Numerical extremal solutions for a mixed problem with singular ϕ -Laplacian

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Abstract. We are concerned with extremal solutions for the mixed boundary value problem

$$-\left(r^{N-1}\phi(u')\right)' = r^{N-1}g(r, u), \quad u'(0) = 0 = u(R),$$

where $g : [0, R] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $\phi : (-\eta, \eta) \rightarrow \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$. We prove the existence of minimal and maximal solutions in presence of well-ordered lower and upper solutions and develop a numerical algorithm for their approximation. Also, we provide numerical experiments confirming the theoretical aspects.

Mathematics Subject Classification (2010). 34K28, 34L30, 65L10.

Keywords. Singular ϕ -Laplacian, Lower and upper solutions, Minimal and maximal solutions, Monotone iterative method, Shooting method, Euler's method.

1. Introduction

In this paper we deal with the mixed boundary value problem

$$-(r^{N-1}\phi(u'))' = r^{N-1}g(r, u), \quad u'(0) = 0 = u(R), \quad (1.1)$$

where $N \geq 1$ is an integer, $R > 0$, $g : [0, R] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $\phi : (-\eta, \eta) \rightarrow \mathbb{R}$ ($\eta < +\infty$) is an increasing homeomorphism with $\phi(0) = 0$. Such an ϕ is called *singular*.

This study is mainly motivated by the numerical approximation of extremal (minimal/maximal) classical radial solutions for the nonlinear Dirichlet problem associated with the *mean extrinsic curvature operator in Minkowski space*:

$$\operatorname{div} \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + g(|x|, v) = 0 \quad \text{in } \mathcal{B}(R), \quad v = 0 \quad \text{on } \partial\mathcal{B}(R), \quad (1.2)$$

where $\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$ and $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. As usual, setting $r = |x|$ and $v(x) = u(r)$, problem (1.2) becomes

$$-\left(r^{N-1} \frac{u'}{\sqrt{1-u'^2}}\right)' = r^{N-1}g(r, u), \quad u'(0) = 0 = u(R) \quad (1.3)$$

and the solutions of (1.3) are just classical radial solutions of (1.2). Clearly, (1.3) is nothing else but a problem of type (1.1), with

$$\phi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in (-1, 1)). \quad (1.4)$$

We note that Dirichlet problems of type (1.2) are of interest in special relativity and related aspects from Minkowski geometry (see e.g., [1, 6, 13]). In the last years, boundary value problems involving singular ϕ -Laplacian operators have received attention mainly with respect to qualitative aspects, such as existence and multiplicity of the solutions. In this view, mixed problems of type (1.1), but in a slightly different setting, were recently studied in the papers [2–5, 12]. However, effective techniques to compute the solutions seems lagging behind. A first natural way to find numerical solutions is the classical finite differences method. But, as pointed out in [14]—where a simpler Dirichlet problem with singular ϕ -Laplacian is studied, this one encounters serious damages due to the singular character of ϕ . That's why the authors develop a numerical approach relaying on a shooting technique combined with the classical Euler's method, which seems to work successfully for this type of singular problems. This will be the strategy we will adopt here.

The rest of the paper is organized as follows. In Sect. 2 we apply the monotone iterative technique coupled with the method of lower and upper solutions in order to obtain two monotone sequences that uniformly converge to the extremal solutions of problem (1.1) (Theorem 2.5). Notice that the uniqueness of the solution will not be ensured by the assumptions on the data. The reader is referred to [10] for a detailed study of monotone iterative techniques (also, see [8, 9] for Neumann and periodic problems involving a ϕ -Laplacian operator). Then, if ϕ is an increasing diffeomorphism with $\phi(0) = 0$ and $\phi'(y) \geq d > 0$ for all $y \in (-\eta, \eta)$, we find in Sect. 3 the approximate extremal solutions of problem (1.1). With this aim, combining Euler's method with a shooting technique, we need to develop a numerical convergent algorithm for a problem of type

$$-(r^{N-1}\phi(u'))' = r^{N-1}f(r), \quad u'(0) = 0 = u(R),$$

with $f : [0, R] \rightarrow \mathbb{R}$ continuous (Theorem 3.1). Numerical experiments which confirm the theoretical aspects from this section are also provided.

2. Extremal solutions

In this section we construct two monotone sequences which uniformly converge to the extremal solutions of problem (1.1) from an interval described by a lower and an upper solution, under the hypothesis

$(H_\phi)\phi : (-\eta, \eta) \rightarrow \mathbb{R}$ ($0 < \eta < \infty$) is an increasing homeomorphism with $\phi(0) = 0$.

Here and hereafter, C stands for the Banach space of the continuous functions defined on $[0, R]$ endowed with the usual sup-norm $\|\cdot\|$. Recall that, by a *solution* of (1.1) we mean a function $u \in C^1 := C^1[0, R]$ with $\|u'\| < \eta$, such that $r^{N-1}\phi(u') \in C^1$ and (1.1) is satisfied.

