On small perturbations of a nonlinear periodic system with degenerate equilibrium

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Abstract. This paper considers the reducibility and existence of periodic solutions for a class of nonlinear periodic system with a degenerate equilibrium point under small perturbations. By introducing some parameter, we consider an equivalent periodic system. Then we prove that by an affine linear periodic transformation the parameterized periodic system is reducible to one with zero as an equilibrium. Topological degree theorem ensures that for some parameter the result can go back to the original system. Then, we obtain a small periodic solution.

Mathematics Subject Classification. Primary 37J40; Secondary 70H12, 34C20.

Keywords. Periodic system, Degenerate equilibrium, Small perturbation, Reducibility.

1. Introduction

To state our main theorem, we need first to give some definitions and notations. Let $f(t)$ be a 2π -periodic function. If $f(t)$ is analytic periodic on a strip domain $\mathbb{T}_s = \{t \in \mathbb{C}/2\pi\mathbb{Z} \mid |\operatorname{Im}(t)| \leq s\}$, then we denote $f(t) \in \mathcal{A}_s$ and expand $f(t)$ as Fourier series $f(t) = \sum_{k \in \mathbb{Z}} f_k e^{ikt}$, where $i = \sqrt{-1}$. Define $\|f(t)\|_s = \sum_{k \in \mathbb{Z}} |f_k| e^{|k|s}$. Denote by $[f] = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$ the average of f .

For $x = (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$, we define $|x| = \max_{1 \leq j \leq n} |x_j|$ and denote by $D_r = \{x \in \mathbb{C}^n \mid |x| \leq r\}$.

Let $f(x, t)$ be real analytic in x and t on the domain $D_r \times \mathbb{T}_s$ and 2π -periodic with respect to t . Then we have $f(x, t) = \sum_{k \in \mathbb{Z}} f_k(x) e^{ikt}$. Define

$$\|f(x, t)\|_{r,s} = \sum_{k \in \mathbb{Z}} \|f_k(x)\|_r e^{|k|s},$$

where $\|f_k(x)\|_r = \sup_{x \in D_r} |f_k(x)|$. It is easy to see $\|fg\|_{r,s} \leq \|f\|_{r,s} \|g\|_{r,s}$.

The work was supported by the NNSF of China (11071038, 11371090) and the NSF of Jiangsu Province (BK2010420). This work was also supported by the Innovative Project of Jiangsu Province (CXLX13.074).

If $f(x, t) = (f_1(x, t), f_2(x, t), \dots, f_n(x, t))$, we define $\|f(x, t)\|_{r,s} = \max_{1 \leq j \leq n} \|f_j(x, t)\|_{r,s}$.

For a matrix $A = (A_{ij})_{1 \leq i, j \leq n}$, define $\|A\| = n \times \max_{ij} |A_{ij}|$. Then we have $\|A \cdot B\| \leq \|A\| \cdot \|B\|$. Similarly, for matrix functions we define $\|A(x, t)\|_{r,s}$ or $\|A(t)\|_s$.

This paper considers a small perturbation of a class of periodic system with degenerate equilibrium:

$$\dot{x} = N(x) + h(x, t) + f(x, t), \quad (1.1)$$

where $N(x) = (x_1^{2l_1+1}, x_2^{2l_2+1}, \dots, x_n^{2l_n+1})^T$ with $l_1, l_2, \dots, l_n$ being non-negative integers. $h = (h_1, \dots, h_n)^T$ denotes a high order term with $h_j = O(x_j^{2l_j+2})$ for $j = 1, 2, \dots, n$ as x tends to 0 and $f(x, t)$ is a small perturbation.

Suppose that $h(x, t)$ and $f(x, t)$ are real analytic in x and t on the domain $D_{r_0} \times \mathbb{T}_{s_0}$ and 2π -periodic with respect to t .

