A \in \mathbb{R}} \lambda_A \leq \inf_{w \in \mathcal{I}^\alpha, \|w\|_2=1} \mathcal{E}^\alpha(w, w). \quad (1.5)$$

The bounds are proved in Sects. 5.1 and 5.2, correspondingly. In Sect. 5.3 we explain that the minimum on the right hand side of (1.3) is attained, and we finish the proof of the theorem.

Comparing our approach with the arguments used in the case of local operators, cf. [4, 21], we note that the use of the Green function seems more robust whenever we lack sufficient differentiability of functions appearing in variational formulas. Recall that in the present case we need to deal with $H_0^{\alpha/2}(D)$, which limits the applicability of the arguments based on the usual differentiation rules of the classical calculus, e.g. the Leibnitz formula or the chain rule. We consider the use of the Green function as one of the major features of our approach. In addition, the non-locality of the quadratic forms forces a substantial modifications of several other arguments, e.g. those involving conditioning of nonlocal operators and quadratic forms in the proof of the upper bound (1.5) in Sect. 5.2. Finally, we stress the fact that the Dirichlet fractional Laplacian on a bounded domain D is *not* a fractional power of the Dirichlet Laplacian on D , e.g. the eigenfunctions of these operators have a different power-type decay at the boundary, see [3, 31, 35] in this connection.

As a preparation for the proof, we recall in Sect. 2 the estimates of [11, 13] for the Green function and transition density of L_A for the Dirichlet problem on D . These functions are defined using Hunt's formula (2.31), which in principle requires the drift $b(x)$ to be defined on the entire $\mathbb{R}^d$. We show however, in Corollary 3.9, that they are determined by the restriction of the drift to the domain D . In Sect. 4 we prove that the corresponding Green's and transition operators are compact, see Lemmas 4.1 and 4.2. This result is used to define the principal eigen-pair of L_A , via the Krein–Rutman theorem. In Theorem 3.6 of Sect. 3 we prove that the domains of $\Delta^{\alpha/2}$ and L_A in $L^2(D)$ coincide. In Sect. 5 we employ the bilinear form of L_A to estimate the principal eigenvalue. The technical assumption $\nabla b \in L^{2d/(d+\alpha)}(D)$ is only needed in Sect. 5.2 to characterize the first integrals of b by means of the theory of flows developed by DiPerna and Lions in [17] for Sobolev-regular vector fields.

2. Preliminaries

2.1. Generalities

We start with a brief description of the setting and recapitulation of some of the results of [11, 13]. Further details and references may be found in those papers (see also [8, 10, 26] and the references therein). In what follows, $\mathbb{R}^d$ is the Euclidean space of dimension $d \geq 2$, scalar product $x \cdot y$, norm $|x|$ and Lebesgue measure dx . All sets, measures and functions in $\mathbb{R}^d$ considered throughout this

paper will be Borel. We denote by

$$B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\},$$

the ball of center $x \in \mathbb{R}^d$ and radius $r > 0$.

We will consider nonempty, bounded open set $D \subset \mathbb{R}^d$, whose boundary is of class $C^{1,1}$. The latter means that $r > 0$ exists such that for every $Q \in \partial D$ there are balls $B(x', r) \subset D$ and $B(x'', r) \subset \mathbb{R}^d \setminus D$, which are tangent at Q (the *inner* and *outer* tangent ball, respectively). We will refer to such sets D as to $C^{1,1}$ domains, without requiring connectivity. For an alternative analytic description and localization of $C^{1,1}$ domains we refer to [10, Lemma 1]. We note that each connected component of D contains a ball of radius r , and the same is true for D^c . Therefore D and D^c have a finite number of components, which will play a role in a later discussion of extensions of the vector field to a neighborhood of $\bar{D}$. The distance of a given $x \in \mathbb{R}^d$ to D^c will be denoted by

$$\delta_D(x) = \inf\{|y - x| : y \in \mathbb{R}^d \setminus D\}.$$

Constants mean positive numbers, that do not depend on the considered arguments of the functions being compared. Accordingly, notation $f(x) \approx g(x)$ means that there is a constant C such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$ for all x . As usual, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

We will employ the function space $L^2(D)$, consisting of all square integrable real valued functions, with the usual scalar product

$$(f, g) = \int_D f(x)g(x)dx.$$

Generally, given $1 \leq p \leq \infty$, the norms in $L^p(D)$ shall be denoted by $\|\cdot\|_p$. Customarily, $C_c^\infty(D)$ denotes the space of smooth functions on $\mathbb{R}^d$ with compact support in D . Also, $H_0^1(D)$ denotes the closure of $C_c^\infty(D)$ in the norm $\|\cdot\|_1$, where

$$|f|_1^2 := \int_{\mathbb{R}^d} (|f(x)|^2 + |\nabla f(x)|^2)dx = (2\pi)^{-d} \int_{\mathbb{R}^d} (1 + |\xi|^2) |\hat{f}(\xi)|^2 d\xi. \quad (2.1)$$

In the last equality we have used Plancherel theorem. The Fourier transform of f is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx. \quad (2.2)$$

2.2. Isotropic α -stable Lévy process on $\mathbb{R}^d$

The discussion in Sects. 2.2 and 2.3 is valid for any $\alpha \in (0, 2)$. Let

$$\nu(y) = \frac{\alpha 2^{\alpha-1} \Gamma((d+\alpha)/2)}{\pi^{d/2} \Gamma(1-\alpha/2)} |y|^{-d-\alpha}, \quad y \in \mathbb{R}^d.$$

The coefficient is chosen in such a way that

$$\int_{\mathbb{R}^d} [1 - \cos(\xi \cdot y)] \nu(y) dy = |\xi|^\alpha, \quad \xi \in \mathbb{R}^d. \quad (2.3)$$

We define the *fractional Laplacian* as the $L^2(\mathbb{R}^d)$ -closure of the operator

$$\Delta^{\alpha/2}\phi(x) = \lim_{\varepsilon \downarrow 0} \int_{|y|>\varepsilon} [\phi(x+y) - \phi(x)] \nu(y) dy, \quad x \in \mathbb{R}^d, \phi \in C_c^\infty(\mathbb{R}^d).$$

Its Fourier symbol is given by $\widehat{\Delta^{\alpha/2}\phi}(\xi) = -|\xi|^\alpha \hat{\phi}(\xi)$, cf (2.2). The fractional Laplacian is the generator of the semigroup of the isotropic α -stable Lévy process $(Y_t, \mathbb{P}^x)$ on $\mathbb{R}^d$. Here $\mathbb{P}^x$ and $\mathbb{E}^x$ are the law and expectation for the process starting at $x \in \mathbb{R}^d$. These are defined on the Borel σ -algebra of the canonical càdlàg path space $\mathbf{D}([0, +\infty); \mathbb{R}^d)$ via transition probability densities as follows. We let (Y_t) be the canonical process, i.e.

$$Y_t(\omega) := \omega(t) \quad \text{for } \omega \in \mathbf{D}([0, +\infty); \mathbb{R}^d),$$

and define time-homogeneous transition density $p(t, x, y) := p_t(y - x)$, where

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} e^{-ix \cdot \xi} dx, \quad x \in \mathbb{R}^d. \quad (2.4)$$

According to (2.3) and the Lévy–Khinchin formula, $\{p_t\}$ is a probabilistic convolution semigroup of functions with the Lévy measure $\nu(y)dy$, see e.g. [8]. From (2.4) we have

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x), \quad t > 0, x \in \mathbb{R}^d. \quad (2.5)$$

It is well-known (see [8]) that $p_1(x) \approx 1 \wedge |x|^{-d-\alpha}$, hence

$$p_t(x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \quad t > 0, x \in \mathbb{R}^d. \quad (2.6)$$

2.3. Stable process killed off D

Let $\tau_D = \inf\{t > 0 : Y_t \notin D\}$ be the *time of the first exit* of the (canonical) process from D . For each $(t, x) \in (0, +\infty) \times D$, the measure

$$A \mapsto P_D(t, x, A) := \mathbb{P}^x[Y_t \in A, \tau_D > t]$$

is absolutely continuous with respect to the Lebesgue measure on D . Its density $p_D(t, x, y)$ is continuous in $(t, x, y) \in (0, +\infty) \times D^2$ and satisfies G. Hunt's formula (see [10, 30]),

$$p_D(t, x, y) = p(t, x, y) - \mathbb{E}^x[\tau_D < t; p(t - \tau_D, Y_{\tau_D}, y)]. \quad (2.7)$$

In addition, the kernel is symmetric (see [6, 10] for discussion and references):

$$p_D(t, x, y) = p_D(t, y, x), \quad t > 0, x, y \in D. \quad (2.8)$$

and defines a strongly continuous semigroup on $L^2(D)$,

$$P_t^D f(x) := \int_D p_D(t, x, y) f(y) dy.$$

We shall denote $P_t = P_t^{\mathbb{R}^d}$. The *Green function* of $\Delta^{\alpha/2}$ for D is defined as

$$G_D(x, y) = \int_0^\infty p_D(t, x, y) dt,$$

and the respective *Green operator* is

$$G_D f(x) := \mathbb{E}^x \left[\int_0^{\tau_D} f(Y_t) dt \right] = \int_{\mathbb{R}^d} G_D(x, y) f(y) dy.$$

We have ([11]),

$$G_D(\Delta^{\alpha/2} f) = -f, \quad f \in C_c^\infty(D), \quad (2.9)$$

and

$$\Delta^{\alpha/2}(G_D f) = -f, \quad f \in L^2(D), \quad (2.10)$$

see [7, Lemma 5.3], where the fractional Laplacian operator appearing above is defined in the sense of distributions theory. See formula (2.38) below for another statement.

