An existence result for a class of quasilinear elliptic eigenvalue problems in unbounded domains

Kanishka Perera, Patrizia Pucci and Csaba Varga

Abstract. We consider a nonlinear eigenvalue problem under Robin boundary conditions in a domain with (possibly noncompact) smooth boundary. The problem involves a weighted p -Laplacian operator and subcritical nonlinearities satisfying Ambrosetti–Rabinowitz type conditions. Using Morse theory and a cohomological local splitting as in Degiovanni et al. (Commun Contemp Math 12:475–486, 2010), we prove the existence of a nontrivial weak solution for all (real) values of the eigenvalue parameter. Our result is new even in the semilinear case $p = 2$ and complements some recent results obtained in Autuori et al. (Adv Anal Equ 18:1–48, 2013).

Mathematics Subject Classification (2000). Primary 35J62, 35J66; Secondary 35J20, 35J70.

Keywords. Nonlinear eigenvalue problems, Unbounded domains, Weighted p -Laplacian, Robin boundary conditions, Nontrivial solutions, Morse theory, Cohomological local splitting.

1. Introduction

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$ with (possibly noncompact) smooth boundary $\partial\Omega$ and consider the problem

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x)|u|^{p-2}u + g(x, u) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\partial_\nu u + b(x)|u|^{p-2}u = h(x, u) & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $1 < p < N$ and $\partial_\nu = \partial/\partial\nu$ is the exterior normal derivative on $\partial\Omega$. We assume that $a \in C^1(\Omega) \cap L^\infty(\Omega)$, with $a \geq a_0$ for some constant $a_0 > 0$, that $b \in C(\partial\Omega)$ verifies

$$\frac{1}{C}(1 + |x|)^{1-p} \leq b(x) \leq C(1 + |x|)^{1-p} \quad \text{for all } x \in \partial\Omega,$$

and that f is a measurable function on Ω satisfying

$$0 < f(x) \leq C w_1(x) \quad \text{for a.a. } x \in \Omega,$$

where $w_1(x) = (1 + |x|)^{-\alpha_1}$, $p < \alpha_1 < N$. Here and in the sequel C denotes a generic positive constant.

If w is a weight on Ω (i.e., an a.e. positive measurable function), let $L^\sigma(\Omega, w)$, $\sigma \geq 1$, denote the weighted Lebesgue space with the norm

$$\|u\|_{\sigma, w} = \left(\int_{\Omega} w(x) |u(x)|^\sigma dx \right)^{1/\sigma}.$$

Similarly, if $\tilde{w}$ is a weight on $\partial\Omega$, $L^\sigma(\partial\Omega, \tilde{w})$ denotes the weighted space with the norm

$$\|u\|_{\sigma, \tilde{w}, \partial\Omega} = \left(\int_{\partial\Omega} \tilde{w}(x) |u(x)|^\sigma dS \right)^{1/\sigma}.$$

We assume that

(g₁) g is a Carathéodory function on $\overline{\Omega} \times \mathbb{R}$ satisfying

$$|g(x, s)| \leq g_0(x) |s|^{p-1} + g_1(x) |s|^{r-1} \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$

where $p < r < p^* := Np/(N-p)$, g_0 and g_1 are measurable functions on Ω satisfying

$$0 < g_0(x) \leq C w_2(x), \quad 0 \leq g_1(x) \leq C g_0(x) \quad \text{for a.a. } x \in \Omega,$$

$w_2(x) = (1 + |x|)^{-\alpha_2}$, $N - (N-p)r/p < \alpha_2 < N$, $g_0/w_2 \in L^{r/(r-p)}(\Omega, w_2)$, and $g_0 \in L^{\tilde{r}/(\tilde{r}-p)}(\Omega, w_2)$, $r < \tilde{r} < \min\{pr, p^*\}$;

(g₂) $G(x, s) := \int_0^s g(x, t) dt$ satisfies uniformly for a.a. $x \in \Omega$

$$\lim_{s \rightarrow 0} \frac{G(x, s)}{g_0(x) |s|^p} = 0, \quad \lim_{|s| \rightarrow \infty} \frac{G(x, s)}{g_0(x) |s|^p} = \infty;$$

