Stability of equilibria for a two-phase osmosis model

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Abstract. For a two-phase moving boundary problem modelling the motion of a semipermeable membrane by osmotic pressure and surface tension, we prove that the manifold of equilibria is locally exponentially attractive. Our method relies on maximal regularity results for parabolic systems with relaxation type boundary dynamics.

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1. Introduction

In this paper we consider the two-phase version of a moving boundary problem modelling osmosis: A closed semipermeable membrane $\Gamma(t)$ moves freely in an incompressible fluid, which is situated inside some fixed container $C \subset \mathbb{R}^N$. The membrane encloses a (by definition connected) domain $\Omega_+(t)$ and separates it from a second domain $\Omega_-(t)$ between $\Gamma(t)$ and the container wall ∂C . In both regions a certain amount of a solute is dissolved in the fluid. Its concentration at position $x \in \Omega_\pm(t)$ and at time t is denoted by $u_\pm = u_\pm(t, x)$. The membrane is impermeable to the solute but permeable to the solvent. In dimensionless variables, diffusion of the solute (according to constant diffusivities) and the motion of the membrane is described by the following moving boundary problem:

$$\left. \begin{aligned} \partial_t u_\pm - \kappa_\pm \Delta u_\pm &= 0 && \text{in } \Omega_\pm(t), \ t > 0, \\ u_\pm V_n + \kappa_\pm \partial_n u_\pm &= 0 && \text{on } \Gamma(t), \ t > 0, \\ V_n &= H + \llbracket u \rrbracket && \text{on } \Gamma(t), \ t > 0, \\ \partial_n u_- &= 0 && \text{on } \partial C, \ t > 0. \end{aligned} \right\} \quad (1.1)$$

Here and throughout, we denote by $\llbracket \cdot \rrbracket$ the jump of the quantity in brackets across $\Gamma(t)$, e.g. $\llbracket u \rrbracket := u_+ - u_-$. Furthermore, $H = H(t)$ denotes the $(N - 1)$ -fold mean curvature of the surface $\Gamma(t)$, taken to be negative where $\Omega(t)$ is convex, and V_n is the normal velocity of the family $\{\Gamma(t); t > 0\}$ with respect to the unit normal field on $\Gamma(t)$ pointing outward $\Omega_+(t)$. In addition, the operator ∂_n denotes the derivative in the direction of the same normal field on $\Gamma(t)$, and in the normal direction exterior to C on ∂C .

For technical reasons we prefer to work with the equivalent formulation

$$\left. \begin{aligned} \partial_t u_{\pm} - \kappa_{\pm} \Delta u_{\pm} &= 0 && \text{in } \Omega_{\pm}(t), t > 0, \\ u_{\pm}(H + \llbracket u \rrbracket) + \kappa_{\pm} \partial_n u_{\pm} &= 0 && \text{on } \Gamma(t), t > 0, \\ V_n &= H + \llbracket u \rrbracket && \text{on } \Gamma(t), t > 0, \\ \partial_n u_- &= 0 && \text{on } \partial C, t > 0. \end{aligned} \right\} \quad (1.2)$$

The problem has to be complemented by appropriate initial conditions

$$u_{\pm}(0) = u_{\pm,0}, \quad \Gamma(0) = \Gamma_0. \quad (1.3)$$

We make the natural assumptions that $u_{\pm,0}$ is nonnegative and $u_{+,0} \not\equiv 0$.

Our problem can be interpreted as a multidimensional generalization of the closed osmometer problem that has been studied in one space dimension in [6, 12]. It also appears as a model for the osmotic swelling of egg cells [10, 14, 15]. Detailed descriptions of the modelling aspects can be found in [9, 10, 14–16]. Essentially, the motion of the membrane is governed by osmotic pressure determined by the concentration jump $\llbracket u \rrbracket$ of the solute across the interface and by surface tension. In our formulation the dimensionless parameters $\kappa_{\pm}$ can be interpreted as the ratio of the typical time scales of diffusion and relaxation of the membrane without solute.

We remark that the problem even has an intriguing variational structure which can be seen as a combination of the well-known variational structures of the diffusion equation (involving the entropy functional and the Wasserstein metric) and of mean curvature flow [17], but we will not use this here.

In [9], the one-phase version of problem (1.2), (1.3) (with $u_- \equiv 0$ and C replaced by $\mathbb{R}^N$) has been investigated. Existence and uniqueness of classical solutions are shown there for a large class of initial geometries. Moreover, solutions are shown to exist for arbitrary long time, provided they start sufficiently close to an equilibrium.

In the present paper we will prove a stronger result for the two-phase problem near equilibrium, namely, local attractivity of the manifold of equilibrium solutions of problem (1.2). It should be mentioned that a strategy of answering stability questions for problems of the above type has been established only recently in [11], where several versions of the Stefan problem are discussed. This relies on maximal regularity results for linear parabolic systems with inhomogeneous boundary data and so-called boundary dynamics of relaxation type, cf. [2], i.e. coupled systems of evolution equations inside the domain and on its boundary that are, in an appropriate sense, “jointly parabolic”.

It is worthwhile to mention the following qualitative difference between the one-phase and the two-phase version of the osmosis problem we discuss here: In the one-phase case, translational invariance of the problem can be used to modify the problem so that it has sufficiently many conserved quantities to determine a priori the equilibrium to which the evolution will converge. In this case, the convergence result can be proved in a considerably simpler fashion. Thus, the full power of the method used here is needed if and only if the two-phase problem is considered.

This paper is organized as follows: in Sect. 2 we use the Hanzawa transformation to reformulate the system (1.2) as a nonlinear PDE system given on a fixed time-space cylinder of infinite length in the time direction. In Sect. 3 we shall linearize at an equilibrium solution and study the spectral properties of the linearization. The proof of stability for the fully nonlinear problem is contained in Sect. 4.

The general approach to the proof of our result coincides with that given in [11]. In particular, we construct and apply local parameterizations for the phase manifold and its submanifold of equilibria (Lemmas 4.3–4.5 in the present paper) and use a splitting of the evolution problem in stable and neutral directions in the same way as in [11]. This leads to the identification of a fixed point problem (cf. 4.8) in a completely parallel fashion. Furthermore, our use of the maximal regularity results from [2] and spaces of functions with exponential time decay is also analogous.

However, our proof differs from the one given in [11] in a number of respects. Obviously, the discussion of the linearization and its spectral properties for the osmosis problem in Sect. 3 is independent from the parallel considerations for Stefan problems. Moreover, when constructing the parameterization of the submanifold of equilibria in Lemma 4.5 we provide a dimension argument whose counterpart for the Stefan problem is not expressly given in [11]. We also make some arguments more explicit by introducing and discussing Nemytskii operators. Furthermore, to solve (4.8) we do not apply a direct fixed point iteration but the Implicit Function theorem. This provides an additional result concerning smooth dependence of the solution on the initial data. In general, we tried to present the basic approach as clearly as possible. For this purpose, we listed the operators involved in the fixed point argument and formulated the crucial observation explicitly as Remark 4.6.

We close this section by collecting some basic observations concerning equilibria of (1.2), and by giving an informal version of our main result.

Our analysis is restricted to the case where the membrane is at a positive distance from the container boundary. In this case, all equilibria are given by spheres whose radius is uniquely determined by the concentration jump (observe, however, that the geometry of the container and the total amount of solute might exclude the existence of such equilibria).

Let ω_N denote the volume of the N -dimensional unit ball and $|C|$ the volume of C .