We associate to problem (1.1) a fixed point operator (see [2, 4, 11] for slightly different operators). In this view, setting $\sigma(r) := -1/r^{N-1}$ ($r > 0$), we first define

$$S : C \rightarrow C, \quad Su(r) = \sigma(r) \int_0^r t^{N-1} u(t) dt \quad (r \in (0, R]), \quad Su(0) = 0;$$

$$K : C \rightarrow C, \quad Ku(r) = \int_R^r u(t) dt, \quad (r \in [0, R]).$$

It is easy to see that K is bounded and standard arguments, invoking the Arzela-Ascoli theorem, show that S is compact. This implies that the nonlinear operator $K \circ \phi^{-1} \circ S : C \rightarrow C$ is compact. Also, an easy computation shows that, for any $\ell \in C$, the mixed problem

$$-(r^{N-1}\phi(u'))' = r^{N-1}\ell(r), \quad u'(0) = 0 = u(R),$$

has an unique solution u given by

$$u = K \circ \phi^{-1} \circ S\ell.$$

We denote by N_g the Nemytskii operator associated to g ; recall that this is defined by

$$N_g(u) = g(\cdot, u(\cdot)), \quad \text{for all } u \in C$$

and $N_g : C \rightarrow C$ is continuous and takes bounded sets into bounded sets.

The following result is proved in [2].

Proposition 2.1. (i) A function $u \in C$ is a solution of problem (1.1) if and only if it is a fixed point of the compact operator

$$\mathcal{M} : C \rightarrow C, \quad \mathcal{M} := K \circ \phi^{-1} \circ S \circ N_g.$$

(ii) Problem (1.1) has at least a solution for any g continuous. If, in addition, $g(r, \cdot)$ is nonincreasing for each fixed $r \in [0, R]$, then the solution of (1.1) is unique.

Definition 2.2. Given a class of functions $\mathcal{A} \subset C$, a solution $u \in \mathcal{A}$ of problem (1.1) is called *minimal* (resp. *maximal*) in $\mathcal{A}$ if $u \leq v$ (resp. $v \leq u$), for any other solution $v \in \mathcal{A}$.

Definition 2.3. A *lower solution* of problem (1.1) is a function $\alpha \in C^1$ such that $\|\alpha'\| < \eta$, $r^{N-1}\phi(\alpha') \in C^1$ and

$$-(r^{N-1}\phi(\alpha'(r)))' \leq r^{N-1}g(r, \alpha(r)) \quad (r \in [0, R]), \quad \alpha(R) \leq 0. \quad (2.1)$$

Similarly, an *upper solution* of (1.1) is defined by reversing the inequalities in (2.1).

We shall need the following comparison principle.

Proposition 2.4. *Let $M > 0$ and $v_1, v_2 \in C^1$ be such that $r^{N-1}\phi(v'_i) \in C^1$ ($i = 1, 2$). If*

$$-(r^{N-1}\phi(v'_1))' + Mr^{N-1}v_1 \leq -(r^{N-1}\phi(v'_2))' + Mr^{N-1}v_2 \quad \text{on } [0, R] \quad (2.2)$$

and $v_1(R) \leq v_2(R)$, then $v_1(r) \leq v_2(r)$, for all $r \in [0, R]$.

Proof. Suppose, by contradiction, that there exists some $r_0 \in [0, R]$ such that

$$\max_{[0, R]}(v_1 - v_2) = v_1(r_0) - v_2(r_0) > 0.$$

If $r_0 \in (0, R)$ then $v'_1(r_0) = v'_2(r_0)$ and there is a sequence $\{r_k\}$ in $(0, r_0)$ converging to r_0 such that $v'_1(r_k) - v'_2(r_k) \geq 0$. As ϕ is increasing, one has

$$r_k^{N-1}\phi(v'_1(r_k)) - r_0^{N-1}\phi(v'_1(r_0)) \geq r_k^{N-1}\phi(v'_2(r_k)) - r_0^{N-1}\phi(v'_2(r_0)),$$

implying that

$$(r^{N-1}\phi(v'_1(r)))'_{r=r_0} \leq (r^{N-1}\phi(v'_2(r)))'_{r=r_0}. \quad (2.3)$$

