Some remarks on quasilinear parabolic problems with singular potential and a reaction term

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Abstract. In this paper we deal with the following quasilinear parabolic problem

$$\begin{cases} (u^\theta)_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q + f, & u \geq 0 \quad \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

where θ is either 1 or $(p - 1)$, $N \geq 3$, $\Omega \subset \mathcal{R}^N$ is either a bounded regular domain containing the origin or $\Omega \equiv \mathcal{R}^N$, $1 < p < N$, $q > 0$ and $u_0 \geq 0$, $f \geq 0$ with suitable hypotheses. The aim of this work is to get natural conditions to show the existence or the nonexistence of nonnegative solutions. In the case of nonexistence result, we analyze blow-up phenomena for approximated problems in connection with the classical Harnack inequality, in the Moser sense, more precisely in connection with a *strong maximum principle*. We also study when finite time extinction ($1 < p < 2$) and finite speed propagation ($p > 2$) occur related to the reaction power.

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1. Introduction

This paper deals with the following quasilinear parabolic problems

$$\begin{cases} (u^\theta)_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q + f(x, t) & \text{in } \Omega_T \equiv \Omega \times (0, T), \\ u \geq 0 \text{ and } u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where Ω is either a bounded regular domain containing the origin or $\Omega = \mathbb{R}^N$, $1 < p < N$, $q > 0$, θ is either $\theta = 1$ or $\theta = (p - 1)$, $\lambda \geq 0$, f and u_0 are nonnegative under additional hypotheses that we will specify below.

These problems are related to the following classical Hardy inequality

$$\Lambda_{N,p} \int_{\mathbb{R}^N} \frac{|\phi|^p}{|x|^p} dx \leq \int_{\mathbb{R}^N} |\nabla \phi|^p dx, \quad \text{for all } \phi \in W^{1,p}(\mathbb{R}^N), \quad (1.2)$$

where $\Lambda_{N,p} = \left(\frac{N-p}{p}\right)^p$ is optimal and is not attained.

The same inequality holds in the space $W_0^{1,p}(\Omega)$ where Ω is any domain containing the origin with the same optimal constant. As in the case of $W^{1,p}(\mathbb{R}^N)$, the optimal constant $\Lambda_{N,p}$ is not achieved in $W_0^{1,p}(\Omega)$. We refer to [28] for more details about the Hardy inequality.

Quasilinear parabolic equations are a classical subject widely studied in the literature. They are used in models involving nonlinear diffusion that appear, for instance, in Newtonian Fluid Mechanic, diffusion of high temperatures, some biological phenomena, etc.

The common fact for these class of operators is that all the *rheological laws* considered are assumed to verify the so called *power law*. We refer, for instance, to [31, 35, 38] and the references therein for more details about the use of this class of operators. It is worthy to point out that in general these kind of modelizations give only a rough approach to the real problems, but in many times this is sufficient to understand some qualitative behaviors such as finite time blow-up in a suitable sense, finite time extinction, finite speed of propagation, etc.

Existence and asymptotic behavior of solution to problem (1.1) without the singular term are well studied in the literature, we refer to [6, 9, 21, 29], and mainly the book [37] and the references therein. For $\lambda < 0$, without any reaction term, existence of global solution was obtained in [34], it is clear that, in this case, the singular term has a regularizing effect on the problem and then *a priori* estimates are easily obtained.

In the present paper, we deal with the case $\lambda > 0$, under the presence of a reaction term. In this case, the situation is more delicate and needs to be carefully analyzed.

Precedents with the Hardy potential in this framework are, among other references [4, 5, 19, 28].

1.1. Some previous results

If $p = 2$, without the reaction term u^q , problem (1.1) has been studied in [10]. The authors prove existence and nonexistence results related to the fact that $\lambda \leq \Lambda_{N,2}$ or $\lambda > \Lambda_{N,2}$, respectively.

The semilinear case with $\lambda > 0$, under the presence of the reaction term u^q , was studied in [3]. The authors prove the existence of a critical exponent $q_+(\lambda)$ depending only on λ such that existence holds if and only if $q < q_+(\lambda)$. The existence of a Fujita type exponent was also proved.

Notice that the proofs in the semilinear cases are based on the behavior of the solution near the origin $x_0 = 0$ which is obtained using a suitable change of function. In the quasilinear case, this method fails and must be modified.

The problem (1.1) with $p \neq 2$ and $\theta = 1$ was considered by several authors.

The case $\lambda > 0$ and $\theta = 1$ was studied in [28] where the problem is analyzed without the presence of the reaction term. In [4], the authors prove that if $p < \frac{2N}{N+2}$, then there exists a global solution for all $\lambda > 0$ under suitable conditions on f and u_0 . The extension of the above results to quasilinear parabolic problems with Caffarelli–Kohn–Nirenberg weights without reaction term, was obtained in [19].

For $\theta \neq 1$ and $\lambda = 0$, concerning the *doubly nonlinear problem*, we can quote the paper [24] where the one dimensional case is studied without the reaction term. For $\theta = p - 1$, a *classical Harnack inequality* is established in [32] for the homogeneous equation. This property will be very relevant for our study.

1.2. Main results and organization of the paper

We start in Sect. 2 by giving some auxiliary results related to the quasilinear associated elliptic problem. In particular we prove the existence of a critical exponent $q_+(p, \lambda)$ which is the threshold for existence, that is, there is no solution, in a suitable sense, for $q > q_+(p, \lambda)$ and there exists a solution if $q < q_+(p, \lambda)$ under some convenient conditions on the data.

In Sect. 3 we will consider the case $\theta = p - 1$, which will be called the doubly nonlinear parabolic problem. The main result in this part is to study existence versus complete and instantaneous blow-up according to $q < q_+(p, \lambda)$ or $q > q_+(p, \lambda)$ respectively. We must overcome some difficulties produced by the nonlinearity of the differential operator. This is a significative difference with respect to the heat equation.

In Sect. 4 we analyze the Cauchy problem, namely $\Omega = \mathbb{R}^N$, related to the doubly nonlinear parabolic equation, and $q < q_+(p, \lambda)$. In Sect. 4.1 we prove one of the main results in the paper. We find an exponent $F(p, \lambda)$ such that the Cauchy problem has a family of subsolutions blowing up in a finite time with respect to a suitable weighted norm if $p - 1 < q < F(p, \lambda) < q_+(\lambda)$. If $F(\lambda) < q < q_+(\lambda)$, we find a global in time family of supersolutions. These two results show that $F(p, \lambda)$ is the *Fujita exponent* for $\lambda \geq 0$. Notice that this result is also new for $\lambda = 0$. The Fujita exponent for the case $\theta = 1$ and $\lambda = 0$ was obtained in [26].

The case $\theta = 1$ and $p < 2$ is studied in Sect. 5, in this case the strong maximum principle does not hold in general. We start by proving the existence of solution if $q_+(p, \lambda) \leq q \leq 1$, under suitable hypotheses on u_0 and f . This make the difference between the cases $p = 2$ and $p < 2$.

One of the main results in this section is to prove that the critical value of the reaction power to have extinction in finite time is $q = p - 1$. Under this value there exists a positive global in time solution with null initial value, while if $p - 1 < q \leq 1$, we are able to prove extinction in finite time.

In Sect. 6 we give some results for the case $p > 2$ and $\theta = 1$. Here also one of the more relevant facts is to prove that the threshold of the reaction power to have finite speed of propagation, is $q = 1$. If $q < 1$, we find a positive solution with zero initial value and we have finite speed of propagation, at least, for $1 < q < p - 1$.

1.3. Some preliminaries

To deal with quasilinear equation problems we precise the sense in which a solution is considered. Namely we have the following definitions.

Definition 1.1. We say that $u \in \mathcal{T}_0^{1,p}(\Omega \times (0, T))$ if $T_k(u) \in L^p((0, T); W_0^{1,p}(\Omega))$ for all $k > 0$ where $T_k(u)$, the truncating, is defined by $T_k(u) = \max\{\min\{k, u\}, -k\}$.

For simplicity of typing, we set

$$b(s) = |s|^{p-2}s \quad (1.3)$$

Definition 1.2. Assume that $F \in L^p(\Omega_T)$ and $b(u_0) \in L^p(\Omega)$, where b is defined in (1.3). We say that u is a weak solution to the following doubly nonlinear parabolic problem

$$\begin{cases} (b(u))_t - \Delta_p u = F & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ b(u(x, 0)) = b(u_0(x)) & \text{in } \Omega, \end{cases} \quad (1.4)$$

if $b(u) \in \mathcal{C}([0, T], L^p(\Omega))$, $(b(u))_t \in L^{p'}([0, T]); W^{-1,p'}(\Omega)$ and for every $\phi \in \mathcal{C}_0^1(\Omega \times [0, T])$ with $\phi(x, T) = 0$, we have

$$- \iint_{\Omega_T} b(u) \phi_t + \iint_{\Omega_T} |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle = \int_{\Omega} b(u_0(x)) \phi(x, 0) dx + \iint_{\Omega_T} F \phi. \quad (1.5)$$

If $(F, b(u_0)) \in L^1(\Omega_T) \times L^1(\Omega)$, we can define the entropy solution to problem (1.4).

Definition 1.3. Assume that $(F, b(u_0)) \in L^1(\Omega_T) \times L^1(\Omega)$. We say that u is an entropy solution to (1.4) if $b(u) \in \mathcal{C}([0, T]; L^1(\Omega))$, $T_k(u) \in L^p((0, T); W_0^{1,p}(\Omega))$ for all $k > 0$ and for every $\phi \in L^p((0, T); W_0^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$ with $\phi_t \in L^{p'}((0, T); W^{-1,p'}(\Omega)) + L^1(\Omega_T)$, it holds that

$$\begin{aligned} & \int_{\Omega} \Theta_k(b(u(T)) - \phi(T)) + \iint_{\Omega_T} |\nabla u|^{p-2} \nabla u \cdot \nabla T_k(b(u) - \phi) \\ &= - \int_0^T \langle \phi_t, T_k(b(u) - \phi) \rangle + \iint_{\Omega_T} \left(F T_k(b(u) - \phi) + \int_{\Omega} \Theta_k(b(u_0) - \phi(0)) \right). \end{aligned}$$

Remark 1.4. For problem (1.1), we say that u is an entropy solution to (1.1) if:

- i) $\lambda \frac{u^{p-1}}{|x|^p}$, u^q , $f \in L^1(\Omega_T)$ and
- ii) (1.5) holds with $F \equiv \lambda \frac{u^{p-1}}{|x|^p} + u^q + f$.

The following Theorem is proved, for instance, in [8, 11, 14, 18].

Theorem 1.5. Let Ω be a bounded regular domain. Assume that $(F, b(u_0)) \in L^1(\Omega_T) \times L^1(\Omega)$, then problem (1.4) has one and only one entropy solution u such that $b(u) \in C([0, T], L^1(\Omega))$.

As a consequence of the monotonicity of b and the operator, we get the following comparison principle.

Lemma 1.6. Let u , v be entropy solutions of (1.4) with data (f, u_0) and (g, v_0) respectively. Suppose that $g \leq f$ and $v_0 \leq u_0$, then $v \leq u$.

To overcome the difficulty caused by the nonlinearity of the principal part of the operator, we will use the next extension of Picone inequality, see [1, 7].

Theorem 1.7. (Picone inequality) Let $v \in W_0^{1,p}(\Omega)$ be such that $v \geq 0$. Suppose that $-\Delta_p v$ is a bounded Radon measure with $-\Delta_p v \geq 0$ and $-\Delta_p v \neq 0$. Then for all $u \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u|^p dx \geq \int_{\Omega} \left(\frac{|u|^p}{v^{p-1}} \right) (-\Delta_p v) dx.$$

As a direct application of the Picone inequality, we have the next comparison principle proved in [1] that extends the classical comparison result obtained by Brezis-Kamin [17].

Theorem 1.8. (Comparison Principle) Assume that $1 < p$ and let f be a non-negative continuous function such that $\frac{f(x, s)}{s^{p-1}}$ is decreasing for $s > 0$. Suppose that $u, v \in W_0^{1,p}(\Omega)$ are such that

$$\begin{cases} -\Delta_p u \geq f(x, u), & u > 0 \text{ in } \Omega, \\ -\Delta_p v \leq f(x, v), & v > 0 \text{ in } \Omega. \end{cases} \quad (1.6)$$

Then $u \geq v$ in Ω .

2. Analysis of the elliptic equation and some auxiliary results

Next, we will give the results that we need about the stationary problem, we refer to [36] for more details.

$$\begin{cases} -\Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q + g & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where Ω is a bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$ and g is a nonnegative function with suitable hypotheses. To analyze the behavior of the solution to problem (2.1), we follow closely the arguments used in [2, 16].

Let us look for radial solutions to the following associated elliptic problem

$$-\Delta_p w = \lambda \frac{w^{p-1}}{|x|^p}, \quad (2.2)$$

by setting $w(x) = C|x|^{-\alpha}$. Using the expression of the p-laplacian for radial functions

$$-\Delta_p w = \frac{-1}{r^{N-1}} (|w'|^{p-2} w' r^{N-1})'$$

we get the next algebraic equation

$$h(\alpha) \equiv (p-1)\alpha^p - (N-p)\alpha^{p-1} + \lambda = 0. \quad (2.3)$$

If $\lambda < \Lambda_{N,p}$, the Eq. (2.3) possesses exactly two solutions

$$\alpha_1 < \frac{N-p}{p} < \alpha_2. \quad (2.4)$$

See the details in [2]. If $\lambda = 0$, then $\alpha_1 = 0$ and $\alpha_2 = \frac{N-p}{p-1}$, while if $\lambda = \Lambda_{N,p}$, $\alpha_1 = \alpha_2 = \frac{N-p}{p}$. Indeed, in this case, the unique strict minimum of h is $\alpha = \frac{N-p}{p}$ with $h(\frac{N-p}{p}) = 0$.