Lemma 1.1 (Equilibria). *Let $m_{\pm} := \int_{\Omega_{\pm}(0)} u_{\pm,0}$.*

- (i) *There is a unique $0 < R < R_{\max} := (|C|/\omega_N)^{1/N}$ such that*

$$(c_+, c_-) := (m_+ / (\omega_N R^N), m_- / (|C| - \omega_N R^N))$$

satisfies $c_+ - c_- = (N - 1)/R$.
- (ii) *If a (sufficiently smooth) solution of (1.2), (1.3) converges to an equilibrium solution $(u_{\infty,+}, u_{\infty,-}, \Gamma_\infty)$, then $u_{\infty,\pm} \equiv c_\pm$, and $\Gamma_\infty = \partial\mathbb{B}(x, R)$ for some $x \in C$ such that $\partial\mathbb{B}(x, R) \subset C$.*

(As our considerations here are still informal, we refrain from strictly defining the convergence in (ii). The convergence which is shown later will be sufficiently strong).

Proof. (i) Consider the equation $c_+ - c_- = (N - 1)/R$, i.e.

$$G(R) := \frac{m_+}{\omega_N R^{N-1}} - \frac{m_- R}{|C| - \omega_N R^N} - (N - 1) = 0.$$

Obviously, $\lim_{R \rightarrow 0} G(R) = +\infty$, while $\lim_{R \rightarrow R_{\max}} G(R) = -\infty$. Since $G' < 0$, (i) is proved.

- (ii) It follows from straightforward calculations and the strong elliptic maximum principle that equilibrium concentrations must be constant. Hence, Γ_∞ is a closed, compact hypersurface with constant mean curvature, i.e. a sphere. This, together with the observation that the quantities

$$m_\pm = \int_{\Omega_\pm(t)} u_\pm$$

are conserved in time, yields (ii). □

If $S_{R,\xi} \subset C$ is some fixed sphere of radius R with center ξ and $\rho : S_{R,\xi} \rightarrow \mathbb{R}$, let $S_{R,\xi,\rho}$ denote the image of the mapping $\theta : S_{R,\xi} \rightarrow \mathbb{R}^N$, $x \mapsto x + \rho(x)/R \cdot (x - \xi)$. By W_p^s with $s \notin \mathbb{N}$ we will denote fractional order Sobolev spaces (on various domains of definition), see Sect. 2.

Theorem 1.2. (Exponential stability) *Let $p > N + 2$ and $(c_+, c_-, S_{R,\xi})$ be an equilibrium solution of (1.2). Assume $\Gamma_0 = S_{R,\xi,\rho}$ and $\|\rho\|_{W_p^{3-3/p}(S_R)} + \|c_+ - u_{+,0}\|_{W_p^{2-2/p}(\Omega_+(0))} + \|c_- - u_{-,0}\|_{W_p^{2-2/p}(\Omega_-(0))}$ sufficiently small. If the compatibility conditions*

$$u_{\pm,0}(H(\Gamma_0) + \llbracket u_0 \rrbracket) + \kappa_\pm \partial_n u_{\pm,0} = 0 \text{ on } \Gamma_0, \quad \partial_n u_- = 0 \text{ on } \partial C$$

are satisfied, then there exists a unique global strong solution of (1.2), (1.3). It converges to some equilibrium $(\bar{c}_+, \bar{c}_-, S_{\bar{R},\bar{\xi}})$ at an exponential rate. The solution depends continuously differentiable on the initial geometry.

Observe that $\bar{c}_+$, $\bar{c}_-$ and $\bar{R}$ can be calculated from the initial data by Lemma 1.1 but that the center $\bar{\xi}$ of the approached equilibrium sphere is (apparently) not to be determined a priori by conserved quantities. For the details of the definition of a global strong solution we refer to Sect. 4, Definition 4.1. The precise formulation of the statements in Theorem 1.2 is given in Theorem 4.2.

2. Transformation to fixed domains

At this point we fix a domain C with smooth boundary ∂C and assume that there exist constants $\tilde{v}_+, \tilde{v}_-, R > 0$ and $x \in C$ with the two properties $\llbracket \tilde{v} \rrbracket = (N-1)/R$ and $\mathbb{B}(x, R) \subset C$. We shift and scale to obtain $x = 0$, $R = 1$. Letting

$$S := \partial \mathbb{B}(0, 1)$$

we find that $(\tilde{v}_\pm, S)$ is an equilibrium solution of (1.2), $\text{dist}(S, \partial C) > 0$ and

$$\llbracket \tilde{v} \rrbracket = N - 1. \quad (2.1)$$

Let $\Omega_+ := \mathbb{B}(0, 1) \subset \mathbb{R}^N$, $\Omega_- := C \setminus \bar{\Omega}_+$. We are going to transform system (1.2) into a set of equations given over $\Omega_\pm$ as fixed reference domains. The unknown family of surfaces $\{\Gamma(t)\}$ will be described by a signed distance function with respect to the unit sphere. The mapping

$$X : S \times (-1, 1) \rightarrow \mathbb{R}^N, \quad (x, s) \mapsto (1 + s)x,$$

is a smooth diffeomorphism onto its range. Fix $0 < a < 1$ small enough that $\bar{D} \subset C$, where $D := \text{range}(X|_{S \times (-a, a)})$. As it is convenient, we decompose the inverse of $X := X|_{S \times (-a, a)}$ into $X^{-1} = (P, \Lambda) : D \rightarrow S \times (-a, a)$, where P is the metric projection onto S , and Λ is the signed distance function with respect to S , i.e. $P(x) = x/|x|$, $\Lambda(x) = |x| - 1$. Let $\tilde{a} \in (0, a/4)$ and

$$\text{Ad} := \{\sigma \in C^1(S); \|\sigma\|_{C(S)} < \tilde{a}\}.$$

It is well-known that, given $\sigma \in \text{Ad}$, the mapping $\theta_\sigma(x) := (1 + \sigma(x))x$ is a diffeomorphism mapping S onto $S_\sigma := \theta_\sigma[S]$. We extend this diffeomorphism to the whole of $\mathbb{R}^N$: let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ satisfy $\chi|_{[-\tilde{a}, \tilde{a}]} \equiv 1$, $\chi|_{(-\infty, -3\tilde{a}]} \equiv 0$, $\chi|_{[3\tilde{a}, \infty)} \equiv 0$, $\|\chi'\|_\infty < 1/\tilde{a}$. Then the mapping

$$y \mapsto \begin{cases} X(P(y), \Lambda(y) + \chi(\Lambda(y)) \cdot \sigma(P(y))), & \text{if } y \in D \\ y, & \text{if } y \notin D \end{cases} \quad (2.2)$$

($\sigma \in \text{Ad}$), again denoted by θ_σ , is an appropriate extension, the so-called Han-zawa diffeomorphism. We have $\theta_\sigma \in \text{Diff}(\mathbb{R}^N, \mathbb{R}^N)$. Moreover, $\theta_\sigma \equiv \text{id}$ outside D , in particular in a sufficiently small open neighborhood of ∂C . Moreover, denoting by $\Omega_{\sigma,+}$ the domain enclosed by S_σ and letting $\Omega_{\sigma,-} := C \setminus \bar{\Omega}_{\sigma,+}$, we have that

$$\theta_\sigma|_{\Omega_\pm} \in \text{Diff}(\Omega_\pm, \Omega_{\sigma,\pm}),$$

$\sigma \in \text{Ad}$, and $\partial \Omega_{\sigma,+} = S_\sigma$, $\partial \Omega_{\sigma,-} = S_\sigma \cup \partial C$. Finally note that the surface S_σ is the zero level set of the function φ_σ defined by

$$\varphi_\sigma(x) = \Lambda(x) - \sigma(P(x)),$$

$x \in D$, $\sigma \in \text{Ad}$, i.e. $S_\sigma = \varphi_\sigma^{-1}[\{0\}]$. For later use we set

$$L_\sigma(x) := |\nabla \varphi_\sigma|(\theta_\sigma(x)).$$

It can be shown that $L_\sigma > 0$ on S for all $\sigma \in \text{Ad}$.