(g₃) there exists a $\mu > p$ such that for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$

$$\mu G(x, s) \leq s g(x, s) + \gamma_0(x) + \gamma_1(x) |s|^p,$$

where $\gamma_0 \in L^1(\Omega)$ and γ_1 is a measurable function on Ω satisfying

$$0 < \gamma_1(x) \leq C w_1(x) \quad \text{for a.a. } x \in \Omega;$$

(h₁) h is a Carathéodory function on $\partial\Omega \times \mathbb{R}$ satisfying

$$|h(x, s)| \leq h_0(x) |s|^{p-1} + h_1(x) |s|^{q-1} \quad \text{for a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R},$$

where $p < q < \bar{p} := (N-1)p/(N-p)$, h_0 and h_1 are measurable functions on $\partial\Omega$ satisfying

$$0 < h_0(x) \leq C w_3(x), \quad 0 \leq h_1(x) \leq C h_0(x) \quad \text{for a.a. } x \in \partial\Omega,$$

$w_3(x) = (1 + |x|)^{-\alpha_3}$, $N - 1 - (N-p)q/p < \alpha_3 < N$, $h_0/w_3 \in L^{q/(q-p)}(\partial\Omega, w_3)$, and $h_0 \in L^{\tilde{q}/(\tilde{q}-p)}(\partial\Omega, w_3)$, $q < \tilde{q} < \min\{qr, \bar{p}\}$;

(h_2) $H(x, s) := \int_0^s h(x, t) dt$ satisfies uniformly for a.a. $x \in \partial\Omega$

$$\lim_{s \rightarrow 0} \frac{H(x, s)}{h_0(x) |s|^p} = 0, \quad \lim_{|s| \rightarrow \infty} \frac{H(x, s)}{h_0(x) |s|^p} = \infty;$$

(h_3) there exists a $\tilde{\mu} > p$ such that for a.a. $x \in \partial\Omega$ and all $s \in \mathbb{R}$

$$\tilde{\mu} H(x, s) \leq s h(x, s) + \tau_0(x) + \tau_1(x) |s|^p,$$

where $\tau_0 \in L^1(\partial\Omega)$, τ_1 is a measurable functions on $\partial\Omega$ satisfying

$$0 < \tau_1(x) \leq C w_4(x) \quad \text{for a.a. } x \in \partial\Omega,$$

and $w_4(x) = (1 + |x|)^{-\alpha_4}$, $p - 1 < \alpha_4 < N$.

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)$ -functions restricted to Ω , and define the weighted Sobolev space E to be the completion of $C_\delta^\infty(\Omega)$ with respect to the norm

$$\|u\|_E = \left(\int_\Omega |\nabla u(x)|^p + \frac{|u(x)|^p}{(1 + |x|)^p} dx \right)^{1/p}.$$

By Pflüger [12, Lemma 2],

$$\|u\| = \left(\int_\Omega a(x) |\nabla u(x)|^p dx + \int_{\partial\Omega} b(x) |u(x)|^p dS \right)^{1/p}$$

is an equivalent norm on E .

A weak solution of problem (1.1) is a function $u \in E$ satisfying

$$\begin{aligned} & \int_\Omega a(x) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_{\partial\Omega} b(x) |u|^{p-2} u \varphi dS \\ &= \int_\Omega (\lambda f(x) |u|^{p-2} u + g(x, u)) \varphi dx + \int_{\partial\Omega} h(x, u) \varphi dS \end{aligned}$$

for all $\varphi \in E$.

Since $g(x, 0) = 0$ for a.a. $x \in \Omega$ by (g_1) and $h(x, 0) = 0$ for a.a. $x \in \partial\Omega$ by (h_1), problem (1.1) admits the trivial solution $u = 0$. The following existence result was recently obtained by Autuori et al. [1, Theorem 4.3].

Theorem 1.1. *In addition to (g_1)–(g_3) and (h_1)–(h_3), assume that*

$$\begin{aligned} & G(x, s) \geq 0 \text{ for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R} \text{ and } H(x, s) \geq 0 \text{ for} \\ & \text{a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R}. \end{aligned}$$

Then problem (1.1) has a nontrivial solution $u \in E$ for all $\lambda \in \mathbb{R}$.

