Dissipative solutions for the modified two-component Camassa–Holm system

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Abstract. Camassa–Holm model is capable of characterizing the dynamic behavior of shallow water wave, thus has been extensively studied. This paper is concerned with shallow water wave behavior after wave breaking. To better reflect the whole process, the modified two-component Camassa–Holm system is considered. The continuation of solutions of such system after wave braking is investigated. By introducing a skillfully defined characteristic, together with a set of newly defined variables, the original system is converted into a Lagrangian equivalent system, from which global dissipative solutions are obtained. The results obtained herein are deemed useful in understanding the dynamic behavior of shallow water wave during and after wave breaking.

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1. Introduction

It is important to understand the dynamic behavior of shallow water wave during and after wave breaking in view of its potential application in fluid dynamics. As the modified two-component Camassa–Holm (MCH2) system is capable of reflecting the shallow water wave dynamic behavior [1], this work investigates the solutions and the associated features for the following MCH2 system, namely

$$\begin{cases} u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + g\eta\bar{\eta}_x = 0, & t > 0, x \in R, \\ \eta_t + (\eta u)_x = 0, & t > 0, x \in R, \\ \eta = (1 - \partial_x^2)(\bar{\eta} - \bar{\eta}_0), & t > 0, x \in R, \end{cases} \quad (1.1)$$

where u denotes the velocity field, g is the downward constant acceleration of gravity in applications to shallow water waves, and $\bar{\eta}_0$ is taken to be a constant. We are particularly interested in the global existence of the dissipative

solutions of the system, such issue, although important, has not been adequately addressed in the literature.

Note that system (1.1), a modified version of the two-component Camassa–Holm (CH2) system, has been established recently by Holm et al. in [1] to allow a dependence on the average density $\bar{\eta}$ and the pointwise density η , where the CH2 system is an extension of the Camassa–Holm (CH) equation by combining its integrability property with compressibility, or free-surface elevation dynamics in its shallow-water interpretation [2, 3]. From the geometric points of view, the MCH2 system can be defined as geodesic motion on the corresponding semidirect product Lie group with respect to a certain metric and be given as a set of Euler–Poincaré equations on the dual of the corresponding Lie algebra [1, 4]. This MCH2 system does admit peaked solutions in the velocity and average density, but it may not be integrable unlike the CH2 system.

Note that when $\eta = 0$, system (1.1) is reduced to the CH equation, which models the unidirectional propagation of shallow water waves over a flat bottom, has a bi-Hamiltonian structure [5] and is completely integrable [6–8]. The CH equation has attracted considerable attention because it has peaked solitons [6, 9] and experiences wave breaking [6, 10]. The presence of breaking waves means that the solution remains bounded while its slope becomes unbounded in finite time [10, 11]. After wave breaking the solutions of the CH equation can be continued uniquely as either global conservative [12–15] or global dissipative solutions [16–18].

Recently, the MCH2 system has been studied extensively in [19–29] from many different points of view. It has been shown that system (1.1) is locally well-posed and also has global strong solutions which blow-up in finite time [20, 29]. The aim of this paper is to study the problem of continuation beyond wave breaking and to obtain the existence of global dissipative solutions for system (1.1) with the case of $g = 1$ just for simplicity. the problem of continuation beyond wave breaking for the MCH2 system, where the existence issue for the global conservative and global dissipative solutions was addressed, has also been studied in [21–23] by applying an approach that reformulates the system (1.1) as a semilinear system of O.D.E. taking values in a Banach space. However, the results here are more detailed and precise. In particular, we establish the semigroup property of the solutions and the continuity of the semigroup with respect to a new distance which is introduced. Furthermore, we show that the corresponding dissipative solution satisfies the following entropy condition

$$u_x(t, x) \leq 1/t + \sqrt{5}/2 \cdot \|z_0\|_{H^1},$$

for almost every x and all $t \geq 0$, which is not discussed in [21].

In this paper we develop a new approach to establish a global and stable dissipative solution for the MCH2 system, by making use of a skillfully defined characteristic and a set of newly introduced variables, in which the associated energy is introduced as an additional variable so as to obtain a well-posed initial-value problem, facilitating the study on the behavior of wave breaking.

Our inspiration of investing the underlying issue mainly stems from the early work [14, 15] in the study of the global dissipative solution of the CH equation. Here we deal with a coupled system where the mutual effect between two components makes the analysis more complicated than a single one as considered in [14] and [15]. The key to circumvent the difficulty is to use the newly defined characteristic and a set of variables, which allows for the establishment of the global dissipative solutions of Eq. (1.1).

The remainder of this paper is organized as follows. In Sect. 2, we transform the original system into an equivalent semilinear system by introducing a new set of Lagrangian variables. The global solution of the equivalent system is derived in Sect. 3. A global continuous semigroup of weak dissipative solutions for the original system is established and the entropy condition which the dissipative solution satisfies is also obtained in Sect. 4. The paper is closed in Sect. 5.

2. The Lagrangian equivalent system

Let us start by defining P_1, P_2, P_3 as

$$\begin{aligned} P_1(t, x) &= G * (u^2 + u_x^2/2 + v^2/2 - v_x^2/2) \\ &= 1/2 \cdot \int_R e^{-|x-x'|} (u^2 + u_x^2/2 + v^2/2 - v_x^2/2)(t, x') dx', \\ P_2(t, x) &= G * (u_x v_x) = 1/2 \cdot \int_R e^{-|x-x'|} (u_x v_x)(t, x') dx', \\ P_3(t, x) &= G * (v u_x) = 1/2 \cdot \int_R e^{-|x-x'|} (v u_x)(t, x') dx', \end{aligned}$$

where we have taken that $v = \bar{\eta} - \bar{\eta}_0$, and $*$ denotes the spatial convolution and $G = e^{-|x|}/2$ such as $G * f(x) = 1/2 \cdot \int_R e^{-|x-x'|} f(x') dx'$ for all $f \in L^2(R)$. Thus, system (1.1) can be rewritten as

$$\begin{cases} u_t + uu_x + P_{1,x} = 0, & t > 0, x \in R, \\ v_t + uv_x + P_{2,x} + P_3 = 0, & t > 0, x \in R. \end{cases} \quad (2.1)$$

Note that $z = (u, v) \in H^1 \times H^1$, Young's inequality ensures that $P_1, P_2, P_3 \in H^1$.

Let us introduce the Lagrangian variables with which we reformulate system (2.1). For a given initial data $y(0, \xi)$, let $y(t, \xi)$, which can be decomposed as $\varsigma(t, \xi) = y(t, \xi) - \xi$, denote the corresponding characteristic such as

$$y_t(t, \xi) = u(t, y(t, \xi)), \quad (2.2)$$

where the variable ξ is identified with a "particle". The change in Lagrangian energy distribution along the particle path is given by

$$h(t, \xi) = (u^2 + u_x^2 + v^2 + v_x^2) \circ yy_\xi. \quad (2.3)$$

It is not hard to check that

$$(u^2 + u_x^2 + v^2 + v_x^2)_t + (u(u^2 + u_x^2 + v^2 + v_x^2))_x = (u^3 - 2uP_1 + 2vP_2 - 2vP_{3,x})_x. \quad (2.4)$$

Throughout the following, we use the notation

$$U(t, \xi) = u(t, y(t, \xi)), \quad V(t, \xi) = v(t, y(t, \xi)), \quad N(t, \xi) = v_x(t, y(t, \xi)).$$

After the change of variables $x = y(t, \xi)$ and $x' = y(t, \xi')$, we obtain the following expressions for P_i and $P_{i,x}$ ($i = 1, 2, 3$):

$$\begin{aligned} P_1(t, \xi) &= P_1(t, y(\xi)) = 1/4 \cdot \int_R e^{-|y(\xi)-y(\xi')|} [h + (U^2 - 2N^2)y_\xi](\xi') d\xi', \\ P_{1,x}(t, \xi) &= P_{1,x}(t, y(\xi)) \\ &= -1/4 \cdot \int_R \operatorname{sgn}(\xi - \xi') e^{-|y(\xi)-y(\xi')|} [h + (U^2 - 2N^2)y_\xi](\xi') d\xi', \\ P_2(t, \xi) &= P_2(t, y(t, \xi)) = 1/2 \cdot \int_R e^{-|y(\xi)-y(\xi')|} (U_\xi N)(\xi') d\xi', \\ P_{2,x}(t, \xi) &= P_{2,x}(t, y(t, \xi)) = -1/2 \cdot \int_R \operatorname{sgn}(\xi - \xi') e^{-|y(\xi)-y(\xi')|} (U_\xi N)(\xi') d\xi', \\ P_3(t, \xi) &= P_3(t, y(t, \xi)) = 1/2 \cdot \int_R e^{-|y(\xi)-y(\xi')|} (U_\xi V)(\xi') d\xi', \\ P_{3,x}(t, \xi) &= P_{3,x}(t, y(t, \xi)) = -1/2 \cdot \int_R \operatorname{sgn}(\xi - \xi') e^{-|y(\xi)-y(\xi')|} (U_\xi V)(\xi') d\xi', \end{aligned} \tag{2.5}$$

where we have dropped the variable t for simplification and taken that y is an increasing function for any fixed time t for granted (Later we will prove this) and used the definition of (2.3). From the definition of the characteristic, it follows that

$$\begin{aligned} U_t(t, \xi) &= u_t(t, y) + u_x(t, y)y_t(t, \xi) = -P_{1,x} \circ y(t, \xi), \\ V_t(t, \xi) &= v_t(t, y) + v_x(t, y)y_t(t, \xi) = (-P_{2,x} - P_3) \circ y(t, \xi), \\ N_t(t, \xi) &= v_{xt}(t, y) + v_{xx}(t, y)y_t(t, \xi) = (-P_2 - P_{3,x}) \circ y(t, \xi). \end{aligned} \tag{2.6}$$