Given $\sigma \in \text{Ad}$, let θ_σ^* , θ_σ^σ denote the pull-back and push-forward operators induced by θ_σ , i.e. $\theta_\sigma^* f = f \circ \theta_\sigma$, $\theta_\sigma^\sigma g = g \circ \theta_\sigma^{-1}$. If suitable functions b, ρ are time

dependent, i.e. $b = b(t, x)$, $\rho = \rho(t, x)$, we define $[\theta_\rho^* b](t, x) := [\theta_{\rho(t)}^* b(t, \cdot)](x)$, analogue for θ_ρ^* .

Using this notation, for $\rho : J \subset [0, \infty) \rightarrow \text{Ad} \cap C^2(S)$ and sufficiently smooth $w_\pm \in \mathbb{R}^{\Omega_\pm}$ we introduce the transformed operators

$$\begin{aligned}\mathcal{A}_\pm(\rho)w_\pm &:= \theta_\rho^*(\Delta(\theta_\rho^* w_\pm)) \\ \mathcal{B}_\pm(\rho)w_\pm &:= \theta_\rho^*(\nabla(\theta_\rho^* w_\pm) \cdot \nabla \varphi_\rho / |\nabla \varphi_\rho|) \\ H(\rho) &:= \theta_\rho^* H_{[S_\rho]}.\end{aligned}$$

Letting $v_\pm := u_\pm \circ \theta_\rho$, $v_{\pm,0} := u_{\pm,0} \circ \theta_{\rho_0}$, instead of (1.2), (1.3), we study the following problem on $\Omega_\pm$ as fixed reference domains:

$$\left. \begin{aligned} \partial_t v_\pm - \kappa_\pm \mathcal{A}_\pm(\rho) v_\pm - \mathcal{R}_\pm(v_\pm, \rho) &= 0 & \text{in } \Omega_\pm \times (0, \infty), \\ \kappa_\pm \mathcal{B}_\pm(\rho) v_\pm + v_\pm(H(\rho) + \llbracket v \rrbracket) &= 0 & \text{on } S \times (0, \infty), \\ \partial_n v_- &= 0 & \text{on } \partial C \times (0, \infty), \\ \partial_t \rho - L_\rho H(\rho) - L_\rho \llbracket v \rrbracket &= 0 & \text{on } S \times (0, \infty), \\ v_\pm(0) &= v_{\pm,0} & \text{in } \Omega_\pm, \\ \rho(0) &= \rho_0 & \text{on } S. \end{aligned} \right\} \quad (2.3)$$

The terms $\mathcal{R}_\pm$ arise from the transformation of the time derivative v_t and are determined by

$$\mathcal{R}_\pm(w_\pm, \sigma)(y) = r_0(L_\sigma[H(\sigma) + \llbracket w \rrbracket], B_\mu(\sigma)w_\pm)(y), \quad y \in \Omega_\pm,$$

where $w_\pm \in C^1(\bar{\Omega}_\pm)$, $\sigma \in \text{Ad}$ and

$$r_0(h, k)(y) := \begin{cases} \chi(\Lambda(y)) \cdot h(P(y)) \cdot k(y), & \text{if } y \in D \\ 0, & \text{if } y \in \bar{C} \setminus D, \end{cases} \quad (2.4)$$

$$B_\mu(\sigma)w_\pm(y) = \theta_\sigma^* \nabla(\theta_\sigma^* w_\pm)(y) \cdot (\mu_S \circ P)(y), \quad y \in \Omega_\pm$$

(μ_S being the exterior unit normal field of S). The explicit calculation of $\mathcal{R}_\pm$ is straightforward, cf. again [5].

Let $p > N + 2$ and $T > 0$. For $s \geq 0$ and a Banach space Y , for $M \in \{\Omega_\pm, S, J\}$, $J \in \{[0, T], \mathbb{R}^+\}$, $\mathbb{R}^+ := [0, \infty)$, we denote by $W_p^s(M, Y)$ the L^p -based Sobolev space of order s . In particular, if $s \notin \mathbb{N}$, this fractional-order Sobolev space coincides with the Besov space $B_{pp}^s(M, Y)$ (cf. [13]). For the sake of brevity we write $W_p^s(M) := W_p^s(M, \mathbb{R})$.

We close this section by an observation which is crucial for our analysis: we consider the time independent problem

$$\left. \begin{aligned} -\kappa_\pm \mathcal{A}_\pm(\rho) v_\pm &= \mathcal{R}_\pm(v_\pm, \rho) & \text{in } \Omega_\pm, \\ \kappa_\pm \mathcal{B}_\pm(\rho) v_\pm + v_\pm(H(\rho) + \llbracket v \rrbracket) &= 0 & \text{on } S, \\ \partial_n v_- &= 0 & \text{on } \partial C, \\ -H(\rho) &= \llbracket v \rrbracket & \text{on } S. \end{aligned} \right\} \quad (2.5)$$

and the set

$$\tilde{\mathcal{E}} := \{e := (\eta_+, \eta_-, \varepsilon) \in W_p^2(\Omega_+) \times W_p^2(\Omega_-) \times W_p^{3-1/p}(S); e \text{ is a solution of (2.5)}\}.$$

Lemma 2.1. *In a neighborhood of $(\tilde{v}_+, \tilde{v}_-, 0)$, $\tilde{\mathcal{E}}$ is a smooth $(N+2)$ -dimensional submanifold of $W_p^2(\Omega_+) \times W_p^2(\Omega_-) \times W_p^{3-1/p}(S)$.*

Proof. As $W_p^{3-1/p}(S) \hookrightarrow C(S)$, we may assume that $\tilde{\mathcal{E}} \cap U \subset W_p^2(\Omega_+) \times W_p^2(\Omega_-) \times (W_p^{3-1/p}(S) \cap \text{Ad})$, U being a sufficiently small neighborhood of $(\tilde{v}_+, \tilde{v}_-, 0)$ in $W_p^2(\Omega_+) \times W_p^2(\Omega_-) \times W_p^{3-1/p}(S)$. Observe that $\mathcal{R}_\pm(e) = 0$ for any $e \in \tilde{\mathcal{E}} \cap U$. Moreover, since $\mathcal{A}(\sigma)$ is uniformly elliptic and $\mathcal{B}(\sigma)$ is nontangential for any $\sigma \in \text{Ad}$, we conclude from the strong elliptic maximum principle that $\eta_\pm \equiv \text{const}$ for any $e = (\eta_+, \eta_-, \varepsilon) \in \tilde{\mathcal{E}}$ close enough to $(\tilde{v}_+, \tilde{v}_-, 0)$. Thus, $H(\varepsilon) = H(\theta_\varepsilon[S]) \equiv -\llbracket \eta \rrbracket$. Therefore, $\theta_\varepsilon[S]$ is a sphere of radius $(N-1)/\llbracket \eta \rrbracket$. On the other hand, if $\eta_\pm \equiv \text{const}$, $\llbracket \eta \rrbracket = (N-1)/R$ and $\theta_\varepsilon[S]$ is a sphere of radius R , then $(\eta_+, \eta_-, \varepsilon) \in U$ belongs to $\tilde{\mathcal{E}}$. This shows that we can use arguments parallel to those in the proof of Proposition 6.4 in [3] to see that if $\gamma > 0$ is small enough, the mapping