From (2.2) and (2.3), we obtain

$$0 < Mr_0^{N-1}(v_1(r_0) - v_2(r_0)) \leq (r^{N-1}\phi(v'_1(r)))'_{r=r_0} - (r^{N-1}\phi(v'_2(r)))'_{r=r_0} \leq 0,$$

a contradiction. If $r_0 = 0$ then there exists $r_1 \in (0, R]$ such that $v_1(r) - v_2(r) > 0$ for all $r \in [0, r_1]$ and $v'_1(r_1) - v'_2(r_1) \leq 0$. From (H_ϕ) and (2.2), we get

$$0 \geq r_1^{N-1}\phi(v'_1(r_1)) - r_1^{N-1}\phi(v'_2(r_1)) \geq \int_0^{r_1} Mr^{N-1}(v_1(r) - v_2(r))dr > 0,$$

a contradiction, again. Consequently, $v_1(r) \leq v_2(r)$ for all $r \in [0, R]$. $\square$

Remark 2.5. Note that the comparison principle does not require to impose any condition on the derivative of v_1 or v_2 at $r = 0$. This property is translated to the above definition of the lower and the upper solutions.

Now, we are in position to apply a monotone iterative method to provide the existence of extremal solutions for problem (1.1).

Theorem 2.6. If (1.1) has a lower solution α and an upper solution β such that

$$\alpha \leq \beta \quad \text{on } [0, R] \quad (2.4)$$

and, for all $r \in [0, R]$, g satisfies

$$g(r, v) - g(r, w) \leq -L(v - w), \quad \forall v, w \in [\alpha(r), \beta(r)] \text{ with } v \leq w, \quad (2.5)$$

where L is a positive constant, then problem (1.1) has a minimal solution u_{min} and a maximal solution u_{max} in $\mathcal{A}_{\alpha, \beta} = \{v \in C : \alpha \leq v \leq \beta\}$.

Proof. We construct u_{min} by monotone iterations. Let $u_0 = \alpha$. On account of Proposition 2.1, we can define u_{n+1} ($n = 0, 1, 2, \dots$) as being the unique solution of the problem

$$\begin{cases} -(r^{N-1}\phi(u'_{n+1}))' + Lr^{N-1}u_{n+1} = r^{N-1}(Lu_n + g(r, u_n)), \\ u'_{n+1}(0) = 0 = u_{n+1}(R). \end{cases} \quad (2.6)$$

Step 1. We claim that

$$\alpha(r) \leq u_n(r) \leq \beta(r) \quad (n = 0, 1, 2, \dots), \quad \forall r \in [0, R]. \quad (2.7)$$

We proceed by induction. Clearly, (2.7) is valid for $n = 0$ by hypothesis (2.4). Then, assuming $\alpha \leq u_n \leq \beta$, we have to show that $\alpha \leq u_{n+1} \leq \beta$. From (2.6), (2.5) and because α is a lower solution of problem (1.1), we get

$$\begin{aligned} -(r^{N-1}\phi(u'_{n+1}(r)))' + Lr^{N-1}u_{n+1}(r) &= r^{N-1}(Lu_n(r) + g(r, u_n(r))) \\ &\geq r^{N-1}(L\alpha(r) + g(r, \alpha(r))) \\ &\geq -(r^{N-1}\phi(\alpha'(r)))' + Lr^{N-1}\alpha(r), \end{aligned}$$

and in view of Proposition 2.4, one has

$$\alpha(r) \leq u_{n+1}(r), \quad \forall r \in [0, R].$$

Similarly, for all $r \in [0, R]$, we deduce

$$-(r^{N-1}\phi(u'_{n+1}(r)))' + Lr^{N-1}u_{n+1}(r) \leq -(r^{N-1}\phi(\beta'(r)))' + Lr^{N-1}\beta(r)$$

and, again from Proposition 2.4, we obtain

$$u_{n+1}(r) \leq \beta(r), \quad \forall r \in [0, R]$$

and (2.7) is proved, as claimed.

Step 2. We show that

$$u_n(r) \leq u_{n+1}(r) \quad (n = 0, 1, 2, \dots), \quad \forall r \in [0, R]. \quad (2.8)$$

Again, we proceed by induction. The case $n = 0$ follows from Step 1. Next, assume $u_{n-1} \leq u_n$. From (2.5)–(2.7), we successively have

$$\begin{aligned} -(r^{N-1}\phi(u'_{n+1}(r)))' + Lr^{N-1}u_{n+1}(r) &= r^{N-1}(Lu_n(r) + g(r, u_n(r))) \\ &\geq r^{N-1}(Lu_{n-1}(r) + g(r, u_{n-1}(r))) = -(r^{N-1}\phi(u'_n(r)))' + Lr^{N-1}u_n(r), \end{aligned}$$

for all $r \in [0, R]$. Then (2.8) follows from Proposition 2.4.