We start by proving the following local behavior result.

Lemma 2.1. *Assume that $u \in W_{loc}^{1,p}(\Omega)$ is a nonnegative supersolution to problem (2.2), then if $u \not\equiv 0$, there exists a positive constant C and a small ball $B_r(0) \subset\subset \Omega$ such that*

$$u \geq C|x|^{-\alpha_1} \quad \text{in } B_r(0) \quad (2.5)$$

where α_1 is defined above.

Proof. The proof is a simple modification of the arguments used in [2], for the reader convenience we include here a short proof. Notice that, since $u \not\equiv 0$, then by the strong maximum principle we obtain that $u \geq \eta > 0$ in a small ball $B_r(0) \subset\subset \Omega$. Fix $B_r(0)$ and consider the problem

$$\begin{cases} -\Delta_p w_n = \lambda \frac{w_n^{p-1}}{|x|^p} & \text{in } B_r(0), \\ w_n = \eta & \text{on } \partial B_r(0) \end{cases} \quad (2.6)$$

with $w_0 \equiv 0$. By the iteration process and comparison principle, we obtain $u \geq w_n$. On the other hand, it is not difficult to show that w_n converges to w , the solution of the problem

$$\begin{cases} -\Delta_p w = \lambda \frac{w^{p-1}}{|x|^p} & \text{in } B_r(0), \\ w = \eta & \text{on } \partial B_r(0). \end{cases} \quad (2.7)$$

Notice that we can choose $C > 0$ such that $w_1(x) = C|x|^{-\alpha_1}$, is also a solution to (2.7). It is clear that $w \leq w_1$, to show that $w = w_1$, we follow closely the argument used in [1]. Then we get

$$\frac{-\Delta_p w_1}{w_1^{p-1}} - \frac{-\Delta_p w}{w^{p-1}} = 0.$$

Using $(w_1 - w)_+$ as a test function in the previous equation and following the arguments in [1] we reach that $(w_1 - w)_+ = 0$. Thus $w = w_1$ and then

$$u \geq C|x|^{-\alpha_1} \quad \text{in } B_r(0). \quad (2.8)$$

In particular any entropy positive solution is not bounded. $\square$

Assume $0 < \lambda < \Lambda_{N,p}$, we look now for radial solutions to the elliptic equation

$$-\Delta_p w = \lambda \frac{w^{p-1}}{|x|^p} + w^q, \quad x \in \mathbb{R}^N. \quad (2.9)$$

Then by setting $w(x) = A|x|^{-\alpha}$, it follows that

$$A^{p-1}((p-1)\alpha^p - (N-p)\alpha^{p-1} - \lambda)r^{-\alpha(p-1)-p} = A^q r^{-\alpha q}, \quad (2.10)$$

so by identification, we have $\alpha = \frac{p}{q - (p-1)}$, $q > p-1$, $A = ((p-1)\alpha^p - (N-p)\alpha^{p-1} - \lambda)^{\frac{1}{q-(p-1)}}$ and $\alpha < \frac{N-p}{p-1}$. Since $\frac{w^{p-1}}{|x|^p} \in L^1_{loc}(\Omega)$ and $w^q \in L^1_{loc}(\Omega)$, then by the result of Lemma 2.1 we obtain that $\alpha_1 < \alpha < \alpha_2$ which is equivalent to

$$\frac{p}{\alpha_2} + p - 1 < q < \frac{p}{\alpha_1} + p - 1, \quad (2.11)$$

We set

$$q_+(p, \lambda) \equiv \frac{p}{\alpha_1} + p - 1 \quad \text{and} \quad q_-(p, \lambda) \equiv \frac{p}{\alpha_2} + p - 1. \quad (2.12)$$

Since $0 < \lambda \leq \Lambda_{N,p}$, then $\alpha_i > 0$ and $q_{\pm}(p, \lambda)$ are well defined. If $\lambda = 0$ we recover $q_-(p, 0) = \frac{(p-1)N}{N-p}$ and $q_+(p, 0) = \infty$.

Since we are considering an equation with right hand side in L^1 , then we will use the concept of entropy solutions as in Definition 1.3, we refer to [12] for more details.

Let prove now the following nonexistence result.

Theorem 2.2. Assume that $q > q_+(p, \lambda) \equiv ((p-1) + \frac{p}{\alpha_1})$. Then the problem (2.1) has no positive entropy solution.

Proof. We argue by contradiction. Let u be an entropy solution to (2.1), then using an approximation argument as in [1] we get the existence of a minimal entropy solution obtained as a limit of approximating problems. We denote u the minimal solution. Let $\varphi \in \mathcal{C}_0^\infty(B_r(0))$, then applying the Picone inequality as in Theorem 1.7, for u and φ , it follows that

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^p &\geq \int_{B_r(0)} \frac{-\Delta_p u}{u^{p-1}} |\varphi|^p \geq \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} + \int_{B_r(0)} u^{q-p+1} |\varphi|^p \\ &\geq \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} + \int_{B_r(0)} \frac{|\varphi|^p}{|x|^{\alpha_1(q-p+1)}} \\ &\geq \int_{B_r(0)} \frac{|\varphi|^p}{|x|^{\alpha_1(q-p+1)}}. \end{aligned}$$

If $q > q_+(p, \lambda)$, then $\alpha_1(q-p+1) > p$, thus we get a contradiction with the Hardy inequality (1.2), hence the non existence result holds. $\square$

Remark 2.3. If $p = 2$ a stronger non existence result is proved in [16]. For $p \neq 2$ the existence results and the behavior of the positive solution near the singular point can be seen in [36].

3. Existence and nonexistence result for the homogeneous doubly nonlinear parabolic problem with a reaction term

In this section we deal with the following doubly nonlinear parabolic problem

$$\begin{cases} (u^{p-1})_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q & \text{in } \Omega \times (0, T), \\ u(x, t) > 0 & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (3.1)$$

with $\lambda < \Lambda_{N,p}$. In [32] it is proved that the operator $(u^{p-1})_t - \Delta_p u$ satisfies the following classical Harnack inequality.

Lemma 3.1. (The classical Harnack Inequality) *Let u be a positive weak supersolution to the problem*

$$(u^{p-1})_t - \Delta_p u = 0 \quad \text{in } R = B_\delta(x_0) \times (t_0 - \beta, t_0 + \beta) \subset \Omega \times (0, T)$$

in the sense that $u \in L^p(0, T, W_{loc}^{1,p}(\Omega))$ and for all $\varphi \in \mathcal{C}_0^\infty(\Omega_T)$, $\varphi \geq 0$, we have

$$-\iint_{\Omega_T} u^{p-1} \varphi_t + \iint_{\Omega_T} |\nabla u|^{p-2} \nabla u \nabla \varphi \geq 0.$$

Then, there exists a positive constant $C = C(N, \delta, t_0, \beta)$ such that

$$\iint_{R^-} u \leq C \operatorname{ess\,inf}_{R^+} u,$$

with $R^- = B_{\frac{\delta}{2}}(x_0) \times (t_0 - \frac{3}{4}\beta, t_0 - \frac{1}{4}\beta)$ and $R^+ = B_{\frac{\delta}{2}}(x_0) \times (t_0 + \frac{1}{4}\beta, t_0 + \beta)$.

Notice that, if $\lambda = 0$, by direct calculation, a selfsimilar solution to the equation,

$$(u^{p-1})_t - \Delta_p u = 0 \quad \text{in } \mathcal{R}^N, \quad (3.2)$$

is easily obtained, more precisely, for $p > 1$, we find that the selfsimilar solution is given by,

$$v(x, t) = t^{-\frac{N}{p(p-1)}} e^{\left\{ - \left(\frac{p-1}{p^{\frac{p}{p-1}}} \right) \left(\frac{|x|^p}{t} \right)^{\frac{1}{p-1}} \right\}}. \quad (3.3)$$

In the case where $\lambda > 0$, we have the following results.

Lemma 3.2. Assume that $u_0 \not\geq 0$ and let u be an entropy positive solution to problem (3.1) in the sense of Definition 1.3, then $u(x, t) \rightarrow \infty$ as $|x| \rightarrow 0$ for all $t > 0$.

Proof. Using the weak Harnack inequality obtained in [32], it follows that $u \geq C$ in $B_r(0) \times (t_1, t_2)$ for all $t_1 > 0$. Notice that

$$-\Delta_p \left(\log \frac{r}{|x|} \right) = \frac{C_1}{|x|^p},$$

then by setting $v(t, x) = \varepsilon(t - t_1)^a \left(\log \frac{r}{|x|} \right)$, it follows that

$$\begin{cases} (v^{p-1})_t - \Delta_p v \leq \varepsilon(t_2 - t_1)^{a(p-1)-1} \frac{C_2}{|x|^p} & \text{in } B_r(0) \times (t_1, t_2), \\ v = 0 & \text{on } \partial B_r(0) \times (t_1, t_2), \\ v(t_1, x) = 0 & \text{in } B_r(0), \end{cases}$$

therefore, choosing ε small, we reach that

$$(v^{p-1})_t - \Delta_p v \leq (u^{p-1})_t - \Delta_p u \quad \text{in } B_r \times (t_1, t_2).$$

Using the weak comparison principle we easily obtain that $v \leq u$ in $B_r \times (t_1, t_2)$. Hence

$$\lim_{|x| \rightarrow 0} u(x, t) = \infty.$$

and then the result follows. To find the behavior of the solution close to $(0, t)$, we have the following result.

Lemma 3.3. Let u be an entropy solution to problem (3.1), then for all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$u(x, t) \geq C(\varepsilon) |x|^{-\alpha_1 + \varepsilon} \quad \text{in } B_r \times (t_1, t_2).$$

Proof. We follow closely the arguments used in [5, 28]. Assume that $\lambda < \Lambda_N$, up to a translation in time, we can assume that $t_1 = 0$. Fix $\varepsilon > 0$ and pick $\alpha > \max\{1, \frac{1}{p-1}\}$ such that $\frac{(p-1)(\alpha_1 - \varepsilon) + p}{p(p-1)} < \alpha$. Define $a = \frac{\alpha(p-1)^2}{\alpha(p-1)-1}$, in particular we have, $a > p - 1$.

Consider $w_n = t^\alpha z_n$ where $z_0 = \eta$ and, for $n \geq 1$, z_n solves

$$\begin{cases} z_n^a - \Delta_p z_n = \lambda \frac{z_{n-1}^{p-1}}{|x|^p} & \text{in } B_r, \\ z_n = \eta & \text{on } \partial B_r. \end{cases} \quad (3.4)$$

By straightforward calculation we have

$$(w_n^{p-1})_t - \Delta_p w_n = \alpha^{p-1} t^{\alpha(p-1)-1} z_n^{p-1} + t^{\alpha(p-1)} (-\Delta_p z_n),$$

since $a > p - 1$, using the Young inequality, there exist positive constants c_1 and c_2 such that

$$\begin{aligned} (w_n^{p-1})_t - \Delta_p w_n &\leq \alpha^{p-1} (c_1 t^{\alpha(p-1)} z_n^a + c_2) + t^{\alpha(p-1)} (-\Delta_p z_n) \\ &\leq \lambda t^{\alpha(p-1)} \frac{z_{n-1}^{p-1}}{|x|^p} + \alpha c_2 \leq \lambda \frac{w_{n-1}^{p-1}}{|x|^p} + \alpha c_2. \end{aligned}$$

Since u is a solution to (3.1) and by the fact that $u(x, t) \rightarrow \infty$ as $|x| \rightarrow 0$ uniformly for $t \in (t_2, t_3)$, we can choose $B_R(0)$ small such that $u^q(x, t) \geq M \geq \alpha c_2$ for all $(x, t) \in B_R(0) \times (t_2, t_3)$. By a translation in t , we can assume that $t_2 = 0$. Therefore we conclude that, $w_0(x, t) = t^\alpha \eta$ and

$$\begin{cases} (w_n^{p-1})_t - \Delta_p w_n \leq \lambda \frac{w_{n-1}^{p-1}}{|x|^p} + \alpha c_2 & \text{in } B_R(0) \times (0, T_1), \\ w_n(x, t) = t^\alpha \eta & \text{on } \partial B_R(0) \times (0, T_1), \\ w_n(x, 0) = 0 & \text{in } B_R(0). \end{cases} \quad (3.5)$$

for $n \geq 1$. Choosing η small enough, by iteration and weak comparison principle, we conclude that

$$u \geq w_n \quad \text{in } B_R(0) \times (0, T_1) \quad \text{for all } n \in \mathbb{N}. \quad (3.6)$$

Let analyze now the behavior of the sequence $\{z_n\}_n$, the solution to problem (3.4). Putting $v_n = z_n - \eta$, then $v_0 \equiv 0$ and

$$\begin{cases} -\Delta_p v_n = \lambda \frac{(v_{n-1} + \eta)^{p-1}}{|x|^p} - c(v_n + \eta)^a & \text{in } B_R(0), \\ v_n = 0 & \text{on } \partial B_R(0). \end{cases} \quad (3.7)$$

It is not difficult to prove that $v_n \rightarrow v$, the solution of

$$\begin{cases} -\Delta_p v = \lambda \frac{(v + \eta)^{p-1}}{|x|^p} - (v + \eta)^a & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0), \end{cases} \quad (3.8)$$

in the energy space (and even in $\mathcal{C}^{1,\alpha}(B_R \setminus \{0\})$). Notice that by Lemma 1.8, problem (3.8) has a unique positive solution v . It is clear that $z = v + \eta$ solves

$$\begin{cases} z^{\frac{\alpha(p-1)}{\alpha-1}} - \Delta_p z = \lambda \frac{z^{p-1}}{|x|^p} & \text{in } B_R(0), \\ z = \eta & \text{in } \partial B_R(0). \end{cases} \quad (3.9)$$

Consider for $A > 0$, $z_A(x) = A|x|^{-\alpha_1 + \varepsilon}$. From the condition on a and choosing A small we obtain that z_A is a subsolution to problem (3.9) for all $\varepsilon > 0$. Using again Lemma 1.8, it follows that $z_A \leq z$, thus passing to the limit in

the sequence $\{w_n\}_n$, we conclude that $u(x, t) \geq C|x|^{-\alpha_1+\epsilon}$ in $B_r(0) \times (t_2, \tilde{t})$ for some $r \leq R$. $\square$

As a consequence of the local behavior obtained above, we are able to prove the following nonexistence Theorem for the parabolic problem.