$$\mathcal{P}_e : \mathbb{B}((0_{\mathbb{R}^N}, 0, \tilde{v}_-), \gamma) \subset \mathbb{R}^{N+2} \rightarrow \tilde{\mathcal{E}} \subset W_p^2(\Omega_+) \times W_p^2(\Omega_-) \times W_p^{3-1/p}(S)$$

given by

$$\mathcal{P}_e(z_1, \dots, z_N, r, c_-) = \left(c_+, c_-, \sum_{k=1}^N z_k Y_k - 1 + \sqrt{\left(\sum_{k=1}^N z_k Y_k \right)^2 + (1+r)^2 - \sum_{k=1}^N z_k^2} \right),$$

$c_+ := c_- + \frac{N-1}{1+r}$, is a local parameterization of $\tilde{\mathcal{E}}$ near $(\tilde{v}_+, \tilde{v}_-, 0)$. Here, $Y_k : S \rightarrow \mathbb{R}$, $k = 1, \dots, N$, denote the coordinate functions $x \mapsto x_k$. $\square$

3. Linearization at equilibria and spectral analysis

For the sake of brevity, we write from now $m := N - 1$.

In this section we are interested in the eigenvalue problem

$$\left. \begin{aligned} \lambda w_\pm - \kappa_\pm \Delta w_\pm &= 0 && \text{in } \Omega_\pm, \\ \tilde{v}_\pm (\llbracket w \rrbracket + \Delta_S \sigma + m\sigma) + \kappa_\pm \partial_n w_\pm &= 0 && \text{on } S, \\ \partial_n w_- &= 0 && \text{on } \partial C, \\ \lambda \sigma - (\Delta_S \sigma + m\sigma) - \llbracket w \rrbracket &= 0 && \text{on } S. \end{aligned} \right\} \quad (3.1)$$

It corresponds to the linearization of (2.3) at the equilibrium $(\tilde{v}_\pm, 0)$. This will be worked out in detail at the beginning of Sect. 4. Observe that the second and the fourth equation of (3.1) can be equivalently rewritten as

$$\left. \begin{aligned} \kappa_\pm \partial_n w_\pm &= -\tilde{v}_\pm \lambda \sigma && \text{on } S, \\ \llbracket w \rrbracket &= \lambda \sigma - (\Delta_S \sigma + m\sigma) && \text{on } S. \end{aligned} \right\} \quad (3.2)$$

We want to consider the eigenvalue problem (3.1) as an abstract operator equation. For this let

$$X := L^p(\Omega_+) \times L^p(\Omega_-) \times W_p^{1-1/p}(S), \quad \alpha_\pm := \kappa_\pm / \tilde{v}_\pm.$$

We formally introduce an operator $\hat{L}$ by

$$\hat{L}(w_+, w_-, \sigma) = (\kappa_+ \Delta w_+, \kappa_- \Delta w_-, -\alpha_+ \partial_n w_+).$$

This operator can be defined on various subsets of X . Of particular interest will be the set

$$\begin{aligned} D(L) := & \{(w_+, w_-, \sigma) \in W_p^2(\Omega_+) \\ & \times W_p^2(\Omega_-) \times W_p^{3-1/p}(S); \\ & \partial_n w_- = 0 \text{ on } \partial C; \\ & (\Delta_S + m)\sigma + \llbracket w \rrbracket + \alpha_+ \partial_n w_+ = 0 \text{ on } S; \\ & \alpha_+ \partial_n w_+ = \alpha_- \partial_n w_- \text{ on } S\}, \end{aligned}$$

which defines the unbounded linear operator $L : D(L) \subset X \rightarrow X$, $L(z) = \hat{L}(z)$, where $z = (w_+, w_-, \sigma)$. Our goal is to characterize the spectrum of L . We observe:

$$L(w_+, w_-, \sigma) = \lambda(w_+, w_-, \sigma) \quad \text{iff} \quad (3.1) \text{ holds true.}$$

Lemma 3.1. (i) *The spectrum of L consists purely of isolated eigenvalues having eigenspaces of finite dimension.*

(ii) *The value $\lambda = 0$ is an eigenvalue of L with $\dim \mathcal{N}(L) = N + 2$.*

(iii) *All other eigenvalues of L are real and negative.*

Corollary 3.2. *We have*

$$\sup \operatorname{Re}(\sigma(L) \setminus \{0\}) < 0.$$

Proof. (i) The operator L has nonempty resolvent set (Theorem 2.2 in [2]) and its domain of definition is compact in X . Hence the statement follows by Kato [7], Theorem III. 6.29.

(ii) Suppose $L(w_+, w_-, \sigma) = 0$ for $(w_+, w_-, \sigma) \in D(L)$. This implies that $w_\pm$ are constant on $\Omega_\pm$, and $\llbracket w \rrbracket$ is constant on S . From elementary properties of Δ_S we get

$$\mathcal{N}(L) = \operatorname{span}\{(m, 0, -1), (0, m, 1), (0, 0, x_1), \dots, (0, 0, x_N)\}. \quad (3.3)$$

(iii) On X we define the bilinear form $\langle \cdot, \cdot \rangle$ by

$$\begin{aligned} \langle (w_+, w_-, \sigma), (u_+, u_-, \theta) \rangle = & \tilde{v}_- \int_{\Omega_+} w_+ u_+ dx + \tilde{v}_+ \int_{\Omega_-} w_- u_- dx \\ & + \tilde{v}_+ \tilde{v}_- \int_S \sigma (-\Delta_S - m) \theta dS. \end{aligned}$$

Let $(\bar{w}_+, \bar{w}_-, \bar{\sigma})$ denote the componentwise complex conjugate of (w_+, w_-, σ) . By Green's formulas and the definition of $D(L)$,

$$\begin{aligned} -\langle L(w_+, w_-, \sigma), (\bar{w}_+, \bar{w}_-, \bar{\sigma}) \rangle = & -\tilde{v}_- \kappa_+ \int_{\Omega_+} \Delta w_+ \bar{w}_+ dx \\ & -\tilde{v}_+ \kappa_- \int_{\Omega_-} \Delta w_- \bar{w}_- dx \\ & + \tilde{v}_+ \tilde{v}_- \alpha_+ \int_S \partial_n w_+ (\alpha_+ \partial_n \bar{w}_+ + \llbracket \bar{w} \rrbracket) dS \end{aligned}$$

$$\begin{aligned}
&= \tilde{v}_- \kappa_+ \|\nabla w_+\|_{L^2(\Omega_+)}^2 + \tilde{v}_+ \kappa_- \|\nabla w_-\|_{L^2(\Omega_-)}^2 \\
&\quad + \kappa_+ \alpha_+ \tilde{v}_- \|\partial_n w_+\|_{L^2(S)}^2 \geq 0
\end{aligned} \tag{3.4}$$

for all $(w_+, w_-, \sigma) \in D(L)$.