Step 3 (Existence of u_{min}). On account of (2.7) and (2.8), we have

$$\alpha(r) = u_0(r) \leq u_1(r) \leq \dots \leq u_n(r) \leq u_{n+1}(r) \leq \dots \leq \beta(r), \quad \forall r \in [0, R].$$

Therefore

$$u_{min}(r) := \lim_{n \rightarrow \infty} u_n(r) \quad (2.9)$$

exists for all $r \in [0, R]$. Since $\{u_n\}_{n \geq 1}$ and $\{u'_n\}_{n \geq 1}$ are bounded in C , using Arzela-Ascoli theorem we infer that $u_{min} \in C$. Then by (2.9) and Dini's theorem, $u_n \rightarrow u_{min}$, uniformly in C .

We rewrite problem (2.6) as

$$\begin{cases} -(r^{N-1}\phi(u'_{n+1}(r)))' = r^{N-1}g_n(r, u_{n+1}), \\ u'_{n+1}(0) = 0 = u_{n+1}(R), \end{cases} \quad (2.10)$$

with $g_n(r, s) := -Ls + Lu_n(r) + g(r, u_n(r))$. Associated to (2.10) is the operator $\mathcal{M}_n$, defined as in Proposition 2.1 (i), i.e.,

$$\mathcal{M}_n = K \circ \phi^{-1} \circ S \circ N_{g_n}.$$

On account of $\mathcal{M}_n(u_{n+1}) = u_{n+1}$, the continuity of the operator $K \circ \phi^{-1} \circ S$ and

$$N_{g_n}(u_{n+1}) = g_n(\cdot, u_{n+1}(\cdot)) \rightarrow g(\cdot, u_{\min}(\cdot)) = N_g(u_{\min})$$

yield $\mathcal{M}_n(u_{n+1}) \rightarrow \mathcal{M}(u_{\min})$ in C . This implies $\mathcal{M}(u_{\min}) = u_{\min}$, thus $u_{\min} \in \mathcal{A}_{\alpha, \beta}$ is a solution of problem (1.1).

Step 4. If u is another solution of (1.1) in $\mathcal{A}_{\alpha, \beta}$, we shall show that

$$u_n(r) \leq u(r) \quad (n = 0, 1, 2, \dots), \quad \forall r \in [0, R]. \quad (2.11)$$

Clearly, (2.11) holds true for $n = 0$. By induction, if we assume that $u_n \leq u$, then we have to prove that $u_{n+1} \leq u$. As in Step 2, from (2.5) and (2.6), one has

$$\begin{aligned} -(r^{N-1}\phi(u'(r)))' + Lr^{N-1}u(r) &= r^{N-1}(Lu(r) + g(r, u(r))) \\ &\geq r^{N-1}(Lu_n(r) + g(r, u_n(r))) = -(r^{N-1}\phi(u'_{n+1}(r)))' + Lr^{N-1}u_{n+1}(r), \end{aligned}$$

and (2.11) follows from Proposition 2.4.

Hence, from (2.11) we get $u_{\min} \leq u$, i.e., $u_{\min}$ is the minimal solution of problem (1.1) in $\mathcal{A}_{\alpha, \beta}$.

The proof of the existence of a decreasing sequence $\{v_n\}$, starting at $v_0 = \beta$, which converges uniformly to the maximal solution $u_{\max}$ of problem (1.1) in $\mathcal{A}_{\alpha, \beta}$, is similar. $\square$

Remark 2.7. It is worth to point out that if α is a lower solution and β an upper solution of problem (1.1) such that (2.4) holds true and, for all $r \in [0, R]$, g satisfies

$$|g(r, v) - g(r, w)| \leq L|v - w|, \quad \forall v, w \in [\alpha(r), \beta(r)], \quad (2.12)$$

then (2.5) is fulfilled, hence problem (1.1) has minimal and maximal solutions in $\mathcal{A}_{\alpha, \beta}$. However, there are functions g verifying (2.5), which does not satisfy the L -Lipschitz condition (2.12)—see Example 3.3 below.

3. Numerical minimal and maximal solutions

Throughout this section, instead of (H_ϕ) we assume the stronger hypothesis (H'_ϕ) $\phi : (-\eta, \eta) \rightarrow \mathbb{R}$ ($0 < \eta < \infty$) is an increasing diffeomorphism with $\phi(0) = 0$ and there is some $d > 0$ such that $\phi'(y) \geq d$ for all $y \in (-\eta, \eta)$.