Theorem 3.4. (1) Assume that $\lambda > \Lambda_{N,p}$, then problem (3.1) has no positive entropy solution.

(2) If $\lambda \leq \Lambda_{N,p}$ and $q > q_+(p, \lambda)$, then problem (3.1) has no positive solution.

Proof. If $\lambda > \Lambda_{N,p}$, then the nonexistence result follows using a Picone type inequality as in the semilinear case (see [5]).

In the second case we will argue by contradiction. Assume that u is a nonnegative solution to problem (3.1). By Lemma 3.3, there exists a ball $B_r(0)$ such that

$$u \geq C|x|^{-\alpha_1+\epsilon} \quad \text{in } B_r(0) \times (t_1, t_2). \quad (3.10)$$

Choosing a r smaller we can moreover assume that, $u(x, t) > 1$ in $B_r(0) \times (t_1, t_2)$.

Let $\varphi \in C_0^\infty(B_r)$, applying the Picone type inequality in Theorem 1.7, we get

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^p dx &\geq \int_{B_r(0)} \frac{-\Delta_p u}{u^{p-1}} |\varphi|^p dx \\ &\geq \int_{B_r(0)} u^{q-(p-1)} |\varphi|^p dx + \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} dx - \int_{B_r(0)} \frac{(u^{p-1})_t}{u^{p-1}} |\varphi|^p dx. \end{aligned}$$

Integrating in time and using (3.10), there results

$$\begin{aligned} (t_2 - t_1) \int_{B_r(0)} |\nabla \varphi|^p dx &\geq C'(t_2 - t_1) \int_{B_r(0)} |x|^{-\alpha_1(q-(p-1))} |\varphi|^p dx \\ &+ \lambda(t_2 - t_1) \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} dx - (p-1) \int_{B_r(0)} \log(u(x, t_2)) |\varphi|^p dx. \end{aligned}$$

Notice that $\log(u(x, t_2))$ is well defined since $u(x, t) > 1$ in $B_r(0) \times (t_1, t_2)$. Therefore it follows that

$$\int_{B_r(0)} \log(u(x, t_2)) |\varphi|^p dx \leq \left(\int_{B_r(0)} |\varphi|^p dx \right)^{\frac{p}{p^*}} \left(\int_{B_r(0)} (\log(u(x, t_2)))^{\frac{N}{p}} dx \right)^{\frac{p}{N}}.$$

Observing that for all $s > 1$ and for any $\gamma > 0$, we have $\log(s) \leq c_0 s^\gamma + c_1$ where c_0, c_1 are positive fixed constants. Then choosing $\gamma < \frac{p}{N}$, we get $\int_{B_r(0)} (\log(u(x, t_2)))^{\frac{N}{p}} dx \leq C(t_2)$. Hence using Sobolev inequality we conclude that

$$\int_{B_r(0)} |\nabla \varphi|^p dx \geq C(t_1, t_2, r) \int_{B_r(0)} \frac{|\varphi|^p}{|x|^{\alpha_1(q-(p-1))}}.$$

Since $q > q_+(p, \lambda) \equiv (p-1 + \frac{p}{\alpha_1})$, then $\alpha_1(q-(p-1)) > p$ and we reach a contradiction with Hardy inequality. Thus the nonexistence result follows. $\square$

It is clear that as a consequence of the nonexistence result we can prove the complete and instantaneous blow-up for the approximated problems,

$$\begin{cases} \frac{\partial(u_n^{p-1})}{\partial t} - \Delta_p u_n = \lambda a_n(x) u_n^{p-1} + g_n(u_n) & \text{in } \Omega \times (0, T), \\ u_n(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (3.11)$$

where $a_n(x) = \min\{n, \frac{1}{|x|^p}\}$ and $g_n(s) = \min\{n, s^q\}$. Namely using the Harnack inequality in Lemma 3.1 we can prove that for any $(x_0, t_0) \in \Omega \times (0, T)$, $u_n(x_0, t_0) \rightarrow \infty$ as $n \rightarrow \infty$.

If $q < q_+(p, \lambda)$ we have the following existence result, which prove the optimality of $q_+(p, \lambda)$.

Theorem 3.5. *Assume that $q < q_+(p, \lambda)$, then under suitable hypotheses on u_0 , problem (3.1) has a nonnegative minimal solution.*

Proof. Since $\lambda < \Lambda_{N,p}$, then if $q < p^* - 1$ and $u_0 \in L^p(\Omega)$, there exists a minimal solution in the energy space $L^p((0, T); W_0^{1,p}(\Omega))$.

In the case where $p^* - 1 \leq q < q_+(p, \lambda)$, then to get the existence result we use the following monotonicity argument.

By a direct computation, we obtain a radial supersolution to the stationary problem of the form $w(x) \equiv A|x|^{-\beta}$. Hence, if $u_0(x) \leq w$, by setting $v_0 \equiv 0$ and defining v_n as the unique positive solution to the approximated problem

$$\begin{cases} \frac{\partial(v_n^{p-1})}{\partial t} - \Delta_p v_n = \lambda a_n(x) v_n^{p-1} + g_{n-1}(v_{n-1}) & \text{in } \Omega \times (0, T), \\ v_n(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ v_n(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (3.12)$$

where $a_n(x) = \min\{n, \frac{1}{|x|^p}\}$ and $g_n(s) = \min\{n, s^q\}$, we conclude that $v_n \leq v_{n+1} \leq w$. Since $\{\lambda a_n(x) v_{n-1}^{p-1} + g_{n-1}(v_{n-1})\}_n$ is an increasing sequence which converges strongly in $L^1(\Omega_T)$, then we easily conclude that the limit is a minimal positive entropy solution. $\square$

4. Cauchy problem for the doubly nonlinear parabolic problem: Fujita exponent

Fujita [25] considered the initial value problem

$$\begin{cases} u_t - \Delta u = u^q, & x \in \mathcal{R}^N, t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \mathcal{R}^N, \end{cases} \quad (4.1)$$

where $1 < q < \infty$. He showed that if $1 < q < 1 + \frac{2}{N}$, then there exists $T > 0$ such that the solution to problem (4.1) satisfies $\|u(\cdot, t_n)\|_\infty \rightarrow \infty$ as $t_n \rightarrow T$. However, if $q > 1 + \frac{2}{N}$, then both global solutions (for small data) and non-global solutions (for large data) exist. The number $1 + \frac{2}{N}$ is often called the

critical Fujita blow-up exponent for the heat equation. Moreover it is proved that for $q = 1 + \frac{2}{N}$, a suitable norm of the solution goes to infinity in a finite time, we refer to [40] for a simple proof of this last fact.

Notice that in the case $p \neq 2$, $\theta = 1$, the existence of the Fujita exponent was obtained by Galaktionov [26], we refer to [27] and the survey by Levine [33], for more detailed references about related problems.

4.1. Fujita type exponent for $\lambda \geq 0$

Assume that $0 \leq \lambda < \Lambda_{N,p}$ and consider the Cauchy problem

$$\begin{aligned} (u^{p-1})_t - \Delta_p u &= \lambda \frac{u^{p-1}}{|x|^p} + u^q \quad \text{in } \mathcal{R}^N, \\ u(x, 0) &= u_0(x) \quad \text{in } \mathcal{R}^N, \end{aligned} \quad (4.2)$$

with $p-1 < q < p_+(\lambda)$.

The semilinear problem, that is $p = 2$, has been discussed in [3].

Existence of local in time solution, to (4.2), under suitable condition on the datum u_0 , can be obtained using a sub-supersolution argument as above. Indeed, if $p-1 + \frac{p(p-1)}{N-p} < q < p_+(\lambda)$, we can take $w(x) \equiv A|x|^{-\alpha}$, with $\alpha = \frac{p}{q-(p-1)}$, as a supersolution if $u_0 \leq w$. In the case where $p-1 < q \leq p-1 + \frac{p(p-1)}{N-p}$, we consider the approximated problem

$$\begin{cases} (u_n^{p-1})_t - \Delta_p u_n = \lambda \frac{u_n^{p-1}}{|x|^{p+\frac{1}{n}}} + \min\{u_n^q, n\} & \text{in } B_n(0) \times (0, T), \\ u_n(x, t) = 0 & \text{on } \partial B_R \times (0, T), \\ u_n(x, 0) = T_n(u_0(x)) & \text{in } B_n(0), \end{cases} \quad (4.3)$$

Denoting u_n , the minimal solution of (4.3), we get easily that $\{u_n\}_n$ is increasing in n . Hence, following the same kind of computation as in [30], we can prove the existence of $T_1 > 0$ such that for all $R > 0$, $\|u_n\|_{L^q(B_R(0) \times (0, T_1))} \leq C(T_1, R)$. As a consequence we get the existence of $u = \limsup_{n \rightarrow \infty} u_n$, a local solution to the Cauchy problem (4.2).

It is clear that $u \notin L^\infty(B_\eta(0) \times (t_1, t_2))$ for any $\eta > 0$ and $t_1 > 0$. Then a non-trivial blow-up will be obtained in a suitable weighted Lebesgue space. More precisely we have the next definition.

Definition 4.1. Consider $u(x, t)$ a positive solution to (4.2), then we say that u blows-up in a finite time if there exists $T^* < \infty$ such that

$$\lim_{t \rightarrow T^*} \int_{B_r(0)} |x|^{-\alpha_1} u^{p-1}(x, t) dx = \infty,$$

for any ball $B_r(0)$ and where α_1 is defined in (2.4).

In order to get the critical Fujita type exponent in the sense of definition 4.1, we look for a family of subsolutions (4.2) in the form $w(r, t, T) = (T - t)^{-\theta} \zeta \left(\frac{r}{(T - t)^\beta} \right)$, with $\theta, \beta > 0$ to be chosen and $\zeta > 0$ a smooth function.

Denoting $s = \frac{r}{(T-t)^\beta}$, it follows that

$$\begin{aligned} w_t &= (T-t)^{-\theta-1} \left(\theta \zeta + \frac{\beta r}{(T-t)^\beta} \zeta'(s) \right), \\ w_r(r, t) &= (T-t)^{-\theta-\beta} \zeta'(s), \\ w_{rr}(r, t) &= (T-t)^{-\theta-2\beta} \zeta''(s). \end{aligned} \quad (4.4)$$

We need to have

$$(w_t)^{p-1} - (p-1)|w'|^{p-2}w'' - \left(\frac{N-1}{r} \right) |w'|^{p-2}w' - \lambda \frac{w^{p-1}}{r^p} \leq w^q. \quad (4.5)$$

By setting $\theta = \frac{1}{q-(p-1)}$, $\beta = \frac{1}{p}$ and replacing these values in (4.4), then (4.5) becomes

$$\begin{aligned} &-(p-1)|\zeta'|^{p-2}\zeta'' - \left(\frac{N-1}{s} \right) |\zeta'|^{p-2}\zeta' + \frac{p-1}{p}s\zeta^{p-2}\zeta' \\ &+ \left(\theta(p-1) - \frac{\lambda}{s^p} \right) \zeta^{p-1} \leq \zeta^q. \end{aligned} \quad (4.6)$$

Assume that $\zeta(s) = A\phi(s)$ with $\phi(s) = s^{-\alpha_1} \exp \left\{ -\left(\frac{p-1}{p^{p-1}} \right) s^{\frac{p}{p-1}} \right\}$, $A > 0$. Then

$$\zeta'(s) = A\phi(s)D(s), \text{ and } \zeta''(s) = A\phi(s)K(s)$$

where

$$D(s) = -\frac{\alpha_1}{s} - \frac{s^{\frac{1}{p-1}}}{p^{\frac{1}{p-1}}} \quad \text{and} \quad K(s) = \frac{\alpha_1^2 + \alpha_1}{s^2} + \frac{s^{\frac{2}{p-1}}}{p^{\frac{2}{p-1}}} + \frac{2\alpha_1(p-1) - 1}{(p-1)p^{\frac{1}{p-1}}} s^{\frac{2-p}{p-1}}.$$

From (4.6) it follows that

$$\begin{aligned} &-(p-1)|D|^{p-2}K - \left(\frac{N-1}{s} \right) |D|^{p-2}D + \frac{p-1}{p}sD + \left(\theta(p-1) - \frac{\lambda}{s^p} \right) \\ &\leq (A\phi)^{q-(p-1)}. \end{aligned} \quad (4.7)$$

As $s \rightarrow 0$, the first term in (4.7) is equivalent to

$$-\frac{(p-1)\alpha_1^p - (N-p)\alpha_1^{p-1} + \lambda}{s^p} - \frac{p-1}{p}\alpha_1 + \frac{p-1}{q-(p-1)}.$$

Since $(p-1)\alpha_1^p - (N-p)\alpha_1^{p-1} + \lambda = 0$, then (4.7) is satisfied at least as $s \rightarrow 0$.

Let consider now the case where $s \rightarrow \infty$, in this case the first term in (4.7) is equivalent to

$$-(p-1) \left\{ \left(\frac{p-1}{p^{\frac{p}{p-1}}} \right)^p + \frac{p-1}{p^{\frac{p}{p-1}}} \right\} s^{\frac{p}{p-1}} \rightarrow -\infty \quad \text{as } s \rightarrow \infty.$$

Choosing A large enough, it follows that (4.7) holds for all $s \geq 0$. Therefore we conclude that

$$w(x, t, T) = A(T - t)^{-\frac{1}{q-(p-1)}} \left(\frac{r}{(T - t)^{\frac{1}{p}}} \right)^{-\alpha_1} \\ \times \exp \left\{ - \left(\frac{p-1}{p^{\frac{p}{p-1}}} \right) \left(\frac{r}{(T - t)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}} \right\},$$

with $r = |x|$ is a family of subsolutions to problem (4.2).