Let now $\lambda \neq 0$ be a (possibly complex) eigenvalue of L and $(w_+, w_-, \sigma) \in D(L) \setminus \{0\}$ a corresponding eigenvector. Let

$$\underline{w}_\pm = \frac{1}{|\Omega_\pm|} \int_{\Omega_\pm} w_\pm dx, \quad \underline{\sigma} = \frac{1}{|S|} \int_S \sigma dS$$

be the averages of $w_\pm$ and σ . Define further $\tilde{w}_\pm = w_\pm - \underline{w}_\pm$, $\tilde{\sigma} = \sigma - \underline{\sigma}$ so that $\tilde{w}_\pm \perp \underline{w}_\pm$, $\tilde{\sigma} \perp \underline{\sigma}$ in $L^2(\Omega_\pm)$ and $L^2(S)$, respectively.

We have

$$\begin{aligned}
\lambda \underline{\sigma} &= -\frac{\alpha_\pm}{|S|} \int_S \partial_n w_\pm dS = \mp \frac{\alpha_\pm}{|S|} \int_{\Omega_\pm} \Delta w_\pm dx = \mp \frac{\alpha_\pm}{|S|} \lambda \int_{\Omega_\pm} w_\pm dx \\
&= \mp \tilde{v}_\pm^{-1} \frac{|\Omega_\pm|}{|S|} \lambda \underline{w}_\pm
\end{aligned}$$

and as $\lambda \neq 0$

$$|\Omega_\pm| |\underline{w}_\pm|^2 = \tilde{v}_\pm^2 \frac{|S|^2}{|\Omega_\pm|} |\underline{\sigma}|^2. \tag{3.5}$$

Observe further that $|S|/|\Omega_+| = N > 1$ and $\tilde{v}_+ = \tilde{v}_- + m$. By these observations, orthogonality, and the spectral properties of Δ_S we have

$$\begin{aligned}
\langle (w_+, w_-, \sigma), (\bar{w}_+, \bar{w}_-, \bar{\sigma}) \rangle &= \tilde{v}_- \|\tilde{w}_+\|_{L^2(\Omega_+)}^2 + \tilde{v}_+ \|\tilde{w}_-\|_{L^2(\Omega_-)}^2 \\
&\quad + v_+ v_- \int_S \tilde{\sigma} (-\Delta_S - m) \bar{\tilde{\sigma}} dS \\
&\quad + v_- |\Omega_+| |\underline{w}_+|^2 + v_+ |\Omega_-| |\underline{w}_-|^2 \\
&\quad - m v_+ v_- |S| |\underline{\sigma}|^2 \\
&= \tilde{v}_- \|\tilde{w}_+\|_{L^2(\Omega_+)}^2 + \tilde{v}_+ \|\tilde{w}_-\|_{L^2(\Omega_-)}^2 \\
&\quad + v_+ v_- \int_S \tilde{\sigma} (-\Delta_S - m) \bar{\tilde{\sigma}} dS \\
&\quad + v_+ v_- |S| \left((\tilde{v}_- + m) \frac{|S|}{|\Omega_+|} \right. \\
&\quad \left. + \tilde{v}_- \frac{|S|}{|\Omega_-|} - m \right) |\underline{\sigma}|^2 \geq 0
\end{aligned}$$

with equality only if $(w_+, w_-, \sigma) \in \mathcal{N}(L)$ which is excluded by $\lambda \neq 0$. Hence we have strict positivity. On the other hand, by (3.4) we find

$$-\lambda \langle (w_+, w_-, \sigma), (\bar{w}_+, \bar{w}_-, \bar{\sigma}) \rangle = -\langle L(w_+, w_-, \sigma), (\bar{w}_+, \bar{w}_-, \bar{\sigma}) \rangle \geq 0,$$

hence $\lambda < 0$. □

Lemma 3.3. *The zero eigenvalue of L is semisimple, i.e. $X = \mathcal{N}(L) \oplus \mathcal{R}(L)$.*

Proof. For $i = 1, \dots, N$, let $p_i \in W_2^1(\Omega_-)$ be a solution of the BVP

$$\Delta p_i = 0 \quad \text{in } \Omega_-, \quad \partial_n p_i = x_i \quad \text{on } S, \quad \partial_n p_i = 0 \quad \text{on } \partial C. \quad (3.6)$$

(The solvability of this problem follows from $\int_S x_i dS = 0$.) Define the auxiliary map $\Phi \in \text{Hom}(X, \mathbb{R}^{N+2})$ by

$$\begin{aligned} \Phi(f_+, f_-, g) &= (\Phi_+, \Phi_-, \Phi_1, \dots, \Phi_N)(f_+, f_-, g), \\ \Phi_\pm(f_+, f_-, g) &= \int_{\Omega_\pm} f_\pm dx \pm \tilde{v}_\pm \int_S g dS, \\ \Phi_i(f_+, f_-, g) &= \kappa_+^{-1} \int_{\Omega_+} x_i f_+ dx + \kappa_-^{-1} \int_{\Omega_-} p_i f_- dx \\ &\quad + (\alpha_+^{-1} + 1) \int_S x_i g dS - \alpha_-^{-1} \int_S p_i g dS, \end{aligned}$$

$i = 1, \dots, N$. We claim

$$\Phi|_{\mathcal{N}(L)} \in \text{Isom}(\mathcal{N}(L), \mathbb{R}^{N+2}), \quad (3.7)$$

$$\Phi|_{\mathcal{R}(L)} = 0. \quad (3.8)$$

Indeed, with respect to the basis of $\mathcal{N}(L)$ given in (3.3) (and the standard basis of $\mathbb{R}^{N+2}$) $\Phi|_{\mathcal{N}(L)}$ has a matrix representation

$$\begin{pmatrix} A & 0 \\ * & C + D \end{pmatrix},$$

where

$$A = \begin{pmatrix} m|\Omega_+| - \tilde{v}_+|S| & \tilde{v}_+|S| \\ \tilde{v}_-|S| & m|\Omega_-| - \tilde{v}_-|S| \end{pmatrix}.$$

It follows from $\tilde{v}_\pm > 0$, $[\tilde{v}] = m = N - 1$, $|\Omega_+| = |S|N$ that $\det(A) < 0$, i.e. A is regular. Furthermore,

$$\begin{aligned} C &= (c_{ij})_{i,j=1,\dots,N}, \quad D = (d_{ij})_{i,j=1,\dots,N}, \\ c_{ij} &= (\alpha_+^{-1} + 1) \int_S x_i x_j dS, \\ d_{ij} &= -\alpha_-^{-1} \int_S p_i x_j dS = -\alpha_-^{-1} \int_S p_i \partial_n p_j dS = \alpha_-^{-1} \int_{\Omega_-} \nabla p_i \cdot \nabla p_j dx, \end{aligned}$$

where (3.6) and Green's identity have been used. So, the matrix C is a positive multiple of the identity matrix, and for $\xi \in \mathbb{R}^N$ we have, summing over double indices,

$$\xi^i \xi^j d_{ij} = \alpha_-^{-1} \int_{\Omega_-} \nabla(\xi^i p_i) \cdot \nabla(\xi^j p_j) dx \geq 0,$$

i.e. D is positive semidefinite. Therefore $C + D$ is positive definite, and hence invertible. This proves (3.7).