With these new variables, we now derive an equivalent system of Eqs. (2.1),

$$\begin{cases} \varsigma_t = U, \\ U_t = -P_{1,x}, \\ V_t = -P_{2,x} - P_3, \\ N_t = -P_2 - P_{3,x}, \\ h_t = (3U^2 - 2P_1 + 2V^2)U_\xi - (2P_2 + 2P_{3,x})V_\xi - (2UP_{1,x} + 2VP_{2,x} + 2VP_3)y_\xi, \end{cases} \tag{2.7}$$

where P_i and $P_{i,x}$ ($i = 1, 2, 3$) are given in (2.5). Differentiating the first three equations in (2.7) w. r. t. the variable ξ yields

$$\begin{cases} \varsigma_{\xi t} = U_{\xi}, \\ U_{\xi t} = h/2 + (U^2/2 - N^2 - P_1)y_{\xi}, \\ V_{\xi t} = U_{\xi}N - (P_2 + P_{3,x})y_{\xi}, \\ h_t = (3U^2 - 2P_1 + 2V^2)U_{\xi} - (2P_2 + 2P_{3,x})V_{\xi} - (2UP_{1,x} + 2VP_{2,x} + 2VP_3)y_{\xi}, \end{cases} \quad (2.8)$$

which is semilinear with respect to the variables $y_{\xi}, U_{\xi}, V_{\xi}$ and h . By solving (2.7) and (2.8) we can obtain the conservative solution. While dissipative solutions differ from conservative solutions when particles collide, that is, when $y_{\xi}(t, \xi) = 0$ for ξ in an interval of positive length. To obtain the dissipative solution, we impose that when particles collide, they lose their energy, that is, if $y_{\xi}(\tau, \xi) = 0$ for some τ , then we set $h(\tau, \xi) = 0$. One can show that $y_{\xi}(\tau, \xi) = 0$ implies $U_{\xi}(\tau, \xi) = V_{\xi}(\tau, \xi) = 0$ so that $(y_{\xi}, U_{\xi}, V_{\xi}, h)(t, \xi) = 0$ for $t \geq \tau$ from system (2.8). Thus we can define $\tau(\xi)$ be the first time when $y_{\xi}(t, \xi)$ vanishes, namely

$$\tau(\xi) = \sup \{ t \in R^+ \mid y_{\xi}(t', \xi) > 0 \text{ for all } 0 \leq t' < t \}. \quad (2.9)$$

However, from this definition $\tau(\xi)$ in (2.9) it is not clear whether τ is measurable or not. We now replace this definition by the following one. Let $A_t = \cup_{n \geq 1} \cap_{t_i \leq t} \{ \xi \in R \mid y_{\xi}(t, \xi) > 1/n \}$, which are measurable for all t with $A_{t'} \subset A_t$ for $t < t'$, where $\{t_i\}$ is a dense countable subset of $[0, T]$. We define

$$\tau(\xi) = \lim_{n \rightarrow \infty} \tau^n(\xi)$$

where $\tau^n(\xi) = \sum_{i=0}^{2^n} 2^{-n} iT \chi_{i,n}(\xi)$ with the set $\{2^{-n} iT\}_{i=0}^{2^n}$ a dyadic partition of the interval $[0, T]$ and $\chi_{i,n}$ the indicator function of the set $A_{2^{-n} iT} / A_{2^{-n}(i+1)T}$. One can check that this definition $\tau(\xi)$ is measurable as $\tau^n(\xi)$ is by construction measurable and increasing w. r. t. n , and moreover, this definition coincides with (2.9).

Hence, the expressions for P_i and $P_{i,x}$ ($i = 1, 2, 3$) in (2.5) become

$$\begin{aligned} P_1(t, \xi) &= 1/4 \cdot \int_{t < \tau(\xi)} e^{-|y(\xi) - y(\xi')|} [h + (U^2 - 2N^2)y_{\xi}] (\xi') d\xi', \\ P_{1,x}(t, \xi) &= -1/4 \cdot \int_{t < \tau(\xi)} \operatorname{sgn}(\xi - \xi') e^{-|y(\xi) - y(\xi')|} [h + (U^2 - 2N^2)y_{\xi}] (\xi') d\xi', \\ P_2(t, \xi) &= 1/2 \cdot \int_{t < \tau(\xi)} e^{-|y(\xi) - y(\xi')|} (U_{\xi}N)(\xi') d\xi', \\ P_{2,x}(t, \xi) &= -1/2 \cdot \int_{t < \tau(\xi)} \operatorname{sgn}(\xi - \xi') e^{-|y(\xi) - y(\xi')|} (U_{\xi}N)(\xi') d\xi', \\ P_3(t, \xi) &= 1/2 \cdot \int_{t < \tau(\xi)} e^{-|y(\xi) - y(\xi')|} (U_{\xi}V)(\xi') d\xi', \\ P_{3,x}(t, \xi) &= -1/2 \cdot \int_{t < \tau(\xi)} \operatorname{sgn}(\xi - \xi') e^{-|y(\xi) - y(\xi')|} (U_{\xi}V)(\xi') d\xi', \end{aligned} \quad (2.10)$$

and the modified system to be solved here reads

$$\varsigma_t = U, \quad U_t = -P_{1,x}, \quad V_t = -P_{2,x} - P_3, \quad N_t = -P_2 - P_{3,x} \quad (2.11)$$

$$\begin{cases} \varsigma_{\xi t} = \begin{cases} U_{\xi} & \text{if } t < \tau(\xi), \\ 0 & \text{otherwise,} \end{cases} \\ U_{\xi t} = \begin{cases} h/2 + (U^2/2 - N^2 - P_1)y_{\xi} & \text{if } t < \tau(\xi), \\ 0 & \text{otherwise,} \end{cases} \\ V_{\xi t} = \begin{cases} U_{\xi}N - (P_2 + P_{3,x})y_{\xi} & \text{if } t < \tau(\xi), \\ 0 & \text{otherwise,} \end{cases} \\ h_t = \begin{cases} (3U^2 - 2P_1 + 2V^2)U_{\xi} - (2P_2 + 2P_{3,x})V_{\xi} \\ \quad - (2UP_{1,x} + 2VP_{2,x} + 2VP_3)y_{\xi} & \text{if } t < \tau(\xi), \\ 0 & \text{otherwise.} \end{cases} \end{cases} \quad (2.12)$$

3. Global solutions of the Lagrangian equivalent system

To obtain the global existence of solutions, we start from a contraction argument that offers the local existence of solutions, which is proved in the Banach space E given by

$$E = L^{\infty} \cap [L^2 \cap L^{\infty}]^7 \cap L^1.$$

Let $W = L^2 \cap L^{\infty}$. Note that global solutions of (2.11) may not exist for all initial data in E . However they exist when the initial data $X_0 = (\varsigma_0, U_0, V_0, N_0, h_0)$ belongs to the set Γ which is defined as follows:

Definition 3.1. *The set Γ is composed of all (ς, U, V, N, h) such that*

$$X = (\varsigma, U, V, N, \varsigma_{\xi}, U_{\xi}, V_{\xi}, h) \in E, \quad (3.1a)$$

$$y_{\xi} \geq 0, h \geq 0 \text{ almost everywhere,} \quad (3.1b)$$

$$y_{\xi}h = y_{\xi}^2U^2 + U_{\xi}^2 + y_{\xi}^2V^2 + V_{\xi}^2 \text{ almost everywhere,} \quad (3.1c)$$

$$1/(y_{\xi} + h) \in L^{\infty}(R), \quad (3.1d)$$

$$\lim_{\xi \rightarrow -\infty} \varsigma(\xi) = 0, \quad (3.1e)$$

$$g(y, U, V, y_{\xi}, U_{\xi}, V_{\xi}, h) - 1 \in W, \quad (3.1f)$$

with $y(\xi) = \varsigma(\xi) + \xi$, and $g(\mathbf{x})$ is given by

$$g(\mathbf{x}) = \begin{cases} |x_5| + |x_6| + 2(1 + x_2^2 + x_3^2)x_4, & \text{if } x \in \Omega, \\ x_4 + x_7, & \text{otherwise,} \end{cases} \quad (3.2)$$

for $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in R^7$, where Ω is the following subset of R^7

$$\Omega = \{ \mathbf{x} \in R^7 \mid |x_5| + |x_6| + 2(1 + x_2^2 + x_3^2)x_4 \leq x_4 + x_7 \text{ and } x_5, x_6 \leq 0 \}.$$

The system (2.12) can be rewritten in the compact form

$$Y_t(t, \xi) = \chi_{\{t < \tau(\xi)\}}(\xi)F(X)Y(t, \xi) \quad (3.3)$$

with χ_B the indicator function of the $\sigma\epsilon\tau gB$ where $X = (y, U, V, N, h)$ and $Y = (y_\xi, U_\xi, V_\xi, h)$ and the definition of the 4×4 matrix $F(X)$ follows directly from (2.12). The equation $Y_t = F(X)Y$ is linear and therefore it admits a unique global solution in $[0, T]$. Hence, for $t \in [0, \tau]$ we can define $Y(t, \xi)$ and for $t \geq \tau(\xi)$, $Y(t, \xi) = Y(\tau(\xi), \xi)$. The system (3.3) is thus well-defined. In this definition, we do not reset the energy density h to zero after collision but keep the value it reached just before the collision, which has the advantage of rendering the right hand of (2.12) continuous across the value $t = \tau(\xi)$ and the behavior of the system remain unchanged.

Let us define B_M (the subset of E) as follows

$$B_M = \left\{ X = (\varsigma, U, V, N, \varsigma_\xi, U_\xi, V_\xi, h) \in E \mid \|X\|_E + \|g(X) - 1\|_W + \|1/(y_\xi + h)\|_{L_T^\infty L_R^\infty} \leq M \right\}.$$

Lemma 3.1. Let $\bar{M} = \sum_{i=1}^3 (\|P_i\|_{L_T^\infty L_R^\infty} + \|P_{i,x}\|_{L_T^\infty L_R^\infty}) + M_0$. Given $X_0 \in B_{M_0}$ for some constant M_0 , and $X = (\varsigma, U, V, N, \varsigma_\xi, U_\xi, V_\xi, h) \in C([0, T], B_M)$ Then it holds

(i)

$$y_\xi h = y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2 \quad (3.4)$$

$$y_\xi(t, \xi) \geq 0, \quad h(t, \xi) \geq 0, \quad (3.5)$$

for all t and almost all ξ .

(ii)

$$\|1/(y_\xi + h)(t, \cdot)\|_{L_R^\infty} \leq 8e^{C(\bar{M}, T)T} \|1/(y_\xi + h)(0, \cdot)\|_{L_R^\infty} \quad (3.6)$$

for all $t \in [0, T]$ and a constant $C(\bar{M}, T)$.