Let $B_r(0)$ be the ball in $\mathbb{R}^N$ centered at the origin, then

$$\int_{B_r(0)} |x|^{-\alpha_1} w^{p-1}(x, t, T) dx = C(T - t)^{-\frac{1}{q-(p-1)} + \frac{N}{p} - \frac{\alpha_1}{p}} \\ \times \int_0^{\frac{r}{(T-t)^{1/p}}} \exp \left\{ -(p-1) \left(\frac{p-1}{p^{\frac{p}{p-1}}} \right) s^{\frac{p}{p-1}} \right\} s^{N-p\alpha_1-1} ds.$$

If $q < p-1 + \frac{p(p-1)}{N-\alpha_1}$, we have $-\frac{1}{q-(p-1)} + \frac{N}{p} - \frac{\alpha_1}{p} < 0$. Hence,

$$\lim_{t \rightarrow T} \int_{B_r(0)} |x|^{-\alpha_1} w^{p-1}(x, t, T) dx = \infty.$$

Thus we take $F(p, \lambda) = p-1 + \frac{p(p-1)}{N-\alpha_1}$ as the candidate to Fujita type exponent to problem (4.2).

Notice that for $0 < \lambda \leq \Lambda_{N,p}$,

$$p-1 < p-1 + \frac{p(p-1)}{N} = F(0) < F(p, \lambda) < q_-(p, \lambda) \leq p^*-1 \leq q_+(p, \lambda) < \infty.$$

It is clear that $F(p, \lambda) < q_+(p, \lambda)$. To show that $F(p, \lambda) < q_-(p, \lambda)$, we just have to prove that

$$(p-1)\alpha_2 + \alpha_1 < N - p. \quad (4.8)$$

To get the above estimate we use the definitions of α_2 and α_1 . Recall that $h(\alpha_i) = 0$ for $i = 1, 2$ and $\alpha_1 < \alpha_2$ for $\lambda > 0$, then

$$(p-1)\alpha_2 + \alpha_1 = (N-p) - \frac{\lambda}{\alpha_2^{p-1}} + \alpha_1 < (N-p) - \frac{\lambda}{\alpha_1^{p-1}} + \alpha_1 \\ = p\alpha_1 < N - p.$$

Thus (4.8) follows and then $F(p, \lambda) < q_-(p, \lambda)$.

Remark 4.2. Notice that the threshold exponent for existence only depends on the elliptic part but the Fujita exponent also depends on θ . Recall that, for $\theta = 1$ $\lambda = 0$, the Fujita exponent is $F_1 = 1 + \frac{2}{N} + (p-2)(1 + \frac{1}{N})$ (see [26]) while, for $\theta = (p-1)$ and $\lambda = 0$, we obtain that $F(p, 0) = (p-1) \left(1 + \frac{p}{N} \right)$. It is clear that both exponents agree for $p = 2$, which is the classical Fujita exponent.

Let now u be a positive supersolution to problem (4.2) such that $u(x, T) \geq C|x|^{-\alpha_1}$ for some $T > 0$. We set $\bar{u}(x, t) = u(x, t + T)$, then $\bar{u}$ solves (4.2) with $\bar{u}(x, 0) \geq C|x|^{-\alpha_1}$. Notice that

$$w(x, 0, T) = AT^{-\frac{1}{q-(p-1)}} \left(\frac{|x|}{T^{\frac{1}{p}}}} \right)^{-\alpha_1} e^{\left\{ - \left(\frac{p-1}{p^{\frac{p}{p-1}}} \right) \left(\frac{|x|^p}{T} \right)^{\frac{1}{p-1}} \right\}},$$

hence we can choose A small enough such that $w(x, 0, T) \leq \bar{u}(x, 0)$.

Fixed A as above, we claim that $\bar{u}(x, t) \geq w(x, t, T), \forall t < T$. Let $k(s) = s^{p-1}$ for $s \geq 0$, hence by subtracting we get

$$k(w)_t - k(\bar{u})_t - \Delta_p w + \Delta_p \bar{u} \leq \lambda \left(\frac{k(w)}{|x|^p} - \frac{k(\bar{u})}{|x|^p} \right) + (k(w))^{\frac{q}{p-1}} - (k(\bar{u}))^{\frac{q}{p-1}}.$$

To overcome the difficulty caused by the Hardy potential we will proceed by approximation. Define $w_n(x, t, T) = w(|x| + \frac{1}{n}, t, T)$, then

$$\begin{aligned} w_n(x, t, T) &= A(T-t)^{-\frac{1}{q-(p-1)}} \left(\frac{|x| + \frac{1}{n}}{(T-t)^{\frac{1}{p}}} \right)^{-\alpha_1} \\ &\quad \times e^{\left\{ - \left(\frac{p-1}{p^{\frac{p}{p-1}}} \right) \left(\frac{(|x| + \frac{1}{n})^p}{(T-t)} \right)^{\frac{1}{p-1}} \right\}}. \end{aligned}$$

Notice that $w_n \in L^\infty$ and

$$k(w_n)_t - \Delta_p w_n \leq \lambda \frac{k(w_n)}{(|x| + \frac{1}{n})^p} + (k(w_n))^{\frac{q}{p-1}}.$$

Since

$$k(\bar{u})_t - \Delta_p \bar{u} \geq \lambda \frac{k(\bar{u})}{(|x| + \frac{1}{n})^p} + (k(\bar{u}))^{\frac{q}{p-1}},$$

then

$$\begin{aligned} k(w_n)_t - k(\bar{u})_t - \Delta_p w_n + \Delta_p \bar{u} &\leq \frac{\lambda}{(|x| + \frac{1}{n})^p} (k(w_n) - k(\bar{u})) + (k(w_n))^{\frac{q}{p-1}} \\ &\quad - (k(\bar{u}))^{\frac{q}{p-1}}. \end{aligned} \quad (4.9)$$

Define

$$\psi_\sigma(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ 1 & \text{if } s \geq \sigma, \\ \frac{s}{\sigma} & \text{if } 0 < s < \sigma, \end{cases} \quad (4.10)$$

using $\psi_\sigma((w_n - \bar{u})_+)$ as a test function in (4.9), by the monotonicity of the p -laplacian and the boundedness of w_n , it follows that

$$\int_0^T \int_{\mathbb{R}^N} (k(w_n) - k(\bar{u}))_t \psi_\sigma((w_n - \bar{u})_+) \leq C \int_0^T \int_{\mathbb{R}^N} (k(w_n) - k(\bar{u})) \psi_\sigma((w_n - \bar{u})_+).$$

Letting $\sigma \rightarrow 0$, we get

$$\int \int_{w_n > \bar{u}} (k(w_n) - k(\bar{u}))_t \leq C \int \int_{w_n > \bar{u}} (k(w_n) - k(\bar{u})).$$

Since $\{w_n > \bar{u}\} = \{k(w_n) > k(\bar{u})\}$, then by integration in time,

$$\int_{\mathbb{R}^N} (k(w_n(x, T)) - k(\bar{u}(x, T)))_+ dx \leq \int_0^T \int_{\mathbb{R}^N} (k(w_n(x, t)) - k(\bar{u}(x, t)))_+ dx dt.$$

Using Gronwall's inequality we get $k(w_n(x, t)) - k(\bar{u}(x, t))_+ \equiv 0$, hence $\bar{u}(x, t) \geq w_n(x, t), \forall (x, t) \in \mathbb{R}^N \times (0, T)$ and for all $n \in \mathbb{N}$. Thus $\bar{u} \geq w$ and the claim follows.

As a conclusion we have proved the following result.

Theorem 4.3. Assume that $q < F(p, \lambda) \equiv p - 1 + \frac{p(p-1)}{N - \alpha_1}$ and let u be a solution to problem (4.2) such that $u(x, T) \geq C|x|^{-\alpha_1}$ for some $T > 0$, then u blows-up in a finite time in the sense of Definition 4.1.

Remark 4.4. It is clear that if $u_0(x) \geq C|x|^{-\alpha_1}$, then we reach the complete blow-up in the sense of Definition 4.1.

The condition on the behavior of u near the origin seems to be natural since we have proved in Lemma (3.3) that $u(x, t) \geq C|x|^{-\alpha_1 + \varepsilon}$ for any $\varepsilon > 0$. Notice that in the case $p = 2$, using a change of the function and by a weighted Harnack inequality we reach exactly that $u(x, t) \geq C|x|^{-\alpha_1}$ for all $t > 0$.

4.2. Global existence for $F(p, \lambda) < q < q_+(p, \lambda)$ and suitable data

To show that $F(p, \lambda)$ is the Fujita exponent, we will prove the next existence result.

Theorem 4.5. Assume that $F(p, \lambda) < q < q_+(p, \lambda)$, then under suitable condition on u_0 , the problem (4.2) has a global positive solution.

Proof. We have just to construct a suitable global supersolution. Define

$$w(r, t, T) = (T + t)^{-\theta} g\left(\frac{r}{(T + t)^\beta}\right) \text{ with } \theta, \beta > 0 \text{ to be chosen later.}$$

If w is a supersolution w to (4.2), then it satisfies

$$(w^{p-1})_t - (p-1)|w'|^{p-2}w'' - \left(\frac{N-1}{r}\right)|w'|^{p-2}w' - \lambda \frac{w^{p-1}}{r^p} \geq w^q \quad (4.11)$$

We set $s = \frac{r}{(T + t)^\beta}$, it follows that

$$\begin{aligned} w_t &= -(T + t)^{-\theta-1} \left(\theta g + \frac{\beta r}{(T + t)^\beta} g'(s) \right), \quad w_r(r, t) = (T + t)^{-\theta-\beta} g'(s), \\ w_{rr}(r, t) &= (T + t)^{-\theta-2\beta} g''(s). \end{aligned} \quad (4.12)$$

As in the previous subsection, in order to get homogeneity in the equation, we chose $\theta = \frac{1}{q - (p-1)}$ and $\beta = \frac{1}{p}$. Therefore (4.11) gives

$$(p-1)|g'|^{p-2}g'' + \left(\frac{N-1}{s}\right)|g'|^{p-2}g' + \frac{p-1}{p}sg^{p-2}g' + \left(\frac{\lambda}{s^p} + \theta(p-1)\right)g^{p-1}(s) + g^q \leq 0. \quad (4.13)$$

Assume that $g(s) = A\phi(cs)$ with $\phi(s) = s^{-\gamma}\exp\left\{-\left(\frac{p-1}{p^{p-1}}\right)s^{\frac{p}{p-1}}\right\}$, $A > 0$, where $\gamma > \alpha_1$, to be chosen later. Then

$$g'(s) = Ac\phi(cs)D(cs) \quad \text{and} \quad g''(s) = Ac^2\phi(cs)K(cs)$$

where

$$D(s) = -\frac{\gamma}{s} - \frac{s^{\frac{1}{p-1}}}{p^{\frac{1}{p-1}}} \quad \text{and} \quad K(s) = \frac{\gamma^2 + \gamma}{s^2} + \frac{s^{\frac{2}{p-1}}}{p^{\frac{2}{p-1}}} + \frac{2\gamma(p-1) - 1}{(p-1)p^{\frac{1}{p-1}}}s^{\frac{2-p}{p-1}}.$$

Thus by (4.13), it follows that

$$(p-1)c^p|D(cs)|^{p-2}K(cs) + c^{p-1}\left(\frac{N-1}{s}\right)|D(cs)|^{p-2}D(cs) + \frac{p-1}{p}csD(cs) + (\theta(p-1) + \frac{\lambda}{s^p}) + A^{q-(p-1)}\phi^{q-(p-1)}(cs) \leq 0. \quad (4.14)$$

As $s \rightarrow 0$, the left hand term in (4.14) is equivalent to

$$\frac{1}{s^p} \left\{ (p-1)\gamma^p + (p-N)\gamma^{p-1} + \lambda \right\} - \frac{p-1}{p}\gamma + \frac{p-1}{q-(p-1)} \times A^{q-(p-1)}s^{-\gamma(q-(p-1))}.$$

Since $\gamma > \alpha_1$, then $(p-1)\gamma^p + (p-N)\gamma^{p-1} + \lambda < 0$. Using the fact that $q < q_+(p, \lambda)$, it follows that $\alpha_1 < \frac{p}{q-(p-1)}$, therefore we can choose γ close to α_1 such that $\alpha_1 < \gamma < \frac{p}{q-(p-1)}$. Thus as $s \rightarrow 0$,

$$(p-1)c^p|D(cs)|^{p-2}K(cs) + c^{p-1}\left(\frac{N-1}{s}\right)|D(cs)|^{p-2}D(cs) + \frac{p-1}{p}csD(cs) + \left(\theta(p-1) + \frac{\lambda}{s^p}\right) + A^{q-(p-1)}\phi^{q-(p-1)}(cs) \leq 0.$$

We deal now with the case where $s \rightarrow \infty$. In this case the left hand side of (4.14) is equivalent to

$$(p-1)s^{\frac{p}{p-1}}c^{\frac{p}{p-1}} \left\{ \left(\frac{p-1}{p^{\frac{p}{p-1}}}\right)^p c^p - (p-1)\frac{p-1}{p^{\frac{p}{p-1}}} \right\}.$$

Choosing $c < 1$ small we reach that $\left(\frac{p-1}{p^{\frac{p}{p-1}}}\right)^p c^p - (p-1)\frac{p-1}{p^{\frac{p}{p-1}}}$, thus

$$(p-1)c^p|D(cs)|^{p-2}K(cs) + c^{p-1}\left(\frac{N-1}{s}\right)|D(cs)|^{p-2}D(cs) + \frac{p-1}{p}csD(cs) + \left(\theta(p-1) + \frac{\lambda}{s^p}\right) + A^{q-(p-1)}\phi^{q-(p-1)}(cs) \leq 0 \text{ as } s \rightarrow \infty.$$

Since $\alpha_1 < \gamma < \frac{p}{q-(p-1)}$ and $\beta = \frac{1}{p}$, then $-\frac{p-1}{p}\gamma - \theta(p-1) < 0$. Choosing A small enough we get

$$(p-1)c^p|D(cs)|^{p-2}K(cs) + c^{p-1}\left(\frac{N-1}{s}\right)|D(cs)|^{p-2}D(cs) + \frac{p-1}{p}csD(cs) \\ + (\theta(p-1) + \frac{\lambda}{s^p}) + A^{q-(p-1)}\phi^{q-(p-1)}(cs) \leq 0$$

for all $s \geq 0$.