Thus the generator of (P_t^D) is $\Delta^{\alpha/2}$ with zero (Dirichlet) exterior conditions, i.e. with the domain equal to the range of G_D ,

$$\mathcal{D}(\Delta^{\alpha/2}) = G_D(L^2(D)) =: R(G_D). \quad (2.11)$$

The following estimate has been proved by Kulczycki [30] and Chen and Song [14] (see also [24, Theorem 21]),

$$\begin{aligned} G_D(x, y) &\approx |x - y|^{\alpha-d} \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{[\delta_D(x) \vee |x - y| \vee \delta_D(y)]^\alpha} \\ &\approx |x - y|^{\alpha-d} \left(\frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{|x - y|^\alpha} \wedge 1 \right), \quad x, y \in D. \end{aligned} \quad (2.12)$$

In particular,

$$G_D(x, y) \leq C|x - y|^{\alpha-d}, \quad x, y \in D.$$

From (2.12) it also follows that

$$\mathbb{E}_x \tau_D = G_D 1(x) \approx \delta_D^{\alpha/2}(x), \quad x \in D. \quad (2.13)$$

From [12, Corollary 3.3] we have the following gradient estimate,

$$|\nabla_x G_D(x, y)| \leq d \frac{G_D(x, y)}{\delta_D(x) \wedge |x - y|}, \quad x, y \in D, \quad x \neq y. \quad (2.14)$$

We define $H^{\alpha/2} = H^{\alpha/2}(\mathbb{R}^d)$ as the subspace of $L^2(\mathbb{R}^d)$ made of those elements for which

$$|f|_{\alpha/2} := \{\|f\|_2^2 + \mathcal{E}^\alpha(f, f)\}^{1/2} < +\infty. \quad (2.15)$$

Here (the Dirichlet form) $\mathcal{E}^\alpha$ is given as follows (cf [22]):

$$\begin{aligned} \mathcal{E}^\alpha(f, f) &:= \lim_{t \rightarrow 0+} \frac{1}{t} ((I - P_t)f, f) \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [f(x) - f(y)]^2 \nu(y - x) dx dy, \end{aligned} \quad (2.16)$$

By polarization we can define bilinear form $\mathcal{E}_\alpha(f, g)$ on $H^{\alpha/2} \times H^{\alpha/2}$. We also have

$$|f|_{\alpha/2}^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} (1 + |\xi|^\alpha) |\hat{f}(\xi)|^2 d\xi, \quad (2.17)$$

therefore

$$|f|_{\alpha/2} \leq \sqrt{2} |f|_1, \quad f \in L^2(\mathbb{R}^d). \quad (2.18)$$

We define $H_0^{\alpha/2}(D)$ as the closure of $C_c^\infty(D)$ in the norm $|\cdot|_{\alpha/2}$. By Theorems 4.4.2, A.2.10 and formula (4.3.1) of [22], we have [cf. (2.16)],

$$\mathcal{E}^\alpha(f, g) = \lim_{t \rightarrow 0+} \frac{1}{t} ((I - P_t^D)f, g), \quad f, g \in H_0^{\alpha/2}(D).$$

If $f \in H_0^{\alpha/2}(D)$ and g belongs to the domain of the fractional Laplacian, then

$$\mathcal{E}^\alpha(f, g) = - \int_{\mathbb{R}^d} f(x) \Delta^{\alpha/2} g(x) dx. \quad (2.19)$$

2.4. Gradient perturbations of $\Delta^{\alpha/2}$ on $\mathbb{R}^d$

Throughout the remainder of the paper we always assume that $1 < \alpha < 2$, and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a bounded vector field. For $t > 0$ and $x, y \in \mathbb{R}^d$ we let

$$p_0(t, x, y) := p(t, x, y)$$

and for each $n \geq 1$

$$p_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} p_{n-1}(t-s, x, z) b(z) \cdot \nabla_z p(s, z, y) dz ds. \quad (2.20)$$

Let

$$\tilde{p}(t, x, y) = \sum_{n=0}^{\infty} p_n(t, x, y). \quad (2.21)$$

It follows from [26, Theorem 2 and Example 2] that series (2.21) converges uniformly on compact subsets of $(0, +\infty) \times (\mathbb{R}^d)^2$. From the results of [10], we know that $\tilde{p}(t, x, y)$ is a transition probability density function, i.e. it is non-negative and

$$\int_{\mathbb{R}^d} \tilde{p}(t, x, y) dy = 1, \quad t > 0, x \in \mathbb{R}^d, \quad (2.22)$$

In addition, $\tilde{p}$ is continuous on $(0, +\infty) \times (\mathbb{R}^d)^2$, and

$$c_T^{-1} p(t, x, y) \leq \tilde{p}(t, x, y) \leq c_T p(t, x, y), \quad x, y \in \mathbb{R}^d, \quad 0 < t \leq T, \quad (2.23)$$

where $c_T \rightarrow 1$ if $T \rightarrow 0$. In fact, this holds under much weaker, Kato-type condition on b , see [10, Theorems 1 and 2].

We denote by $\tilde{\mathbb{P}}^x$ and $\tilde{\mathbb{E}}^x$ the law and expectation on $\mathbf{D}([0, +\infty); \mathbb{R}^d)$ for the (canonical) Markov process Y starting at x and defined by the transition probability density $\tilde{p}$,

Remark 1. $\tilde{\mathbb{P}}^x$ may also be defined by solving stochastic differential equation $dY_t = dY_t^{(0)} + b(Y_t)dt$, where the isotropic α -stable Lévy process is now denoted by $Y^{(0)}$. Such equations have been studied in dimension 1 in [37] under the assumptions of boundedness and continuity of the vector field; also for $\alpha = 1$. We refer the reader to [34, formula (13)], for a closer description of a connection to (2.21) and (2.20).

Remark 2. We may also define the perturbation series for $-b(x)$. In what follows, objects pertaining to $-b$ will be marked with the superscript hash ($\#$), e.g. $\tilde{p}^\#(t, x, y) = \sum_{n=0}^{\infty} (-1)^n p_n(t, x, y)$.

2.5. Antisymmetry of the perturbation

If $\operatorname{div} b = 0$ on D in the sense of distributions theory, and $f \in C_c^\infty(D)$, then

$$\int_D f(x) b(x) \cdot \nabla f(x) dx = \frac{1}{2} \int_D b(x) \cdot \nabla f^2(x) dx = 0. \quad (2.24)$$

If also $g \in C_c^\infty(D)$ and we substitute $f + g$ for f in (2.24), then we obtain

$$\int_D f(x) b(x) \cdot \nabla g(x) dx = - \int_D g(x) b(x) \cdot \nabla f(x) dx. \quad (2.25)$$

The last equality extends to arbitrary $f, g \in H_0^1(D)$.

Proposition 2.1. *If $\operatorname{div} b = 0$ on $\mathbb{R}^d$, then we have $p_n^\#(t, x, y) = p_n(t, y, x)$ and $\tilde{p}^\#(t, x, y) = \tilde{p}(t, y, x)$ for all $t > 0$, $x, y \in \mathbb{R}^d$, and $n \geq 0$.*

Proof. Let $s, t > 0$, $x, y \in \mathbb{R}^d$. By (2.25) (see also [25, Lemma 4]),

$$\int_{\mathbb{R}^d} p(t, x, z) b(z) \cdot \nabla_z p(s, z, y) dz = - \int_{\mathbb{R}^d} p(s, y, z) b(z) \cdot \nabla_z p(t, z, x). \quad (2.26)$$

From (2.26) we conclude that $p_1^\#(t, x, y) = p_1(t, y, x)$. For a general $n \geq 1$, by (2.20) we have

$$p_n(t, x, y) = \int_{D_n(t)} ds \int_{(\mathbb{R}^d)^n} P_n(\mathbf{s}, x, \mathbf{z}, y) d\mathbf{z}. \quad (2.27)$$

where $\mathbf{z} := (z_1, \dots, z_n)$, $\mathbf{s} := (s_1, \dots, s_n)$,

$$D_n(t) := [\mathbf{s} : 0 \leq s_1 \leq \dots \leq s_n \leq t],$$

and

$$P_n(\mathbf{s}, x, \mathbf{z}, y) = p(s_1, x, z_1) \prod_{i=1}^n [b(z_i) \cdot \nabla_{z_i} p(s_{i+1} - s_i, z_i, z_{i+1})], \quad (2.28)$$

with the convention that $(s_{n+1}, z_{n+1}) = (t, y)$, cf. [25, (4.6)]. Using formula (2.26) n times in space, and then integrating in time we see that

$$p_n(t, x, y) = (-1)^n p_n(t, y, x), \quad (2.29)$$

which yields the identities stated in the lemma. $\square$

Remark 3. A strengthening of Proposition 2.1 will be given in Corollary 3.11 below.