Proof. (i) It is not hard to know that (3.4)-(3.6) hold for almost every $\xi \in R$ as $X_0 \in \Gamma$ at $t = 0$. We consider a fixed ξ that we ignore in the notation. From (2.12) we can obtain that,

$$\begin{aligned} (y_\xi h)_t &= y_{\xi t} h + y_\xi h_t \\ &= U_\xi h + y_\xi \{ (3U^2 - 2P_1 + 2V^2)U_\xi - (2UP_{1,x} + 2VP_{2,x} + 2VP_3)y_\xi \\ &\quad - (2P_2 + 2P_{3,x})V_\xi \}, \end{aligned}$$

and on the other hand,

$$\begin{aligned} (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2)_t &= 2y_\xi y_{\xi t} U^2 + 2y_\xi^2 U U_t + 2U_\xi U_{\xi t} + 2y_\xi y_{\xi t} V^2 + 2y_\xi^2 V V_t + 2V_\xi V_{\xi t} \\ &= U_\xi h - (2UP_{1,x} + 2VP_{2,x} + 2VP_3)y_\xi^2 + (3U^2 - 2P_1 + 2V^2)U_\xi y_\xi \\ &\quad - (2P_2 + 2P_{3,x})V_\xi y_\xi, \end{aligned}$$

which implies $(y_\xi h)_t = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2)_t$ and therefore $y_\xi h(t) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2)(t)$ for all $t \in [0, T]$ as $y_\xi h(0) = (y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2)(0)$, and we have proved (3.4). It is not hard to know that $y_\xi(t) \geq 0$ for $t \geq 0$ from the definition of τ and y_ξ . For $t \in [0, \tau]$, it follows from (3.4) that $h(t) =$

$(y_\xi^2 U^2 + U_\xi^2 + y_\xi^2 V^2 + V_\xi^2) / y_\xi(t) \geq 0$. Since the continuity (w.r.t. t) of h implies $h(\tau(\xi)) \geq 0$ and the variable does not change for $t \geq \tau(\xi)$, we have $h(t) \geq 0$ for all $t \geq 0$.

(ii) We consider a fixed ξ and drop it in the notation. We denote by $|Y|_2 = (y_\xi^2 + U_\xi^2 + V_\xi^2 + h^2)^{1/2}$ the Euclidean norm of $Y = (y_\xi, U_\xi, V_\xi, h)$. Since $Y_t = F(X)Y$, we obtain

$$\begin{aligned} d/dt \cdot |Y|_2^{-2} &= -2 |Y|_2^{-4} Y \cdot dY/dt = -2 |Y|_2^{-4} Y \cdot F(X)Y \\ &\leq 2 \sup_{t \in [0, T]} |F(X)(t)| |Y|_2^{-2}. \end{aligned}$$

We have $|F(X)(t)| \leq K(\sum_{i=1}^3 (\|P_i\|_{L_T^\infty L_R^\infty} + \|P_{i,x}\|_{L_T^\infty L_R^\infty}) + \|U\|_{L_T^\infty L_R^\infty} + \|V\|_{L_T^\infty L_R^\infty} + \|N\|_{L_T^\infty L_R^\infty})$ for a constant K depending on the norm chosen for the matrix F . From (2.11) we infer that

$$\begin{aligned} \|U\|_{L_T^\infty L_R^\infty} &\leq U_0 + T \|P_{1,x}\|_{L_T^\infty L_R^\infty} \leq C(\bar{M}, T), \\ \|V\|_{L_T^\infty L_R^\infty} &\leq V_0 + T(\|P_{2,x}\|_{L_T^\infty L_R^\infty} + \|P_{3,x}\|_{L_T^\infty L_R^\infty}) \leq C(\bar{M}, T), \\ \|N\|_{L_T^\infty L_R^\infty} &\leq N_0 + T(\|P_2\|_{L_T^\infty L_R^\infty} + \|P_{3,x}\|_{L_T^\infty L_R^\infty}) \leq C(\bar{M}, T), \end{aligned}$$

for a constant depending only on $\bar{M}$ and T . We denote generically by $C(\bar{M}, T)$ such constant. Thus, $|F(X)(t)| \leq C(\bar{M}, T)$. It follows from Gronwall's lemma that $|Y(t)|_2^{-2} \leq e^{2C(\bar{M}, T)T} |Y(0)|_2^{-2}$, which yields

$$1/(|y_\xi| + |U_\xi| + |V_\xi| + |h|)(t) \leq 4/(|y_\xi| + |U_\xi| + |V_\xi| + |h|)(0) e^{C(\bar{M}, T)T}.$$

Since y_ξ and h are positive, it follows from (3.4) that

$$|U_\xi(t)| \leq \sqrt{y_\xi(t)h(t)} \leq (y_\xi + h)(t)/2 \quad \text{and} \quad |V_\xi(t)| \leq \sqrt{y_\xi(t)h(t)} \leq (y_\xi + h)(t)/2,$$

and therefore $(|y_\xi| + |U_\xi| + |V_\xi| + |h|)(t) \leq 2(y_\xi + h)(t)$, which yields

$$1/(|y_\xi| + |h|)(t) \leq 8e^{C(\bar{M}, T)T} \cdot 1/(|y_\xi| + |h|)(0).$$

□

Lemma 3.2. (i) Let $X \in C([0, T], B_M)$ be given, then we have $\|P_i(t, \cdot)\|_W$, $\|P_{i,x}(t, \cdot)\|_W \leq C(M)$ ($i = 1, 2, 3$), for some constant $C(M)$ depending only on M .

(ii) Let $G_\varepsilon = A_\varepsilon \cup K_{1/\varepsilon}$, where

$$A_\varepsilon = \{\xi \in R \mid 0 < y_{0,\xi}(\xi) \leq \varepsilon \text{ and } -\varepsilon \leq U_{0,\xi}(\xi) < 0\}, \quad K_\gamma = \{\xi \in R \mid h_0(\xi) \geq \gamma\}.$$

Thus, for T small enough, we have

$$\{\xi \in R \mid 0 < \tau(\xi) < T\} \subset G_\varepsilon. \quad (3.7)$$

Proof. (i) Let us begin with some a priori estimates on $P_{1,x}$ given by (2.10).

The estimates for other P_i and $P_{i,x}$ ($i = 1, 2, 3$) can be obtained in exactly the same way. Let us write $P_{1,x}(\xi)$ as $P_{1,x}(\xi) = P_{1,x}^1(\xi) + P_{1,x}^2(\xi)$ with

$$P_{1,x}^1(\xi) = -e^{-\varsigma(\xi)} / 4 \cdot (f_1 * [\chi_{\{t < \tau(\xi)\}} r_1])(t, \xi),$$

$$P_{1,x}^2(\xi) = e^{\varsigma(\xi)} / 4 \cdot (f_2 * [\chi_{\{t < \tau(\xi)\}} r_2])(t, \xi)$$

where

$$f_1(\xi) = \chi_{\{\xi > 0\}}(\xi) e^{-\xi}, \quad r_1 = r_1^1 + r_1^2, \quad r_1^1 = e^\varsigma (U^2 - 2N^2)(1 + \varsigma_\xi), \quad r_1^2 = e^\varsigma h,$$

$$f_2(\xi) = \chi_{\{\xi < 0\}}(\xi) e^{-\xi}, \quad r_2 = r_2^1 + r_2^2, \quad r_2^1 = e^{-\varsigma} (U^2 - 2N^2)(1 + \varsigma_\xi), \quad r_2^2 = e^{-\varsigma} h.$$

Young's inequalities imply that

$$\begin{aligned} \|P_{1,x}^1(t, \cdot)\|_W &\leq 1/4 \cdot \|e^{-\varsigma}\|_{L_{\tilde{R}}^\infty} (\|f_1 * [\chi_{\{t < \tau(\xi)\}} r_1^1]\|_W + \|f_1 * [\chi_{\{t < \tau(\xi)\}} r_1^2]\|_W) \\ &\leq C(M) (\|f_1\|_{L_R^1} + \|f_1\|_{L_R^2}) \|r_1^1\|_{L_R^2} + (\|f_1\|_{L_R^2} + \|f_1\|_{L_{\tilde{R}}^\infty}) \|r_1^2\|_{L_R^1} \leq C(M), \end{aligned} \quad (3.8)$$

- where we have denoted by $C(M)$ a generic constant only depending on M .
- (ii) We have $\lim_{\varepsilon \rightarrow 0} \text{meas}(A_\varepsilon) = 0$ as $\cap_{n>0} A_{1/n} = \emptyset$, and $\lim_{\varepsilon \rightarrow 0} \text{meas}(K_\gamma) = 0$ as $\text{meas}(K_\gamma) \leq 1/\gamma \cdot \int_R h_0(\xi) d\xi \leq 1/\gamma \cdot \|h_0\|_{L_R^1}$. Without loss of generality we assume $T \leq 1$. We consider a fixed ξ and drop it in the notation, such that $\tau(\xi) < T$ and $\xi \in K_\gamma^c$. From lemma 3.1 (ii), there exists $\varepsilon'(\bar{M})$ such that $[(y_\xi + h) \cdot (2(1 + U^2 + V^2))^{-1}](t, \xi) \geq \varepsilon'$ for all $t \in [0, T]$ and ξ . Let $\bar{\varepsilon} = \min(\varepsilon, \varepsilon')$. As $T \leq 1$ by assumption, we get that $|Y(t)| \leq C(\bar{M}, T) |Y(0)| \leq C(\bar{M}, \gamma)$ for a constant $C(\bar{M}, \gamma)$ by applying Gronwall's lemma on (3.3). Let us define

$$t_0 = \inf \{ t \in [0, \tau) \mid 0 < y_\xi \leq \bar{\varepsilon}, -\bar{\varepsilon} < U_\xi(\bar{t}, \xi) \leq 0, V_\xi(\bar{t}, \xi) \neq 0 \text{ for all } \bar{t} \in [t, \tau) \},$$

which is well-posed when $\tau > 0$ as $y_{\xi tt}(\tau) = U_{\xi t}(\tau) = h(\tau)/2 > 0$ and $y_{\xi t}(\tau) = U_\xi(\tau) = y_\xi(\tau) = 0$, then we have $t_0 < \tau$. We claim that $t_0 = 0$ and therefore $\xi \in G_{\bar{\varepsilon}} \subset G_\varepsilon$. Assume that $t_0 > 0$, then $y_\xi(t_0) = 0$ or $y_\xi(t_0) = \bar{\varepsilon}$ or $U_\xi(t_0) = -\bar{\varepsilon}$ or $V_\xi(t_0) = -\bar{\varepsilon}$ or $U_\xi(t_0) = V_\xi(t_0) = 0$. By the definition of τ , we can not have $y_\xi(t_0) = 0$. If $y_\xi(t_0) = \bar{\varepsilon}$, then $0 = y_\xi(\tau) = y_\xi(t_0) + \int_{t_0}^\tau U_\xi(t) dt$ implies $0 \geq \bar{\varepsilon} - \bar{\varepsilon}(\tau - t_0)$ which leads to a contradiction if we choose $T < 1$. If $U_\xi(t_0) = -\bar{\varepsilon}$, the second equation in (2.12) implies $0 = U_\xi(\tau) \leq U_\xi(t_0) + C(\bar{M}, \gamma)(\tau - t_0) \leq -\bar{\varepsilon} + C(\bar{M}, \gamma)(\tau - t_0)$ which leads to a contradiction if T is chosen small enough. At last, if $U_\xi(t_0) = V_\xi(t_0) = 0$, it follows from (3.4) that $h(t_0) = [(U^2 + V^2)y_\xi](t_0)$ and therefore $\bar{\varepsilon} \geq y_\xi(t_0) \geq (y_\xi + h)(t_0)/(1 + U^2 + V^2)(t_0) \geq 2\varepsilon'$, which contradicts the definition of $\bar{\varepsilon}$ and (3.7) has been proved.