Hence we get the existence of a family of globally defined in time supersolutions. Notice that

$$w(x, 0) = AT^{-\frac{1}{q-(p-1)}}|cx|^{-\gamma}e^{\left\{-\left(\frac{c^{\frac{p}{p-1}}}{T^{\frac{1}{p-1}}}\right)\left(\frac{p-1}{p^{\frac{p}{p-1}}}\right)|x|^{\frac{p}{p-1}}\right\}}.$$

Consider the set

$$\mathcal{A} = \left\{u_0 \in L^1(\mathcal{R}^N) : \text{ such that } u_0(x) \leq |x|^{-\gamma} \right. \\ \left. \times e^{\left\{-D\left(\frac{p-1}{p^{\frac{p}{p-1}}}\right)|x|^{\frac{p}{p-1}}\right\}} \text{ for some } D > 0 \right\}.$$

It is clear that for all $u_0 \in \mathcal{A}$, we can find $T > 0$ and $c < 1$ close to 1 such that $w(x, 0) \geq u_0(x)$. Thus w is a supersolution to problem (4.2). Since $v(x, t) = 0$ is a strict subsolution, then by an iterative argument we get the existence of a global positive solution u to problem (4.2). $\square$

Remark 4.6. Assume that $p-1 < q < q_+(p, \lambda)$ and consider an initial datum such that $u_0(x) \geq \phi$, where $\phi \in \mathcal{C}_0^\infty(\mathcal{R}^N)$ is a non negative function with $\text{supp}(\phi) \subset B_0(R)$ and

$$\frac{1}{q+1} \int_{\mathbb{R}^N} \phi^{q+1} dx > \frac{1}{p} \int_{\mathbb{R}^N} \left(|\nabla \phi|^p - \lambda \frac{\phi^p}{|x|^p} \right) dx. \quad (4.15)$$

If u is a positive solution to problem (4.2) with datum $u_0(x)$, then there exists $T^* < \infty$ such that

$$\int_{B_R(0)} u^p(x, t) dx \rightarrow \infty \quad \text{as } t \rightarrow T^*. \quad (4.16)$$

To proof (4.16), we proceed by contradiction. We consider the solution v with datum ϕ , then by comparison principle we conclude that $u \geq v$. Hence if (4.16) does not hold we reach a contradiction with (4.15).

5. Existence result for the case $\theta = 1$ and $p < 2$: solution with and without finite time of extinction

In this section we consider the problem

$$\begin{cases} u_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & \text{in } \Omega \end{cases} \quad (5.1)$$

with $p < 2$, the main goal of this section is to show that under suitable conditions on u_0 , we get existence of a nonnegative solution for some $q > q_+(p, \lambda)$. This will make the difference between problem (5.1) and problem (1.1) and clarify the role of the classical Harnack inequality in the nonexistence results. More precisely, for $\theta = p - 1$ the strong maximum principle for the parabolic equation holds, while for $\theta = 1$, only holds for $p = 2$.

We start by proving the following global existence result.

Theorem 5.1. *Assume $1 < p < \frac{2N}{N+2}$. Let $u_0 \in L^1(\Omega)$ be a nonnegative function such that $u_0(x) \leq \frac{c}{|x|^{\frac{p}{2-p}}}$, then for all $\lambda > 0$ and $q < 1$, problem (5.1) has a nonnegative minimal energy solution u such that $u \in L^p(0, T; W_0^{1,p}(\Omega))$.*

Proof. Without loss of generality we can assume that $\lambda > (\frac{p}{2-p})^{p-1}(N - \frac{p}{2-p})$. To obtain the existence result we will show that the above problem has a good supersolution. Then the existence result follows by using monotonicity. Using a rescaling argument we can assume that $\Omega \subset\subset B_0(1)$.

Define

$$S_{\lambda_0}(x, t) = A(\lambda_0) \left(\frac{t + T_0}{|x|^p} \right)^{\frac{1}{2-p}}$$

where $\lambda_0 > \lambda$ will be chosen later, then if $1 < p < \frac{2N}{N+1}$, using the result of [4], we know that S_{λ_0} satisfies

$$(S_{\lambda_0})_t - \Delta_p S_{\lambda_0} = \lambda_0 \frac{S_{\lambda_0}^{p-1}}{|x|^p}$$

where $A(\lambda_0) = \left(\left(\frac{p}{2-p} \right)^{p-1} (p - N(2-p)) + \lambda(2-p) \right)^{\frac{1}{2-p}}$. Fixed $\lambda_1 = \lambda_0 + \lambda$, then

$$(S_{\lambda_0})_t - \Delta_p S_{\lambda_0} = \lambda \frac{S_{\lambda_0}^{p-1}}{|x|^p} + \lambda_1 \frac{S_{\lambda_0}^{p-1}}{|x|^p}.$$

Since $q < 1$, under suitable condition on λ_1 and T_0 we reach that

$$\lambda_1 \frac{S_{\lambda_0}^{p-1}}{|x|^p} \geq S_{\lambda_0}^q.$$

Thus to show that S_{λ_0} is a supersolution to the problem (5.1), we have just to compare the initial conditions. It is clear that $S_{\lambda_0}(x, 0) = \lambda_0 \left(\frac{T_0}{|x|^p} \right)^{\frac{1}{2-p}}$.

Choosing λ_0 such that $\lambda_0 T_0^{\frac{1}{2-p}} \geq c$, it follows that $S_{\lambda_0}(x, 0) \geq u_0(x)$. Since $p < \frac{2N}{N+2}$, then $\lambda \frac{S_{\lambda_0}^{p-1}}{|x|^p} + \lambda_1 \frac{S_{\lambda_0}^{p-1}}{|x|^p} \in L^1(\Omega \times (0, T))$. Hence we conclude that S_{λ_0} is a supersolution to (5.1).

Consider now the following iteration scheme, $u_0 \equiv 0$ and define u_n to be the nonnegative solution to problem

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta_p u_n = g_n(x, u_{n-1}) & \text{in } \Omega \times (0, T), \\ u_n(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) = T_n(u_0) & \text{in } \Omega, \end{cases} \quad (5.2)$$

where, for $s \geq 0$, $g_n(x, s) = \lambda \frac{s^{p-1}}{|x|^p + \frac{1}{n}} + s^q$. It is clear that $u_0 \leq S_{\lambda_0}$ and then by iteration we reach that $u_n \leq u_{n+1} \leq S_{\lambda_0}$ for all n . Define $u \equiv \limsup_{n \rightarrow \infty} u_n$, then $u \leq S_{\lambda_0}$.

Since $\frac{S_{\lambda_0}^p}{|x|^p} \in L^1(\Omega \times (0, T))$, then using u_n as a test function in (5.2), we conclude that the sequence $\{u_n\}_n$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$. Hence u is an energy solution to (5.1). $\square$

Remark 5.2. For the nonhomogeneous problem

$$\begin{cases} u_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q + f & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & \text{in } \Omega, \end{cases} \quad (5.3)$$

under the same condition on u_0 as in Theorem 5.1, and if we suppose that $f(x, t) \leq \frac{C}{|x|^{\frac{p}{2-p}}}$, then we can prove that for all $\lambda > 0$, the problem (5.3) has a nonnegative energy solution.

In the case $\lambda \leq \Lambda_{N,p}$, we have the next finite time extinction result.

Theorem 5.3. *Let $1 < p < 2$ and assume that $\lambda \leq \Lambda_{N,p}$, then for all $u_0 \in L^2(\Omega)$ and for all $(p-1) < q \leq 1$, problem (5.1) has a nonnegative minimal solution $u \in L^p(0, T; W_0^{1,p}(\Omega))$, moreover if $p > \frac{2N}{N+2}$, then under a smallness condition on $\|u_0\|_{L^2}$, there exists a finite time T^* such that $u(\cdot, t) \equiv 0$ for all $t \geq T^*$.*

Proof. We begin by the case $q = 1$. Let u_n be the minimal solution of the approximated problem

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta_p u_n = \lambda \frac{u_n^{p-1}}{|x|^p + \frac{1}{n}} + u_n & \text{in } \Omega \times (0, T), \\ u_n(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) = u_0 & \text{in } \Omega, \end{cases} \quad (5.4)$$

taking u_n as test function in (5.4) and using Hardy inequality we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx - \int_{\Omega} u^2 dx + \left(1 - \frac{\lambda}{\Lambda}\right) \int_{\Omega} |\nabla u|^p dx \leq 0. \quad (5.5)$$

Thus by the Gronwall inequality we conclude that

$$\int_{\Omega} u^2(x, T) dx + \left(1 - \frac{\lambda}{\Lambda}\right) \int_0^T \int_{\Omega} |\nabla u|^p dx dt \leq \|u_0\|_{L^2} e^{\frac{1}{2}T}.$$

Therefore we reach that $\{u_n\}_n$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^2(\Omega \times (0, T))$, then the existence result follows using an iteration process and a monotonicity argument.

Let us assume now that $\frac{2N}{N+2} < p < 2$, using Sobolev inequality in (5.5) there result that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx - \int_{\Omega} u^2 dx + C \left(1 - \frac{\lambda}{\Lambda}\right) \left(\int_{\Omega} |u|^{p^*} dx\right)^{\frac{p}{p^*}} \leq 0$$

where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. Thus

$$\frac{d}{dt} \left(e^{-2t} \int_{\Omega} u^2 dx \right) + C \left(1 - \frac{\lambda}{\Lambda}\right) e^{-(2-p)t} \left(\int_{\Omega} (e^{-t} |u|)^{p^*} dx \right)^{\frac{p}{p^*}} \leq 0.$$

Since $\frac{2N}{N+2} < p < 2$, then $p^* > 2$, therefore by setting $F(t) = e^{-2t} \int_{\Omega} u^2 dx$, it follows that

$$\frac{F'(t)}{F^{\frac{p}{2}}(t)} \leq -C \left(1 - \frac{\lambda}{\Lambda}\right) e^{-(2-p)t},$$

integrating in time, we reach

$$F(t) \leq \left\{ F^{1-\frac{p}{2}}(0) + C \left(1 - \frac{\lambda}{\Lambda}\right) \left[\frac{1}{2-p} e^{-(2-p)t} - \frac{1}{2-p} \right] \right\}^{\frac{2}{2-p}}.$$

Recalling that $F(0) = \int_{\Omega} u^2(x, 0) dx = \|u_0\|_{L^2}$; so if

$$\|u_0\|_{L^2} \leq \left[C \left(1 - \frac{\lambda}{\Lambda}\right) \frac{1}{2-p} \right]^{\frac{2}{2-p}}$$

we obtain $F(t) \leq 0$ for some $T^* = T^*(\lambda, p)$ and then the extinction result follows.

We consider now the case $(p-1) < q < 1$. Since $p > \frac{2N}{N+2}$ and $\lambda < \Lambda_{N,p}$, then we can choose a constant $s \in (0, 1)$, close to 1, such that $\frac{sp^p}{(s+p-1)^p} - \frac{\lambda}{\Lambda} > 0$ and

$$(s+p-1) \frac{N}{N-p} > s+1 > q+s.$$

Now using $(u+\epsilon)^s - \epsilon^s$ as test function in (5.1), it follows that

$$\begin{aligned} & \frac{1}{s+1} \frac{d}{dt} \int_{\Omega} (u+\epsilon)^{s+1} dx + \frac{sp^p}{(s+p-1)^p} \int_{\Omega} \left| \nabla (u+\epsilon)^{\frac{(s+p-1)}{p}} \right|^p dx \\ & \leq \lambda \int_{\Omega} \frac{(u+\epsilon)^{s+p-1}}{|x|^p} dx + \int_{\Omega} (u+\epsilon)^{q+s} dx. \end{aligned}$$

By Hardy inequality and the condition on s we get

$$\begin{aligned} & \frac{1}{s+1} \frac{d}{dt} \int_{\Omega} (u+\epsilon)^{s+1} dx + \left(\frac{sp^p}{(s+p-1)^p} - \frac{\lambda}{\Lambda} \right) \int_{\Omega} \left| \nabla (u+\epsilon)^{\frac{(s+p-1)}{p}} \right|^p dx \\ & \leq \int_{\Omega} (u+\epsilon)^{q+s} dx. \end{aligned}$$

Using now Sobolev inequality, it follows that

$$\begin{aligned} \frac{1}{s+1} \frac{d}{dt} \int_{\Omega} (u + \epsilon)^{s+1} dx + c \left(\frac{sp^p}{(s+p-1)^p} - \frac{\lambda}{\Lambda} \right) \left(\int_{\Omega} (u + \epsilon)^{\frac{(s+p-1)}{p} p^*} dx \right)^{\frac{p}{p^*}} \\ \leq \int_{\Omega} (u + \epsilon)^{q+s} dx. \end{aligned}$$