Remark 4. If $\operatorname{div} b = 0$ on $\mathbb{R}^d$, then (2.22) and Proposition 2.1 yield

$$\int_{\mathbb{R}^d} \tilde{p}(t, x, y) dx = 1, \quad t > 0, y \in \mathbb{R}^d. \quad (2.30)$$

2.6. Gradient perturbations with Dirichlet conditions

We recall that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Hunt's formula may be used to define the transition probability density of the (first) perturbed and (then) killed process [11]. Thus, for $t > 0$, $x, y \in D^2$ we let

$$\tilde{p}_D(t, x, y) = \tilde{p}(t, x, y) - \tilde{\mathbb{E}}^x[\tau_D < t; \tilde{p}(t - \tau_D, Y_{\tau_D}, y)]. \quad (2.31)$$

We have

$$\tilde{\mathbb{E}}^x[\tau_D < t; f(Y_t)] = \int_D \tilde{p}_D(t, x, y) f(y) dy, \quad t > 0, x \in D, f \in L^\infty(D),$$

and

$$0 \leq \tilde{p}_D(t, x, y) \leq \tilde{p}(t, x, y) \leq c_t p(t, x, y), \quad t > 0, x, y \in D, \quad (2.32)$$

where $\sup_{t \in [0, T]} c_t < +\infty$ for any $T > 0$.

By [11, formula (40)], there exist constants $c, C > 0$ such that

$$\tilde{p}_D(t, x, y) \leq C e^{-ct}, \quad t \geq 1, \quad x, y \in \mathbb{R}^d. \quad (2.33)$$

We define the Green function of D for L :

$$\tilde{G}_D(x, y) = \int_0^\infty \tilde{p}_D(t, x, y) dt. \quad (2.34)$$

Clearly $\tilde{G}_D$ is nonnegative. By Blumenthal's 0-1 law, $\tilde{p}_D(t, x, y) = 0$ for all $t > 0$ [and thus $\tilde{G}_D(x, y) = 0$] if $x \in D^c$ or $y \in D^c$, see (2.23). It follows from (2.36) and (2.12) (see [11, Lemma 7]), that

$$\tilde{G}_D(x, y) \leq C_0 |x - y|^{\alpha-d}, \quad x, y \in D. \quad (2.35)$$

The main result of [11] asserts that

$$C^{-1} G_D(x, y) \leq \tilde{G}_D(x, y) \leq C G_D(x, y), \quad \text{if } x, y \in D, \quad (2.36)$$

and $\tilde{G}_D(x, y)$ is continuous for $x \neq y$ (for estimates of $\tilde{p}_D$ see [13]). We consider the integral operators

$$\tilde{P}_t^D f(x) = \int_{\mathbb{R}^d} \tilde{p}_D(t, x, y) f(y) dy, \quad t > 0,$$

and

$$\tilde{G}_D f(x) = \int_{\mathbb{R}^d} \tilde{G}_D(x, y) f(y) dy. \quad (2.37)$$

In light of (2.32) and (2.36), the above operators are bounded on every $L^p(D)$, $p \in [1, +\infty)$. The analogous operators $\tilde{P}_t^{D\#}$ and $\tilde{G}_D^\#$, defined for $-b(x)$, turn out to be mutually adjoint on $L^2(D)$, as follows from Corollary 3.11 below. From (2.22), (2.30) and (2.31) we obtain that $\{\tilde{P}_t^D\}$ is a semigroup of contractions in every $L^p(D)$, $p \in [1, +\infty]$. It is strongly continuous for

$p \in [1, +\infty)$. Let L be the $L^2(D)$ generator of the semigroup, with the domain $\mathcal{D}(L) = R(\tilde{G}_D) = \tilde{G}_D(L^2(D))$. We have

$$L\tilde{G}_D f = -f, \quad f \in L^2(D). \quad (2.38)$$

It has been shown in [11] that the following crucial recursive formula holds

$$\tilde{G}_D = G_D + \tilde{G}_D(b \cdot \nabla)G_D, \quad (2.39)$$

and for all $\varphi \in C_c^\infty(D)$ and $x \in D$ we have

$$\int_D \tilde{G}_D(x, z) \left(\Delta^{\alpha/2} \varphi(z) + b(z) \cdot \nabla \varphi(z) \right) dz = -\varphi(x). \quad (2.40)$$

Later on we shall also consider the operator

$$L_A = \Delta^{\alpha/2} + Ab \cdot \nabla, \quad (2.41)$$

corresponding to the vector field Ab , where $A \in \mathbb{R}$ and we let $A \rightarrow \infty$. Clearly, if A is fixed, then there is no loss of generality to focus on $L = L_1$.

3. Comparison of the domains of generators

The following pointwise version of (2.39) is proved in [11, Lemma 12],

$$\tilde{G}_D(x, y) = G_D(x, y) + \int_D \tilde{G}_D(x, z) H(z, y) dz, \quad x, y \in D, \quad (3.1)$$

with

$$H(x, y) := b(x) \cdot \nabla_x G_D(x, y), \quad x, y \in D. \quad (3.2)$$

We define

$$Hf(x) = \int_D H(x, y) f(y) dy. \quad (3.3)$$

After a series of auxiliary estimates, we will prove H to be compact on $L^2(D)$.

Lemma 3.1. *There exist $C > 0$ such that*

$$|H(x, y)| \leq C \delta_D^{\alpha/2-1}(x) |y - x|^{\alpha-1-d} \delta_D^{1-\alpha/2}(y), \quad x, y \in D. \quad (3.4)$$

Proof. Using (2.14) we obtain that

$$|H(x, y)| \leq c \frac{G_D(x, y)}{\delta_D(x) \wedge |x - y|} = c \frac{G_D(x, y) [\delta_D(x) \vee |x - y|]}{\delta_D(x) |x - y|},$$

which we bound from above, thanks to (2.12), by

$$\begin{aligned} & c_1 \delta_D(x)^{\alpha/2-1} |x - y|^{\alpha-1-d} \frac{[\delta_D(x) \vee |x - y|] \delta_D(y)^{\alpha/2}}{[\delta_D(x) \vee |x - y| \vee \delta_D(y)]^\alpha} \\ & \leq c_1 \delta_D(x)^{\alpha/2-1} |x - y|^{\alpha-1-d} \frac{\delta_D(y)^{\alpha/2}}{[\delta_D(x) \vee |x - y| \vee \delta_D(y)]^{\alpha-1}}, \end{aligned} \quad (3.5)$$

and this yields (3.4). $\square$

Lemma 3.2. *If $0 < 1 - \beta < \gamma < 1$, then there is $c = c(d, \beta, \gamma, D)$ such that*

$$\int_D |y - x|^{-(d-1)-\beta} \delta_D(y)^{-\gamma} dy \leq c \delta_D(x)^{1-\beta-\gamma}, \quad x \in D. \quad (3.6)$$

Proof. We only need to examine points close to ∂D . Given such a point we consider the integral over its neighborhood $\mathcal{O}$. The neighborhood can be chosen in such a way that, after a bi-Lipschitz change of variables (see [11, formula (75)], [12, formula (11)]), we can reduce our consideration to the case when $\partial D \cap \mathcal{O} \subset [x_1 = 0]$, $\mathcal{O} \subset [x_1 > 0]$ and $\delta_D(x) = x_1$ for $x \in \mathcal{O}$. The respective integral over $\mathcal{O}$ is then estimated by

$$\begin{aligned} & c \int_0^\infty \frac{dy_1}{y_1^\gamma} \int_{\mathbb{R}^{d-1}} \frac{dx'}{(|x_1 - y_1| + |x'|)^{d-1+\beta}} \\ &= c_1 \int_0^\infty |y_1 - x_1|^{-\beta} y_1^{-\gamma} dy_1 = c x_1^{1-\beta-\gamma}. \end{aligned}$$

□

Remark 5. The result is valid for all bounded open *Lipschitz* sets [12] in all dimensions $d \in \mathbb{N}$.

Lemma 3.2 yields the following.

Corollary 3.3. *Let $1 < \alpha' \leq \alpha < 2$ and $\alpha' - 1 < \iota, \kappa < 1$. Suppose also that*

$$\begin{aligned} K(x, y) &:= \delta_D(x)^{\alpha/2-1} |y - x|^{-(d-1)-(2-\alpha')} \delta_D(y)^{1-\alpha/2}, \\ p(x) &:= \delta(x)^{1-\alpha/2-\iota} \quad \text{and} \quad q(y) := \delta(y)^{\alpha/2-1-\kappa}, \quad x, y \in D. \end{aligned} \quad (3.7)$$

Denote

$$pK(y) = \int_D p(x)K(x, y)dx, \quad Kq(x) = \int_D K(x, y)q(y)dy.$$

Then,

$$pK(y) \leq c_1 \delta_D(y)^{\alpha'-\alpha-\iota}, \quad Kq(x) \leq c_2 \delta_D(x)^{-2+\alpha'+\alpha/2-\kappa}, \quad x, y \in D.$$

Constants c_1, c_2 depend only on $\alpha, \alpha', \iota, d, D$.