To show (3.8), fix $(f_+, f_-, g) \in \mathcal{R}(L)$. Then

$$(f_+, f_-, g) = (\kappa_+ \Delta w_+, \kappa_- \Delta w_-, -\alpha_+ \partial_n w_+)$$

for some $(w_+, w_-, \sigma) \in D(L)$. Using Green's identity, (3.6), the conditions defining $D(L)$, and the symmetry of Δ_S we get

$$\begin{aligned}\Phi_{\pm}(f_+, f_-, g) &= \kappa_{\pm} \left(\int_{\Omega_{\pm}} \Delta w_{\pm} dx \mp \int_S \partial_n w_{\pm} dS \right) = 0, \\ \Phi_i(f_+, f_-, g) &= \int_{\Omega_+} x_i \Delta w_+ dx + \int_{\Omega_-} p_i \Delta w_- dx \\ &\quad - (\alpha_+ + 1) \int_S x_i \partial_n w_+ dS + \int_S p_i \partial_n w_- dS \\ &= \int_S (x_i \partial_n w_+ - \partial_n x_i w_+) dS - \int_S (p_i \partial_n w_- - \partial_n p_i w_-) dS \\ &\quad - (\alpha_+ + 1) \int_S x_i \partial_n w_+ dS + \int_S p_i \partial_n w_- dS \\ &= - \int_S x_i (\llbracket w \rrbracket + \alpha_+ \partial_n w_+) dS = \int_S x_i (\Delta_S + m) \sigma dS \\ &= \int_S (\Delta_S + m) x_i \sigma dS = 0.\end{aligned}$$

Hence (3.8) is proved.

Observe now that by (3.7) the mapping $P := (\Phi|_{\mathcal{N}(L)})^{-1} \circ \Phi$ is a projection of X onto $\mathcal{N}(L)$, hence

$$X = \mathcal{N}(L) \oplus K,$$

where $K = \mathcal{N}(P) = \mathcal{N}(\Phi)$. By (3.8) we have $\mathcal{R}(L) \subseteq K$. As L has nonempty resolvent set and $D(L)$ is compactly embedded in X , L (considered as bounded operator from $D(L)$ to X) is Fredholm and has index zero. Therefore

$$\dim X/\mathcal{R}(L) = \dim \mathcal{N}(L) = \dim X/K = N + 2,$$

and thus

$$\dim K/\mathcal{R}(L) = \dim X/\mathcal{R}(L) - \dim X/K = 0,$$

hence $K = \mathcal{R}(L)$. □

Remark The projection P constructed in the proof is the spectral projection corresponding to the eigenvalue $\lambda = 0$ of L , see Lunardi [8], Appendix A.

We introduce the notation

- $X_s := \mathcal{R}(L)$, $X_c := \mathcal{N}(L)$;
- $P^s := P$, $P^c := \text{id} - P$.

Observe that we have $P^s|_{X_c} = 0$, $P^c|_{X_s} = 0$, $P^s|_{X_s} = \text{id}$, $P^c|_{X_c} = \text{id}$.

4. Nonlinear stability

Let $T > 0$, $p > N + 2$ and $s \geq 0$. Let Y be a Banach space and $J \in \{[0, T], \mathbb{R}^+\}$. Our further analysis will use the following function spaces:

- $W_p^{s,s/2}(J, M) := L_p(J, W_p^s(M)) \cap W_p^{s/2}(J, L_p(M))$, $M \in \{\Omega_\pm, S\}$;
- $\tilde{W}_p^{s,s/2}(J, \Omega_\pm) := \{u_\pm \in W_p^{s,s/2}(J, \Omega_\pm); \partial_n u_\pm = 0 \text{ on } \partial C\}$,
 $s > 1 + (N+2)/p$; and for $\delta \geq 0$
- $W_p^s(J, \delta, Y) := \{f; e^{\delta t} f \in W_p^s(J, Y)\}$;
- $F(J, \delta, M) := \{f; e^{\delta t} f \in F(J, M)\}$,
 $(F, M) \in \{W_p^{s,s/2} \times \{\Omega_\pm, S\}, \tilde{W}_p^{s,s/2} \times \{\Omega_\pm\}\}$.

Finally,

- $E := \{(u_\pm, \rho) \in W_p^{2-2/p}(\Omega_\pm) \times W_p^{3-3/p}(S); \partial_n u_- = 0 \text{ on } \partial C\}$;
- $E_1 := \{(u_\pm, \rho) \in W_p^2(\Omega_\pm) \times W_p^{3-1/p}(S); \partial_n u_- = 0 \text{ on } \partial C\}$;
- $X := L_p(\Omega_\pm) \times W_p^{1-1/p}(S)$;
- $\mathbb{E}(\delta) := \tilde{W}_p^{2,1}(\mathbb{R}^+, \delta, \Omega_\pm) \times W_p^{3-1/p, (3-1/p)/2}(\mathbb{R}^+, \delta, S)$;
- $\mathbb{F}(\delta) := L_p(\mathbb{R}^+, \delta, \Omega_\pm) \times [W_p^{1-1/p, (1-1/p)/2}(\mathbb{R}^+, \delta, S)]^3$;
- $\mathbb{X}(\delta) := \{f; e^{\delta t} f \in L_p(\mathbb{R}^+, D(L)) \cap W_p^1(\mathbb{R}^+, X)\}$.

Here and in the sequel, we use the short notation $(u_\pm, \rho) := (u_+, u_-, \rho)$ and $F(\Omega_\pm) := F(\Omega_+) \times F(\Omega_-)$, F being a space of functions defined on Ω_+ , Ω_- , respectively.

Recall the standard embedding result

$$W_p^{s,s/2}(J, M) \hookrightarrow W_p^\tau(J, W_p^\sigma(M)), \quad M \in \{\Omega_\pm, S\},$$

whenever $2\tau + \sigma \leq s$. Moreover,

$$W_p^{s,s/2}(J, \delta, M) \hookrightarrow C(J, W_p^{s-2/p}(M)), \quad 2/p < s < 4, \quad M \in \{\Omega_\pm, S\}.$$

cf. [4], Remark 5.3 d). Therefore, for the time trace operator at $t = 0$ we have

$$(u_\pm, \rho) \mapsto (u_\pm, \rho)(\cdot, 0) \in \mathcal{L}(\mathbb{E}(\delta), E).$$

Moreover, if $\|\rho\|_{W_p^{3-1/p, (3-1/p)/2}(\mathbb{R}^+, \delta, S)}$ is small enough, we may assume that $\rho[\mathbb{R}^+] \subset \text{Ad} \cap W_p^{3-3/p}(S) \subset \text{Ad} \cap C^2(S)$.

We fix $\delta > 0$ and set $u_\pm := v_\pm - \tilde{v}_\pm$, $u_{\pm,0} := v_{\pm,0} - \tilde{v}_\pm$.