As we have seen in the proof of Theorem 2.6, finding the minimal and maximal solutions of problem (1.1) by the monotone iterative technique, requires to solve at step $n + 1$ a problem of the type (see (2.6)):

$$-(r^{N-1}\phi(u'_{n+1}))' + Lr^{N-1}u_{n+1} = r^{N-1}\tilde{\ell}_n(r), \quad u'_{n+1}(0) = 0 = u_{n+1}(R), \quad (3.1)$$

with $\tilde{\ell}_n(r) = Lu_n(r) + g(r, u_n(r))$ known from step n . The fixed point operator associated with (3.1) is (see Proposition 2.1 (i)):

$$\widetilde{\mathcal{M}}_{n+1} = K \circ \phi^{-1} \circ S \circ N_{\tilde{g}_{n+1}},$$

where $\tilde{g}_{n+1}(r, s) = -Ls + \tilde{\ell}_n(r)$ $((r, s) \in [0, R] \times \mathbb{R})$, with L entering in (2.5). If, in addition, one has that

$$L < \frac{2Nd}{R^2}, \quad (3.2)$$

then it is straightforward to see that $\widetilde{\mathcal{M}}_{n+1}$ is a contraction, hence by virtue of Banach's fixed point theorem, the sequence $\{u_{n+1}^k\}_{k \geq 0}$ defined by

$$u_{n+1}^{k+1} = \widetilde{\mathcal{M}}_{n+1}(u_{n+1}^k) \quad (k = 0, 1, 2, \dots) \quad (3.3)$$

converges in C to the unique solution of problem (3.1), regardless of the initial value $u_{n+1}^0 \in C$. It is worth to point out that condition (3.2) does not imply the unique solvability of the original problem (1.1). Setting $\bar{w} := u_{n+1}^{k+1}$, (3.3) is equivalent to

$$-(r^{N-1}\phi(\bar{w}'))' = r^{N-1}[-Lu_{n+1}^k(r) + \tilde{\ell}_n(r)], \quad \bar{w}'(0) = 0 = \bar{w}(R). \quad (3.4)$$

Thus, assuming (3.2), finding $\bar{w}$ in (3.4) reduces to solving a mixed problem of type:

$$-(r^{N-1}\phi(\bar{w}'))' = r^{N-1}f(r), \quad \bar{w}'(0) = 0 = \bar{w}(R), \quad (3.5)$$

with $f: [0, R] \rightarrow \mathbb{R}$ continuous.

In the sequel, combining Euler's method with a shooting technique, we derive an algorithm to find numerical solutions for problem (3.5). To this aim, setting $\psi := \phi^{-1}$ and $\bar{v} := -r^{N-1}\phi(\bar{w}')$, problem (3.5) is transformed into

$$\begin{cases} \bar{v}' = r^{N-1}f(r); \\ \bar{w}' = \psi(\sigma(r)\bar{v}(r)); \\ \bar{w}(R) = 0, \quad \bar{v}(R) = a_*, \end{cases} \quad (3.6)$$

with

$$a_* = \int_0^R \zeta^{N-1} f(\zeta) d\zeta.$$

By the change of variable $t = R - r$ and setting $\bar{w}(r) = w(t)$, $\bar{v}(r) = v(t)$ in (3.6), we arrive to the initial value problem

$$\begin{cases} v' = -\gamma(t); \\ w' = -\psi(\sigma(R-t)v(t)); \\ w(0) = 0, \quad v(0) = a_*, \end{cases} \quad (3.7)$$

where, we have denoted

$$\gamma(t) := (R-t)^{N-1}f(R-t). \quad (3.8)$$

Obviously, a_* can be expressed as

$$a_* = \int_0^R \gamma(t) dt. \quad (3.9)$$

Notice that both of the systems (3.6) and (3.7) have unique solutions.

Next, let $m \in \mathbb{N}$ ($m \geq 2$), $h = R/m$ and $t_i = ih$, $r_i = (m - i)h$ for $i \in \{0, 1, \dots, m\}$. Applying the standard Euler's method (see e.g. [7]) to problem (3.7) and the rectangles quadrature formula to approximate a_* in (3.9), we get

$$\begin{cases} v_{i+1} = v_i - h\gamma(t_i), & v_0 = a_m; \\ w_{i+1} = w_i - h\psi(\sigma(R - t_i)v_i), & w_0 = 0, \end{cases} \quad (3.10)$$

for $i \in \{0, 1, \dots, m - 1\}$, with

$$a_m = h \sum_{j=0}^{m-1} \gamma(t_j). \quad (3.11)$$

From (3.10), it follows

$$v_i = a_m - h \sum_{j=0}^{i-1} \gamma(t_j), \quad (i \in \{1, 2, \dots, m\}),$$

which implies that

$$w_1 = -h\psi(\sigma(R)a_m) \quad (3.12)$$

and

$$\begin{aligned} w_i &= w_{i-1} - h\psi(\sigma(R - t_{i-1})v_{i-1}) \\ &= w_{i-1} - h\psi\left(\sigma(R - t_{i-1})\left(a_m - h \sum_{j=0}^{i-2} \gamma(t_j)\right)\right), \end{aligned} \quad (3.13)$$

for $i \in \{2, 3, \dots, m\}$.

Now, to compute the approximate values $\bar{w}_i$ of $\bar{w}(r_i)$ ($\bar{w}$ = the exact solution of problem (3.5)) we have the following:

Algorithm A.