We also have that, there exists some ε , such that for $\xi \in G_\varepsilon$, $y_\xi(t, \xi)$ is a decreasing function and $U_\xi(t, \xi)$ increasing functions of time. Indeed, If $y_\xi(t) \leq \varepsilon$, we have

$$\begin{aligned} U_{\xi t} &= (h + y_\xi)/2 + (U^2/2 - N^2 - P_1 - 1/2)y_\xi \\ &\geq (2C_2(M, T))^{-1} - C_3(M, T)y_\xi \geq (2C_2(M, T))^{-1} - 2C_3(M, T)\varepsilon. \end{aligned} \quad (3.9)$$

By choosing $\varepsilon(M, T) \leq ((4C_2C_3)(M, T))^{-1}$, we get $U_{\xi t} \geq 0$ and therefore $U_{\xi}(t, \xi)$ is increasing function of time. While $y_{\xi t} = U_{\xi} \leq 0$ implies that $y_{\xi}(t, \xi)$ is a decreasing function of time. $\square$

Theorem 3.1. *For any initial data $X_0 = (\varsigma_0, U_0, V_0, N_0, h_0) \in \Gamma$, there exists a unique local solution of (2.11) and (2.12) in $C([0, T], E)$ for some time $T > 0$.*

Proof. To obtain the local existence of solutions, it suffices to show that the transformation $\mathfrak{R}$

$$\mathfrak{R}(X) = \bar{Y} + \int_0^t F(X)(t', \cdot) Y(t', \cdot) dt'$$

is a contraction, and therefore, there exists a unique fixed point $(\varsigma, U, V, N, \varsigma_{\xi}, U_{\xi}, V_{\xi}, h) \in C([0, T], E)$.

By choosing $T > 0$ small enough, there exists a $\bar{M}$ depending only on M such that $\mathfrak{R}$ is a mapping from $C([0, T], B_{\bar{M}})$ to $C([0, T], B_{\bar{M}})$. Let us denote by D the image of $\mathfrak{R}$, namely

$$D = \{ \mathfrak{R}(X) \mid X \in C([0, T], B_{\bar{M}}) \}.$$

Let $\mathbf{X} = \mathfrak{R}(X) = (\varsigma, \mathbf{U}, \mathbf{V}, \mathbf{N}, \varsigma_{\xi}, \mathbf{U}_{\xi}, \mathbf{V}_{\xi}, \mathbf{h})$ and $\bar{\mathbf{X}} = \mathfrak{R}(\bar{X}) = (\bar{\varsigma}, \bar{\mathbf{U}}, \bar{\mathbf{V}}, \bar{\mathbf{N}}, \bar{\varsigma}_{\xi}, \bar{\mathbf{U}}_{\xi}, \bar{\mathbf{V}}_{\xi}, \bar{\mathbf{h}})$ be two elements of D . We denote $\mathbf{Y} = (\varsigma_{\xi}, \mathbf{U}_{\xi}, \mathbf{V}_{\xi}, \mathbf{h})$ and $\bar{\mathbf{Y}} = (\bar{\varsigma}_{\xi}, \bar{\mathbf{U}}_{\xi}, \bar{\mathbf{V}}_{\xi}, \bar{\mathbf{h}})$. We now consider ξ such that $\tau(\xi) \neq \bar{\tau}(\xi)$. Without loss of generality we assume $0 < \bar{\tau}(\xi) < \tau(\xi) \leq T$. For $t \in [\bar{\tau}(\xi), \tau(\xi)]$, we can get from (2.12) that

$$0 \geq U_{\xi}(t, \xi) = U_{\xi}(\bar{\tau}, \xi) + 1/2 \cdot \int_{\bar{\tau}}^t h(t', \xi) dt' + \int_{\bar{\tau}}^t ((U^2/2 - N^2 - P_1)y_{\xi})(t', \xi) dt'$$

which implies that

$$\begin{aligned} \int_{\bar{\tau}}^{\tau} h(t, \xi) dt &\leq -2U_{\xi}(\bar{\tau}, \xi) + C(M) \int_{\bar{\tau}}^{\tau} y_{\xi}(t, \xi) dt \leq 2(\bar{U}_{\xi} - U_{\xi})(\bar{\tau}, \xi) \\ &\quad + C(M) \int_{\bar{\tau}}^{\tau} (y_{\xi} - \bar{y}_{\xi})(t, \xi) dt \\ &\leq C(M) (\|\bar{U}_{\xi} - U_{\xi}\|_{L_T^{\infty} W} + \|y_{\xi} - \bar{y}_{\xi}\|_{L_T^{\infty} W}), \end{aligned} \quad (3.10)$$

where we have used the fact that $\bar{U}_{\xi}(\bar{\tau}, \xi) = \bar{y}_{\xi}(\bar{\tau}, \xi) = 0$ for $t \in [\bar{\tau}(\xi), \tau(\xi)]$. Since $X(t) \in B_{\bar{M}}$, $1/(y_{\xi} + h)(t, \xi) \leq \bar{M}$, and we have

$$\begin{aligned} \tau(\xi) - \bar{\tau}(\xi) &\leq 1/\bar{M} \cdot \int_{\bar{\tau}}^{\tau} (y_{\xi} + h)(t, \xi) dt \leq 1/\bar{M} \cdot \int_{\bar{\tau}}^{\tau} (y_{\xi} - \bar{y}_{\xi})(t, \xi) dt \\ &\quad + C(M) (\|\bar{U}_{\xi} - U_{\xi}\|_{L_T^{\infty} W} + \|y_{\xi} - \bar{y}_{\xi}\|_{L_T^{\infty} W}) \leq C(M) (\|\bar{U}_{\xi} - U_{\xi}\|_{L_T^{\infty} W} \\ &\quad + \|y_{\xi} - \bar{y}_{\xi}\|_{L_T^{\infty} W}). \end{aligned} \quad (3.11)$$

It then follows from (3.10) and (3.11) that

$$\begin{aligned} \|\tau - \bar{\tau}\|_{L_R^1} &+ \int_R \left| \int_{\tau}^{\bar{\tau}} (|\bar{h}(t, \xi)| \chi_{\tau < \bar{\tau}(\xi)} + |h(t, \xi)| \chi_{\bar{\tau} < \tau(\xi)}) dt \right| d\xi \\ &\leq C(M) \text{meas}(G_{\varepsilon}) \|X - \bar{X}\|_{L_T^{\infty} W}. \end{aligned} \quad (3.12)$$

For $\xi \in G_\varepsilon^c$, we have $\mathbf{Y}_t = F(X)\mathbf{Y}$ and $\bar{\mathbf{Y}}_t = F(\bar{X})\bar{\mathbf{Y}}$ for all $t \in [0, T]$. Thus

$$\begin{aligned} \|(\mathbf{Y} - \bar{\mathbf{Y}})(t, \cdot)\|_{W(G_\varepsilon^c)} &\leq \int_0^t \|(F(X) - F(\bar{X}))(t', \cdot)\mathbf{Y}(t', \cdot)\|_{W(G_\varepsilon^c)} dt' \\ &+ \int_0^t \|(F(\bar{X})(\mathbf{Y} - \bar{\mathbf{Y}}))(t', \cdot)\|_{W(G_\varepsilon^c)} dt'. \end{aligned} \quad (3.13)$$

After applying Gronwall's lemma to (3.13), as $\|F(\bar{X})\|_{L_T^\infty L^\infty} \leq C(M)$, we have

$$\begin{aligned} \|(\mathbf{Y} - \bar{\mathbf{Y}})\|_{L_T^\infty W(G_\varepsilon^c)} &\leq C(M) \|(F(X) - F(\bar{X}))\mathbf{Y}\|_{L_T^1 W} \\ &\leq C(\bar{M}, T) \left(\sum_{i=1}^3 \|P_{i,x} - \bar{P}_{i,x}\|_{L_T^1 W} + \sum_{i=1}^3 \|P_i - \bar{P}_i\|_{L_T^1 W} \right), \end{aligned} \quad (3.14)$$

which together with (3.12) implies that

$$\|(\mathbf{Y} - \bar{\mathbf{Y}})\|_{L_T^\infty W(G_\varepsilon^c)} \leq C(M)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}. \quad (3.15)$$