The above result shall be used to establish that operator K (thus also H) is L^2 bounded via Schur's test, see [23, Theorem 5.2]. Note that we have $pK \leq cq$, for some constant $c > 0$, provided that $\iota - \kappa \leq \alpha' - \alpha + 1$. Likewise, $Kq \leq cp$, provided that $\iota - \kappa \geq 3 - \alpha' - \alpha$. Summarizing, we will require the following conditions:

$$1 < \alpha' \leq \alpha < 2, \quad (3.8)$$

$$\alpha' - 1 < \kappa, \iota < 1, \quad \text{and} \quad (3.9)$$

$$3 - \alpha' - \alpha \leq \iota - \kappa \leq \alpha' - \alpha + 1. \quad (3.10)$$

Lemma 3.4. *Conditions (3.8–3.10) hold, if $1 < \alpha < 2$ and*

$$\alpha' := \alpha - \gamma(\alpha), \quad \kappa := \alpha - 1 + \gamma(\alpha), \quad \iota := 1 - \gamma(\alpha),$$

where $\gamma(\alpha) := (\alpha - 1)(2 - \alpha)/3$.

Proof. We have $\iota - \kappa - (3 - \alpha' - \alpha) = 1 - \alpha(2 - \alpha) \geq 0$, etc. □

Proposition 3.5. *Operator H is compact on $L^2(D)$.*

Proof. We note that $|H(x, y)| \leq cK(x, y)$, where $K(x, y)$ is given by (3.7) and $c > 0$ is some constant. Let α', κ, ι be as in Lemma 3.4. By Corollary 3.3 and the discussion preceding Lemma 3.4, there are constants $c_1(\alpha'), c_2(\alpha')$ such that

$$pK \leq c_1(\alpha')q \quad \text{and} \quad Kq \leq c_2(\alpha')p \quad (3.11)$$

Using Schur's test for boundedness of integral operators on L^2 spaces (see [23, Theorem 5.2]) we obtain

$$\|H\| \leq c\|K\| \leq c\sqrt{c_1(\alpha')c_2(\alpha')} < \infty.$$

For $N > 0$, $r \in \mathbb{R}$ we let $\Phi_N(r) = (r \wedge N) \vee (-N)$, $r \in \mathbb{R}$, and

$$H_N(x, y) := \delta_D^{\alpha/2-1}(x)\Phi_N\left(\delta_D(x)^{1-\alpha/2}H(x, y)\delta_D(y)^{\alpha/2-1}\right)\delta_D^{1-\alpha/2}(y).$$

Each operator $H_N f(x) := \int_D H_N(x, y)f(y)dy$ is Hilbert–Schmidt, hence compact ([32, Theorem 4, p. 247]), due to the fact that

$$\int_{D^2} H_N^2(x, y)dx dy \leq N^2 \int_D \delta_D(x)^{\alpha-2}dx \int_D \delta_D(y)^{2-\alpha}dy < \infty.$$

We have $H(x, y) = H_N(x, y)$, unless $\delta_D(x)^{1-\alpha/2}|H(x, y)|\delta_D(y)^{\alpha/2-1} > N$. The latter may only happen if x, y satisfy $C|y - x|^{-(d+1-\alpha)} > N$, with $C > 0$ [cf (3.4)], or equivalently when $|y - x| < (C/N)^{1/(d+1-\alpha)}$, and then there are constants $C, C_1 > 0$ such that

$$\begin{aligned} |H(x, y) - H_N(x, y)| &\leq |H(x, y)| \leq C|x - y|^{\alpha'-\alpha}K(x, y) \\ &\leq C_1N^{(\alpha'-\alpha)/(d+1-\alpha)}K(x, y). \end{aligned}$$

This estimate for $H(x, y) - H_N(x, y)$ actually holds for all $x, y \in D$, hence (3.11) yields

$$\begin{aligned} p|H - H_N| &\leq C_1N^{(\alpha'-\alpha)/(d+1-\alpha)}c_1(\alpha')q, \\ |H - H_N|q &\leq C_1N^{(\alpha'-\alpha)/(d+1-\alpha)}c_2(\alpha')p. \end{aligned}$$

Applying Schur's test we get $\|H - H_N\| \leq C_1N^{(\alpha'-\alpha)/(d+1-\alpha)}\sqrt{c_1(\alpha')c_2(\alpha')}$. Since H is a norm limit of compact operators, it is compact, too. $\square$

In view of Proposition 3.5 we may regard the gradient operator $b \cdot \nabla$ as a small perturbation of $\Delta^{\alpha/2}$ when $\alpha \in (1, 2)$. In fact $b \cdot \nabla$ is relatively compact with respect to $\Delta^{\alpha/2}$ with Dirichlet conditions in the sense of [27, IV.1.3].

Theorem 3.6. $\mathcal{D}(L) = \mathcal{D}(\Delta^{\alpha/2})$.

Proof. By virtue of (3.1), we can write

$$G_D = \tilde{G}_D(I - H) \quad \text{on } L^2(D). \quad (3.12)$$

By (2.10), G_D is injective, therefore so is $I - H$. In particular, 1 is not an eigenvalue of H . Since H is compact, by the Riesz–Schauder theory [38, Theorem X.5.1, p. 283], $I - H$ is invertible. Thus,

$$\mathcal{D}(\Delta^{\alpha/2}) = R(G_D) = R(\tilde{G}_D(I - H)) = R(\tilde{G}_D) = \mathcal{D}(L).$$

For future reference we remark that $I - H^*$ is invertible, too. $\square$

Corollary 3.7. $\mathcal{D}(L) \subset H_0^1(D)$.

Proof. By [11, Lemma 10],

$$\nabla G_D g(x) = \int_D \nabla_x G_D(x, y) g(y) dy, \quad (3.13)$$

for any bounded function g . Invoking the argument used in Proposition 3.5, we conclude that there is a number c independent of g for which

$$\|\nabla G_D g\|_2 \leq c \|g\|_2. \quad (3.14)$$

By approximation, (3.13) and (3.14) extend to all $g \in L^2(D)$. Furthermore, if $\{g_j\} \subset C_c^\infty(D)$ and $\lim_{j \rightarrow +\infty} g_j = g$ in $L^2(D)$, then $\lim_{j \rightarrow +\infty} |G_D g - G_D g_j|_1 = 0$, cf. (2.1). Each $G_D g_j$ is continuous on $\bar{D}$ and $G_D g_j = 0$ on ∂D , hence $G_D g_j \in H_0^1(D)$ (see [1, Theorem 5.37]). Therefore, $G_D g \in H_0^1(D)$. $\square$

The following result justifies our notation $L = \Delta^{\alpha/2} + b \cdot \nabla$.

Proposition 3.8. If $f \in \mathcal{D}(L)$, then $Lf = \Delta^{\alpha/2} f + b \cdot \nabla f$.

Proof. Let $f \in \mathcal{D}(L)$ and $h = b \cdot \nabla f$. Applying $\tilde{G}_D$ to $\Delta^{\alpha/2} f + h$ and using (2.39), we obtain

$$\begin{aligned} \tilde{G}_D(\Delta^{\alpha/2} f + h) &= (G_D + \tilde{G}_D(b \cdot \nabla) G_D)(\Delta^{\alpha/2} f + h) \\ &= -f + G_D h - \tilde{G}_D h + \tilde{G}_D(b \cdot \nabla) G_D h = -f, \end{aligned} \quad (3.15)$$

which, thanks to (2.38), concludes the proof. $\square$

We shall also observe the following localization principle for our perturbation problem.

Corollary 3.9. Operators $\tilde{G}_D$ and $(\tilde{P}_t^D, t > 0)$ do not depend on the values of b on D^c .

Proof. If b_1 and b_2 are two (bounded) vector fields on $\mathbb{R}^d$ equal to b on D , and $\tilde{G}_D^{(1)}$ and $\tilde{G}_D^{(2)}$ are the corresponding Green's operators, then, by (3.12),

$$\tilde{G}_D^{(1)}(I - H) = \tilde{G}_D^{(2)}(I - H),$$

which identifies $\tilde{G}_D := \tilde{G}_D^{(1)} = \tilde{G}_D^{(2)}$ as operators on $L^2(D)$, hence also as functions defined in Sect. 2.6. A similar conclusion for $\tilde{P}_t^D$ follows from the fact that $\tilde{G}_D^{-1}$ is the generator of the semigroup. $\square$

We will identify the adjoint operator of H on $L^2(D)$.

Lemma 3.10. $H^* = -G_D b \cdot \nabla$.

Proof. If $f, g \in L^2(D)$, then by Corollary 3.7 and (2.25),

$$(Hf, G_D g) = (b \cdot \nabla G_D f, G_D g) = -(b \cdot \nabla G_D g, G_D f) = -(f, G_D b \cdot \nabla G_D g).$$

Functions $\{G_D g, g \in L^2(D)\}$ form a dense set in $L^2(D)$, which ends the proof. $\square$

When defining $\tilde{P}_t^D$, via (2.31), we may encounter the situation when b is given only on D . We may extend the field outside D by letting e.g. $b = 0$ on D^c . Of course such an extension needs not satisfy $\operatorname{div} b = 0$, even though the condition may hold on D . However, we still have a local analogue of Proposition 2.1. Recall that the transition probability densities and Green function corresponding to the vector field $-b$ are marked with a hash (#).