Our purpose is to prove that there exists a small constant $\epsilon > 0$ such that if $\|f(x, t)\|_{r_0, s_0} \leq \epsilon$, (1.1) can be reduced to a suitable form with zero as an equilibrium.

Theorem 1.1. *There exists a small constant $\epsilon > 0$ such that if $\|f(x, t)\|_{r_0, s_0} \leq \epsilon$, there exists an affine linear periodic transformation $x = (I + u(t))y + v(t)$, where I is the identity matrix, $u(t)$ and $v(t)$ are both real analytic periodic on $\mathbb{T}_{s_0}$, such that the system (1.1) is changed into the following form:*

$$\dot{y} = N(y) + \hat{\Omega}y + f_*(y, t)$$

where $\hat{\Omega}$ is a constant matrix with $\|\hat{\Omega}\| = O(\epsilon^{\frac{1}{2}})$ and $f_*(y, t) = O(y^2)$ as y tends to zero. Thus $x = v(t)$ is a periodic solution of the system (1.1).

The study of dynamical properties near the equilibrium in the dynamical systems is a very important subject. Usually, we wonder that: what is the normal form near the equilibrium? Which property near the equilibrium can persist under a small perturbation? The results depend on whether the equilibrium is degenerate or not. When the equilibrium is non-degenerate, that is, the Jacobian matrix at the equilibrium is non-degenerate, we usually change it to some normal form or suitable form by a transformation to obtain some dynamical behaviors near the equilibrium. See [1, 8, 10, 11]. The reducibility of linear systems is a significant problem and many authors are dedicated to the study of periodic or quasi-periodic systems. For periodic systems, the classical result is the well known Floquet theorem [8, 10], which asserts that linear periodic systems are always reducible, that is, they can be changed to linear constant coefficients systems by periodic transformations.

More recently, much progress has been made in quasi-periodic systems with small perturbations [2–6, 9]. In [5], Jorba and Simo proved the reducibility for a class of linear quasi-periodic systems with a small parameter under some non-resonance assumptions and non-degenerate conditions. Later the result in [5] is generalized to a class of nonlinear quasi-periodic systems [6].

Note that in the case of quasi-periodic systems, there are much more difficulties due to the notorious small divisors. Hence the KAM method becomes a strong tool for quasi-periodic systems and we refer to [2, 3, 5–7]. As for nonlinear periodic systems, of course there are plenty of similar results since there is no problem of small divisors.

When the equilibrium is degenerate, the problem becomes much more complicated. The previous idea in the non-degenerate case is not valid any more. More recently, we considered a class of one dimensional quasi-periodic system with a degenerate equilibrium. Under small perturbations, using the KAM method we proved that the system can be reduced to one with zero as an equilibrium and hence admits a small quasi-periodic solution [12]. We also extend the results to two dimensional quasi-periodic systems [13]. However, the method can not be used to higher dimensional cases of quasi-periodic systems.

We find that for periodic systems, the similar results can hold for high dimensional cases. This motivates us to study the periodic system (1.1).

For the proof of our theorem, we can follow the same outline of the KAM method used in [5, 6, 12, 13]. At each step of the KAM iteration, we need to kill some lower order terms as many as possible so that the system is changed into normal form or suitable form with zero as an equilibrium. The important thing is to construct an appropriate transformation. In the case of quasi-periodic systems, because of small divisors, we need the KAM technique to overcome some difficulty; while, for periodic systems, since there is no problem of small divisors, this can be done by the implicit function theorem. The troubling term is the zero-order average term. When the equilibrium is non-degenerate, it is easy to be eliminated by a translation of coordinates. However, the technique does not apply to the degenerate case since we can not have some good estimates for the shift of the equilibrium. To overcome this difficulty, we shall introduce some parameter and instead consider a parameterized system. Finally, the result for parameterized system can go back to the original one at some special parameter by topological degree theorem.

Below we introduce a parameter by a translation of coordinates and change the system (1.1) to an equivalent parameterized system.