It follows from (3.4) that $\mathbf{h} = (\mathbf{U}^2 + \mathbf{V}^2)(1 + \varsigma_\xi)^2 + \mathbf{U}_\xi^2 + \mathbf{V}_\xi^2 - \varsigma_\xi \mathbf{h}$ and $\bar{\mathbf{h}} = (\bar{\mathbf{U}}^2 + \bar{\mathbf{V}}^2)(1 + \bar{\varsigma}_\xi)^2 + \bar{\mathbf{U}}_\xi^2 + \bar{\mathbf{V}}_\xi^2 - \bar{\varsigma}_\xi \bar{\mathbf{h}}$, which yield

$$\begin{aligned} \|h - \bar{h}\|_{L^1(G_\varepsilon^c)} &\leq C(M) \|X - \bar{X}\|_{E(G_\varepsilon^c)} + \|\varsigma_\xi \mathbf{h} - \bar{\varsigma}_\xi \bar{\mathbf{h}}\|_{L^1(G_\varepsilon^c)} \\ &\leq C(M) \left(\|X - \bar{X}\|_{E(G_\varepsilon^c)} + \|\mathbf{h} - \bar{\mathbf{h}}\|_{L^1(G_\varepsilon^c)} \right), \end{aligned} \quad (3.16)$$

and therefore

$$\|(\mathbf{Y} - \bar{\mathbf{Y}})\|_{L_T^\infty V(G_\varepsilon^c)} \leq C(M)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}, \quad (3.17)$$

where $V = W \cap W \cap W \cap L^1$. For $t \in [\bar{\tau}(\xi), T]$, we have that

$$|(\mathbf{y}_\xi - \bar{\mathbf{y}}_\xi)(t, \xi)| \leq |(\mathbf{y}_\xi - \bar{\mathbf{y}}_\xi)(\bar{\tau}, \xi)|, \quad |(\mathbf{U}_\xi - \bar{\mathbf{U}}_\xi)(t, \xi)| \leq |(\mathbf{U}_\xi - \bar{\mathbf{U}}_\xi)(\bar{\tau}, \xi)|, \quad (3.18)$$

as y_ξ is positive decreasing and U_ξ is negative decreasing and $\bar{y}_\xi(t, \xi) = \bar{U}_\xi(t, \xi) = 0$ for $t \in [\bar{\tau}(\xi), T]$. We have $\mathbf{h}_t = (3U^2 - 2P_1 + 2V^2)\mathbf{U}_\xi - (2P_2 + 2P_{3,x})\mathbf{V}_\xi - (2UP_{1,x} + 2VP_{2,x} + 2VP_3)\mathbf{y}_\xi$ and $\bar{\mathbf{h}}_t = 0$ for $t \in [\bar{\tau}(\xi), T]$, which implies from (3.18) that

$$\begin{aligned} |(\mathbf{h} - \bar{\mathbf{h}})(t, \xi)| &\leq |(\mathbf{h} - \bar{\mathbf{h}})(\bar{\tau}, \xi)| + C(M)(|(\mathbf{y}_\xi - \bar{\mathbf{y}}_\xi)(\bar{\tau}, \xi)| + |(\mathbf{U}_\xi - \bar{\mathbf{U}}_\xi)(\bar{\tau}, \xi)| \\ &+ |(\mathbf{V}_\xi - \bar{\mathbf{V}}_\xi)(\bar{\tau}, \xi)|). \end{aligned} \quad (3.19)$$

For $t \in [0, \bar{\tau}(\xi)]$, we have $\mathbf{Y}_t = F(X)\mathbf{Y}$ and $\bar{\mathbf{Y}}_t = F(\bar{X})\bar{\mathbf{Y}}$. We proceed as we obtained (3.15) and get

$$|(\mathbf{Y} - \bar{\mathbf{Y}})(\bar{\tau}, \xi)| \leq C(M)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}. \quad (3.20)$$

By combining (3.18), (3.19) and (3.20), we then obtain that (3.20) holds for all $t \in [0, T]$, which implies that

$$\|(\mathbf{Y} - \bar{\mathbf{Y}})\|_{L_T^\infty V(G_\varepsilon^c)} \leq C(M)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}, \quad (3.21)$$

and therefore

$$\|(\mathbf{Y} - \bar{\mathbf{Y}})\|_{L_T^\infty V} \leq C(M)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}. \quad (3.22)$$

From (2.11) and (3.12), we have that

$$\|\mathbf{U} - \bar{\mathbf{U}}\|_{L_T^\infty W} \leq T \|P_{1,x} - \bar{P}_{1,x}\|_{L_T^\infty W} \leq C(\bar{M}, T)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}, \quad (3.23)$$

$$\begin{aligned} \|\mathbf{V} - \bar{\mathbf{V}}\|_{L_T^\infty W} &\leq T(\|P_{2,x} - \bar{P}_{2,x}\|_{L_T^\infty W} \\ &\quad + \|P_3 - \bar{P}_3\|_{L_T^\infty W}) \leq C(\bar{M}, T)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}, \end{aligned} \quad (3.24)$$

$$\|\varsigma - \bar{\varsigma}\|_{L_T^\infty L^\infty} \leq T \|\mathbf{U} - \bar{\mathbf{U}}\|_{L_T^\infty L^1} \leq C(M)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}. \quad (3.25)$$

It thus follows from (3.22)–(3.25) that

$$\|\mathbf{X} - \bar{\mathbf{X}}\|_{L_T^\infty E} \leq C(M)(T + \text{meas}(G_\varepsilon)) \|X - \bar{X}\|_{L_T^\infty E}.$$

By choosing $\varepsilon > 0$ and $T > 0$ small enough, as $\lim_{\varepsilon \rightarrow 0} \text{meas}(G_\varepsilon) = 0$, we thus obtain

$$\|\mathbf{X} - \bar{\mathbf{X}}\|_{L_T^\infty E} \leq 1/2 \|X - \bar{X}\|_{L_T^\infty E}$$

and therefore $\mathfrak{R}$ is a contraction, and the local existence of solutions follows from the standard contraction argument of ordinary differential equations. $\square$

Theorem 3.2. *Let any initial data $(y_0, U_0, V_0, N_0, h) \in \Gamma$ be given. The system (2.11) and (2.12) admit a unique global solution $(y, U, V, N, h) \in C(R^+, \Gamma)$.*

Proof. The local solution described in Theorem 3.1 does not provide any lower bound on the time of existence of the solution. We here introduce the maximum existing time $T_{\max}$, namely

$$T_{\max} = \sup \{t \in R^+ \mid \text{the solutions } X(t) \text{ to (2.11) exists in } [0, t]\}.$$

Let us assume that $T_{\max} < \infty$. We are now to obtain a global bound on the solution X .

Let us start by showing that, for given $M_0, T_0 > 0$, there exists a constant $M(M_0, T_0)$ such that, for any $X_0 \in B_{M_0}$, if $h_0 \in W(B)$ for some set B , then $\|X\|_{L_T^\infty W(B)}$ is bounded by a constant $C(M, \|h_0\|_{W(B)})$. We know that $\lim_{\xi \rightarrow \pm\infty} U(t, \xi) = \lim_{\xi \rightarrow \pm\infty} V(t, \xi) = 0$ as $U(t, \cdot), V(t, \cdot) \in H^1(R)$, it thus follows from (2.12) that

$$\begin{aligned} \|h(t, \cdot)\|_{L^1} &= \lim_{N \rightarrow \infty} \int_{-N}^N h(t, \xi) d\xi = \lim_{N \rightarrow \infty} \left(\int_{-N}^N h(0, \xi) d\xi \right. \\ &\quad + \int_0^t ((U^3 - 2P_1U - 2P_2V - 2P_{3,x}V)(t', N) - (U^3 - 2P_1U - 2P_2V \\ &\quad \left. - 2P_{3,x}V)(t', -N)) dt' = \lim_{N \rightarrow \infty} \int_{-N}^N h(0, \xi) d\xi = \|h(0, \cdot)\|_{L^1} \end{aligned} \quad (3.26)$$

for all $t \in [0, T]$. Since $U_\xi(\xi) = 0$ when $y_\xi(\xi) = 0$ from (3.1c) and then we have

$$\begin{aligned} U^2(\xi) &= 2 \int_{-\infty}^{\xi} U(\eta) U_\xi(\eta) d\eta \leq \int_R U(\eta) U_\xi(\eta) d\eta = \int_{\{y_\xi(\eta) > 0\}} U(\eta) U_\xi(\eta) d\eta \\ &\leq 1/2 \cdot \int_{\{y_\xi(\eta) > 0\}} (y_\xi U^2 + U_\xi^2 / y_\xi)(\eta) d\eta = 1/2 \cdot \int_{\{y_\xi(\eta) > 0\}} h(\eta) d\eta \\ &\leq 1/2 \|h(t, \cdot)\|_{L^1(R)}, \end{aligned}$$

which implies that

$$\|U(t, \cdot)\|_{L^\infty(R)} \leq 1/\sqrt{2} \|h(t, \cdot)\|_{L^1(R)}^{1/2} = 1/\sqrt{2} \|h_0\|_{L^1(R)}^{1/2} \leq M, \quad (3.27)$$

where we have dropped t for simplification. We denote by M such a general constant depending only on M_0 , T_0 and similarly, we get $\|V(t, \cdot)\|_{L_R^\infty} \leq M$. We can obtain from the governing eqnarray (2.11) that

$$|\varsigma(t, \xi)| \leq |\varsigma(0, \xi)| + \|U(t, \cdot)\|_{L^\infty} T \leq M, \quad (3.28)$$

and then $\|\varsigma(t, \cdot)\|_{L_R^\infty} \leq M$. From the identity $h = U^2 y_\xi + V^2 y_\xi + N^2 y_\xi + U_\xi^2 / y_\xi$, we know that

$$|(U^2 - 2N^2)y_\xi| \leq 2h,$$

which implies that

$$\begin{aligned} |P_{1,x}| &\leq 1/4 \cdot \left| \int_R e^{-|y(\xi) - y(\xi')|} [h + (U^2 - 2N^2)y_\xi](\xi') d\xi' \right| \\ &\leq 1/4 \cdot \left| \int_R e^{-|y(\xi) - y(\xi')|} 3h(\xi') d\xi' \right| \leq 3/4 \cdot \|h(t, \cdot)\|_{L^1(R)} = 3/4 \cdot \|h_0\|_{L^1(R)} \leq M. \end{aligned} \quad (3.29)$$

The similar bounds hold for other P_i and $P_{i,x}$ ($i = 1, 2, 3$). We can also get from the governing eqnarray (2.11) that

$$\|N(t, \cdot)\|_{L_R^\infty} \leq |N(0, \xi)| + (\|P_2(t, \cdot)\|_{L^\infty} + \|P_{3,x}(t, \cdot)\|_{L^\infty}) T \leq M. \quad (3.30)$$

To summarize, we have established

$$\begin{aligned} &\|\varsigma(t, \cdot)\|_{L_R^\infty} + \|U(t, \cdot)\|_{L_R^\infty} + \|V(t, \cdot)\|_{L_R^\infty} + \|N(t, \cdot)\|_{L_R^\infty} + \|h(t, \cdot)\|_{L_R^\infty} \\ &+ \sum_{i=1}^3 (\|P_i(t, \cdot)\|_{L_R^\infty} + \|P_{i,x}(t, \cdot)\|_{L_R^\infty}) \leq M. \end{aligned} \quad (3.31)$$

Let us consider a set B such that $\|h_0\|_{W(B)}$ is finite. Thus it is not hard to obtain that

$$\begin{aligned} \|P_{1,x}\|_{L^2(B)} &\leq 3/4 \cdot \|h(t, \cdot)\|_{L^1(B)} \left\| e^{-y(\xi)} \right\|_{L^2(B)} \\ &\leq 3/4 \cdot \|h_0\|_{L^1(B)} \left\| e^{-y(\xi)} \right\|_{L^2(B)} \leq M. \end{aligned} \quad (3.32)$$

Similarly, one can obtain that the same bounds hold for other P_i and $P_{i,x}$ ($i = 1, 2, 3$). Let

$$Z(t) = \|U(t, \cdot)\|_{W(B)} + \|U_\xi(t, \cdot)\|_{W(B)} + \|V(t, \cdot)\|_{W(B)} + \|V_\xi(t, \cdot)\|_{W(B)} \\ + \|\varsigma_\xi(t, \cdot)\|_{W(B)} + \|h(t, \cdot)\|_{W(B)}.$$

We can obtain, after taking the $W(B)$ -norms on both sides of (2.11) and (2.12), that

$$Z(t) \leq Z(0) + M \int_0^t Z(t') dt',$$

which, by using Gronwall's lemma, implies that

$$Z(t) \leq Z(0)e^{MT} \leq C \quad (3.33)$$

for some constant C depending only on M and $\|h_0\|_{W(B)}$.