Linearizing our problem (2.3) around the equilibrium $(\tilde{v}_\pm, 0)$, we rewrite it equivalently as

$$\left. \begin{aligned} \partial_t u_\pm - \kappa_\pm \Delta u_\pm &= F_\pm(u_\pm, \rho) && \text{in } \Omega_\pm \times (0, \infty), \\ \tilde{v}_\pm(\llbracket u \rrbracket + \Delta S \rho + m \rho) + \kappa_\pm \partial_n u_\pm &= G_\pm(u_\pm, \rho) && \text{on } S \times (0, \infty), \\ \partial_n u_- &= 0 && \text{on } \partial C \times (0, \infty), \\ \partial_t \rho - (\Delta S \rho + m \rho) - \llbracket u \rrbracket &= \tilde{H}(u_\pm, \rho) && \text{on } S \times (0, \infty), \\ u_\pm(0) &= u_{\pm,0} && \text{in } \Omega_\pm, \\ \rho(0) &= \rho_0 && \text{on } S, \end{aligned} \right\}$$

where $(F_\pm, G_\pm, \tilde{H})$ represent the nonlinear remainder terms. Therefore we have

$$(F_\pm, G_\pm, \tilde{H})(0) = 0, \quad (F_\pm, G_\pm, \tilde{H})'(0) = 0.$$

Moreover, it is not difficult to verify (cf. [9], Lemma 4.2) that $(F_\pm, G_\pm, \tilde{H}) \in C^1(\mathbb{B}_{\mathbb{E}(\delta)}(0, \gamma), \mathbb{F}(\delta))$, for $\gamma > 0$ small enough. (We will use additional mapping properties of the nonlinearities in the proof of Lemma 4.4 but refrain from stating them here for the sake of notational efficiency).

Recall the notation $\alpha_{\pm} := \kappa_{\pm}/\tilde{v}_{\pm}$. We want to consider the equivalent system

$$\left. \begin{aligned} \partial_t u_{\pm} - \kappa_{\pm} \Delta u_{\pm} &= F_{\pm}(u_{\pm}, \rho) && \text{in } \Omega_{\pm} \times (0, \infty), \\ \llbracket u \rrbracket + \Delta_S \rho + m\rho + \alpha_+ \partial_n u_+ &= G_1(u_{\pm}, \rho) && \text{on } S \times (0, \infty), \\ \alpha_+ \partial_n u_+ - \alpha_- \partial_n u_- &= G_2(u_{\pm}, \rho) && \text{on } S \times (0, \infty), \\ \partial_n u_- &= 0 && \text{on } \partial C \times (0, \infty), \\ \partial_t \rho + \alpha_+ \partial_n u_+ &= H(u, \rho) && \text{on } S \times (0, \infty), \\ u_{\pm}(0) &= u_{\pm,0} && \text{in } \Omega_{\pm}, \\ \rho(0) &= \rho_0 && \text{on } S, \end{aligned} \right\} \quad (4.1)$$

where

$$\begin{aligned} G_1(u_{\pm}, \rho) &= G_+(u_{\pm}, \rho)/\tilde{v}_+; \\ G_2(u_{\pm}, \rho) &= G_+(u_{\pm}, \rho)/\tilde{v}_+ - G_-(u_{\pm}, \rho)/\tilde{v}_-; \\ H(u_{\pm}, \rho) &= \tilde{H}(u_{\pm}, \rho) + G_1(u_{\pm}, \rho). \end{aligned}$$

Obviously,

$$G_i(0) = 0, \quad G'_i(0) = 0, \quad i = 1, 2,$$

thus $H(0) = 0$, $H'(0) = 0$. Letting

$$B(u_{\pm}, \rho) := (\llbracket u \rrbracket + \Delta_S \rho + m\rho + \alpha_+ \partial_n u_+, \llbracket \alpha \partial_n u \rrbracket),$$

$\mathbf{F} := (F_{\pm}, H)$, $\mathbf{G} := (G_1, G_2)$ and $\mu := (u_{\pm}, \rho)$, $\mu_0 := (u_{\pm,0}, \rho_0)$, the last problem can be written more compactly as

$$(\partial_t - \hat{L}, B)\mu = (\mathbf{F}, \mathbf{G})(\mu), \quad \mu(0) = \mu_0. \quad (4.2)$$

Solutions of (4.2) are paths within the state manifold

$$\mathcal{M} = \{\mu \in E; B\mu = \mathbf{G}(\mu)\}$$

whose tangent space in 0 is

$$\mathcal{Z} = \{z \in E; Bz = 0\}$$

Observe that $\mathcal{N}(L) \subset D(L) = \mathcal{Z} \cap E_1 \subset \mathcal{Z} \subset E \subset X$. The set of equilibria of (4.2) is given by

$$\mathcal{E} := \tilde{\mathcal{E}} - (\tilde{v}_{\pm}, 0) = \{e \in E_1; -\hat{L}e = \mathbf{F}(e); Be = \mathbf{G}(e)\} \subset \mathcal{M}.$$

We can now give a precise statement of our main result:

Definition 4.1. A global strong solution of the evolution problem (4.2) is a solution $\mu = (u_{\pm}, \rho) : [0, \infty) \rightarrow E$ such that

$$\mu|_{[0,T]} \in \tilde{W}_p^{2,1}([0,T], \Omega_{\pm}) \times W_p^{3-1/p, (3-1/p)/2}([0,T], S) \quad \forall T > 0.$$

If $\mu = (u_{\pm}, \rho)$ is a global strong solution of (4.2), $(\theta_*^p(u_{\pm} + \tilde{v}_{\pm}), \theta_{\rho}[S])$ is said to be a global strong solution of (1.2), (1.3).

Theorem 4.2. There exist $\gamma, \delta > 0$ such that, given $\mu_0 \in \mathbb{B}_E(0, \gamma) \cap \mathcal{M}$, problem (4.2) admits a unique global strong solution $\mu = \xi + e$, where $(\xi, e) \in \mathbb{E}(\delta) \times \mathcal{E}$. Moreover, $\mu_0 \mapsto (\xi, e) \in C^1(\mathbb{B}_E(0, \gamma) \cap \mathcal{M}, \mathbb{E}(\delta) \times \mathcal{E})$.

In order to parameterize the manifold $\mathcal{M}$ over the tangent space $\mathcal{Z}$ we need the following

Lemma 4.3. *Let $\omega > 0$. Then*

$$(\omega - \hat{L}, B) \in \mathcal{L}_{is}(E, W_p^{-2/p}(\Omega_{\pm}) \times [W_p^{1-3/p}(S)]^3) \\ \cap \mathcal{L}_{is}(E_1, L_p(\Omega_{\pm}) \times [W_p^{1-1/p}(S)]^3).$$

In accordance with our general notation, we denote by $W_p^{-2/p}(\Omega_{\pm}) := B_{pp}^{-2/p}(\Omega_{\pm})$ a Besov space of negative differentiability order, see [13].

Proof. We concentrate on the first statement. The proof of the second one is completely analogous. We have to show that for any $(f_{\pm}, g_1, g_2, h) \in W_p^{-2/p}(\Omega_{\pm}) \times [W_p^{1-3/p}(S)]^3$, there is precisely one $(u_{\pm}, \rho) \in W_p^{-2/p}(\Omega_{\pm}) \times W_p^{3-3/p}(S)$ that satisfies

$$\left. \begin{aligned} (\omega - \kappa_{\pm} \Delta) u_{\pm} &= f_{\pm} && \text{in } \Omega_{\pm}, \\ \llbracket u \rrbracket + (\Delta_S + m) \rho + \alpha_+ \partial_n u_+ &= g_1 && \text{on } S, \\ \alpha_+ \partial_n u_+ - \alpha_- \partial_n u_- &= g_2 && \text{on } S, \\ \omega \rho + \alpha_+ \partial_n u_+ &= h && \text{on } S, \\ \partial_n u_- &= 0 && \text{on } \partial C, \end{aligned} \right\} \quad (4.3)$$

and that there is a constant $C = C_{\omega}$ independent of $f_{\pm}$, g_1 , g_2 and h such that

$$\|(u_{\pm}, \rho)\|_E \leq C \left(\|(f_{\pm}, g_1, g_2, h)\|_{W_p^{-2/p}(\Omega_{\pm}) \times [W_p^{1-3/p}(S)]^3} \right). \quad (4.4)$$