Since $p-1+s < q+s < 1+s$, then for all $\eta > 0$, there exists $C(\eta) > 0$ such that

$$s^{q+s} \leq \eta s^{s+p-1} + C(\eta) s^{s+1} \text{ for all } \eta > 0.$$

Thus by integration,

$$\int_{\Omega} (u + \epsilon)^{q+s} dx \leq C_{\eta} \int_{\Omega} (u + \epsilon)^{s+1} dx + \eta \int_{\Omega} (u + \epsilon)^{s+p-1} dx.$$

Using Hölder inequality we reach that

$$\int_{\Omega} (u + \epsilon)^{q+s} dx \leq C_{\eta} \int_{\Omega} (u + \epsilon)^{s+1} dx + \eta C(\Omega) \left(\int_{\Omega} (u + \epsilon)^{\frac{s+p-1}{p} p^*} dx \right)^{\frac{p}{p^*}}.$$

Choosing η small enough and substituting in the inequality above, we conclude that

$$\begin{aligned} \frac{1}{s+1} \frac{d}{dt} \int_{\Omega} (u + \epsilon)^{s+1} dx + c \left(\frac{sp^p}{(s+p-1)^p} - \frac{\lambda}{\Lambda} \right) \left(\int_{\Omega} (u + \epsilon)^{\frac{(s+p-1)}{p} p^*} dx \right)^{\frac{p}{p^*}} \\ \leq \bar{C} \int_{\Omega} (u + \epsilon)^{1+s} dx. \end{aligned}$$

Passing to the limit as $\epsilon \rightarrow 0$, and by setting $F(t) = e^{-\bar{C}t} \int_{\Omega} u^{s+1} dx$, as in the case $q = 1$, there results that

$$F' + C e^{-c_1 t} F^{\frac{s+p-1}{s+1}} \leq 0$$

where $c_1 > 0$ depends only on s, N, p and $\bar{C}$. Thus as in the first case, if $F(0) \equiv \int_{\Omega} u_0^{s+1} dx$ is small then we get a finite time extinction. Since Ω is bounded then under a smallness condition on $\|u_0\|_{L^2}$, we get the same conclusion. Hence the proof is complete. $\square$

We consider now the case where $1 < p < \frac{2N}{N+2}$, then we have the next finite time extinction result that extends some results in [4].

Theorem 5.4. Assume that $\lambda < (\frac{N(2-p)-p}{p})(\frac{p}{2-p})^p$, $p-1 < q \leq 1$ and $u_0 \in L^{\frac{N(2-p)}{p}}(\Omega)$. Then under smallness condition on $\|u_0\|_{L^{\frac{N(2-p)}{p}}(\Omega)}$, problem (5.1) has a nonnegative minimal solution $u \in L^p(0, T; W_0^{1,p}(\Omega))$ such that $u(., t) \equiv 0$ for all $t \geq T^*$.

Proof. We can use u^s as a test function in (5.1) with $s = \frac{N(2-p)-p}{p} > 1$, hence by Hardy inequality we reach that

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + \left(\frac{sp^p}{(s+p-1)^p} - \frac{\lambda}{\Lambda} \right) \int_{\Omega} \left| \nabla u^{\frac{(s+p-1)}{p}} \right|^p dx \leq \int_{\Omega} u^{q+s} dx.$$

Since $p < \frac{2N}{N+2}$, then $(\frac{sp^p}{(s+p-1)^p} - \frac{\lambda}{\Lambda}) > 0$, thus, using Sobolev inequality, there result

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + c \left(\frac{sp^p}{(s+p-1)^p} - \frac{\lambda}{\Lambda} \right) \left(\int_{\Omega} u^{\frac{(s+p-1)}{p} p^*} dx \right)^{\frac{p}{p^*}} \leq \int_{\Omega} u^{q+s} dx.$$

Since $(s+p-1) < s+q < s+1 < (s+p-1)\frac{N}{N-p}$, then using Hölder and Young inequality we get

$$\int_{\Omega} u^{q+s} dx \leq C_{\varepsilon} \int_{\Omega} u^{s+1} dx + \varepsilon \left(\int_{\Omega} u^{\frac{s+p-1}{p} p^*} dx \right)^{\frac{p}{p^*}}.$$

Choosing ε small enough, we reach that

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + c \left(\int_{\Omega} u^{\frac{(s+p-1)}{p} p^*} dx \right)^{\frac{p}{p^*}} \leq \bar{C} \int_{\Omega} u^{1+s} dx.$$

Recall that $s = \frac{N(2-p)-p}{p}$, then $\frac{(s+p-1)}{p} p^* = s+1$, hence by setting $F(t) = e^{-\bar{C}t} \int_{\Omega} u^{s+1} dx$, we obtain

$$F' + C e^{-c_1 t} F^{\frac{N-p}{N}} \leq 0$$

where $c_1 > 0$ depends only on s, N, p and $\bar{C}$. Hence, if $F(0) \equiv \int_{\Omega} u_0^{s+1} dx$ is small, we get a finite time extinction. $\square$

On the opposite to the results in Theorems 5.3 and 5.4, a completely different behavior appears when $q < p-1$. More precisely we have the following result.

Theorem 5.5. *Assume that $1 < p < 2$, $0 < \lambda < \Lambda_{N,p}$ and let $q < p-1$, then the problem*

$$\begin{cases} u_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = 0 & \text{in } \Omega, \end{cases} \quad (5.6)$$

has a global solution u such that $u(x, t) > 0$ for all $t > 0$ and $x \in \Omega$. Moreover, $u(\cdot, t) \uparrow w$ as $t \rightarrow \infty$ where w is the unique positive solution to the problem

$$\begin{cases} -\Delta_p w = \lambda \frac{w^{p-1}}{|x|^p} + w^q & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.7)$$

As a consequence there is not finite time extinction.

Proof. The proof uses the sub-supersolution argument. Without loss of generality we can assume that $\Omega \subset B_1(0)$. Since $p < 2$, then $q < 1$. Let us begin by the construction of a suitable subsolution. Define $\mu(t) = (1-q)t^{\frac{1}{1-q}}$, then μ solves $\mu' = \mu^q$ and $\mu(0) = 0$. Consider w_1 the unique positive solution to problem

$$\begin{cases} -\Delta_p w_1 = \lambda \frac{w_1^{p-1}}{|x|^p + 1} + w_1^q & \text{in } \Omega, \\ w_1 = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.8)$$

It is clear that $\mu^{p-1}(\varepsilon t) \leq c_0 \mu^q(\varepsilon t)$ for all $t \in [0, T]$ and $w_1 \leq c_1 w_1^q$ in Ω . Let $\vartheta(x, t) = \mu(\varepsilon t) w_1$, then we can choose ε and C , depending only on T and c_1 such that

$$\vartheta_t - \Delta_p \vartheta \leq \lambda \frac{\vartheta^{p-1}}{|x|^p} + \vartheta^q \quad \text{in } \Omega \times (0, T).$$

Thus ϑ is a subsolution to problem (5.6). Using the fact that w is a supersolution to (5.6) with $\vartheta \leq cw$ in $\Omega \times (0, T)$, then by the sub-supersolution argument we reach the existence of a global solution $u \geq \vartheta$ in $\Omega \times (0, T)$. It is clear that $u(x, t) > 0$ for all $(x, t) \in \Omega \times (0, T)$.

To prove that $u(\cdot, t) \uparrow w$ as $t \rightarrow \infty$, for $t_0 > 0$, define $v(x, t, t_0) = u(x, t - t_0)$ where $t \geq t_0$, then v solves

$$\begin{cases} v_t - \Delta_p v = \lambda \frac{v^{p-1}}{|x|^p} + v^q & \text{in } \Omega \times (t_0, T), \\ v(x, t) = 0 & \text{on } \partial\Omega \times (t_0, T), \\ v(x, t_0) = u(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

Since $u(x, t_0) > 0$, by a weak comparison principle we reach that $v \leq u$ for all $t > t_0$. As t_0 is arbitrary, then u cannot converge to 0.

Now, using u as a test function in (5.6), it follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \left(1 - \frac{\lambda}{\Lambda}\right) \int_{\Omega} |\nabla u|^p dx \leq \int_{\Omega} u^{q+1} dx.$$

Since $q < p - 1$, by Hölder's and Poincaré's inequalities, we reach that $\|u(\cdot, t)\|_{W_0^{1,p}(\Omega)} \leq C$ uniformly in t . Moreover $u(x, t) \leq w(x)$. Classical results implies that $\lim_{t \rightarrow \infty} u(x, t) = \bar{u}(x)$ exists and solves (5.7), hence by uniqueness we conclude that $\bar{u} = w$ and the result follows. $\square$

Remark 5.6. If $\lambda > \Lambda_{N,p}$ and $p < 2$, then following the same arguments as in [4] and [19], we can prove that solutions to problem (5.1) are strictly positive for all $t > 0$. The main argument of the proof is to construct a suitable subsolution using an eigenvalue elliptic problem.

It is clear that if $p > 2$ and $u_0(x) > c$ for all $x \in B_r(0) \subset\subset \Omega$, and taking a suitable Barenblatt function as subsolution, the finite time extinction is impossible.

6. Extensions and further results: Case $p > 2$

Following the same arguments as in Sect. (3), we can deal with the case $\theta = 1$ and $p > 2$. More precisely, under some additional conditions on u_0 or f , we prove a nonexistence result for $q > q_+(p, \lambda)$ and as a consequence, we get a

regional blow-up result for approximated problems. Consider the problem

$$\begin{cases} u_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q + f & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (6.1)$$

where $\lambda < \Lambda_{N,p}$. Then we have the following nonexistence result.

Theorem 6.1. *Assume that $q > q_+(p, \lambda)$ and let v be a solution to the problem*

$$\begin{cases} v_t - \Delta_p v = 0 & \text{in } \Omega_1 \times (0, T), \\ v(x, t) = 0 & \text{on } \partial\Omega_1 \times (0, T), \\ v(x, 0) = v_0(x) & \text{in } \Omega_1, \end{cases} \quad (6.2)$$

where $v_0 \in L^\infty(\Omega_1)$ and $0 \in \Omega_1 \subset \Omega$. Assume that $v(x, t) \geq C > 0$ in $B_r(0) \times (t_1, t_2) \subset \Omega_1 \times (0, T)$, then problem (6.1) has no positive entropy solution u such that $u \geq v$ in $\Omega_1 \times (0, T)$.

Proof. We argue by contradiction. Assume that problem (6.1) has a solution u such that $u \geq v$ in $\Omega \times (0, T)$, then $u(x, t) \geq C > 0$ in $B_r(0) \times (t_1, t_2)$. Since $-\Delta_p \left(\log \frac{r}{|x|} \right) = \frac{C_N}{|x|^p}$, then by setting $w(x, t) = \varepsilon (t - t_1)^a \left(\log \frac{r}{|x|} \right)$, we reach that w solves

$$\begin{cases} w_t - \Delta_p w \leq \varepsilon (\tilde{t} - t_1)^{a(p-1)} \frac{C_2}{|x|^p} & \text{in } B_r(0) \times (t_1, t_2), \\ w(x, t) = 0 & \text{in } \partial B_r(0) \times (t_1, t_2), \\ w(x, t_1) = 0 & \text{in } B_r. \end{cases}$$

Choosing ε small it follows that $\varepsilon (\tilde{t} - t_2)^{a(p-1)} \frac{C_2}{|x|^p} \leq \lambda \frac{C^{p-1}}{|x|^p}$, thus

$$w_t - \Delta_p w \leq u_t - \Delta_p u \text{ in } B_r(0) \times (t_1, t_2).$$

Therefore, by the comparison principle we reach that $w \leq u$. Hence $\lim_{|x| \rightarrow 0} u(x, t) = \infty$ uniformly for $t \in (t_1, t_2)$.

We claim that

$$u \geq C|x|^{-\alpha_1+\varepsilon} \text{ in } B_\rho(0) \times (t_1, \tilde{t}) \subset B_r(0) \times (t_1, t_2). \quad (6.3)$$

Modulo a translation, we can assume that $t_1 = 0$. Fixed $\varepsilon \leq 1$ very small and consider $1 < \alpha < \frac{p-1}{p-2}$ be such that $\frac{(p-1)(\alpha_1-\varepsilon)+p}{(\alpha_1-\varepsilon)(p-2)+p} < \alpha$. Define $w_n = t^\alpha z_n$ where z_n is the unique positive solution to the problem

$$\begin{cases} cz_n^a - \Delta_p z_n = \lambda \frac{z_{n-1}^{p-1}}{|x|^p} & \text{in } B_\rho(0), \\ z_n = \eta & \text{in } \partial B_\rho(0), \\ z_0 = \eta, \end{cases} \quad (6.4)$$

with $a = \frac{\alpha}{\alpha-1}$, $\rho \leq 1$ and c is a positive constant to be chosen later. As in the proof of Lemma 3.3 and using Young inequality we reach that

$$(w_n)_t - \Delta_p w_n \leq \lambda \frac{w_{n-1}^{p-1}}{|x|^p} + \alpha c_2.$$

Since u is a solution to (6.1) and by the fact that $u(x, t) \rightarrow \infty$ as $|x| \rightarrow 0$ uniformly for $t \in (t_1, t_2)$, then we can choose ρ small such that $u^q(x, t) \geq \alpha c_2$ in $B_\rho(0) \times (0, t_2)$. Therefore we conclude that

$$\begin{cases} (w_n)_t - \Delta_p w_n \leq \lambda \frac{w_n^{p-1}}{|x|^p} + \alpha c_2 & \text{in } B_\rho(0) \times (0, T_1), \\ w_n(x, t) = t^\alpha \eta & \text{on } \partial B_\rho(0) \times (0, T_1), \\ w_n(x, 0) = 0 & \text{in } B_\rho(0). \end{cases} \quad (6.5)$$

Choosing η small, using the iteration process and the comparison principle, we conclude that

$$u \geq w_n \quad \text{in } B_\rho(0) \times (0, T_1) \quad \text{and for all } n \in \mathbb{N}. \quad (6.6)$$

Using energy estimates, we can show that $z_n \rightarrow z$ strongly in $W_0^{1,p}(B_\rho(0))$ where z is the unique positive solution of problem

$$\begin{cases} cz^{\frac{\alpha}{\alpha-1}} - \Delta_p z = \lambda \frac{z^{p-1}}{|x|^p} & \text{in } B_\rho(0), \\ z = \eta & \text{in } \partial B_\rho(0). \end{cases} \quad (6.7)$$

Notice that $z_A(x) = A|x|^{-\alpha_1+\varepsilon}$, for A small, is a subsolution to (6.7), thus by comparison result in Lemma 1.8, we get $z \geq z_A$, thus passing to the limit in (6.6), there results that

$$u(x, t) \geq C|x|^{-\alpha_1+\varepsilon} \quad \text{in } B_\rho(0) \times (t_1, \tilde{t})$$

and the claim follows.