For this purpose we define $\Pi = [-\delta_1, \delta_1] \times \cdots \times [-\delta_n, \delta_n]$, where $\delta_j \leq r_0/4$. Let $r \leq r_0/4$. For all $\xi \in \Pi$, make a transformation $x = y + \xi$, $|y| \leq 2r$. Set $h(y, t; \xi) = h(y + \xi, t)$, $f(y, t; \xi) = f(y + \xi, t)$ for simplicity. Then (1.1) is changed into

$$\dot{y} = N(\xi + y) + h(y, t; \xi) + f(y, t; \xi). \quad (1.2)$$

We take the expansion to obtain that:

$$(y_j + \xi_j)^{2l_j+1} = \xi_j^{2l_j+1} + (2l_j + 1)\xi_j^{2l_j} y_j + g_j(y; \xi),$$

where

$$g_j(y; \xi) = \frac{(2l_j + 1)2l_j}{2} \xi_j^{2l_j-1} y_j^2 + \cdots + (2l_j + 1)\xi_j y_j^{2l_j} + y_j^{2l_j+1}.$$

Let $\Omega(\xi) = \text{diag}(\Omega_1(\xi), \Omega_2(\xi), \dots, \Omega_n(\xi))$, where $\Omega_j(\xi) = (2l_j + 1)\xi_j^{2l_j}$ and $g = (g_1, g_2, \dots, g_n)^T$. Thus the system (1.2) can be written as

$$\dot{y} = N(\xi) + \Omega(\xi)y + P(y, t; \xi)$$

where $P(y, t; \xi) = g(y; \xi) + h(y, t; \xi) + f(y, t; \xi)$.

It is easy to see that P is real analytic in $D_{2r} \times \mathbb{T}_s \times \Pi$ with $s = s_0$. Let $P(x, t; \xi) = \sum_{k \in \mathbb{Z}} P_k(x; \xi) e^{ikt}$. Define

$$\|P\|_{2r, s; \delta} = \sum_{k \in \mathbb{Z}} \|P_k\|_{2r; \delta} e^{|k|s},$$

where $\delta = (\delta_1, \delta_2, \dots, \delta_n)$ and $\|P_k\|_{2r; \delta} = \sup_{(x, \xi) \in D_{2r} \times \Pi} |P_k(x; \xi)|$.

Let $\delta_j = \epsilon^{\frac{1}{2l_j+2}}$ and $r = \epsilon^{\frac{1}{2}}$. If ϵ is sufficiently small, we have $\delta_j \leq r_0/4$ and $r \leq r_0/4$. This yields $\|g\|_{2r; \delta} \leq c\epsilon$ apparently, where c is a constant. For simplicity we use c to indicate various constants in estimates.

Similarly, we have $\|h_j\|_{2r, s; \delta} \leq c\delta_j^{2l_j+2}$ and $\|h\|_{2r, s; \delta} \leq c\epsilon$. Note that $\|f\|_{2r, s; \delta} \leq \epsilon$. Thus we also have

$$\|P\|_{2r, s; \delta} \leq c\epsilon.$$

Based on the above discussion, we consider the equivalent differential equations with parameter

$$\dot{y} = N(\xi) + \Omega(\xi)y + P(y, t; \xi) \quad (1.3)$$

where $P(y, t; \xi)$ is real analytic in $D_{2r} \times \mathbb{T}_s \times \Pi$ and satisfies $\|P\|_{2r, s; \delta} \leq c\epsilon$.

We want to prove that there exists a parameter $\xi \in \Pi$ such that (1.3) can be reduced to a desired form. For this purpose we introduce an external parameter λ and consider the following equation:

$$\dot{y} = N(\xi) - \lambda + \Omega(\xi)y + P(y, t; \xi) \quad (1.4)$$

where λ is an external parameter.

Obviously, if $\lambda = 0$ the system (1.4) becomes the form of (1.3).