Corollary 3.11. *If $\operatorname{div} b = 0$ on D , then for all $t > 0$ and $x, y \in D$,*

$$\tilde{p}_D^\#(t, x, y) = \tilde{p}_D(t, y, x), \quad (3.16)$$

and

$$\tilde{G}_D^\#(x, y) = \tilde{G}_D(y, x). \quad (3.17)$$

Proof. We shall first prove (3.17), or, equivalently, that

$$\tilde{G}_D^\# = \tilde{G}_D^*, \quad (3.18)$$

where $\tilde{G}_D^*$ is adjoint to $\tilde{G}_D$ on $L^2(D)$. From (2.38) applied to $\tilde{G}_D^\#$ we have,

$$(\Delta^{\alpha/2} - b \cdot \nabla) \tilde{G}_D^\# = -I.$$

Applying the operator G_D from the left to both sides of the equality we obtain

$$\tilde{G}_D^\# + G_D(b \cdot \nabla) \tilde{G}_D^\# = G_D. \quad (3.19)$$

By Lemma 3.10,

$$G_D = (I - H^*) \tilde{G}_D^\#. \quad (3.20)$$

Taking adjoints of both sides of (3.12) we also obtain,

$$G_D = (I - H^*) G_D^*. \quad (3.21)$$

We have already noted in the proof of Theorem 3.6 that $I - H^*$ is a linear automorphism of $L^2(D)$. Therefore (3.20) and (3.21) give (3.18). Furthermore, let L^* denote the adjoint of L on $L^2(D)$. We have

$$L^\# = (\tilde{G}_D^\#)^{-1} = (G_D^*)^{-1} = L^*, \quad (3.22)$$

where the last equality follows from [18, Lemma XII.1.6]. Let $(\tilde{P}_t^{D*})$ be the semigroup adjoint to $(\tilde{P}_t^D)$. By [33, Corollary 4.3.7], the generator of $(\tilde{P}_t^{D*})$ is L^* . Since $L^\# = L^*$, the semigroups are equal. The corresponding kernels are defined pointwise, therefore they satisfy (3.16). $\square$

4. Krein–Rutman eigen-pair

Lemma 4.1. *Operator $\tilde{P}_t^D$ is Hilbert-Schmidt on $L^2(D)$ for each $t > 0$.*

Proof. Let $\{e_n, n \geq 0\}$ be an orthonormal base in $L^2(D)$. By (2.32) and Plancherel's identity, we bound the Hilbert-Schmidt norm as follows. Using (2.23) and (2.6)

$$\begin{aligned} & \sum_{n=0}^{+\infty} \|\tilde{P}_t^D e_n\|_2^2 \\ &= \int_D \sum_{n=0}^{+\infty} \left(\int_D \tilde{p}_D(t, x, y) e_n(y) dy \right)^2 dx \\ &= \int_D dx \int_D \tilde{p}_D^2(t, x, y) dy \leq \int_{\mathbb{R}^d} dx \int_D \tilde{p}^2(t, x, y) dy \\ &\leq c_t^2 |D| \int_{\mathbb{R}^d} p_t^2(x) dx < +\infty. \end{aligned}$$

□

Lemma 4.2. $\tilde{G}_D$ is compact on $L^2(D)$.

Proof. Let $N > 0$, and $\tilde{G}_D^{(N)}(x, y) = \tilde{G}_D(x, y) \wedge N$. The integral operator $\tilde{G}_D^{(N)}$ on $L^2(D)$ corresponding to the kernel $\tilde{G}_D^{(N)}(x, y)$ is compact. Indeed, it has a finite Hilbert-Schmidt norm:

$$\int_D \int_D [\tilde{G}_D^{(N)}(x, y)]^2 dx dy \leq N^2 |D|^2 < \infty.$$

The norm of $\tilde{G}_D - \tilde{G}_D^{(N)}$ on $L^2(D)$ may be directly estimated as follows,

$$\begin{aligned} \|\tilde{G}_D - \tilde{G}_D^{(N)}\|^2 &\leq \left\{ \sup_x \int_D \left| \tilde{G}_D(x, y) - \tilde{G}_D^{(N)}(x, y) \right| dy \right\} \\ &\quad \times \left\{ \sup_y \int_D \left| \tilde{G}_D(x, y) - \tilde{G}_D^{(N)}(x, y) \right| dx \right\}, \end{aligned} \quad (4.1)$$

see, e.g., Theorem 3, p. 176 of [32]. We let $N \rightarrow \infty$. The functions $\tilde{G}_D(x, \cdot)$ (and $\tilde{G}_D(\cdot, y)$) are uniformly integrable on D by (2.35). Since $\tilde{G}_D$ is approximated in the norm topology by compact operators, it is compact. □

In the special case when $b(x) \equiv 0$, $(\tilde{P}_t^D)$ equals (P_t^D) , a symmetric contraction semigroup on $L^2(D)$, whence the Green operator G_D is symmetric, compact and positive definite. The spectral theorem yields the following.

Corollary 4.3. G_D^β is symmetric and compact for every $\beta > 0$.

By (2.36) and (2.12), $\tilde{G}_D$ is irreducible. Krein-Rutman theorem (see [29]) implies that there exists a unique nonnegative $\phi \in L^2(D)$ and a number $\lambda > 0$ such that $\|\phi\|_2 = 1$ and

$$\tilde{G}_D \phi = \frac{1}{\lambda} \phi, \quad (4.2)$$

We shall call (λ, ϕ) the *principal eigenpair* corresponding to L . From (2.38) we have $\phi \in \mathcal{D}(L)$ and $L\phi = -\lambda\phi$. Furthermore,

$$\tilde{P}_t^D \phi = e^{-\lambda t} \phi, \quad t \geq 0.$$

The formula (2.37) yields extra regularity of ϕ , as follows.

Lemma 4.4. *If (4.2) holds for some $\phi \in L^1(D)$ and $\lambda \neq 0$, then $\phi \in C(\bar{D})$ and there is $C = C(\alpha, b, D, \lambda)$ such that*

$$|\phi(x)| \leq C \|\phi\|_1 \delta_D^{\alpha/2}(x), \quad x \in D. \quad (4.3)$$

Proof. Starting from (4.2), for an arbitrary integer $n \geq 1$ we obtain

$$\phi(x) = \lambda^n \int_D \tilde{G}_D^{(n)}(x, y) \phi(y) dy,$$

where $\tilde{G}_D^{(1)}(x, y) := \tilde{G}_D(x, y)$ and

$$\tilde{G}_D^{(n+1)}(x, y) := \int_D \tilde{G}_D(x, z) \tilde{G}_D^{(n)}(z, y) dz.$$

To estimate $\tilde{G}_D^{(n)}$, we use basic properties of the Bessel potentials, which can be found in [2, Ch.II.§4]. Recall that for $\alpha > 0$ the Bessel potential kernel $\mathcal{G}_\alpha$ is the unique, extended-continuous, probability density function on $\mathbb{R}^d$, whose Fourier transform is

$$\hat{\mathcal{G}}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}^d.$$

Thus, $\mathcal{G}_\alpha * \mathcal{G}_\beta = \mathcal{G}_{\alpha+\beta}$ for all $\alpha, \beta > 0$, see (4.7) of ibid. If $\alpha < d$, then by [2, (4.2)], $\mathcal{G}_\alpha(x)$ is locally comparable with $|x|^{\alpha-d}$. By (2.35) there is a constant $c > 0$ such that

$$\tilde{G}_D(x, y) \leq c \mathcal{G}_\alpha(y - x), \quad x, y \in \mathbb{R}^d,$$

Hence, $\tilde{G}_D^{(n)}(x, y) \leq c^n \mathcal{G}_{n\alpha}(y - x)$, which is bounded if $n\alpha > d$, see [2, (4.2)] again. Considering such n we conclude that ϕ is bounded. The boundary decay of ϕ follows from (4.2), (2.36) and (2.13). The continuity of ϕ is a consequence of the continuity of $\tilde{G}_D(x, y)$ for $y \neq x$, and the uniform integrability of the kernel, which stems from (2.35). $\square$

5. Proof of Theorem 1.1

We say that w is a *first integral* of b if

$$\int_D w b \cdot \nabla \psi = 0, \quad \psi \in C_c^\infty(D), \quad (5.1)$$

cf. (1.2). We write $w \in \mathcal{I}_0^\alpha$ if $w \in H_0^{\alpha/2}(D)$ and w is not equal to 0 a.e.