The proof of this theorem was based on the sequence of eigenvalues $\lambda_k \nearrow \infty$ of the associated Robin boundary eigenvalue problem

$$\begin{cases} -\operatorname{div}(a(x) |\nabla u|^{p-2} \nabla u) = \lambda f(x) |u|^{p-2} u & \text{in } \Omega, \\ a(x) |\nabla u|^{p-2} \partial_\nu u + b(x) |u|^{p-2} u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

defined using the $\mathbb{Z}_2$ -cohomological index. In the present paper we obtain the following extension.

Theorem 1.2. *Assume (g_1) – (g_3) and (h_1) – (h_3) . Then problem (1.1) has a non-trivial solution $u \in E$ in each of the following cases:*

- (i) $\lambda \notin \{\lambda_k : k \geq 1\}$;
- (ii) $\lambda \in \{\lambda_k : k \geq 1\}$, $G(x, s) \geq 0$ for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$, and $H(x, s) \geq 0$ for a.a. $x \in \partial\Omega$ and all $s \in \mathbb{R}$;
- (iii) $\lambda \in \{\lambda_k : k \geq 1\}$, $G(x, s) \leq 0$ for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$, and $H(x, s) \leq 0$ for a.a. $x \in \partial\Omega$ and all $s \in \mathbb{R}$.

In (i) it is only assumed that λ is not an eigenvalue from the particular sequence $(\lambda_k)_k$, so we have the following corollary.

Corollary 1.3. *Assume (g_1) – (g_3) and (h_1) – (h_3) . Then problem (1.1) has a nontrivial solution $u \in E$ for a.a. $\lambda \in \mathbb{R}$.*

Weak solutions of problem (1.1) coincide with critical points of the C^1 -functional

$$\Phi(u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{p} \|u\|_{p,f}^p - \int_{\Omega} G(x, u) dx - \int_{\partial\Omega} H(x, u) dS, \quad u \in E. \quad (1.3)$$

Theorem 1.1 was proved in Autuori et al. [1] using a linking argument as in Degiovanni and Lancelotti [4]. The proof of Theorem 1.2 in this paper is based on the Morse theory as in Perera [9] and Degiovanni et al. [5].

2. Preliminaries

Let $I(u) = \|u\|^p$, $J(u) = \|u\|_{p,f}^p$, $u \in E$. Then the eigenfunctions of the problem (1.2) on the manifold

$$\mathcal{M} = \{u \in E : J(u) = 1\}$$

and the corresponding eigenvalues coincide with the critical points and the critical values, respectively, of the constrained functional $\tilde{I} := I|_{\mathcal{M}}$. It was shown in Autuori et al. [1, Proposition 3.4] that $\tilde{I}$ satisfies the (PS) condition, so we can define an increasing and unbounded sequence of eigenvalues by a minimax scheme. The standard scheme based on the Krasnosel'skiĭ's genus does not provide sufficient Morse theoretical information to prove Theorem 1.2, so we use the cohomological index as in [1]. Eigenvalues of the p -Laplacian based on a cohomological index were first introduced in Perera [9] for bounded domains (see also Perera and Szulkin [11]).

Let us recall the definition of the $\mathbb{Z}_2$ -cohomological index of Fadell and Rabinowitz [6]. For a symmetric subset M of $E \setminus \{0\}$, let $\bar{M} = M/\mathbb{Z}_2$ be the quotient space of M with each u and $-u$ identified, let $f : \bar{M} \rightarrow \mathbb{R}P^\infty$ be the classifying map of $\bar{M}$, and let $f^* : \mathcal{H}^*(\mathbb{R}P^\infty) \rightarrow \mathcal{H}^*(\bar{M})$ be the induced homomorphism of the Alexander–Spanier cohomology rings. Then the cohomological index of M is defined by

$$i(M) = \begin{cases} \sup \{m \geq 1 : f^*(\omega^{m-1}) \neq 0\}, & M \neq \emptyset \\ 0, & M = \emptyset, \end{cases}$$

where $\omega \in \mathcal{H}^1(\mathbb{R}P^\infty)$ is the generator of the polynomial ring $\mathcal{H}^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$, is the inclusion $\mathbb{R}P^{m-1} \subset \mathbb{R}P^\infty$, which induces isomorphisms on $\mathcal{H}^\kappa$ for $\kappa \leq m-1$, so $i(S^{m-1}) = m$.

Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$ and set

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \tilde{I}(u), \quad k \geq 1.$$

Hence $(\lambda_k)_k$ is a sequence of eigenvalues of (1.2), $\lambda_k \nearrow \infty$, and if $\lambda_k < \lambda_{k+1}$, then

$$\begin{aligned} i(\{u \in E : I(u) \leq \lambda_k J(u)\} \setminus \{0\}) \\ = i(E \setminus \{u \in E : I(u) \geq \lambda_{k+1} J(u)\}) = k \end{aligned} \quad (2.1)$$

(see, e.g., Perera et al. [10, Propositions 3.52 and 3.53]).

Recall that the cohomological critical groups at 0 of the functional Φ , defined in (1.3), are given by

$$C^\kappa(\Phi, 0) = \mathcal{H}^\kappa(\Phi^0 \cap U, \Phi^0 \cap U \setminus \{0\}), \quad \kappa \geq 0, \quad (2.2)$$

where $\Phi^0 = \{u \in E : \Phi(u) \leq 0\}$, U is any neighborhood of 0, and $\mathcal{H}$ denotes Alexander–Spanier cohomology with $\mathbb{Z}_2$ -coefficients. They are independent of U by the excision property.

In the absence of a direct sum decomposition, the main technical tool we use to get an estimate of the critical groups at zero is the notion of a cohomological local splitting introduced in Perera et al. [10], which is a variant of the homological local linking of Perera [8] (see also Li and Willem [7]). The following slightly different form of this notion was given in Degiovanni et al. [5].

Definition 2.1. We say that $\Phi \in C^1(E, \mathbb{R})$ has a cohomological local splitting near 0 in dimension $k \geq 1$ if there are symmetric cones E_+ , $E_- \subset E$, with $E_+ \cap E_- = \{0\}$, and $\rho > 0$ such that

$$i(E_- \setminus \{0\}) = i(E \setminus E_+) = k \quad (2.3)$$

and

$$\Phi(u) \leq \Phi(0) \text{ for all } u \in B_\rho \cap E_-, \quad \Phi(u) \geq \Phi(0) \text{ for all } u \in B_\rho \cap E_+, \quad (2.4)$$

where $B_\rho = \{u \in E : \|u\| \leq \rho\}$.

Proposition 2.2 ([5, Proposition 2.1]). *If $\Phi \in C^1(E, \mathbb{R})$ has a cohomological local splitting near 0 in dimension k and 0 is an isolated critical point of Φ , then $C^k(\Phi, 0) \neq 0$.*

3. Proof of Theorem 1.2

First we prove some lemmas. By Autuori et al. [1, Lemma 4.1],

$$\Phi(u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{p} \|u\|_{p,f}^p + o(\|u\|^p) \text{ as } \|u\| \rightarrow 0. \quad (3.1)$$

Lemma 3.1. *If 0 is an isolated critical point of Φ , then $C^\kappa(\Phi, 0) \approx \delta_{\kappa 0} \mathbb{Z}_2$ for all $\kappa \geq 0$ in the following cases:*

- (i) $\lambda < \lambda_1$;
- (ii) $\lambda = \lambda_1$, $G(x, s) \leq 0$ for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$, and $H(x, s) \leq 0$ for a.a. $x \in \partial\Omega$ and all $s \in \mathbb{R}$.

Proof. We show that Φ has a local minimizer at 0. Since $\lambda_1 = \inf_{u \in \mathcal{M}} \tilde{I}(u)$,

$$\|u\|^p \geq \lambda_1 \|u\|_{p,f}^p \quad \text{for all } u \in E. \quad (3.2)$$

- (i) For sufficiently small $\rho > 0$,

$$\Phi(u) \geq \left(1 - \frac{\max\{\lambda, 0\}}{\lambda_1} + o(1)\right) \frac{\|u\|^p}{p} \geq 0 \quad \text{for all } u \in B_\rho$$

by (3.1) and (3.2).