Step 1: use (3.11) to find a_m ;

Step 2: compute w_1 with (3.12) and w_i by means of (3.13) for $i \in \{2, 3, \dots, m\}$;

Step 3: $\bar{w}_i := w_{m-i}$ for $i \in \{0, 1, \dots, m\}$.

The convergence of this algorithm is proved in the following

Theorem 3.1. Assume (H'_ϕ) and $f \in C$. Then it holds

$$\lim_{m \rightarrow \infty} \left(\max_{0 \leq i \leq m} |\bar{w}(r_i) - \bar{w}_i| \right) = 0. \quad (3.14)$$

Proof. First, it is clear that (3.14) means nothing else but

$$\lim_{m \rightarrow \infty} \left(\max_{0 \leq i \leq m} |w(t_i) - w_i| \right) = 0. \quad (3.15)$$

Also, since $f \in C$, there is a constant $M > 0$ such that

$$|f(s)| \leq M, \quad \text{for all } s \in [0, R]. \quad (3.16)$$

From (3.7), one has

$$v(t) = a_* - \int_0^t \gamma(\zeta) d\zeta$$

and

$$w(t) = - \int_0^t \psi(\sigma(R - \tau)v(\tau)) \, d\tau.$$

By the mean value theorem for Riemann integral, we obtain

$$\begin{aligned} w(t_i) - w(t_{i-1}) &= - \int_{t_{i-1}}^{t_i} \psi(\sigma(R - \tau)v(\tau)) \, d\tau = -h\psi(\sigma(R - \tau_i)v(\tau_i)) \\ &= -h\psi\left(\sigma(R - \tau_i)\left(a_* - \int_0^{\tau_i} \gamma(\zeta) \, d\zeta\right)\right), \end{aligned} \quad (3.17)$$

with some $\tau_i \in (t_{i-1}, t_i)$. Therefore, with $\varphi : [0, R] \rightarrow \mathbb{R}$ given by

$$\varphi(t) = \sigma(R - t) \int_t^R \gamma(s) \, ds \quad (t \in [0, R]), \quad \varphi(R) = 0,$$

from (3.17), (3.9), (3.13) and (3.11), we successively have

$$\begin{aligned} w(t_i) - w(t_{i-1}) &= -h\psi(\varphi(\tau_i)) \\ &= (w_i - w_{i-1}) - h \left[\psi(\varphi(\tau_i)) - \psi\left(\sigma(R - t_{i-1})h \sum_{j=i-1}^{m-1} \gamma(t_j)\right) \right], \end{aligned}$$

for all $i \in \{1, 2, \dots, m\}$. Then, using that

$$|\psi'(\xi)| \leq \frac{1}{d}, \quad \text{for all } \xi \in \mathbb{R} \quad (\text{see hypothesis } (H'_\phi)),$$

we get the estimate

$$|w(t_i) - w_i| \leq |w(t_{i-1}) - w_{i-1}| + \frac{h}{d} \left| \varphi(\tau_i) - \sigma(R - t_{i-1})h \sum_{j=i-1}^{m-1} \gamma(t_j) \right|, \quad (3.18)$$

for $i \in \{1, 2, \dots, m\}$.

Now, let $\varepsilon > 0$ be arbitrary. Since φ and f are uniformly continuous, we can find $\delta_\varepsilon > 0$ such that

$$|\varphi(s') - \varphi(s'')| < d\varepsilon \quad (3.19)$$

and

$$|f(R - s') - f(R - s'')| < d\varepsilon, \quad (3.20)$$

for all $s', s'' \in [0, R]$ with $|s' - s''| < \delta_\varepsilon$.

If $N \geq 2$, we choose $m_\varepsilon \in \mathbb{N}$, so that

$$h = \frac{R}{m} < \min \left\{ \delta_\varepsilon, \frac{d\varepsilon}{M(N-1)} \right\} \quad (3.21)$$

for all $m \geq m_\varepsilon$. Denoting

$$q_i := |w(t_i) - w_i| \quad (i \in \{1, 2, \dots, m\}),$$

from (3.18) and (3.19), it follows

$$\begin{aligned} q_i &\leq q_{i-1} + \frac{h}{d} \left(\left| \varphi(\tau_i) - \varphi(t_{i-1}) \right| + \left| \varphi(t_{i-1}) - \sigma(R - t_{i-1})h \sum_{j=i-1}^{m-1} \gamma(t_j) \right| \right) \\ &< q_{i-1} + h\varepsilon + \frac{h}{d(R - t_{i-1})^{N-1}} \left| \int_{t_{i-1}}^R \gamma(\zeta) d\zeta - h \sum_{j=i-1}^{m-1} \gamma(t_j) \right|. \end{aligned}$$