Theorem 1.2. *If $\epsilon > 0$ is sufficiently small, there exist a real analytic function $\lambda(\xi) = N(\xi) + \hat{N}(\xi)$ on Π and an affine linear analytic periodic transformation $x = (I + u(t; \xi))y + v(t; \xi)$ on $\mathbb{T}_s \times \Pi$, which takes the system (1.4) with $\lambda = \lambda(\xi)$ into:*

$$\dot{y} = \Omega_*(\xi)y + P_*(y, t; \xi) \quad (1.5)$$

where $P_*(y, t; \xi) = O(y^2)$ as y tends to zero, and

$$\|\hat{N}(\xi)\| \leq c\epsilon, \quad \|\Omega_*(\xi) - \Omega(\xi)\| \leq c\epsilon^{\frac{1}{2}}, \quad \forall \xi \in \Pi,$$

where and below c is a constant independent of ϵ . Moreover, u and v are real analytic on $\mathbb{T}_s \times \Pi$ satisfying

$$\epsilon^{\frac{1}{2}} \|u(\cdot; \xi)\|_s + \|v(\cdot; \xi)\|_s \leq c\epsilon, \quad \forall \xi \in \Pi.$$

2. Proof of Theorems

Proof of Theorem 1.1. The proof of Theorem 1.1 is based on Theorem 1.2. We first prove Theorem 1.1 and delay the proof of Theorem 1.2 for later.

In what follows, we are to find a parameter $\xi_* \in \Pi$ such that $\lambda(\xi_*) = 0$. Thus, at $\xi = \xi_*$, the system (1.4) goes back to the original system. Since $N(\xi)$ is degenerate at $\xi = 0$ when $l_j \geq 1$, we use topological degree theory to prove the existence of ξ_* .

By the assumptions we have $\|\hat{N}(\xi)\| \leq c\delta_j^{2l_j+2}$ for all $j = 1, 2, \dots, n$. Recall that $\Pi = [-\delta_1, \delta_1] \times \dots \times [-\delta_n, \delta_n]$, and the boundary is

$$\begin{aligned} \partial\Pi = & \{-\delta_1, \delta_1\} \times [-\delta_2, \delta_2] \dots \times [-\delta_n, \delta_n] \cup \dots \cup [-\delta_1, \delta_1] \\ & \times \dots \times [-\delta_{n-1}, \delta_{n-1}] \times \{-\delta_n, \delta_n\} \end{aligned}$$

If ξ belongs to $\{-\delta_1, \delta_1\} \times [-\delta_2, \delta_2] \times \dots \times [-\delta_n, \delta_n]$, it follows that:

$$|\lambda(\xi)| \geq \delta_1^{2l_1+1} - \sup_{\xi \in \Pi} |\hat{N}(\xi)| \geq \delta_1^{2l_1+1} - c_1 \delta_1^{2l_1+2} > 0.$$

Similarly, we can consider the other case. Then $|\lambda(\xi)| > 0$ holds for all $\xi \in \partial\Pi$.

Let $F_t(\xi) = N(\xi) + t\hat{N}(\xi)$. In the same way as the above we have that for all $t \in [0, 1]$ it follows that $0 \notin F_t(\partial\Pi)$. Note that $F_0 = N(\xi)$ and $F_1 = N(\xi) + \hat{N}(\xi) = \lambda(\xi)$, then F_0 and F_1 are isotopic.

Thus, if ϵ is sufficiently small, we obtain

$$\deg(\lambda, \Pi, 0) = \deg(N, \Pi, 0) = 1.$$

Hence there exists $\xi_* \in \Pi$ satisfying $\lambda(\xi_*) = 0$.

Set $\xi = \xi_*$. Under the transformation $x = (I + u(t; \xi_*))y + v(t; \xi_*)$, the system (1.3) is changed to

$$\dot{y} = \Omega_* y + P_*(y, t; \xi_*)$$

where $P_* = O(y^2)$, $y \rightarrow 0$. Consequently $x = v(t; \xi_*)$ is a periodic solution of (1.1). We have finished the proof of Theorem 1.1.