- (ii) We have

$$\Phi(u) \geq - \int_{\Omega} G(x, u) dx - \int_{\partial\Omega} H(x, u) dS \geq 0 \quad \text{for all } u \in E.$$

□

Lemma 3.2. *If 0 is an isolated critical point of Φ , then $C^k(\Phi, 0) \neq 0$ in the following cases:*

- (i) $\lambda_k < \lambda < \lambda_{k+1}$;
- (ii) $\lambda = \lambda_k < \lambda_{k+1}$, and $G(x, s) \geq 0$ for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$, and similarly $H(x, s) \geq 0$ for a.a. $x \in \partial\Omega$ and all $s \in \mathbb{R}$;
- (iii) $\lambda_k < \lambda_{k+1} = \lambda$, and $G(x, s) \leq 0$ for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$, and similarly $H(x, s) \leq 0$ for a.a. $x \in \partial\Omega$ and all $s \in \mathbb{R}$.

Proof. We show that Φ has a cohomological local splitting near 0 in dimension k and apply Proposition 2.2. Let

$$E_- = \{u \in E : \|u\|^p \leq \lambda_k \|u\|_{p,f}^p\}, \quad E_+ = \{u \in E : \|u\|^p \geq \lambda_{k+1} \|u\|_{p,f}^p\}.$$

Then (2.3) holds by (2.1), so it only remains to show that (2.4) holds for sufficiently small $\rho > 0$.

- (i) For sufficiently small $\rho > 0$,

$$\Phi(u) \leq - \left(\frac{\lambda}{\lambda_k} - 1 + o(1)\right) \frac{\|u\|^p}{p} \leq 0 \quad \text{for all } u \in B_\rho \cap E_-$$

and

$$\Phi(u) \geq \left(1 - \frac{\lambda}{\lambda_{k+1}} + o(1)\right) \frac{\|u\|^p}{p} \geq 0 \quad \text{for all } u \in B_\rho \cap E_+$$

by (3.1).

- (ii) For all $u \in E_-$,

$$\Phi(u) \leq - \int_{\Omega} G(x, u) dx - \int_{\partial\Omega} H(x, u) dS \leq 0,$$

and for sufficiently small $\rho > 0$ we have $\Phi(u) \geq 0$ for all $u \in B_\rho \cap E_+$ as in (i).

- (iii) For sufficiently small $\rho > 0$ we have $\Phi(u) \leq 0$ for all $u \in B_\rho \cap E_-$, as in (i). On the other hand, for all $u \in E_+$,

$$\Phi(u) \geq - \int_{\Omega} G(x, u) dx - \int_{\partial\Omega} H(x, u) dS \geq 0.$$

□

Lemma 3.3. *There exists $\alpha < 0$ such that $\mathcal{H}^\kappa(E, \Phi^\alpha) = 0$ for all $\kappa \geq 0$, where*

$$\Phi^\alpha = \{u \in E : \Phi(u) \leq \alpha\}.$$

Proof. Let us first note that by (g_1) – (g_2) and (h_1) – (h_2) there exists a constant $C > 0$ such that

$$\begin{aligned} |G(x, s)| &\leq Cw_2(x)|s|^p && \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}, \\ |H(x, s)| &\leq Cw_3(x)|s|^p && \text{for a.a. } x \in \partial\Omega \text{ and all } s \in \mathbb{R}. \end{aligned} \quad (3.3)$$

Next we show that for all $u \in E$, with $u \neq 0$,

$$\Phi(tu) \rightarrow -\infty \text{ as } t \rightarrow \infty. \quad (3.4)$$

Indeed, fix $u \in E$, with $u \neq 0$. Hence, either $|u| > 0$ in a subset A of Ω , with $\text{meas}(A) > 0$, or $|u| > 0$ in a subset Γ of $\partial\Omega$, with $\text{meas}_{N-1}(\Gamma) > 0$, being $w > 0$ a.e. in Ω and $\tilde{w} > 0$ a.e. in $\partial\Omega$.