Let $\varphi \in C_0^\infty(B_r)$, with $r \leq \rho$, then using the Picone inequality given in Theorem 1.7 for φ and u , it follows that

$$\begin{aligned} \int_{B_r(0)} |\nabla \varphi|^p dx &\geq \int_{B_r(0)} \frac{-\Delta_p u}{u^{p-1}} |\varphi|^p dx \\ &\geq \int_{B_r(0)} u^{q-(p-1)} |\varphi|^p dx + \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} dx - \int_{B_r(0)} \frac{u_t}{u^{p-1}} |\varphi|^p dx. \end{aligned}$$

Integrating in time and using (6.3), it follows that

$$\begin{aligned} (t_2 - t_1) \int_{B_r(0)} |\nabla \varphi|^p dx &\geq C' (t_2 - t_1) \int_{B_r(0)} \frac{|\varphi|^p}{|x|^{\alpha_1(q-(p-1))}} + \lambda (t_2 - t_1) \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} dx \\ &\quad - \frac{1}{2-p} \int_{B_r(0)} \left(\frac{1}{u^{p-2}(t_2, x)} \right) |\varphi|^p dx. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{B_r(0)} |\nabla \varphi|^p dx + \frac{1}{(2-p)(t_2 - t_1)} \int_{B_r(0)} \left(\frac{1}{u^{p-2}(t_2, x)} \right) |\varphi|^p dx \\ &\geq C' \int_{B_r(0)} \frac{|\varphi|^p}{|x|^{\alpha_1(q-(p-1))}} + \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} dx. \end{aligned}$$

Notice that $u(x, t) \geq c$ in $B_r(0) \times (t_1, t_2)$, then

$$\begin{aligned} & \frac{1}{(2-p)(t_2-t_1)} \int_{B_r(0)} \left(\frac{1}{u^{p-2}(t_2, x)} \right) |\varphi|^p dx \\ & \leq C_1(t_2, t_1, r) \int_{B_r(0)} |\varphi|^p dx \leq C_2(t_2, t_1, r) \int_{B_r(0)} |\nabla \varphi|^p dx. \end{aligned}$$

Therefore we get the existence of a positive constant C' depending only on r, t_1, t_2 such that

$$\int_{B_r(0)} |\nabla \varphi|^p dx \geq C' \int_{B_r(0)} \frac{|\varphi|^p}{|x|^{\alpha_1(q-(p-1))}} dx + \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} dx,$$

Since $q > \left((p-1) + \frac{p}{\alpha_1} \right)$, then $\alpha_1(q-(p-1)) > p$, hence we reach a contradiction with the Hardy inequality (1.2). Thus the nonexistence result follows. $\square$

Corollary 6.2. *Suppose that the initial datum u_0 satisfies*

$$u_0(x) \geq C \quad \text{in } B_r(0) \subset \Omega, \quad (6.8)$$

then problem (6.1) has no-positive solution.

Proof. Let $\mathcal{V}$ the Barenblatt type solution to equation

$$\mathcal{V}_t - \Delta \mathcal{V} = 0,$$

then $\mathcal{V}$ is given by

$$\mathcal{V}(x, t) = t^{-\gamma N} \left(M - \frac{(p-2)\gamma^{\frac{1}{p-1}}}{p} \xi^{\frac{p}{p-1}} \right)_+^{\frac{p-2}{p-1}}$$

where $M > 0$ is a positive constant, $\gamma = \frac{1}{N(p-2)+p}$ and $\xi = \frac{|x|}{t^\gamma}$. We set $v(x, t) = \mathcal{V}(x, t+s)$, then we can chose M, T_1 and s such that

$$\begin{cases} v_t - \Delta_p v = 0 & \text{in } B_r(0) \times (0, T_1), \\ v(x, t) = 0 & \text{on } \partial B_r(0) \times (0, T_1), \\ v(x, 0) \leq \frac{C}{2} & \text{in } B_r(0). \end{cases} \quad (6.9)$$

Since $u_0(x) \geq C$ in $B_r(0) \subset \Omega$, then by the comparison principle we reach that $u \geq v$ in $B_r(0) \times (0, T_1)$. Thus using Theorem 6.1, we conclude. $\square$

Remark 6.3. (1) The same nonexistence holds for the problem

$$\begin{cases} u_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q + f(x, t) & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial \Omega \times (0, T), \\ u(x, 0) = 0 & \text{in } \Omega, \end{cases}$$

if $f(x, t) \geq C > 0$ in $B_r(0) \times (t_1, t_2) \subset \Omega \times (0, T)$.

(2) Under the conditions of Theorem 6.1 and by using the same kind of computations we can prove that problem (6.1) has no supersolution u with

$$u \in \mathcal{T}_0^{1,p}(\Omega \times (0, T)) \text{ and } \lambda \frac{u^{p-1}}{|x|^p} + u^q \in L^1(\Omega_T)$$

- (3) The arguments used to get the nonexistence results are local and then we can prove the nonexistence of local entropy positive supersolution.
- (4) The critical case $q = \frac{p}{\alpha_1} + (p-1)$ seems to be an open problem hence we need to improve the singular behavior of u near the origin.

As a consequence of the nonexistence result we get the following regional blow-up for the approximated problems.

Theorem 6.4. *Let u_n be the minimal positive solution to problem*

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta_p u_n = \lambda a_n(x) u_n^{p-1} + g_n(u_n) + f_n & \text{in } \Omega \times (0, T), \\ u_n(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (6.10)$$

where $a_n(x) = \min\{n, \frac{1}{|x|^p}\}$, $g_n(s) = \min\{n, s^q\}$ and $f_n(x, t) = \min\{n, f(x, t)\}$.

Assume that $u_0 \in L^\infty(\Omega)$ is such that $u_0 \geq c > 0$ in $B_\eta(0) \subset \Omega$. Then for all $r > 0$ with $B_{4r}(0) \subset \Omega$, there exists $T(r) > 0$ such that if $t \geq T(r)$, then $\lim_{n \rightarrow \infty} u_n(x_0, t) = \infty$ for all $x_0 \in B_r(0)$.

For the proof we need the following *weak intrinsic Harnack inequality* obtained in [21].

Theorem 6.5. (Weak Harnack inequality) *Assume that u is a nonnegative solution to problem*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = R(x, t) & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (6.11)$$

where $p > 2$, $(R, u_0) \in L^\infty(\Omega_T) \times L^\infty(\Omega)$. Assume that $u(x, t_0) > 0$, then there exist a constant $B > 1$ depending only upon N and p , such that

$\forall (x_0, t_0) \in \Omega \times (0, T), \forall r, \theta > 0$ such that $B_{4r}(x_0) \subset \Omega$ and $t_0 + \theta < T$ we have

$$\int_{B_r(x_0)} u(x, t_0) dx \leq B \left\{ \left(\frac{r^p}{\theta} \right)^{\frac{1}{p-2}} + \left(\frac{\theta}{r^p} \right)^{\frac{N}{p}} \left[\inf_{B_r(x_0)} u(\cdot, t_0 + \theta) \right] \right\}.$$

Before proving Theorem (6.4), let begin by showing the next lemma.

Lemma 6.6. *Assume u_n be the minimal nonnegative solution to the problem (6.10) with the same hypothesis on u_0 . Then for all $r > 0$ with $B_{4r}(0) \subset \Omega$, there exists $T(r) > 0$ such that*

$$\int_{B_r(0)} u_n(x, t) dx \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad \text{if } t \geq T(r). \quad (6.12)$$

Proof. The existence of the minimal solution follows easily using monotony argument and apriori estimates. It is clear that $u_n \leq u_{n+1}$. We argue by contradiction, assume that for some $r_0 > 0$ with $B_{4r_0}(0) \subset\subset \Omega$, we have

$$\int_{B_{r_0}} u_n(x, \tau) dx \leq C(\tau) \quad \text{for all } n \text{ and for all } \tau > 0.$$

Since $\{u_n\}_n$ is an increasing sequence, we get the existence of a measurable function $u(x, \tau) \in L^1(B_{r_0})$ such that $u_n(x, \tau) \uparrow u(x, \tau)$ for e.a $x \in B_{r_0}(0)$. Choosing r small enough, we know that $u_0 \geq c_0 > 0$ in $B_r(0)$, then as in the proof of Theorem 6.1, we reach that $u \geq u_n \geq c > 0$ in $B_r(0) \times (\frac{\tau}{2}, \tau)$. Using now the *weak Harnack inequality* in Theorem 6.5, we obtain that $\int_{B_{r_0}(0)} u_n(x, t) dx \leq C(r_0, \tau)$ for all $0 < t \leq \tau$. Hence $\int_{B_{r_0}(0)} u(x, t) dx < \infty$ for all $0 < t \leq \tau$.

Following closely the arguments used in the proof of Theorem 6.1, we reach that

$$\lim_{|x| \rightarrow 0} u(x, t) = \lim_{|x| \rightarrow 0} \lim_{n \rightarrow \infty} u_n(x, t) = \infty \text{ uniformly for } t \in (\frac{\tau}{2}, \tau)$$

and

$$u_n(x, t) \geq \widehat{w}_n(x, t) \quad \text{in } B_\rho(0) \times (t_1, t_2) \quad (6.13)$$

with $\widehat{w}_n = t^\alpha \widehat{z}_n$ and $\widehat{z}_n$ is the unique positive solution to the problem

$$\begin{cases} c\widehat{z}_n^a - \Delta_p \widehat{z}_n = \lambda a_n(x) \widehat{z}_n^{p-1} & \text{in } B_\rho(0), \\ \widehat{z}_n = \eta & \text{in } \partial B_\rho(0), \\ \widehat{z}_0 = \eta. \end{cases} \quad (6.14)$$

where η is a small positive constant and a is defined as in the proof of Theorem 6.1.

Let $\varphi \in C_0^\infty(B_{r_2}(0))$ where $r_2 \leq r_0$, then using Picone inequality given in Theorem 1.7, there result that

$$\int_{B_{r_2}(0)} |\nabla \varphi|^p \geq \int_{B_{r_2}(0)} \frac{-\Delta_p u_n}{u_n^{p-1}} |\varphi|^p.$$

Thus

$$\begin{aligned} \int_{B_{r_2}(0)} |\nabla \varphi|^p dx &\geq \lambda \int_{B_{r_2}(0)} a_n(x) |\varphi|^p dx + \int_{B_{r_2}(0)} \frac{g_n(u_n)}{u_n^{p-1}} |\varphi|^p dx \\ &\quad - \int_{B_{r_2}(0)} \frac{\partial_t u_n}{u_n^{p-1}} |\varphi|^p dx. \end{aligned}$$

Let $\frac{\tau}{2} < t_1 < t_2 < \tau$. Integrating in (t_1, t_2) , there results that

$$\begin{aligned} (t_2 - t_1) \int_{B_{r_2}(0)} |\nabla \varphi|^p dx &\geq \lambda (t_2 - t_1) \int_{B_{r_2}(0)} a_n(x) |\varphi|^p dx \\ &\quad + \int_{t_1}^{t_2} \int_{B_{r_2}(0)} \frac{g_n(u_n)}{u_n^{p-1}} \varphi^p - \frac{1}{p-2} \int_{B_{r_2}(0)} \frac{\varphi^p}{u_n(x, t_2)^{p-2}} dx. \end{aligned}$$

Since $\{u_n\}_n$ is an increasing sequence, then using the same computations as in the proof of Theorem 6.1, we conclude that

$$\int_{B_{r_2}(0)} |\nabla \varphi|^p dx \geq C(t_1, t_2, r_2) \int_{B_{r_2}(0)} \frac{|\varphi|^p}{|x|^{-(q-(p-1))(\alpha_1-\varepsilon)}} dx.$$

Choosing ε small such that $(q-(p-1))(\alpha_1-\varepsilon) > p$, we reach a contradiction with the Hardy inequality (1.2). Hence the result follows. $\square$

We are now able to prove the Theorem (6.4).

Proof of Theorem 6.4 It is clear that u_n is globally defined in the time.

Let $x_0 \in \Omega$ be such that $x_0 \in B_r(0)$ with $B_{4r}(0) \subset \Omega$, since $u_0 \geq c > 0$ in $B_\eta(0)$, then by a comparison with a suitable Barenblatt subsolution we get the existence of $\bar{t} > 0$ such that $u_n(x_0, t_0) > 0$ for $t \geq \bar{t} > 0$.

Now, using Theorem 6.5, there results that

$$u_n(x_0, t_0 + \theta) \geq C_2 \int_{B_r(0)} u_n(x, t_0) dx - C_2$$

where $C_1, C_2 > 0$ are independent of n . If $t_0 > \max\{T(r), \bar{t}\}$, then by Lemma (6.6), we obtain that $\int_{B_r(0)} u_n(x, t_0) dx \rightarrow \infty$, hence $\lim_{n \rightarrow \infty} u_n(x_0, t_1) = \infty$ for all $t_1 \geq T(r) + \theta$ and then we conclude.