Proof of Theorem 1.2. Consider the real periodic differential equation with parameters (1.4).

We define a periodic transformation by $y = (I + u(t))z + v(t)$, where u and v are both real periodic on $\mathbb{T}_s$. Under this transformation, the system is changed to

$$\begin{aligned} \dot{z} = & (I + u)^{-1} (N(\xi) - \lambda - \dot{v} + \Omega(\xi)v + P(v, t; \xi)) \\ & + (I + u)^{-1} (-\dot{u} + \Omega(\xi)(I + u) + \partial_y P(v, t; \xi)(I + u))z \\ & + O(z^2) \end{aligned} \quad (2.1)$$

where $O(z^2) = (I + u)^{-1} (P((I + u)z + v, t; \xi) - P(v, t; \xi) - \partial_y P(v, t; \xi)(I + u)z)$.

Then we want to find a set of (u, v, λ) such that

$$\begin{cases} N(\xi) - \lambda - \dot{v} + \Omega(\xi)v + P(v, t; \xi) = 0 \\ (I + u)^{-1} [-\dot{u} + \Omega(\xi)(I + u) + \partial_y P(v, t; \xi)(I + u)] = \Omega_*(\lambda, \xi) \end{cases} \quad (2.2)$$

We begin with the first equation of (2.2), which is equivalent to

$$\begin{cases} -\dot{v} + \Omega(\xi)v + P(v, t; \xi) - [P(v, \cdot; \xi)] = 0 \\ N(\xi) - \lambda + [P(v, \cdot; \xi)] = 0 \end{cases}$$

where $[P(v, \cdot; \xi)]$ is the average of $P(v(t), t; \xi)$ in t .

Below we use Lemma 3.2 to prove the existence of solutions v and λ . Set $\hat{N}(\xi) = [P(v, \cdot; \xi)]$ and $\lambda = N + \hat{N}$. So we only need to solve the first equation for v .

In accordance with the notations appearing in Lemma 3.2, we assume $\mathbb{X}$ to be a subspace made of n -dimensional vector functions of which the components are real analytic periodic on $\mathbb{T}_s$ with zero average and finite norm, written as:

$$\mathbb{X} = \{v(t) = (v_1(t), v_2(t), \dots, v_n(t)) \mid v_i(t) \in \mathcal{A}_s, [v] = 0, \|v\|_s < \infty\}.$$

In view of the definitions given in the introduction, it follows that $\mathbb{X}$ is a Banach space with the analytic sup-norm of complex vectors $\|\cdot\| = \|\cdot\|_s$.

Let $\mathbb{D} = \{\int_0^t v(s) ds \mid v \in \mathbb{X}\}$. Then the differential operator $L = \frac{d}{dt} - \Omega(\xi) : \mathbb{D} \subset \mathbb{X} \rightarrow \mathbb{X}$ is defined well. For $v(t) = \sum_{0 \neq k \in \mathbb{Z}} v_k e^{ik \cdot t} \in \mathbb{X}$, set $L^{-1}v(t) = (w_1, w_2, \dots, w_n)$. It is easy to see

$$w_j(t) = \sum_{0 \neq k \in \mathbb{Z}} \frac{1}{ik - \Omega_j(\xi)} v_{jk} e^{ikt}.$$

Since $\Omega_j(\xi)$ is real, we obtain $|ik - \Omega_j(\xi)| \geq |k| \geq 1$ for $k \neq 0$, thus $L^{-1} : \mathbb{X} \rightarrow \mathbb{D} \subset \mathbb{X}$ is bounded with $\|L^{-1}\| \leq 1$, here $\|\cdot\|$ denotes the operator norm.