Case $|u| > 0$ in A , $\text{meas}(A) > 0$. By (g_2) as $|t| \rightarrow \infty$

$$\frac{G(x, tu(x))}{|t|^p} = \frac{G(x, tu(x))}{g_0(x)|t|^p|u(x)|^p} g_0(x)|u(x)|^p \rightarrow \infty \quad \text{a.e. in } A,$$

and $|t|^{-p}|G(x, tu(x))| \leq Cw_2(x)|u|^p \in L^1(\Omega)$ by (3.3). Hence

$$\lim_{|t| \rightarrow \infty} |t|^{-p} \int_{\Omega} G(x, tu(x)) dx = \infty$$

by the Fatou lemma.

Case $|u| > 0$ in Γ , $\text{meas}_{N-1}(\Gamma) > 0$. Similarly, by (h_2) as $|t| \rightarrow \infty$

$$\frac{H(x, tu(x))}{|t|^p} = \frac{H(x, tu(x))}{h_0(x)|t|^p|u(x)|^p} h_0(x)|u(x)|^p \rightarrow \infty \quad \text{a.e. in } \Gamma,$$

and $|t|^{-p}|H(x, tu(x))| \leq Cw_3(x)|u|^p \in L^1(\partial\Omega)$ by (3.3). Thus

$$\lim_{|t| \rightarrow \infty} |t|^{-p} \int_{\partial\Omega} H(x, tu(x)) dS = \infty$$

again by the Fatou lemma.

In conclusion we have

$$\begin{aligned} \frac{\Phi(tu)}{|t|^p} &= \frac{1}{p} \left(\|u\|^p - \lambda \|u\|_{p,f}^p \right) - \int_{\Omega} \frac{G(x, tu(x))}{|t|^p} dx - \int_{\partial\Omega} \frac{H(x, tu(x))}{|t|^p} dS \\ &\rightarrow -\infty \end{aligned}$$

as $|t| \rightarrow \infty$, along any $u \in E$, with $u \neq 0$. This shows the claim (3.4).

Next we prove that there exists $\alpha < 0$ such that

$$(\Phi'(u), u) < 0 \quad \text{for all } u \in \Phi^\alpha. \quad (3.5)$$

Let us first claim that for all $c \in \mathbb{R}$

$$\limsup_{\|u\| \rightarrow \infty, \Phi(u) \leq c} \frac{(\Phi'(u), u) - p\Phi(u)}{\|u\|^p} < 0. \quad (3.6)$$

Fix $c \in \mathbb{R}$. As in Lemma 3.1 of [5] we assume by contradiction that there exist sequences $(d_n)_n \subset \mathbb{R}$ and $(u_n)_n \subset \Phi^c$ such that $d_n \rightarrow 0$ and

$$(\Phi'(u_n), u_n) - p\Phi(u_n) \geq -d_n \|u_n\|^p \quad \text{for all } n \in \mathbb{N}.$$

Clearly $v_n = u_n/\|u_n\|$ is in the unit sphere of E for all n sufficiently large. Therefore, up to a subsequence, still denoted for simplicity by $(v_n)_n$, there is $v \in E$ such that $v_n \rightharpoonup v$ in E , $v_n \rightarrow v$ in $L^p(\Omega, w)$, $v_n \rightarrow v$ in $L^p(\partial\Omega, \tilde{w})$, $v_n \rightarrow v$ a.e. in Ω and $v_n \rightarrow v$ a.e. in $\partial\Omega$ and $|v_n| \leq \phi$ a.e. in Ω for all n , with $\phi \in L^p(\Omega, w_1) \cap L^p(\partial\Omega, w_3)$.

By (g_3) and (h_3) we have

$$\begin{aligned} -d_n \|u_n\|^p &\leq (\Phi'(u_n), u_n) - p\Phi(u_n) = \int_{\Omega} [\mu G(x, u_n) - u_n g(x, u_n)] dx \\ &\quad + \int_{\partial\Omega} [\tilde{\mu} H(x, u_n) - u_n h(x, u_n)] dS - (\mu - p) \int_{\Omega} G(x, u_n) dx \\ &\quad - (\tilde{\mu} - p) \int_{\partial\Omega} H(x, u_n) dS \\ &\leq \int_{\Omega} (\gamma_0 + \gamma_1 |u_n|^p) dx + \int_{\partial\Omega} (\tau_0 + \tau_1 |u_n|^p) dS \\ &\quad - (\mu - p) \int_{\Omega} G(x, u_n) dx - (\tilde{\mu} - p) \int_{\partial\Omega} H(x, u_n) dS. \end{aligned}$$