Using again the mean value theorem for Riemann integral, we get

$$q_i < q_{i-1} + h\varepsilon + \frac{h^2}{d(R - t_{i-1})^{N-1}} \sum_{j=i-1}^{m-1} |\gamma(\rho_j) - \gamma(t_j)|, \quad (3.22)$$

where $\rho_j \in (t_j, t_{j+1})$. On the other hand, for any $j \in \{i-1, \dots, m-1\}$, one has

$$|(R - \rho_j)^{N-1} - (R - t_j)^{N-1}| \leq (N-1)(R - \varsigma_j)^{N-2}(\rho_j - t_j), \quad (3.23)$$

with some $\varsigma_j \in (t_j, \rho_j)$. Then, taking into account that

$$R - t_{i-1} = h(m - i + 1), \quad (i \in \{1, 2, \dots, m\}),$$

from (3.20), (3.16) and (3.23), we obtain

$$\begin{aligned} |\gamma(\rho_j) - \gamma(t_j)| &= |(R - \rho_j)^{N-1} f(R - \rho_j) - (R - t_j)^{N-1} f(R - t_j)| \\ &\leq (R - \rho_j)^{N-1} |f(R - \rho_j) - f(R - t_j)| + |f(R - t_j)| |(R - \rho_j)^{N-1} - (R - t_j)^{N-1}| \\ &\leq (R - t_j)^{N-1} d\varepsilon + M(N-1)(R - \varsigma_j)^{N-2}(\rho_j - t_j) \\ &\leq (R - t_{i-1})^{N-1} d\varepsilon + M(N-1)(R - t_{i-1})^{N-2}(t_{j+1} - t_j) \\ &= (R - t_{i-1})^{N-1} \left(d\varepsilon + \frac{M(N-1)}{m - i + 1} \right), \end{aligned}$$

which yields

$$q_i < q_{i-1} + h\varepsilon + h^2(m - i + 1)\varepsilon + \frac{h^2}{d} M(N-1) < q_{i-1} + h\varepsilon + hR\varepsilon + h\varepsilon.$$

Hence, for all $m \geq m_\varepsilon$ and $i \in \{1, 2, \dots, m\}$, since $q_0 = 0$ and $ih \leq R$, it follows

$$q_i < q_{i-1} + h(R+2)\varepsilon < \dots < ih(R+2)\varepsilon \leq R(R+2)\varepsilon$$

and obviously (3.15) holds true.

If $N = 1$, we can find some $m_\varepsilon \in \mathbb{N}$, such that

$$h = R/m < \delta_\varepsilon \quad (3.24)$$

for all $m \geq m_\varepsilon$ and, (3.22) becomes

$$q_i < q_{i-1} + h\varepsilon + \frac{h^2}{d} \sum_{j=i-1}^{m-1} |f(R - \rho_j) - f(R - t_j)|$$

TABLE 1. Approximation errors for the test problem in Example 3.2

N	$m = 16$	$m = 32$	$m = 64$	$m = 128$
1	0.0177	0.0088	0.0044	0.0022
2	0.0452	0.0227	0.0114	0.0057
6	0.1129	0.0549	0.0268	0.0132
13	0.2356	0.1102	0.0512	0.0247

and from (3.20), one has

$$q_i < q_{i-1} + h\varepsilon + h^2(m-i+1)\varepsilon \leq q_{i-1} + h\varepsilon + hR\varepsilon.$$

Thus, as $q_0 = 0$ and $ih \leq R$, we obtain

$$q_i < q_{i-1} + h(R+1)\varepsilon < \cdots < ih(R+1)\varepsilon \leq R(R+1)\varepsilon,$$

for all $m \geq m_\varepsilon$ and $i \in \{1, 2, \dots, m\}$. This clearly implies (3.15) and the proof is complete. $\square$

Example 3.2. Let $N \geq 1$, ϕ be given in (1.4) and $f : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(r) = \frac{N}{\sqrt{4-r^2}} + \frac{r^2}{\sqrt{(4-r^2)^3}}, \quad \forall r \in [0, 1].$$

We know that

$$u(r) = \frac{1}{4}(1-r^2)$$

is the exact solution of problem

$$-\left(r^{N-1} \frac{u'}{\sqrt{1-u'^2}}\right)' = r^{N-1} f(r), \quad u'(0) = 0 = u(1).$$

For $N = 1, 2, 6, 13$ and different values of the discretization parameter m , we give in Table 1 the values of the errors computed by

$$\max_{0 \leq i \leq m} |\bar{w}(r_i) - \bar{w}_i| \quad (\text{see (3.14)}).$$

As it can be seen from the table, for a fixed m the error grows as N does. This is in connection with the convergence assumptions on the parameter h (see (3.21)). Using Algorithm A, in Fig. 1 we present the graphics of the exact solution and its corresponding approximations. The numerical experiments are performed for $N = 2, 6$ and $m = 16$, respectively $m = 32$.