Consider the equation $L(v) = F(v)$, where $F(v) = P(v, t; \xi) - [P(v, \cdot; \xi)]$. Denote $\bar{B}(r) = \{v \in \mathbb{X} \mid \|v\| \leq r\}$, we have that $P(y, t; \xi)$ is analytic on $D_{2r} \times \mathbb{T}_s \times \Pi$ with $\|P\|_{2r, s; \delta} \leq c\epsilon$. Then $F : v \in \bar{B}(r) \rightarrow G(v) \in \mathbb{X}$ is continuous. Having in mind $r = \epsilon^{\frac{1}{2}}$, for all $v_1, v_2 \in \bar{B}(r)$, by Cauchy estimate we obtain

$$\|P(v_1, t; \xi) - P(v_2, t; \xi)\| \leq \frac{1}{r} \|P\|_{2r, s; \delta} \cdot \|v_1 - v_2\| \leq c\epsilon^{\frac{1}{2}} \|v_1 - v_2\|.$$

Moreover,

$$\|F(v_1) - F(v_2)\| \leq c\epsilon^{\frac{1}{2}} \|v_1 - v_2\| = b \|v_1 - v_2\| \quad (2.3)$$

We choose $d = \max\{\|L^{-1}\|, 1\} = 1$. If ϵ is sufficiently small, $b = c\epsilon^{\frac{1}{2}} < \frac{1}{4d} < 1$.

Recall that

$$\|F(0)\| \leq \|P(0, t; \xi)\| + \|[P(0, t; \xi)]\| \leq c\epsilon,$$

and choose ϵ sufficiently small to ensure that the following statement holds as well:

$$\|F(0)\| \leq c\epsilon \leq \frac{r}{2}.$$

Applying lemma 3.2 with $d = 1$ and $\tilde{\gamma} = \gamma = r$, hence for all $\xi \in \Pi$ there exists $v(t; \xi) \in \mathbb{X}$ satisfying $\|v\| \leq r$ such that $Lv = F(v)$.

In fact, we succeed in obtaining a more accurate estimate of v by some further computation:

$$\|v\| = \|L^{-1}(P(v, \cdot; \xi) - [P(v, \cdot; \xi)])\| \leq c\epsilon,$$

and $\|\hat{N}(\xi)\| \leq \| [P(v, t; \xi)] \| \leq c\epsilon$ for all $\xi \in \Pi$.

Now we turn to consider the second equation of system (2.2). It is equivalent to

$$-Lu + (I + u)^{-1}(uLu + \partial_y P(v, t; \xi)(I + u)) = \Omega_* - \Omega, \quad (2.4)$$

where, different from the above, the differential operator L here represents:

$$Lu = \dot{u} - \Omega u + u\Omega. \quad (2.5)$$

Then, the equation (2.4) becomes

$$Lu = H(u, Lu) - [H(u, Lu)], \quad \Omega_* - \Omega = \hat{\Omega} = [H(u, Lu)].$$

where

$$H(u, Lu) = (I + u)^{-1}(uLu + \partial_y P(v, t; \xi)(I + u)).$$

Similarly, by a slight abuse of notations, we set $\mathbb{X}$ a subspace consisting of real analytic periodic matrix functions on $\mathbb{T}_s$ with zero average and finite norm, written as:

$$\mathbb{X} = \{u(t) = (u_{ij}(t))_{1 \leq i, j \leq n} \mid u_{ij}(t) \in \mathcal{A}_s, [u] = 0, \|u\|_s < \infty\}.$$

Then $\mathbb{X}$ is a Banach space with the analytic matrix-norm $\|u\| = \|u\|_s = n \max_{ij} |u_{ij}(t)|_s$.

Let $\mathbb{D} = \{\int_0^t u(s) ds \mid u \in \mathbb{X}\}$, and the differential operator $L : \mathbb{D} \subset \mathbb{X} \rightarrow \mathbb{X}$ in (2.5) is well defined.

Note that $\Omega(\xi)$ is a diagonal real matrix. It is easy to see that L is invertible and $L^{-1} : \mathbb{X} \rightarrow \mathbb{D} \subset \mathbb{X}$ is bounded with $\|L^{-1}\| \leq 1$.