Hence

$$\limsup_n \frac{(\mu - p) \int_{\Omega} G(x, u_n) dx + (\tilde{\mu} - p) \int_{\partial\Omega} H(x, u_n) dS}{\|u_n\|^p} < \infty. \quad (3.7)$$

Moreover, by (3.3)

$$\begin{aligned} \frac{|G(x, u_n)|}{\|u_n\|^p} &= \frac{|G(x, \|u_n\| v_n)|}{\|u_n\|^p} \leq C w_2 |v_n|^p \leq C w_2 \phi, \\ \frac{|H(x, u_n)|}{\|u_n\|^p} &\leq C w_3 \phi \end{aligned} \quad (3.8)$$

a.e. in Ω and a.e. on $\partial\Omega$, respectively. Hence by the Fatou lemma and (3.7)

$$\begin{aligned}
-\infty &< (\mu - p) \int_{\Omega} \liminf_n \frac{G(x, u_n)}{\|u_n\|^p} dx + (\tilde{\mu} - p) \int_{\partial\Omega} \liminf_n \frac{H(x, u_n)}{\|u_n\|^p} dS \\
&\leq \liminf_n \frac{(\mu - p) \int_{\Omega} G(x, u_n) dx + (\tilde{\mu} - p) \int_{\partial\Omega} H(x, u_n) dS}{\|u_n\|^p} < \infty.
\end{aligned}$$

On the other hand, as shown above, by (g_2) at any point $x \in \Omega$ at which $v(x) \neq 0$

$$\lim_n \frac{G(x, u_n(x))}{\|u_n\|^p} = \lim_n \frac{G(x, \|u_n\| v_n(x))}{g_0(x) \|u_n\|^p v_n(x)} g_0(x) v_n(x) = \infty,$$

and similarly by (h_2) at any point $x \in \partial\Omega$ at which $v(x) \neq 0$

$$\lim_n \frac{H(x, u_n(x))}{\|u_n\|^p} = \lim_n \frac{H(x, \|u_n\| v_n(x))}{h_0(x) \|u_n\|^p v_n(x)} h_0(x) v_n(x) = \infty.$$

In conclusion the limit v is trivial in E , that is $v = 0$ a.e. in Ω and on $\partial\Omega$ in the sense of the $N - 1$ measure on $\partial\Omega$. Hence by (3.8)

$$\lim_n \frac{\int_{\Omega} G(x, u_n) dx + \int_{\partial\Omega} H(x, u_n) dS}{\|u_n\|^p} = 0. \quad (3.9)$$

Now, passing to the limit as $n \rightarrow \infty$ in both sides of the following inequality

$$\frac{1}{p} - \lambda \|v_n\|_{p,f}^p - \frac{\int_{\Omega} G(x, u_n) dx + \int_{\partial\Omega} H(x, u_n) dS}{\|u_n\|^p} = \frac{\Phi(u_n)}{\|u_n\|^p} \leq \frac{c}{\|u_n\|^p},$$

we get $1/p \leq 0$ by (3.9). This contradiction completes the proof of the claim (3.6).

Now by (3.6), with $c = 0$, there exists $\beta \in \mathbb{R}$ such that $(\Phi'(u), u) - p\Phi(u) \leq \beta$ for all $u \in \Phi^0$. In particular, there exists $\alpha < 0$ such that (3.5) holds.

Fix $u \in E$, with $u \neq 0$. Therefore $\Phi(tu) \leq \alpha$ for all sufficiently large t by (3.4). Set

$$T(u) = \inf \{t \geq 1 : \Phi(tu) \leq \alpha\}.$$

Since

$$\Phi(tu) \leq \alpha \implies \frac{d}{dt} (\Phi(tu)) = (\Phi'(tu), u) = \frac{1}{t} (\Phi'(tu), tu) < 0$$

by (3.5), then

$$\Phi^\alpha = \{tu : u \in E \setminus \{0\}, t \geq T(u)\}$$

and the map $E \setminus \{0\} \rightarrow [1, \infty)$, $u \mapsto T(u)$, is continuous. Thus $E \setminus \{0\}$ radially deformation retracts to Φ^α via

$$(E \setminus \{0\}) \times [0, 1] \rightarrow E \setminus \{0\}, \quad (u, t) \mapsto (1 - t)u + tT(u)u,$$

and hence

$$\mathcal{H}^\kappa(E, \Phi^\alpha) \approx \mathcal{H}^\kappa(E, E \setminus \{0\}) = 0 \quad \text{for all } \kappa \geq 0.$$

□

We are now ready to prove our main result.