At this stage we can propose the following algorithm to find the approximate minimal (resp. maximal) solution of problem (1.1).

FIGURE 1. Exact (– * –) and approximate (– ◦ –) solutions for the problem in Example 3.2 (left to right) $m = 16$, $m = 32$; Top $N = 2$; Bottom $N = 6$

Algorithm B.

- Given g continuous, ϕ satisfying (H'_ϕ) , one assume that the hypotheses of Theorem 2.6 and (3.2) are fulfilled.
- Let $m \in \mathbb{N}$ ($m \geq 2$) be fixed; we take $u_0 := \alpha$ (resp. $u_0 := \beta$).
- The function u_{n+1} ($n = 0, 1, \dots$), is approximated as follows:
 - Let $u_{n+1}^0 \in C$ be arbitrary.
 - We compute, at step $k+1$ ($k = 0, 1, \dots$), the approximate values of $u_{n+1}^{k+1}(r_i)$ ($i = \{0, 1, \dots, m\}$) using Algorithm A, with f entering in (3.8) given by $f(r) = -Lu_{n+1}^k(r) + Lu_n(r) + g(r, u_n(r))$.

Example 3.3. To illustrate the above theoretical aspects we consider the problem

$$-\left(r^{N-1} \frac{u'}{\sqrt{1-u'^2}}\right)' = r^{N-1} g(r, u), \quad u'(0) = 0 = u(1), \quad (3.25)$$

where $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$g(r, u) = \begin{cases} \text{sign}(u-1)\sqrt{|u-1|}, & u \in (0, 2); \\ -u+3, & u \geq 2; \\ -1-u/2, & u \leq 0, \end{cases} \quad \forall r \in [0, 1].$$

TABLE 2. The number of outer (average inner) iterations for the problem in Example 3.3

N	$m = 8$	$m = 16$	$m = 32$	$m = 64$
1	12 (22)	12 (21)	12 (20)	11 (20)
2	12 (21)	11 (20)	11 (19)	11 (19)
6	10 (17)	10 (15)	9 (14)	9 (14)
11	9 (15)	8 (13)	8 (12)	8 (12)

It is easy to see that $\alpha(r) = -2$ and $\beta(r) = 3$ ($r \in [0, 1]$) are a lower, respectively an upper solution of problem (3.25) and that g fulfills condition (2.5) with $L = 1$. Clearly (3.2) is satisfied; ϕ is given in (1.4) and $d = 1$ in hypothesis (H'_ϕ) . Also, note that g does not verify condition (2.12), hence in this case it can not be applied the classical method of successive approximations directly on the problem. We apply Algorithm B with the following stopping criteria

- for the *inner* iteration

$$|u_{n+1}^{k+1} - u_{n+1}^k| \leq 10^{-15}$$

- for the *outer* iteration

$$|u_{n+1} - u_n| \leq 10^{-10}$$

To find the approximate minimal (resp. maximal) solution, for the outer iteration we consider $u_0 = -2$ (resp. $u_0 = 3$), while for the inner iterations we consider $u_{n+1}^0 = 0$ ($n = 0, 1, \dots$). In the case of maximal solution, in Table 2 we give the number of outer iterations and the average number of inner iterations (in brackets), for different values of N and m (similar values were obtained in the case of minimal solution).

Our numerical experiments showed that problem (3.25) seems to have a unique solution between α and β , which is both minimal and maximal. The behavior of Algorithm B is different if we look for a minimal, respectively a maximal solution, but the corresponding sequences $(u_{n+1})_{n \geq 0}$ converges to the same function; in Fig. 2 we plotted these sequences for $N = 2$ and $m = 32$.

FIGURE 2. Example 3.3, case $N = 2, m = 32$; *Left* sequence $(u_{n+1})_{n \geq 0}$ for $u_0 = -2$; *Right* sequence $(u_{n+1})_{n \geq 0}$ for $u_0 = 3$

FIGURE 3. The approximate (unique) solution of the test problem in Example 3.3; case $m = 32$; (left to right) $N = 2$ and $N = 6$

Also, in Fig. 3 we give the graphics of the approximate (unique) solution of problem (3.25), computed for $m = 32$, $N = 2, 6$.

Acknowledgments

The authors express their gratitude to the anonymous referee for useful comments and remarks which led to the improvement of the paper. The support for P. Jebelean from the grant TE-PN-II-RU-TE-2011-3-0157 (CNCS-Romania) is gratefully acknowledged. Also, the work of C. Șerban was supported by the strategic grant POSDRU/CPP107/DMI1.5/S/78421, Project ID 78421 (2010), co-financed by the European Social Fund Investing in People, within the Sectoral Operational Programme Human Resources Development 2007–2013.

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Received: 15 December 2012.

Revised: 6 August 2013.

Accepted: 20 August 2013.