Recall that for $A \in \mathbb{R}$, the operator $L_A = \Delta^{\alpha/2} + A b \cdot \nabla$ is considered with the Dirichlet exterior condition on D , i.e. it acts on $G_D(L^2(D))$, see Theorem 3.6. The Green operator and Krein–Rutman eigen-pair of L_A shall be denoted by $\tilde{G}_A$ and (λ_A, ϕ_A) , respectively. We also recall that $\phi_A \in \mathcal{D}(L_A)$ and

$$\mathcal{D}(L_A) = \mathcal{D}(\Delta^{\alpha/2}) \subset H_0^1(D) \subset H_0^{\alpha/2}(D). \quad (5.2)$$

The proof of (1.3) shall be obtained by demonstration of lower and upper bounds for λ_A (as $A \rightarrow \infty$).

5.1. Proof of the lower bound (1.4)

Proposition 5.1. *If $f \in \mathcal{D}(L_A)$, then $((-L_A)f, f) = \mathcal{E}^\alpha(f, f)$.*

Proof. Let $f \in \mathcal{D}(L_A)$. According to Proposition 3.8, f belongs to $\mathcal{D}(\Delta^{\alpha/2})$ and $H_0^1(D)$. In addition, $L_A f = \Delta^{\alpha/2} f + Ab \cdot \nabla f$. Taking the scalar product of both sides of the equality against f and using (2.25) we get the result, because the second term vanishes. $\square$

According to Proposition 5.1,

$$\lambda_A = ((-L_A)\phi_A, \phi_A) = \mathcal{E}^\alpha(\phi_A, \phi_A) = (G_D^{-1}\phi_A, \phi_A) = \|G_D^{-1/2}\phi_A\|_2^2. \quad (5.3)$$

Suppose $A_n \rightarrow +\infty$, as $n \rightarrow \infty$, but λ_{A_n} stay bounded. By (5.3) and Corollary 4.3, the sequence $\phi_{A_n} = G_D^{1/2} G_D^{-1/2} \phi_{A_n}$ is pre-compact in $L^2(D)$. Suppose that w is a weak limit of ϕ_{A_n} in $H_0^{\alpha/2}(D)$, thus a strong limit in $L^2(D)$. We have $\|w\|_2 = 1$, and for $\psi \in C_c^\infty(D)$,

$$\lambda_{A_n}(\phi_{A_n}, \psi) = \mathcal{E}^\alpha(\phi_{A_n}, \psi) - A_n \int_D \phi_{A_n} b \cdot \nabla \psi dx.$$

Dividing both sides by A_n and passing to the limit, we obtain that

$$\int_D w b \cdot \nabla \psi dx = 0,$$

thus $w \in \mathcal{I}_0^\alpha$. Fatou's lemma and (5.3) yield $\liminf_{n \rightarrow +\infty} \lambda_{A_n} \geq \mathcal{E}^\alpha(w, w)$, therefore (1.4) follows.

5.2. The proof of the upper bound (1.5)

The proof of (1.5) uses “conditioning” of truncations of w^2 by the principal eigenfunction inspired by [4] and [9] [see (5.21) below]. Here w is a first integral in $H_0^{\alpha/2}(D)$. An important part of the procedure is to prove that the truncation of w^2 is also a first integral in $H_0^{\alpha/2}(D)$. This is true if $w \in H_0^1(D)$: suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ and f' are bounded and $f(0) = 0$. Let $\operatorname{div} b = 0$ and that $\operatorname{div}(wb) = b \cdot \nabla w = 0$ a.e. Then, a.e. we have

$$\operatorname{div}(f(w)b) = f(w) \operatorname{div} b + f'(w)b \cdot \nabla w = 0 \quad \text{on } D, \quad (5.4)$$

thus $f(w)$ is also a first integral of b . However, the a.e. differentiability of $w \in H_0^{\alpha/2}(D)$ is not guaranteed for $\alpha < 2$. In fact, for $w \in H_0^{\alpha/2}(D)$, the condition $\operatorname{div}(wb) = 0$ is understood in the sense of distributions theory, cf (1.2), and the calculations in (5.4) may only serve as a motivation. To build up tools for a rigorous proof of the distributional version of (5.4), in Sect. 5.2.1 for a Sobolev-regular, divergence-free vector field $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, we consider the incompressible flow $\{X(t, \cdot), t \in \mathbb{R}\}$ of mappings on $\mathbb{R}^d$ constructed by DiPerna and Lions in [17, Theorem III.1] (the integral curves of b). The flow is used in Lemma 5.2 to characterize the first integrals in $H_0^\alpha(D)$ as those locally invariant under the flow. Then, we are able to conclude that the composition of a first integral in $H_0^\alpha(D)$ with a Lipschitz function is also a first integral in $H_0^\alpha(D)$, see Corollary 5.3, and, finally, use the conditioning of w^2 .

5.2.1. Flows corresponding to Sobolev regular drifts. Unless stated otherwise, in this section we consider general $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $b \in W_{\text{loc}}^{1,1}(\mathbb{R}^d)$ and $\operatorname{div} b = 0$ a.e. on $\mathbb{R}^d$. According to [17, Theorem III.1], there exists a unique a.e. defined jointly Borelian family of mappings $X(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}^d$ (flow generated by b) with the following properties: first, for a.e. $x \in \mathbb{R}^d$, the function $\mathbb{R} \ni t \mapsto b(X(t, x))$ is continuous and

$$X(t, x) = x + \int_0^t b(X(s, x)) ds, \quad t \in \mathbb{R}, \quad (5.5)$$

so that, in particular,

$$X(0, x) = x \quad \text{and} \quad X(t, X(s, x)) = X(t + s, x), \quad s, t \in \mathbb{R}, \quad (5.6)$$

second, for all $t \in \mathbb{R}$ and Borel measurable sets $A \subset \mathbb{R}^d$,

$$m_d(X(t, A)) = m_d(A), \quad (5.7)$$

where m_d is the d -dimensional Lebesgue measure, and, third, for all $s \in \mathbb{R}^d$,

$$\lim_{t \rightarrow s} \|X(t) - X(s)\|_{L^1(B(0, R))} = 0, \quad R > 0. \quad (5.8)$$

By (5.6) and (5.7), if f, g are nonnegative, then

$$\int_{\mathbb{R}^d} f(X(t, x)) g(x) dx = \int_{\mathbb{R}^d} f(x) g(X(-t, x)) dx. \quad (5.9)$$

We let $p \in [1, +\infty]$, $u_0 \in L^p(\mathbb{R}^d)$, and define

$$u(t, x) = u_0(X(t, x)), \quad t \in \mathbb{R}, x \in \mathbb{R}^d. \quad (5.10)$$

Note that, $(t, u_0(\cdot)) \mapsto u(t, \cdot)$, $t \in \mathbb{R}$, defines a group of isometries on $L^p(\mathbb{R}^d)$. If $v \in C_c^\infty(\mathbb{R}^{d+1})$, then

$$v(t, X(t, x)) = v(0, x) + \int_0^t \partial_s v(s, X(s, x)) ds + \int_0^t \operatorname{div}(vb)(X(s, x)) ds, \quad (5.11)$$

for a.e. $x \in \mathbb{R}^d$ and all $t \in \mathbb{R}$, because $\operatorname{div} b = 0$ implies $b \cdot \nabla v = \operatorname{div}(vb)$. Then for all $t \in \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}^d} u(t, x) v(t, x) dx &= \int_{\mathbb{R}^d} u_0(x) v(t, X(-t, x)) dx = \int_{\mathbb{R}^d} u_0(x) v(0, x) dx \\ &+ \int_0^t \int_{\mathbb{R}^d} u(s, x) \partial_s v(s, x) dx ds - \int_0^t \int_{\mathbb{R}^d} u(s, x) \operatorname{div}[v(s, x)b(x)] dx ds. \end{aligned}$$

Let $g \in C_c^\infty(B(0, 1))$, $g \geq 0$ and $\int_{\mathbb{R}^d} g dx = 1$ (a mollifier). Define

$$u^{(\varepsilon)}(t, x) := \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} u(t, x - y) g\left(\frac{y}{\varepsilon}\right) dy. \quad (5.12)$$

The function is an approximate solution of the transport equation

$$\frac{\partial u}{\partial t} - b \cdot \nabla u = 0, \quad (5.13)$$

in the following sense: if

$$r^{(\varepsilon)}(t, x) := \partial_t u^{(\varepsilon)}(t, x) - b(x) \cdot \nabla_x u^{(\varepsilon)}(t, x), \quad (5.14)$$

$q \in [1, +\infty]$, $1/q + 1/p = 1$ and $b \in W_{\text{loc}}^{1,q}(\mathbb{R}^d)$, then for all $t \in \mathbb{R}$ and finite $R > 0$,

$$\lim_{\varepsilon \rightarrow 0+} \int_0^t \|r^{(\varepsilon)}(s, \cdot)\|_{L^q(B(0,R))} ds = 0, \quad (5.15)$$

This follows from [17, Theorem II.1]. We shall use $u^{(\varepsilon)}$ to characterize the first integrals $b \in H_0^{\alpha/2}(D)$ as those elements of $H_0^{\alpha/2}(D)$, which are constant along the flow $X(t, \cdot)$. A word of explanation may be helpful. Suppose that $b \in W_{\text{loc}}^{1,q}(\mathbb{R}^d)$, $\text{div } b = 0$ a.e., and w is an L^p -integrable first integral, i.e. $b \cdot \nabla w = 0$, as distributions on space-time $\mathbb{R} \times \mathbb{R}^d$, see [17, (13)]. By [17, Corollary II.1], there is a unique solution to the transport equation (5.13) with the initial condition $u(0, \cdot) = w$. The equation is understood in the sense of distributions on space-time, too. By [17, Theorem III.1], the solution has the form $(t, x) \mapsto w(X(t, x))$. However, since w is a first integral, the mapping $(t, x) \mapsto w(x)$ defines another solution. Thus, by uniqueness, $w(X(t, x)) = w(x)$ a.e. In our case this argument needs to be slightly modified since the first integral is defined only on D and not on the entire $\mathbb{R}^d$. In particular, the identity $w(X(t, x)) = w(x)$ is bound to hold only for small times t .