To check the assumption of Lemma 3.2, set $\bar{B}(r^{\frac{1}{2}}) = \{u \in \mathbb{X} \mid \|u\| \leq r^{\frac{1}{2}}\}$.

Note that $\|v\| \leq \epsilon \leq r$ and $\|P\|_{2r, s; \delta} \leq c\epsilon$. So it follows that $\|\partial_y P(v, t; \xi)\| \leq c\epsilon/r \leq c\epsilon^{\frac{1}{2}}$.

If $\epsilon \leq \frac{1}{4}$, then $r = \epsilon^{\frac{1}{2}} \leq \frac{1}{2}$, and $\|(I + u)^{-1}\| \leq 2$ holds by a classical estimate. Thus for $(u_i, Lu_i) \in \bar{B}(r^{\frac{1}{2}}) \times \bar{B}(r^{\frac{1}{2}})$ and $\xi \in \Pi$, we have:

$$\|H(u_1, Lu_1) - H(u_2, Lu_2)\| \leq c\epsilon^{\frac{1}{4}} (\|u_1 - u_2\| + \|Lu_1 - Lu_2\|).$$

Let $F(u, Lu) = H(u, Lu) - [H(u, Lu)]$, then

$$\|F(u_1, Lu_1) - F(u_2, Lu_2)\| \leq b(\|u_1 - u_2\| + \|Lu_1 - Lu_2\|) \quad \text{with } b = 2c\epsilon^{\frac{1}{4}}.$$

In view of $d = \max\{\|L^{-1}\|, 1\} = 1$, when ϵ is sufficiently small, $b = 2c\epsilon^{\frac{1}{4}} < \frac{1}{4d} = \frac{1}{4}$. In the same way as the above, we obtain

$$\|F(0, 0)\| = \|H(0, 0) - [H(0, 0)]\| \leq \|\partial_y P(v, t; \xi)\| \leq c\epsilon^{\frac{1}{2}}. \quad (2.6)$$

Choose ϵ sufficiently small to make sure that $c\epsilon^{\frac{1}{2}} < \frac{1}{2}\epsilon^{\frac{1}{4}} = \frac{1}{2}r^{\frac{1}{2}}$. Applying Lemma 3.2 with $F(u, Lu)$, $d = 1$ and $\tilde{\gamma} = \gamma = r^{\frac{1}{2}}$, we have that for all $\xi \in \Pi$, there exists $u(t; \xi) \in \mathbb{X}$ satisfying $\|u(\cdot; \xi)\| \leq r^{\frac{1}{2}}$ such that $Lu = F(u, Lu) = H(u, Lu) - [H(u, Lu)]$. Moreover, $\|Lu\| \leq r^{\frac{1}{2}}$ holds. So it follows that $\|[H(u, Lu)]\| \leq c\epsilon^{\frac{1}{2}}$. Thus, we have

$$\|\hat{\Omega}(\xi)\| \leq \|[H(u, Lu)]\| \leq c\epsilon^{\frac{1}{2}}, \quad \forall \xi \in \Pi.$$

Similarly, a more accurate estimate for u is obtained with

$$\|u(\cdot; \xi)\| \leq \|L^{-1}\| \cdot \|H(u, Lu) - [H(u, Lu)]\| \leq c\epsilon^{\frac{1}{2}}, \quad \forall \xi \in \Pi.$$

Note that u and v are both analytic in the parameter ξ on Π . This proves Theorem 1.2.

3. Appendix

We first recall Cauchy estimate for analytic functions, which can be referred to [14].

Lemma 3.1. (Cauchy estimate) *Let $D_r = \{z \in \mathbb{C}^n : |z| \leq r\}$ be a closed ball of radius r around the origin. Suppose f is an analytic function on D_r with bounded sup-norm $|f|_r$. Then we have*

$$|f_{z_j}|_{r-\rho} \leq \frac{1}{\rho} |f|_r.$$

for all $0 < \rho < r$ and $1 \leq j \leq n$.