Proof of Theorem 1.2. Suppose 0 is the only critical point of Φ . Since Φ satisfies the (PS) condition by Autuori et al. [1, Lemma 4.2], then Φ^0 and Φ^α are deformation retracts of E and $\Phi^0 \setminus \{0\}$, respectively, by the second deformation lemma. Thus, taking $U = E$ in (2.2) gives

$$C^\kappa(\Phi, 0) = H^\kappa(\Phi^0, \Phi^0 \setminus \{0\}) \approx \mathcal{H}^\kappa(E, \Phi^\alpha) = 0 \quad \text{for all } \kappa \geq 0$$

by Lemma 3.3. However, $C^\kappa(\Phi, 0) \neq 0$ for some $\kappa \geq 0$ by Lemmas 3.1 and 3.2. This contradiction completes the proof. □

References

- [1] Autuori, G., Pucci, P., Varga, C.: Existence theorems for quasilinear elliptic eigenvalue problems in unbounded domains. *Adv. Differ. Equ.* **18**, 1–48 (2013)
- [2] Chang, K. C., Ghoussoub, N.: The Conley index and the critical groups via an extension of Gromoll–Meyer theory. *Topol. Methods Nonlinear Anal.* **7**, 77–93 (1996)
- [3] Corvellec, J. -N., Hantoute, A.: Homotopical stability of isolated critical points of continuous functionals. *Calculus of variations, nonsmooth analysis and related topics. Set Valued Anal.* **10**, 143–164 (2002)
- [4] Degiovanni, M., Lancelotti, S.: Linking over cones and nontrivial solutions for p -Laplace equations with p -superlinear nonlinearity. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **24**, 907–919 (2007)
- [5] Degiovanni, M., Lancelotti, S., Perera, K.: Nontrivial solutions of p -superlinear p -Laplacian problems via a cohomological local splitting. *Commun. Contemp. Math.* **12**, 475–486 (2010)
- [6] Fadell, E. R., Rabinowitz, P. H.: Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.* **45**, 139–174 (1978)
- [7] Li, S. J., Willem, M.: Applications of local linking to critical point theory. *J. Math. Anal. Appl.* **189**, 6–32 (1995)
- [8] Perera, K.: Homological local linking. *Abstr. Appl. Anal.* **3**, 181–189 (1998)
- [9] Perera, K.: Nontrivial critical groups in p -Laplacian problems via the Yang index. *Topol. Methods Nonlinear Anal.* **21**, 301–309 (2003)
- [10] Perera, K., Agarwal, R. P., O’Regan, D.: Morse theoretic aspects of p -Laplacian type operators. *Math. Surv. Monogr.* vol. **161**. American Mathematical Society, Providence. pp. xx+141 (2010)

- [11] Perera, K., Szulkin, A.: p -Laplacian problems where the nonlinearity crosses an eigenvalue. *Discret. Contin. Dyn. Syst.* **13**, 743–753 (2005)
- [12] Pflüger, K.: Existence and multiplicity of solutions to a p -Laplacian equation with nonlinear boundary condition. *Electron. J. Diff. Equ.* **10** p. 13 (1998) (electronic)

Kanishka Perera

Department of Mathematical Sciences

Florida Institute of Technology

Melbourne, FL, 32901, USA

e-mail: kperera@fit.edu

Patrizia Pucci

Dipartimento di Matematica e Informatica

Università degli Studi di Perugia

Via Vanvitelli 1, 06123, Perugia, Italy

e-mail: patrizia.pucci@unipg.it

Csaba Varga

Faculty of Mathematics and Computer Science

Babeş-Bolyai University

400084, Cluj-Napoca, Romania

e-mail: csvarga@cs.ubbcluj.ro

Received: 8 May 2013.

Accepted: 27 October 2013.