We can get from (2.11) that

$$\|\varsigma(t', \cdot) - \varsigma(t, \cdot)\|_{L^\infty(R)} \leq M|t' - t|, \quad \|U(t', \cdot) - U(t, \cdot)\|_W \leq C(M)|t' - t|, \\ \|V(t', \cdot) - V(t, \cdot)\|_W \leq C(M)|t' - t|, \quad \|N(t', \cdot) - N(t, \cdot)\|_W \leq C(M)|t' - t|, \quad (3.34)$$

for a constant $C(M)$ depending only on M and we denote by $C(M)$ such a general constant. For $\xi \in K_\gamma$, we have $(\varsigma_\xi, U_\xi, V_\xi, h)(t', \xi) = (\varsigma_\xi, U_\xi, V_\xi, h)(t, \xi)$ for $t' \geq t \geq T$. For $\xi \in K_\gamma^c$, we get from (3.3) that

$$\|Y(t', \cdot) - Y(t, \cdot)\|_{W(K_\gamma^c)} \leq \int_t^{t'} \|F(X)Y\|_{W(K_\gamma^c)} \leq C(M)|t' - t|,$$

and (3.1c) implies that

$$\|h(t', \cdot) - h(t, \cdot)\|_{L^1(K_\gamma^c)} \\ = \|[U^2(1 + \varsigma_\xi)^2 + U_\xi^2 + V^2(1 + \varsigma_\xi)^2 + V_\xi^2 - \varsigma_\xi h](t', \cdot) \\ - [U^2(1 + \varsigma_\xi)^2 + U_\xi^2 + V^2(1 + \varsigma_\xi)^2 + V_\xi^2 - \varsigma_\xi h](t, \cdot)\|_{L^1(K_\gamma^c)} \\ \leq C(M)(\|U(t', \cdot) - U(t, \cdot)\|_{L^2(K_\gamma^c)} + \|V(t', \cdot) - V(t, \cdot)\|_{L^2(K_\gamma^c)} \\ + \|Y(t', \cdot) - Y(t, \cdot)\|_{L^2(K_\gamma^c)}) \\ \leq C(M)|t' - t|.$$

Thus $\|X(t', \cdot) - X(t, \cdot)\|_{E(K_\gamma^c)} \leq C(M)|t' - t|$ and therefore

$$\|X(t', \cdot) - X(t, \cdot)\|_E \leq C(M)|t' - t|, \quad (3.35)$$

which implies that $\lim_{t \rightarrow T_{\max}} X(t)$ exists as E is a Banach space and we denote it by $\tilde{X}$. We claim that $\tilde{X} \in \Gamma$. One can prove that there exists a constant M such that $X(t) \in B_M$ for all $t \in [0, T_{\max})$, and therefore $\|1/(y_\xi + h)(t, \cdot)\| \leq M$ and $\|1/(\tilde{y}_\xi + \tilde{h})\| \leq M$ and the condition (3.1d) holds. Note that the mapping g defined in (3.2) is lower-semicontinuous. We consider a sequence $t_n \rightarrow T_{\max}$ such that $X(t_n, \xi) \rightarrow \tilde{X}(\xi)$ for almost every ξ . We can check from the definition

of g that $g(X) - y_\xi$ is positive as y_ξ and h are positive. Thus, we get by the lower -semicontinuity of g that

$$0 \leq g(\tilde{X}(\xi)) - \tilde{y}_\xi(\xi) \leq \liminf (g(X(t_n, \cdot)) - y_\xi(t_n, \cdot)) \leq 2M$$

and $g(\tilde{X}) - 1 \in L^\infty(R)$ as $\|g(\tilde{X}) - 1\|_{L^\infty(R)} \leq \|g(\tilde{X}) - \tilde{y}_\xi\|_{L^\infty(R)} + \|\tilde{y}_\xi\|_{L^\infty(R)} \leq 3M$. By using Fatou's Lemma we can get

$$\begin{aligned} \|g(\tilde{X}) - \tilde{y}_\xi\|_{L^2(R)} &= \left(\int_R |g(\tilde{X}(\xi)) - \tilde{y}_\xi(\xi)|^2 d\xi \right)^{1/2} \\ &\leq \left(\int_R \liminf |g(X(t_n, \xi)) - y_\xi(t_n, \xi)|^2 d\xi \right)^{1/2} \\ &\leq \liminf \left(\int_R |g(X(t_n, \xi)) - y_\xi(t_n, \xi)|^2 d\xi \right)^{1/2} \\ &= \liminf \|g(X(t_n, \cdot)) - y_\xi(t_n, \cdot)\|_{L^2(R)} \leq 2M, \end{aligned}$$

which yields

$$\|g(\tilde{X}) - 1\|_{L^2(R)} \leq \|g(\tilde{X}) - \tilde{y}_\xi\|_{L^2(R)} + \|\tilde{y}_\xi\|_{L^2(R)} \leq 3M.$$

Hence, $\tilde{X}$ fulfills (3.1f). Since L^∞ or L^1 convergence implies almost everywhere convergence up to a subsequence, the conditions (3.1b), (3.1c), (3.1e) are fulfilled and therefore $\tilde{X} \in \Gamma$. Thus, the local solution X on $[0, T_{\max})$ from Theorem 3.1 with initial data $\tilde{X}$ can be extended on $[0, T_{\max} + \delta)$ for some $\delta > 0$. The assumption regarding $T_{\max}$ is contradicted, and we have proved the global existence of solutions. $\square$

4. Global dissipative solutions of the original system

One needs to establish the correspondence between the Lagrangian equivalent system and the original system to show that the global solution of the equivalent system (2.11) yields a global dissipative solution of the original system (2.1).

Let us start by introducing the set $\Gamma_0 = \{X \in \Gamma \mid g(X) = 1\}$. For any $X \in \Gamma$, we define

$$\varphi(\xi) = \int_{-\infty}^{\xi} (g(X(\eta)) - y_\xi(\eta)) d\eta + y(\xi) \quad (4.1)$$

such that $\varphi_\xi = g(X)$ and

$$\bar{y}(\xi) = y(\eta), \quad \bar{U}(\xi) = U(\eta), \quad \bar{V}(\xi) = V(\eta), \quad \bar{N}(\xi) = N(\eta) \quad (4.2)$$

for any η such that $\xi = \varphi(\eta)$. From the Radon-Nikodym theorem, there exists a unique function $\bar{h}$ in L^1 such that

$$\varphi_\#(h d\xi) = \nu = \bar{h} d\xi + \nu_s \quad (4.3)$$

where $\varphi_{\#}$ denotes the pushforward and $\bar{h}d\xi$ is the absolutely continuous part of the decomposition of ν and ν_s the singular part. From the definitions (4.1)–(4.3), it is not hard to check that $\bar{X} = (\bar{\varsigma}, \bar{U}, \bar{V}, \bar{N}, \bar{h}) \in \Gamma_0$. Thus we define the mapping $\bar{\Pi} : \Gamma \rightarrow \Gamma_0$ as $\bar{X} = \bar{\Pi}(X)$, which is a continuous mapping w. r. t. the distance $\tilde{d}_R$ on bounded set of Γ .

We claim that the system is invariant w. r. t. relabeling.

Definition 4.1. *If there exists a ψ which satisfies*

$$\psi(\xi) - \xi \in L^\infty(R), \quad \psi_\xi - 1 \in L^\infty(R) \cap L^2(R), \quad \psi_\xi \geq 0, \quad \lim_{\xi \rightarrow \infty} (\psi(\xi) - \xi) = 0 \quad (4.4a)$$

and such that

$$\bar{y} = y \circ \psi, \quad \bar{U} = U \circ \psi, \quad \bar{V} = V \circ \psi, \quad \bar{N} = N \circ \psi, \quad (4.4b)$$

we say that $\bar{X} \in \Gamma$ is a relabeling of $X \in \Gamma$.

Note that $\bar{X} \in \Gamma$ and $X \in \Gamma$ imply that for almost all $\xi \in R$ such that $\bar{y}_\xi(\xi) \neq 0$,

$$\bar{h}(\xi) = h \circ \psi \psi_\xi(\xi). \quad (4.5)$$

In the dissipative case, if $\bar{X}$ is a relabeling of X , then X is not necessary a relabeling of $\bar{X}$, basically because ψ^{-1} is either not well-defined or not sufficiently regular, thus we cannot define an equivalence relation between elements that are equal up to a relabeling. However, we have the result that if X_2 is a relabeling of X_1 , then $\bar{\Pi}(X_2) = \bar{\Pi}(X_1)$. Thus we can define an equivalence relation in Γ as follows: X_1 and X_2 are equivalent if $\bar{\Pi}(X_1) = \bar{\Pi}(X_2)$, that is, if X_2 is a relabeling of X_1 , then X_1 and X_2 are equivalent. The set of equivalent classes is in bijection with Γ_0 on which we can define a semigroup as $\tilde{S}_t = \bar{\Pi} \circ S_t$ with the semigroup property

$$\tilde{S}_{t+t'} = \bar{\Pi} \circ S_{t+t'} = \bar{\Pi} \circ S_t \circ S_{t'} = \bar{\Pi} \circ S_t \circ \bar{\Pi} \circ S_{t'} = \tilde{S}_t \circ \tilde{S}_{t'}.$$

We establish a bijection between Lagrangian equivalent system and original system by introducing two mappings between the Lagrangian variable $X \in \Gamma_0$ and the original variable $z = (u, v) \in H^1 \times H^1$.

In the following, we denote by $z = (u, v)$, $Z = (U, V)$ and $H^1 = H^1 \times H^1$. Let us define the mapping M from Lagrangian equivalent system to original system as follows.

Definition 4.2. *Given $X \in \Gamma$, if $x = y(\xi)$, the function*

$$z(x) = (u, v)(x) = (U, V)(\xi) = Z(\xi) \quad (4.6)$$

is well-defined and belongs to H^1 . We thus denote by M the mapping $X \rightarrow z$ from Γ_0 to H^1 .