The lemma below is about the existence of solutions of an abstract equation in Banach space. It can be proved easily by the Banach fixed point theorem. But we state it in a much more convenient form for our problem.

Lemma 3.2. *Let $\mathbb{X}$ be a Banach space. $L : \mathbb{D} \subset \mathbb{X} \rightarrow \mathbb{X}$ is a linear operator. Assume L is invertible and $L^{-1} : \mathbb{X} \rightarrow \mathbb{D} \subset \mathbb{X}$ is a linear bounded operator. Consider the equation $Lw = F(w, Lw)$, $w \in \mathbb{D}$. Denote $\bar{B}(w_0, \gamma) = \{w \in \mathbb{X} \mid \|w - w_0\| \leq \gamma\}$ and $\bar{B}(\gamma) = \{w \in \mathbb{X} \mid \|w\| \leq \gamma\}$ for simplicity. Let $F : (w, Lw) \in \bar{B}(\gamma) \times \bar{B}(\gamma) \rightarrow F(w, Lw) \in \mathbb{X}$ be a continuous mapping. Here and below Lw indicates a variable for convenience if without confusion. Suppose that there exists $b > 0$ such that*

$$\|F(w_2, Lw_2) - F(w_1, Lw_1)\| \leq b(\|w_2 - w_1\| + \|Lw_2 - Lw_1\|)$$

for all $(w_i, Lw_i) \in \bar{B}(\gamma) \times \bar{B}(\gamma)$, $i = 1, 2$. Set $d = \max\{1, \|L^{-1}\|\}$. Let $\tilde{\gamma} = \gamma/d$. If $b \leq \frac{1}{4d}$ and $\|F(0, 0)\| \leq \tilde{\gamma}/2$, then there exists w_* with $\|Lw_*\| \leq \tilde{\gamma}$ and $\|w_*\| \leq \gamma$ such that $Lw_* = F(w_*, Lw_*)$.

Proof. We prove this lemma by the Banach fixed point theorem. For $z \in \bar{B}(0, \tilde{\gamma})$, we have that $L^{-1}z$ belongs to $\bar{B}(\gamma)$. Set $G(z) = F(L^{-1}z, z)$, $z \in \bar{B}(\tilde{\gamma})$. Then the solution of $Lw = F(w, Lw)$ ($w \in \mathbb{D}$) is equivalent to that of the equation $G(z) = z$ ($z \in \mathbb{X}$) with $w = L^{-1}z$.

Let $z_0 = G(0) = F(0, 0)$. Then we have $\|z_0\| \leq \tilde{\gamma}/2$. For $z \in \bar{B}(z_0, \tilde{\gamma}/2) \subset \bar{B}(\tilde{\gamma})$, we obtain

$$\|G(z) - z_0\| = \|G(z) - G(0)\| \leq 2bd\|z\| \leq \tilde{\gamma}/2.$$

Thus, $G : \bar{B}(z_0, \tilde{\gamma}/2) \rightarrow \bar{B}(z_0, \tilde{\gamma}/2)$.

Moreover, for all $z_1, z_2 \in \bar{B}(z_0, \tilde{\gamma}/2)$, we have

$$\|G(z_2) - G(z_1)\| \leq 2bd\|z_2 - z_1\| \leq \frac{1}{2}\|z_2 - z_1\|.$$

Applying the Banach fixed point theorem, there exists $z_* \in \bar{B}(z_0, \tilde{\gamma}/2) \subset \bar{B}(\tilde{\gamma})$ such that $G(z_*) = z_*$. Set $w_* = L^{-1}z_*$, then $\|w_*\| \leq \gamma$. We have proved this lemma. $\square$

Acknowledgements

The authors are very grateful to the referees for their valuable comments and suggestions which help to improve the presentation of this paper.

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Received: 25 April 2012.

Accepted: 15 September 2013.