Lemma 5.2. *Let $q = 2d/(d + \alpha)$, $b \in L^\infty(\mathbb{R}^d) \cap W_{\text{loc}}^{1,q}(\mathbb{R}^d)$, and $w \in H_0^{\alpha/2}(D)$. Then $w \in \mathcal{I}^\alpha$ if and only for every $\rho > 0$ there is $\kappa > 0$ such that*

$$w(X(t, x)) = w(x) \quad \text{for } |t| < \kappa \text{ and a.e. } x \in D \text{ satisfying } \delta_D(x) > \rho. \quad (5.16)$$

Proof. Suppose that $w \in H_0^{\alpha/2}(D)$ satisfies (5.16) with $\kappa > 0$ and let $u_0 \in C_c^\infty(D)$. Choose $\rho > 0$ so that $\delta_{D^c}(x) > \rho$ for all x in the support of u_0 . Using (5.16) and (5.9), for all $|t| < \kappa$ we obtain,

$$\int_D w(x) u_0(x) dx = \int_D w(X(-t, x)) u_0(x) dx = \int_D w(x) u_0(X(t, x)) dx. \quad (5.17)$$

Applying (5.11), we rewrite the rightmost side of (5.17) to obtain,

$$\begin{aligned} & \int_D w(x) u_0(x) dx \\ &= \int_D w(x) u_0(x) dx + \int_D w(x) \int_0^t b(X(s, x)) \cdot \nabla u_0(X(s, x)) ds dx. \end{aligned}$$

As a result,

$$0 = \int_D w(x) \left\{ \frac{1}{t} \int_0^t b(X(s, x)) \cdot \nabla u_0(X(s, x)) ds \right\} dx.$$

Letting $t \rightarrow 0$ we see that

$$0 = \int_D w(x) b(x) \cdot \nabla u_0(x) dx.$$

The limiting passage is justified by boundedness of b , integrability of w [cf. the discussion preceding (5.5)] and dominated convergence theorem. Since $u_0 \in C_c^\infty(D)$ is arbitrary, we conclude that $w \in \mathcal{I}_0^\alpha$.

Conversely, let us assume (5.1) for some $w \in H_0^{\alpha/2}(D)$. By Sobolev embedding theorem [36, Theorem V.1, p. 119], we have that $w \in L^p(D)$, where $p = 2d/(d - \alpha)$. Let $q = 2d/(d + \alpha)$, $\tilde{w} = w$ on D and $\tilde{w} = 0$ on D^c . Note that $1/p + 1/q = 1$. Let $\varepsilon > 0$ and t be such that

$$\varepsilon + |t| \|b\|_\infty < \text{dist}(\text{supp } u_0, D^c).$$

By (5.5) and (5.12), both $u(t, \cdot)$ and $u^{(\varepsilon)}(t, \cdot)$ are supported in D . Using (5.14) and (5.1) we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \tilde{w}(x) u^{(\varepsilon)}(t, x) dx &= \int_D w(x) r^{(\varepsilon)}(t, x) dx + \int_D w(x) b(x) \cdot \nabla_x u^{(\varepsilon)}(t, x) dx \\ &= \int_D w(x) r^{(\varepsilon)}(t, x) dx. \end{aligned}$$

Therefore,

$$\int_D w(x) u^{(\varepsilon)}(t, x) dx - \int_D w(x) u_0(x) dx = \int_0^t ds \int_D w(x) r^{(\varepsilon)}(s, x) dx. \quad (5.18)$$

Since $b \in W_{\text{loc}}^{1,q}(\mathbb{R}^d)$, the remainder $r_\varepsilon(t, x)$ satisfies (5.15). By Hölder inequality, the right hand side of (5.18) tends to 0, as $\varepsilon \rightarrow 0$. This proves (5.16) for $|t| < \kappa$. $\square$

As an immediate consequence of Lemma 5.2 we obtain the following.

Corollary 5.3. *If f is Lipschitz on $\mathbb{R}$, $f(0) = 0$ and $w \in \mathcal{I}_0^\alpha$, then either $f(w) = 0$ or $f(w) \in \mathcal{I}_0^\alpha$.*

Proof. By [22, Theorem 1.4.2 (v)], $f(w) \in H_0^{\alpha/2}(D)$. We apply f to (5.16), and use Lemma 5.2. $\square$

In particular, we may consider truncations of w at the level $N > 0$,

$$w_N := (w \wedge N) \vee (-N).$$

Corollary 5.4. *If $N > 0$ and $w \in \mathcal{I}_0^\alpha$, then $w_N \in \mathcal{I}_0^\alpha$.*

5.2.2. The upper bound when the drift is defined on entire $\mathbb{R}^d$. We shall first prove the upper bound (1.5) under the assumptions that b is bounded and of zero divergence on the whole of $\mathbb{R}^d$, and $b \in W_{\text{loc}}^{1,2d/(d+\alpha)}(\mathbb{R}^d)$.

Proposition 5.5. *Suppose that $A \in \mathbb{R}$, $\varepsilon > 0$, $w \in \mathcal{I}_0^\alpha$ and w is bounded. Then,*

$$\begin{aligned} \lambda_A \int_D \frac{\phi_A(z) w^2(z)}{\phi_A(z) + \varepsilon} dz \\ = \frac{1}{2} \int_{D^2} \left\{ \frac{w^2(x)}{\phi_A(x) + \varepsilon} - \frac{w^2(y)}{\phi_A(y) + \varepsilon} \right\} [\phi_A(x) - \phi_A(y)] \nu(x - y) dx dy. \end{aligned} \quad (5.19)$$

Proof. We denote $\psi = w^2/(\phi_A + \varepsilon)$. By [22, Theorem 1.4.2 (ii), (iv)], $\psi \in H_0^{\alpha/2}(D)$. Also, $\log(\phi_A + \varepsilon) - \log \varepsilon \in H_0^1(D)$. Considering (2.16), we observe

that the right hand side of (5.19) equals $\mathcal{E}_\alpha(\phi_A, \psi)$. By Proposition 3.8, the left hand side of (5.19) is

$$-\int_D L_A \phi_A(z) \psi(z) dz = -\int_D \Delta^{\alpha/2} \phi_A(z) \psi(z) dz - \int_D b(z) \cdot \nabla \phi_A(z) \psi(z) dz.$$

However, the second term on the right hand side vanishes, because

$$-\int_D w^2(z) b(z) \cdot \nabla \log[\phi_A(z) + \varepsilon] dz = 0,$$

and $w^2 = w_N^2$, for a sufficiently large N , is a first integral by virtue of Corollary 5.4. Thus (5.19) follows from (2.19). $\square$

The following elementary identity holds for functions u, v ,

$$\begin{aligned} [u(x) - u(y)]^2 + u^2(x) \frac{v(y) - v(x)}{v(x)} + u^2(y) \frac{v(x) - v(y)}{v(y)} \\ = v(x)v(y) \left[\frac{u(x)}{v(x)} - \frac{u(y)}{v(y)} \right]^2 \geq 0. \end{aligned} \quad (5.20)$$

Let $N, \varepsilon > 0$, $u = w_N$, and $v = \phi_A + \varepsilon$. By (5.20),

$$[w_N(x) - w_N(y)]^2 \geq \left\{ \frac{w_N^2(x)}{\phi_A(x) + \varepsilon} - \frac{w_N^2(y)}{\phi_A(y) + \varepsilon} \right\} [\phi_A(x) - \phi_A(y)]. \quad (5.21)$$

Multiplying both sides by $\nu(x - y) dx dy$ and integrating over D^2 , we obtain

$$2\mathcal{E}^\alpha(w_N, w_N) \geq \int_{D^2} \left\{ \frac{w_N^2(x)}{\phi_A(x) + \varepsilon} - \frac{w_N^2(y)}{\phi_A(y) + \varepsilon} \right\} [\phi_A(x) - \phi_A(y)] \nu(x - y) dx dy.$$

Using (5.19), we see that

$$\mathcal{E}^\alpha(w_N, w_N) \geq \lambda_A \int_{\mathbb{R}^d} \frac{w_N^2(x) \phi_A(x) dx}{\phi_A(x) + \varepsilon}.$$

Letting $\varepsilon \rightarrow 0+$, we conclude that $\mathcal{E}^\alpha(w_N, w_N) \geq \lambda_A \|w_N\|_2^2$. Letting $N \rightarrow +\infty$ and using [22, part (iii) of Theorem 1.4.2], we obtain $\mathcal{E}^\alpha(w, w) \geq \lambda_A \|w\|_2^2$. This proves (1.5).