The mapping L , which conversely transforms original system into Lagrangian equivalent system, is defined as follows.

Definition 4.3. *Given $z \in H^1$, we denote $\bar{X}(x) = (x, u, v, 1, u_x, v_x, u^2 + u_x^2 + v^2 + v_x^2)$. Let*

$$(U, V, N) = (u, v, v_x) \circ y, \quad h = (u^2 + u_x^2 + v^2 + v_x^2) \circ y y_\xi, \quad (4.7)$$

where y is given by

$$\int_{-\infty}^{y(\xi)} (g(\bar{X}(x)) - 1) dx + y(\xi) = \xi \quad (4.8)$$

Thus $X = (y, U, V, N, h) \in \Gamma_0$. We denote by L the mapping $z \rightarrow X$ from H^1 to Γ_0 , which sends bounded set of H^1 into bounded sets of Γ_0 , that is, for any $M > 0$ and $z \in H^1$, $\|z\|_{H^1} \leq M$ implies $L(z) \in B_{\bar{M}}$ for some constant $\bar{M}$ depending only on M .

The next theorem gives us that the transformation from the original system into the Lagrangian equivalent system is a bijection.

Theorem 4.1. $L \circ M = \text{Id}$ and $M \circ L = \text{Id}$.

Proof. Let $X \in \Gamma_0$ be given. We denote $(u, v) = M(X)$ and $\bar{X} = L(u, v)$. Let $S = \{\xi \in \mathbb{R} | X(\xi) \in \Omega\}$. It is not hard to check that $\text{meas}(S) < \infty$. As we have known that $g(X) - y_\xi \in L^1$ and $g(X) = 1$ and $\lim_{\xi \rightarrow -\infty} \varsigma(\xi) = 0$, we can obtain

$$\begin{aligned} \int_{-\infty}^{\xi} (g(X) - y_\xi) d\eta + y(\xi) &= \int_{-\infty}^{\xi} (1 - y_\xi) d\eta + y(\xi) = - \int_{-\infty}^{\xi} \varsigma_\xi d\eta \\ + y(\xi) &= - \varsigma(\xi) + y(\xi) = \xi, \end{aligned}$$

which can be rewritten as

$$\int_{-\infty}^{\xi} ((|U_\xi| + |V_\xi| + (1 + 2U^2 + 2V^2)y_\xi)\chi_S + h\chi_{S^c})(\xi) d\eta + y(\xi) = \xi$$

or

$$\int_{-\infty}^{\xi} ((|U_\xi| + |V_\xi| + (1 + 2U^2 + 2V^2)y_\xi)\chi_{y^{-1}(A)} + h\chi_{y^{-1}(A^c)})(\xi) d\eta + y(\xi) = \xi. \quad (4.9)$$

Since, $(U_\xi, V_\xi) = (u_x, v_x) \circ yy_\xi$ and $h = (u^2 + u_x^2 + v^2 + v_x^2) \circ yy_\xi$ almost everywhere, we can get after a change of variables in (4.9) that

$$\int_{-\infty}^{y(\xi)} ((|u_x| + |v_x| + (1 + 2u^2 + 2v^2))\chi_{(A)} + (u^2 + u_x^2 + v^2 + v_x^2)\chi_{(A^c)})(\xi) d\eta + y(\xi) = \xi.$$

Hence, by definition y and $\bar{y}$ satisfy (4.8) and therefore $\bar{y} = y$. We have

$$(\bar{U}, \bar{V}) = (\bar{u}, \bar{v}) \circ \bar{y} = (u, v) \circ y = (U, V)$$

and

$$\bar{h} = \bar{U}^2 \bar{y}_\xi + \bar{U}_\xi^2 / \bar{y}_\xi + \bar{V}^2 \bar{y}_\xi + \bar{V}_\xi^2 / \bar{y}_\xi = U^2 y_\xi + U_\xi^2 / y_\xi + V^2 y_\xi + V_\xi^2 / y_\xi = h$$

almost everywhere and thus $L \circ M = \text{Id}$.

Given $(u, v) \in H^1 \times H^1$, we denote $X = L(u, v)$ and $(\bar{u}, \bar{v}) = M(X)$. We have $(U, V) = (u, v) \circ y$. Since $X \in \Gamma_0$, y is invertible and therefore $(\bar{u}, \bar{v}) = (U, V) \circ y^{-1}$. Hence, $(\bar{u}, \bar{v}) = (u, v)$ and $M \circ L = \text{Id}$. $\square$

The bijective mapping L allows us to transport the metric $\tilde{d}_R$ on Γ_0 and the semigroup $\tilde{S}_t$ from Γ_0 to $H^1 \times H^1$. The metric d_{H^1} on H^1 and the semigroup T_t on H^1 are defined as follows

$$d_{H^1}(z, \bar{z}) = \tilde{d}_R(L(z), L(\bar{z})) \quad \text{and} \quad T_t = M \circ \tilde{S}_t \circ L.$$

Then a continuous semigroup of dissipative weak solutions for the original system is obtained as the following Theorem shows.

Theorem 4.2. *Let initial data $z_0 = (u_0, v_0) \in H^1 \times H^1$ be given. If we denote $t \rightarrow z(t) = T_t(z_0)$ the corresponding trajectory, then $z = (u, v)$ is a dissipative weak solution of the modified two-component Camassa-Holm system, which constructs a continuous semigroup w. r. t. the metric d_{H^1} on bounded sets of H^1 , that is, for any $M > 0$ and any sequence $z_n \in H^1$ such that $\|z_n\|_{H^1} \leq M$, we have $\lim_{n \rightarrow \infty} d_{H^1}(z_n, z) = 0$ implies $\lim_{n \rightarrow \infty} d_{H^1}(T_t(z_n), T_t(z)) = 0$.*

Proof. Let us denote $X_0 = L(z_0)$ and $X(t) = S_t(X_0)$. To prove $z = (u, v)$ is a weak solution of the original system (2.1), it suffices to show that, for all $\varphi \in C^\infty(R^+ \times R)$ with compact support,

$$\begin{aligned} \int_{R^+ \times R} [-u\phi_t + uu_x\phi](t, x) dx dt &= - \int_{R^+ \times R} [P_{1,x}\phi](t, x) dx dt, \\ \int_{R^+ \times R} [-v\phi_t + uv_x\phi](t, x) dx dt &= - \int_{R^+ \times R} [(P_{2,x} + P_3)\phi](t, x) dx dt, \end{aligned} \quad (4.10)$$

where $P_{1,x}$, $P_{2,x}$ and P_3 are given by (2.1). On the one hand, since $y(t, \xi)$ is Lipschitz and invertible w. r. t. ξ for almost all t , we then can use the change of variables $x = y(t, \xi)$ and obtain

$$\begin{aligned} &\int_{R^+ \times R} [-u\phi_t + uu_x\phi](t, x) dx dt \\ &= \int_{R^+ \times R} [-(Uy_\xi)(t, \xi)\phi_t(t, y(t, \xi)) + (UU_\xi)(t, \xi)\phi(t, y(t, \xi))] d\xi dt. \end{aligned} \quad (4.11)$$

We have $y_{\xi t}(t, \xi) = U_\xi(t, \xi)$ as $y_{\xi t}(t, \xi) = \chi_{\{t < \tau(\xi)\}}(\xi)U_\xi(t, \xi)$ and $U_\xi(t, \xi) = 0$ for $t \geq \tau(\xi)$. By using $y_t = U$ and $y_\xi(t, \xi) = 0$ for $t \geq \tau(\xi)$, it then follows from (2.11) that

$$\begin{aligned} &\int_{R^+ \times R} [-Uy_\xi\phi_t(t, y) + UU_\xi\phi(t, y)] d\xi dt \\ &= 1/4 \cdot \int_{R^+ \times \{t < \tau(\xi)\} \times \{t < \tau(\xi')\}} \text{sgn}(\xi - \xi') [h + (U^2 - 2N^2)y_\xi](\xi') \\ &\quad \times e^{-|y(\xi) - y(\xi')|} \phi(t, y(\xi)) y_\xi(\xi) d\xi' d\xi dt. \end{aligned} \quad (4.12)$$

On the other hand, using the change of variables $x = y(t, \xi)$ and $x' = y(t, \xi')$, and since y is increasing w. r. t. ξ , we have

$$\begin{aligned} &- \int_{R^+ \times R} (P_{1,x}\phi)(t, x) dx dt \\ &= 1/2 \cdot \int_{R^+ \times R^2} \text{sgn}(\xi - \xi') (u^2 + u_x^2/2 + v^2/2 - v_x^2/2)(t, y(\xi')) \end{aligned}$$

$$\begin{aligned}
& \times e^{-|y(\xi)-y(\xi')|} \phi(t, y(\xi)) y_{\xi}(\xi') y_{\xi}(\xi) d\xi' d\xi dt \\
& = 1/4 \cdot \int_{R^+ \times \{t < \tau(\xi)\} \times \{t < \tau(\xi')\}} \operatorname{sgn}(\xi - \xi') [h + (U^2 - 2N^2)y_{\xi}] (\xi') \\
& \quad \times e^{-|y(\xi)-y(\xi')|} \phi(t, y(\xi)) y_{\xi}(\xi) d\xi' d\xi dt,
\end{aligned} \tag{4.13}$$

where we have restricted the integration domain to $R^+ \times \{t < \tau(\xi)\} \times \{t < \tau(\xi')\}$ as $y_{\xi}(t, \xi) = 0$ for $t \geq \tau(\xi)$ and we have used (3.1c). It thus follows from (4.12) and (4.13) that

$$\int_{R^+ \times R} [-U y_{\xi} \phi_t(t, y) + U U_{\xi} \phi(t, y)] d\xi dt = - \int_{R^+ \times R} (P_{1,x} \phi)(t, x) dx dt.$$

Hence, the first identity in (4.10) holds. The second identity in (4.10) follows in the same way. Given a converging sequence $z_n \in H^1$ such that $\|z_n\|_{H^1} \leq M$, we have $X_n = L(z_n) \rightarrow X = L(z)$ in $\tilde{d}_R$ and $L(z_n) \in B_{\tilde{M}}$ for some $\tilde{M}$ by the definition of L , and $\tilde{S}_t(X_n) \rightarrow \tilde{S}_t(X)$ in $\tilde{d}_R$ and therefore $T_t(z_n) \rightarrow T_t(z)$ in d_{H^1} by the definition of $\tilde{S}_t$. $\square$