Notice that if $0 \in B_r(x_0)$ for some $r > 0$ with $B_{4r}(x_0) \subset \Omega$, then there exists $r_1 < r$ such that $B_{r_1}(0) \subset B_r(0)$. Thus repeating the same computations as above, it follows that

$$u_n(x_0, t_0 + \theta) \geq C_1 \int_{B_r(x_0)} u_n(x, t_0) dx - C_2 \geq C_1 \int_{B_{r_1}(0)} u_n(x, t_0) dx - C_2.$$

Since $\int_{B_{r_1}(0)} u_n(x, t_0) dx \rightarrow \infty$ if $t_0 \geq T(r_1)$, then $u_n(x_0, t) \rightarrow \infty$ as $n \rightarrow \infty$. $\square$

6.1. Finite speed of propagation in the case $p > 2$

We analyze now the question of finite speed propagation to problem (6.1). This phenomenon is deeply analyzed for the quasilinear parabolic problem with absorption term, we refer for instance to [13] where an energy method is developed to get finite speed propagation for a class of parabolic problem with absorption.

In this subsection we prove that the same phenomenon occurs for the problem

$$\begin{cases} u_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (6.15)$$

under suitable hypotheses on p, q and u_0 . The main result is the next Theorem.

Theorem 6.7. Assume that $p > 2$, $\lambda < \Lambda_{N,p}$ and $1 \leq q < p - 1$. Suppose that $u_0 \in L^2(\Omega)$ with $\text{supp } u_0 \subset B_{R_0}(x_0) \subset B_R(x_0) \subset \subset \Omega$ for some $R > 4R_0$. Then there exists a $T_1 > 0$ such that for all $0 < t < T_1$, u the minimal solution of problem (6.15) verifies that $\text{supp } u(\cdot, t) \subset \subset \Omega$.

Proof. Since $1 \leq q < p - 1$, then for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$s^q \leq \varepsilon s^{p-1} + C_\varepsilon s \quad \text{for all } s \geq 0.$$

Fix $\varepsilon > 0$ that we will choose later, then

$$u_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q \leq \lambda \frac{u^{p-1}}{|x|^p} + \varepsilon u^{p-1} + C_\varepsilon u.$$

Since $u_t - C_\varepsilon u = e^{C_\varepsilon t} \frac{d}{dt}(e^{-C_\varepsilon t} u)$, by setting $v = e^{-C_\varepsilon t} u$, it follows that

$$v_t + e^{(p-2)C_\varepsilon t} (-\Delta_p v) = \lambda e^{(p-2)C_\varepsilon t} \frac{v^{p-1}}{|x|^p} + \varepsilon e^{(p-2)C_\varepsilon t} v^{p-1}.$$

We follow closely the arguments used in [39].

Let $R_n = R + \delta 2^{-n} R$, $\bar{R}_n = \frac{1}{2}(R - \delta 2^{-n} R)$ for some $\delta < \frac{1}{2}$, it is clear that $R_{n-1} \geq R_n \geq R$ and $\bar{R}_{n-1} < \bar{R}_n < \frac{R}{2}$ for all $n \geq 1$. Let now $\xi_n \in C_0^\infty(\Omega)$, $n \geq 1$ be such that

$$\xi_n(x) = \begin{cases} 1 & \text{if } x \in B_{R_n} \setminus B_{\bar{R}_n}, \\ 0 & \text{if } x \in \Omega \setminus (B_{\bar{R}_{n-1}} \setminus B_{R_n}) \end{cases}$$

and $|\nabla \xi_n| \leq \frac{c2^n}{\delta R}$.

It is clear that $\text{supp } \xi_n \subset \{\xi_{n-1} \equiv 1\}$ and $\text{supp } u_0 \cap \text{supp } \xi_n = \emptyset$ for all $n \geq 1$. Using an approximation argument we can take $v^s \xi_n^p$, $s > 0$ to be chosen, as a test function in (6.15), then

$$\begin{aligned} & \frac{1}{s+1} \frac{d}{dt} \int_{\Omega} v^{s+1} \xi_n^p dx + s e^{(p-2)C_\varepsilon t} \int_{\Omega} \xi_n^p v^{s-1} |\nabla v|^p dx \\ & \leq e^{(p-2)C_\varepsilon t} \left[\lambda \int_{\Omega} \frac{v^{s+p-1} \xi_n^p}{|x|^p} dx + \varepsilon \int_{\Omega} v^{p-1+s} \xi_n^p dx \right. \\ & \quad \left. + p \int_{\Omega} \xi_n^{p-1} v^s |\nabla v|^{p-1} |\nabla \xi_n| dx \right] \\ & = e^{(p-2)C_\varepsilon t} (I_1 + \varepsilon I_2 + I_3.) \end{aligned} \tag{6.16}$$

Using Hardy and Young inequalities we reach that

$$\begin{aligned} I_1 &= \lambda \int_{\Omega} \frac{(v^{\frac{s+p-1}{p}} \xi_n)^p}{|x|^p} dx \leq \frac{\lambda}{\Lambda_{N,p}} \int_{\Omega} |\nabla (\xi_n v^{\frac{s+p-1}{p}})|^p dx \\ &\leq \frac{\lambda}{\Lambda_{N,p}} \int_{\Omega} |v^{\frac{s+p-1}{p}} \nabla \xi_n + \frac{s+p-1}{p} \xi_n v^{\frac{s-1}{p}} \nabla v|^p dx \\ &\leq \frac{\lambda}{\Lambda_{N,p}} (1 + \varepsilon_1)^p \int_{\Omega} \xi_n^p v^{s-1} |\nabla v|^p dx + \frac{\lambda}{\Lambda_{N,p}} \left(1 + \frac{1}{\varepsilon_1}\right)^p \int_{\Omega} v^{s+p-1} |\nabla \xi_n|^p dx \end{aligned}$$

where $\varepsilon_1 > 0$, to be chosen later.

We deal now with I_2 . Using Poincaré inequality, we get

$$\begin{aligned} I_2 &= \int_{\Omega} v^{p-1+s} \xi_n^p dx \leq \frac{1}{\bar{\Lambda}} \int_{\Omega} |\nabla(\xi_n v^{\frac{s+p-1}{p}})|^p dx \\ &\leq \frac{1}{\bar{\Lambda}} (1 + \varepsilon_2)^p \int_{\Omega} \xi_n^p v^{s-1} |\nabla v|^p dx + \frac{1}{\bar{\Lambda}} \left(1 + \frac{1}{\varepsilon_2}\right)^p \int_{\Omega} v^{s+p-1} |\nabla \xi_n|^p dx \end{aligned}$$

where $\bar{\Lambda}$ is the first eigenvalue of the p -Laplacian and $\varepsilon_2 > 0$, to be determined later.

Let us estimate now I_3 . Using Hölder and Young inequalities it follows that

$$I_3 \leq \varepsilon_3 \int_{\Omega} \xi_n^p v^{s-1} |\nabla v|^p dx + C(\varepsilon_3) \int_{\Omega} v^{s+p-1} |\nabla \xi_n|^p dx.$$

Since $\lambda < \Lambda_{N,p}$, then we can choose s close to 1 and $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3$ small enough such that

$$\frac{\lambda}{\Lambda_{N,p}} (1 + \varepsilon_1)^p + \varepsilon \frac{1}{\bar{\Lambda}} (1 + \varepsilon_2)^p < s - \varepsilon_3.$$

Thus combining the inequalities above and integrating in time, we obtain

$$\begin{aligned} &\sup_{0 < t < T} \int_{\Omega} v^{s+1}(x, t) \xi_n^p dx + C(s, N) \iint_{\Omega_T} e^{(p-2)C_\varepsilon t} \xi_n^p v^{s-1} |\nabla v|^p \\ &\leq C_\varepsilon \iint_{\Omega_T} e^{(p-2)C_\varepsilon t} v^{s+p-1} |\nabla \xi_n|^p, \end{aligned} \quad (6.17)$$

hence,

$$\begin{aligned} &\sup_{0 < t < T} \int_{\Omega} v^{s+1}(x, t) \xi_n^p dx + C(s, N) \iint_{\Omega_T} |\nabla(\xi_n v^{\frac{s+p-1}{p}})|^p \leq C(T) \\ &\quad \times \iint_{\Omega_T} v^{s+p-1} |\nabla \xi_n|^p. \end{aligned} \quad (6.18)$$

It is clear that the same kind of estimate holds if we substitute ξ_n by $\xi_n^{1+\frac{\sigma}{p}}$ for some $\sigma > 0$.

Let $w_n = v^{\frac{s+p-1}{p}} \xi_n^a$, where $a > \frac{s+p-1}{s+1}$ and define

$$Y_n \equiv \sup_{0 < t < T} \int_{\Omega} w_n^{\frac{p(s+1)}{s+p-1}}(x, t) dx + \iint_{\Omega_T} |\nabla w_n|^p.$$

Using estimates (6.18), the properties of ξ_n and the fact that $\text{supp } \xi_n \subset \{\xi_{n-1} \equiv 1\}$, there results that

$$\begin{aligned} Y_n &\leq C \iint_{\Omega_T} v^{s+p-1} |\nabla \xi_n|^p \leq C \frac{2^{pn}}{(\delta R)^p} \iint_{\text{supp } \xi_n} v^{s+p-1} \\ &\leq C \frac{2^{pn}}{(\delta R)^p} \iint_{\text{supp } \xi_n} v^{s+p-1} \xi_{n-1}^{pa} \\ &\leq C \frac{2^{pn}}{(\delta R)^p} \iint_{\Omega_T} w_{n-1}^p. \end{aligned} \quad (6.19)$$

By the Nirenberg–Gagliardo inequality we get

$$\int_{\Omega} w_{n-1}^p dx \leq \left(\int_{\Omega} |\nabla w_{n-1}|^p dx \right)^{\sigma} \left(\int_{\Omega} w_{n-1}^{\beta} dx \right)^{\frac{(1-\sigma)p}{\beta}}$$

where $\sigma = \frac{(p-2)N}{(p-2)N+(s+1)p}$ and $\beta = \frac{p(s+1)}{s+p-1}$. Therefore, integrating in time and using Hölder inequality, we reach

$$\begin{aligned} \iint_{\Omega_T} w_{n-1}^p dx dt &\leq T^{1-\sigma} \left(\iint_{\Omega_T} |\nabla w_{n-1}|^p dx dt \right)^{\sigma} \left(\sup_{0 < t < T} \int_{\Omega} w_{n-1}^{\beta}(x, t) dx \right)^{\frac{(1-\sigma)p}{\beta}} \\ &\leq T^{1-\sigma} Y_{n-1}^{1+(1-\sigma)(\frac{p}{\beta}-1)} \end{aligned}$$

Hence,

$$Y_n \leq C \frac{2^{pn}}{(\delta R)^p} T^{1-\sigma} Y_{n-1}^{1+(1-\sigma)(\frac{p}{\beta}-1)}.$$

By Lemma 4.1 in [21] we know that $Y_n \rightarrow 0$ as $n \rightarrow \infty$ if

$$Y_0 \leq \left(\frac{C}{(\delta R)^p} T^{1-\sigma} \right)^{-\frac{1}{\alpha}} 2^{-\frac{p}{\alpha^2}} \text{ where } \alpha = (1-\sigma) \left(\frac{p}{\beta} - 1 \right). \quad (6.20)$$

Notice that

$$Y_0 \leq \frac{C_0}{(\delta R)^p} \int_0^T \int_{B_R \setminus B_{\frac{R}{2}}} v^{s+p-1}.$$

Using v^s as a test function in (6.15), choosing s close to 1 and applying Poincaré inequality there results that

$$\int_0^T \int_{B_R \setminus B_{\frac{R}{2}}} v^{s+p-1} \leq C \int_{B_R} v_0^{s+1} dx = C \int_{B_R} u_0^{s+1} dx \equiv \bar{C}(u_0).$$

Now, for fixed T_1 we will choose $R > R_0$ such that

$$\frac{C_0 \bar{C}(u_0)}{(\delta R)^p} \leq \left(\frac{C}{(\delta R)^p} T^{1-\sigma} \right)^{-\frac{1}{\alpha}} 2^{-\frac{p}{\alpha^2}}. \quad (6.21)$$

It is clear that if $R \geq 4R_0 + CT_1^{\frac{1-\sigma}{p(\alpha+1)}}$, then (6.21) holds.

Thus going back to (6.20), we obtain that $u(x, t) \equiv 0$ for all $t < T_1$ and $x \in \Omega \setminus \bar{B}_R$. Hence the result follows. $\square$

Remark 6.8. Since Theorem 6.7 holds, we say that problem (6.15) has the *finite speed of propagation* property (FSP property to be short).

The condition $q > 1$ is optimal, since in the case where $q < 1$ we can prove the existence of a solution u such that $u > 0$ in $\Omega \times (0, T)$. More precisely, following the proof of Theorem 5.5 we get the existence of a subsolution in the form $u(x, t) = \mu(\varepsilon t)w(x)$ where $\nu(t) = (1-q)t^{\frac{1}{1-q}}$ and w is the unique solution of the problem (5.8).

If $q = p - 1$, the (FSP) property holds if the term u^{p-1} is replaced by σu^{p-1} , where σ is small enough.

If $p > 2$, $1 < q < (p - 1)$ and the initial datum verifies $0 \leq u_0(x) \leq w(x)$ where w is the unique positive solution to the stationary problem

$$-\Delta_p w = \lambda \frac{w^{p-1}}{|x|^p} + w^q \quad \text{in } \Omega, \quad w \in W_0^{1,p}(\Omega),$$

the solution $u(x, t)$ of the evolutionary problem, verifies $\lim_{t \rightarrow \infty} u(x, t) = w(x)$.

The proof is similar to the one in Theorem 5.5.

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