5.2.3. The upper bound when the drift is defined only on D . Suppose that $b \in W^{1,q}(D)$, with $q = 2d/(d + \alpha)$, and $\operatorname{div} b(x) \equiv 0$ on D . Here, as usual, D is a bounded domain with the $C^{1,1}$ class boundary. By the discussion in Sect. 2.1, $\mathbb{R}^d \setminus \bar{D}$ has finitely many, say, $N + 1$ connected components. Denote them by $\mathcal{O}_0, \dots, \mathcal{O}_N$, and assume that ∞ belongs to the compactification of $\mathcal{O}_0$. We start with the following extension result.

Proposition 5.6. *There exist $\delta > 0$ and $\tilde{b} \in W^{1,q}(\mathbb{R}^d)$ such that*

$$\operatorname{div} \tilde{b} \in L^\infty(\mathbb{R}^d), \quad \tilde{b} = b \quad \text{on } D, \quad (5.22)$$

$\tilde{b}$ has compact support and is smooth outside of $D_\delta := \{x \in \mathbb{R}^d : \operatorname{dist}(x, D) < \delta\}$.

Proof. According to the results of Section 4 of [28] we can find $h_{ij} \in W^{2,q}(\mathbb{R}^d)$, supported in D_δ and such that $h_{ij}(\cdot) = -h_{ji}(\cdot)$ for $i, j = 1, \dots, d$, numbers $\lambda_1, \dots, \lambda_N \in \mathbb{R}$, and points $y^{(1)} \in \mathcal{O}_1, \dots, y^{(N)} \in \mathcal{O}_N$ such that

$$b_i(x) = \sum_{j=1}^d \partial_j h_{ji}(x) + \sum_{j=1}^N \lambda_j \frac{x_i - y_i^{(j)}}{|x - y^{(j)}|^d}, \quad \text{for } x \in D, i = 1, \dots, d.$$

Here $b = (b_1, \dots, b_d)$. We denote by $\hat{b}_i$ the right hand side of the above equality. The field $\hat{b}$ is incompressible on $\mathbb{R}^d \setminus \{y^{(1)}, \dots, y^{(d)}\}$. We choose $\delta > 0$ so that $\text{dist}(y^{(j)}, D) > 4\delta$ for $j = 1, \dots, N$. Let $\phi \in C^\infty$ equal to 1 on $D_{\delta/2}$ and vanish outside of D_δ . The field $\tilde{b} := \hat{b}\phi$ has the desired properties. $\square$

We consider the flow X of measurable mappings generated by $\tilde{b}$, which satisfies (5.5), (5.6) and (5.8), see Theorem III.2 of [17]. Condition (5.7) needs to be modified as follows: there exists $C > 0$, such that

$$e^{t/C} m_d(A) \leq m_d(X(t, A)) \leq e^{Ct} m_d(A), \quad A \subset \mathbb{R}^d, t \in \mathbb{R}. \quad (5.23)$$

Let g be a mollifier, let

$$b^{(\varepsilon)}(x) := \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \tilde{b}(x - y) g\left(\frac{y}{\varepsilon}\right) dy,$$

and let X_ε be the flow generated by $b^{(\varepsilon)}$. It has been shown in Section 3 of [17] that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [-T, T]} \int_{B(0, N)} |\phi(X(t, x)) - \phi(X_\varepsilon(t, x))| \wedge 2^N dx = 0 \quad (5.24)$$

for all $T, N > 0$ and measurable functions $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$. We shall need the following modification of Lemma 5.2.

Lemma 5.7. *If $b \in L^\infty(D) \cap W^{1, 2d/(d+\alpha)}(D)$ then, the conclusion of Lemma 5.2 holds for the flow X generated by $\tilde{b}$.*

Proof. Suppose that $w \in \mathcal{I}_0^\alpha$. Since $\tilde{b} = b$ on D , we can repeat the proof of the respective part of Lemma 5.2. Namely, keeping the notation from that lemma, for every $u_0 \in C_c^\infty(D)$ we have $\kappa > 0$ such that

$$\int_D w(x) u_0(X(-t, x)) dx = \int_D w(x) u_0(x) dx, \quad |t| < \kappa. \quad (5.25)$$

To complete the proof of (5.16) it suffices to conclude that $\kappa > 0$ can be so adjusted that

$$\int_D w(x) u_0(X(-t, x)) dx = \int_D u_0(x) w(X(t, x)) dx, \quad |t| < \kappa. \quad (5.26)$$

This part cannot be guaranteed directly from the definition of the flow, as the extended field $\tilde{b}$ needs not be divergence-free. Equality (5.26) holds however,

when $X(t)$ is replaced by $X_\varepsilon(t)$ for a sufficiently small $\varepsilon > 0$. Indeed, by the Liouville theorem, the Jacobian $\mathcal{J}X_\varepsilon(t, x)$ of $X_\varepsilon(t, x)$ satisfies

$$\begin{aligned} \frac{d}{dt} \mathcal{J}X_\varepsilon(t, x) &= \operatorname{div} b^{(\varepsilon)}(X_\varepsilon(t, x)) \mathcal{J}X_\varepsilon(t, x), \\ \mathcal{J}X_\varepsilon(0, x) &= 1, \quad t \in \mathbb{R}, x \in \mathbb{R}^d. \end{aligned}$$

Since $\operatorname{div} b^{(\varepsilon)} = 0$ in an open neighborhood of $\bar{D}$ we conclude that $\mathcal{J}X_\varepsilon(t, x) \equiv 1$ on D for (sufficiently small) $|t| < \kappa$. Since w and u_0 are supported in D ,

$$\int_D u_0(x) w(X_\varepsilon(t, x)) dx = \int_D w(x) u_0(X_\varepsilon(-t, x)) dx, \quad |t| < \kappa. \quad (5.27)$$

Letting $\varepsilon \rightarrow 0$ and using (5.24) we obtain (5.26). The rest of the proof follows that of Lemma 5.2. $\square$

Having established Lemma 5.7 we proceed with the proofs of Corollaries 5.3 and 5.4 and Proposition 5.5 with no alterations. These results yield (1.5).

Example 1. We consider the principal eigen-pair, say (λ_0, ϕ_0) , of $\Delta^{\alpha/2}$ for the unit ball in $\mathbb{R}^d$. By rotation invariance of $\Delta^{\alpha/2}$ and uniqueness, ϕ_0 is a smooth radial function in the ball. For $i, j = 1, \dots, d$ we take radially symmetric functions $h_{ij}(|x|^2) \in C_c^\infty(\mathbb{R}^d)$, such that $h_{ij} = -h_{ji}$. Let $b_i = \sum_{j=1}^d \partial_j h_{ji}$, $i = 1, \dots, d$. The vector field $b = (b_1, \dots, b_d)$ is of zero divergence and tangent to the spheres $|x| = r$ for all $r > 0$. Indeed, since $h'_{ij} = -h'_{ji}$,

$$b(x) \cdot x = \sum_{i,j=1}^d \partial_j h_{ji}(|x|^2) x_i = 2 \sum_{i,j=1}^d h'_{ji}(|x|^2) x_i x_j \equiv 0.$$

As a result $b(x) \cdot \nabla w(x) = 0$ for any C^1 smooth radially symmetric function $w(\cdot)$. Thus, we conclude that ϕ_0 is the principal eigenfunction of L_A for every A . We have $\lambda_A \equiv \lambda_0$, and $\mathcal{E}^\alpha(w, w)$ attains its infimum, λ_0 , at $w = \phi_0$, see Sect. 5.1. In passing we note that the considered limiting eigenproblems are essentially different for different values of α , in accordance with the fact that the “escape rate” $x \mapsto \int_{D^c} \nu(y-x) dy$ of the isotropic α -stable Lévy processes from D depends on α . We refer the interested reader to [3, 19, 31], for more information on the eigenproblem of $\Delta^{\alpha/2}$, see also [15].

5.3. Existence of a minimizer

To complete the proof of Theorem 1.1 we only need to explain the attainability of infimum appearing on the right hand side of (1.3). This is done in the following.

Lemma 5.8. *If $\{w \in \mathcal{I}_0^{\alpha/2} : \|w\|_2 = 1\} \neq \emptyset$, then $w \mapsto \mathcal{E}^\alpha(w, w)$ attains its infimum on the set.*

Proof. If $e_* < \infty$ is the infimum, then we can choose functions w_n in the set given in the statement of the lemma, such that $\mathcal{E}^\alpha(w_n, w_n) \rightarrow e_*$ and, by choosing a subsequence, that w_n weakly converge to $w_* \in H_0^{\alpha/2}(D)$. Since

$\{w_n\}$ is precompact in $L^2(D)$, we may further assume that $w_n \rightarrow w$ in $L^2(D)$ and a.e. This implies that $\|w_*\| = 1$, w_* is a first integral and, by (2.16) and Fatou's lemma, that $\mathcal{E}^\alpha(w_*, w_*) \leq e_*$. From the definition of e_* , we conclude that equality actually occurs. $\square$

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