Proposition 4.1. *Given any initial data $z_0 \in H^1$, for almost every x and all $t \geq 0$, the corresponding dissipative solution $z = (u, v)$ satisfies the entropy condition of the form*

$$u_x(t, x) \leq 1/t + \sqrt{5}/2 \cdot \|z_0\|_{H^1}. \tag{4.14}$$

Proof. Given $\bar{t} \geq 0$ and we drop it in the notation. Let $B = \{\xi \in R \mid y(\bar{t}, \xi), U(\bar{t}, \xi), V(\bar{t}, \xi) \text{ are differentiable and } y_{\xi}(\bar{t}, \xi) > 0\}$. Thus there exists ξ such that $y_{\xi}(\bar{t}, \xi) > 0$ for all $\bar{t}$ and for a. e. x as $\operatorname{meas}(y(B)^c) = \int_{y(B)^c} dx = \int_{B^c} y_{\xi} d\xi = 0$. It follows from the definition of $\tau(\xi)$ that $\bar{t} < \tau(\xi)$ and $y_{\xi}(t, \xi) > 0$ for all $t \in [0, \bar{t}]$. Thus the variable $\alpha = U_{\xi}/y_{\xi}$ is well-defined on $[0, \bar{t}]$. We have

$$\begin{aligned}
u_x(\bar{t}, x) &= u_x(\bar{t}, y(\bar{t}, \xi)) = U_{\xi}/y_{\xi}(\bar{t}, \xi) = \alpha(\bar{t}, \xi), \\
v_x(\bar{t}, x) &= v_x(\bar{t}, y(\bar{t}, \xi)) = V_{\xi}/y_{\xi}(\bar{t}, \xi) = \beta(\bar{t}, \xi),
\end{aligned}$$

and

$$\alpha_t = (U_{\xi t} y_{\xi} - y_{\xi t} U_{\xi})/y_{\xi}^2 = (U^2 y_{\xi}^2 - U_{\xi}^2/2 + V^2 y_{\xi}^2/2 - V_{\xi}^2/2 - P_1 y_{\xi}^2)/y_{\xi}^2,$$

which imply that

$$\alpha_t + \alpha^2/2 + \beta^2/2 = U^2 + V^2/2 - P_1.$$

For all t and ξ , it holds

$$\begin{aligned}
U^2(\xi) + V^2(\xi) &= 2 \int_{-\infty}^{\xi} U(\eta) U_{\xi}(\eta) d\eta + 2 \int_{-\infty}^{\xi} V(\eta) V_{\xi}(\eta) d\eta \\
&\leq \int_R (U U_{\xi} + V V_{\xi})(\eta) d\eta \\
&\leq 1/2 \cdot \int_{\{y_{\xi}(\eta) > 0\}} (U^2 y_{\xi} + U_{\xi}/y_{\xi} + V^2 y_{\xi} + V_{\xi}/y_{\xi})(\eta) d\eta \\
&\leq 1/2 \cdot \int_R h(\eta) d\eta = \|z_0\|_{H^1}^2/2, \|P_1\|_{L^{\infty}} \leq 3/4 \cdot \|z_0\|_{H^1}^2,
\end{aligned}$$

which imply that

$$U^2 + V^2/2 - P_1 \leq 5/4 \cdot \|z_0\|_{H^1}^2.$$

We claim that

$$\alpha(t, \xi) \leq 1/t + \sqrt{5}/2 \cdot \|z_0\|_{H^1}.$$

Let us assume the opposite. We denote $\gamma(t) = 1/t + \sqrt{5}/2 \cdot \|z_0\|_{H^1}$ and t_0 the first time when $\alpha(t_0) = \beta(t_0) = \gamma(t_0)$. For $t \leq t_0$, we have $\alpha(t) < \gamma(t)$, which implies that $\gamma_t(t_0) \leq \alpha_t(t_0)$. On the other hand,

$$\begin{aligned} \alpha_t(t_0) &= -\alpha^2(t_0)/2 - \beta^2(t_0)/2 + (U^2 + V^2/2 - P_1) \\ &= -\gamma^2(t_0) + (U^2 + V^2/2 - P_1) = \gamma_t(t_0) - 5/4 \cdot \|z_0\|_{H^1}^2 \\ &\quad - \sqrt{5} \|z_0\|_{H^1} / t + (U^2 + V^2/2 - P_1) \leq \gamma_t(t_0) - 5/4 \cdot \|z_0\|_{H^1}^2 \\ &\quad - \sqrt{5} \|z_0\|_{H^1} / t + 5/4 \cdot \|z_0\|_{H^1}^2 = \gamma_t(t_0) - \sqrt{5} \|z_0\|_{H^1} / t < \gamma_t(t_0), \end{aligned}$$

which yields a contradiction. This completes the proof. $\square$

5. Conclusions

This paper investigates the dynamic behavior of the modified two-component Camassa–Holm system arisen from shallow water waves moving. It is established that the solutions of the system continue as global dissipative solutions after wave breaking, which presents an interesting and useful result for better understanding the inevitable phenomenon before and after wave breaking.

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References

- [1] Holm, D., Naraigh, L., Tronci, C.: Singular solution of a modified two-component Camassa–Holm equation. *Phys. Rev. E* **79**, 1–13 (2009)
- [2] Chen, M., Liu, S.-Q., Zhang, Y.: A 2-component generalization of the Camassa–Holm equation and its solutions. *Lett. Math. Phys.* **75**, 1–15 (2006)
- [3] Constantin, A., Ivanov, R.: On an integrable two-component Camassa–Holm shallow water system. *Phys. Lett. A* **372**, 7129–7132 (2008)
- [4] Holm, D., Marsden, J., Ratiu, T.: The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Adv. Math* **137**, 1–81 (1998)

- [5] Constantin, A.: The Hamiltonian structure of the Camassa–Holm equation. *Expo. Math* **15**, 53–85 (1997)
- [6] Camassa, R., Holm, D.: An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett* **71**, 1661–1664 (1993)
- [7] Constantin, A.: On the scattering problem for the Camassa–Holm equation. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **457**, 953–970 (2001)
- [8] Dullin, H., Gottwald, G., Holm, D.: An integral shallow water equation with linear and nonlinear dispersion. *Phys. Rev. Lett.* **87**, 4501–4504 (2001)
- [9] Constantin, A., Lannes, D.: The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. *Arch. Ration. Mech. Anal* **192**, 165–186 (2009)
- [10] Constantin, A., Escher, J.: Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.* **181**, 229–243 (1998)
- [11] Constantin, A.: Global existence of solutions and breaking waves for a shallow water equation: a geometric approach. *Ann. Inst. Fourier (Grenoble)* **50**, 321–362 (2000)
- [12] Bressan, A., Constantin, A.: Global conservative solutions of the Camassa–Holm equation. *Arch. Ration. Mech. Anal* **183**, 215–239 (2007)
- [13] Holden, H., Raynaud, X.: Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view. *Comm. P. D. E* **32**, 11511–1549 (2007)
- [14] Holden, H., Raynaud, X.: Periodic conservative solutions of the Camassa–Holm equation. *Ann. Inst. Fourier (Grenoble)* **58**, 945–988 (2008)
- [15] Holden, H., Raynaud, X.: Global conservative multipeakon solutions of the Camassa–Holm equation. *J. Hyperbolic Differ. Equ.* **4**, 39–64 (2007)
- [16] Bressan, A., Constantin, A.: Global dissipative solutions of the Camassa–Holm equation. *Appl. Anal.* **5**, 1–27 (2007)
- [17] Holden, H., Raynaud, X.: Dissipative solutions for the Camassa–Holm equation. *Discrete Contin. Dyn. Syst.* **24**, 1047–1112 (2009)
- [18] Holden, H., Raynaud, X.: Global dissipative multipeakon solutions of the Camassa–Holm equation. *Communications in P.D.E* **33**, 2040–2063 (2008)
- [19] Lv, G., Wang, M.: Some remarks for a modified periodic Camassa–Holm system. *Discrete Contin. Dyn. Syst.* **30**, 1161–1180 (2011)
- [20] Guan, C., Karlsen, K.H., Yin, Z.: Well-posedness and blow-up phenomena for a modified two-component Camassa–Holm equation, Nonlinear partial differential equations and hyperbolic wave phenomena. *Contemp. Math.* **526**, 199–220 (2010)
- [21] Tan W., Yin Z.: Global dissipative solutions of a modified two-component Camassa–Holm shallow water system. *J. Math. Phys.* 52:033507 (2011)

- [22] Tan, W., Yin, Z.: Global conservative solutions of a modified two-component Camassa–Holm shallow water system. *J. Differ. Equ.* **251**, 3558–3582 (2011)
- [23] Tan, W., Yin, Z.: Global periodic conservative solutions of a periodic modified two-component Camassa–Holm equation. *J. Funct. Anal.* **261**, 1204–1226 (2011)
- [24] He, B.: New peakon, solitary wave and periodic wave solutions for the modified Camassa–Holm equation. *Nonlinear Anal.* **71**, 6011–6018 (2009)
- [25] Wen, Z., Liu, Z.: Bifurcation of peakons and periodic cusp waves for the generalization of the Camassa–Holm equation. *Nonlinear Anal. R.W.A* **12**, 1698–1707 (2011)
- [26] Guo, Z., Zhu, M.: Wave breaking for a modified two-component Camassa–Holm system. *J. Differ. Equ.* **252**, 2759–2770 (2012)
- [27] Jin, L., Guo, Z.: A note on a modified twocomponent CamassaHolm system *Nonlinear Anal R W A* 13887892 (2012)
- [28] Guo, Z., Zhu, M.: Blow-up criteria of solutions to a modified two-component Camassa–Holm system. *Nonlinear Anal. R. W. A.* **12**, 3531–3540 (2011)
- [29] Tian, L., Wang, Y.: Global conservative and dissipative solutions of a coupled Camassa–Holm equations. *J. Math. Phys.* 52:063702 (2011)
- [30] Wang, Y., Huang, J.: Global conservative solutions of the two-component Camassa–Holm shallow water system. *Int. J. Nonlinear Sci.* **9**, 379–384 (2010)
- [31] Guan, C., Yin, Z.: Global existence and blow-up phenomena for an integrable two-component Camassa–Holm shallow water system. *J. Differ. Equ* **248**, 2003–2014 (2010)
- [32] Shen, Z., Wang, Y., Karimi, H., Song, Y.: On the multipeakon dissipative behavior of the modified coupled Camassa–Holm model for shallow water system. *Mathematical Problems in Engineering* <http://dx.doi.org/10.1155/2013/107450> (2013)

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