1. Observe that (4.3) constitutes a (two sided) elliptic boundary system for which the Lopatinskiĭ–Shapiro condition (in an appropriate form) is valid. Therefore, if $f_{\pm} = g_1 = g_2 = h = 0$, any solution (u_+, u_-, ρ) to (4.3) is smooth, and in particular $(\omega - L)(u_+, u_-, \rho) = 0$ which implies $(u_+, u_-, \rho) = 0$ by Lemma 3.1.
2. Assume $f_{\pm}$, g_1 , g_2 and h are smooth. By suitable extensions we construct $z_{\pm} = z_{\pm}(g_1, g_2) \in W_p^2(\Omega_{\pm})$ such that

$$z_+ = g_1, \quad \partial_n z_+ = 0, \quad z_- = 0, \quad \partial_n z_- = -g_2/\alpha_- \text{ on } S, \quad \partial_n z_- = 0 \text{ on } \partial C.$$

We seek a solution to (4.3) in the form

$$(u_+, u_-, \rho) = (v_+, v_-, \rho) + (z_+, z_-, 0)$$

where

$$\left. \begin{aligned} (\omega - \kappa_{\pm} \Delta) v_{\pm} &= -(\omega - \kappa_{\pm} \Delta) z_{\pm} + f_{\pm} && \text{in } \Omega_{\pm}, \\ \llbracket v \rrbracket + (\Delta_S + m) \rho + \alpha_+ \partial_n v_+ &= 0 && \text{on } S, \\ \alpha_+ \partial_n v_+ - \alpha_- \partial_n v_- &= 0 && \text{on } S, \\ \omega \rho + \alpha_+ \partial_n v_+ &= h && \text{on } S, \\ \partial_n v_- &= 0 && \text{on } \partial C. \end{aligned} \right\}$$

This is equivalent to $(v_+, v_-, \rho) \in D(L)$ and

$$(\omega - L)(v_+, v_-, \rho) = (-(\omega - \kappa_{\pm} \Delta) z_{\pm} + f_{\pm}, h)$$

and thus the existence of a solution follows from Lemma 3.1.

3. By elliptic a priori estimates (in weak norms, as in [13], Sect. 4), together with the uniqueness result in one we obtain (4.4), first for $f_{\pm}$, g_1 , g_2 and h smooth. The general case follows from a standard approximation argument. $\square$

Lemma 4.4. (i) *There exist zero-neighborhoods O in $\mathcal{Z}$ and W in E and a function $\varphi \in C^\infty(O, W)$ such that*

- $\varphi(0) = 0$, $\varphi'(0) = 0$;
- $\text{id} + \varphi \in \text{Diff}^\infty(O, \mathcal{M} \cap W)$.

(ii) *The statements in (i) remain true if the spaces $(\mathcal{Z}, E)$ are replaced by $(D(L), E_1)$.*

Proof. Recall that $(\mathbf{F}, \mathbf{G})(0) = (0, 0)$, $(\mathbf{F}, \mathbf{G})'(0) = (0, 0)$. To show (i), let $z \in \mathcal{Z}$ be given. Lemma 4.3 and the Implicit Function theorem imply that the equation

$$(\omega - \hat{L}, B)\xi = (\mathbf{F}, \mathbf{G})(z + \xi)$$

possesses a unique solution $\xi = \varphi(z) \in E$, where $\varphi \in C^\infty(\mathbb{B}_{\mathcal{Z}}(0, \gamma), E)$ satisfies $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\gamma > 0$ is sufficiently small. The regularity of φ follows from the fact that $(\mathbf{F}, \mathbf{G}) \in C^\infty(E, W_p^{-2/p}(\Omega_{\pm}) \times [W_p^{1-3/p}(S)]^3)$ (cf. [9] Lemma 4.2). Moreover, we obviously have $z + \varphi(z) \in \mathcal{M}$. On the other hand, if a sufficiently small $\nu \in \mathcal{M}$ is given and $\xi := (\omega - \hat{L}, B)^{-1}((\mathbf{F}, \mathbf{G})(\nu))$, let $z := \nu - \xi$. Then $z \in \mathcal{Z}$ and $\xi = \varphi(z)$, i.e. $\text{id} + \varphi$ is onto. The proof of (ii) is analogous. $\square$

In a comparable fashion, the next lemma provides a local parametrization of the submanifold $(\text{id} + \varphi)^{-1}(\mathcal{E})$ (and, consequently, of $\mathcal{E}$) over $\mathcal{N}(L)$:

Lemma 4.5. *There exist zero-neighborhoods U in $\mathcal{N}(L)$ and V in E_1 and a function $\psi \in C^\infty(U, D(L) \cap X_s)$ such that*

- $\psi(0) = 0$, $\psi'(0) = 0$;
- $\chi := (\text{id} + \varphi) \circ (\text{id} + \psi) \in \text{Diff}^\infty(U, \mathcal{E} \cap V)$.

Proof. Let $D(L^s) := X_s \cap D(L)$, $L^s := L|_{D(L^s)}$. For a sufficiently small neighborhood V of zero in E_1 we define

$$\begin{aligned} \mathcal{E}' &:= ((\text{id} + \varphi)^{-1}(\mathcal{E} \cap V)) \cap V, \\ \mathcal{E}'' &:= \{z \in D(L) \cap V \mid LP^s z = -\omega P^s \varphi(z)\}. \end{aligned}$$

Observe that $\mathcal{E}' \subset D(L)$ is a manifold of dimension $N + 2$ according to Lemma 2.1. Moreover, for $z \in \mathcal{E}'$ we have

$$\begin{aligned} -\hat{L}(z + \varphi(z)) &= \mathbf{F}(z + \varphi(z)) \\ Bz &= 0 \end{aligned}$$

which is by construction of φ equivalent to

$$Lz = -\omega \varphi(z),$$

hence $\mathcal{E}' \subset \mathcal{E}''$.

After shrinking V if necessary, the manifold $\mathcal{E}''$ can be parametrized near zero over $\mathcal{N}(L)$ via

$$x \mapsto x + \psi(x), \quad x \in U,$$

where U is a neighborhood of zero in $\mathcal{N}(L)$ and $\psi \in C^\infty(U, D(L^s))$ satisfies

$$L(x + \psi(x)) = L^s \psi(x) = -\omega P^s \varphi(x + \psi(x)).$$

The existence of such a ψ follows via the Implicit Function theorem from Lemma 4.4 (ii) and the fact that L^s is an isomorphism from $D(L^s)$ to X_s . Additionally, $\psi(0) = 0$, $\psi'(0) = 0$. As $\dim \mathcal{E}'' = \dim \mathcal{E}' = N + 2$ we actually have $\mathcal{E}' = \mathcal{E}''$, and the assertions of the lemma follow. $\square$

We are ready now to solve (4.2) by a fixed point argument which we will first sketch informally before providing the details. For $\gamma, \delta > 0$ let $\mathbb{B}$ be the open ball around 0 of radius γ in $\mathcal{Z} \times \mathbb{E}(\delta)$. Furthermore, we fix $\omega > 0$ and an extension operator $\text{ext}_\delta \in \mathcal{L}(E, \mathbb{E}(\delta))$ such that $(\text{ext}_\delta w)(0) = w$, $w \in E$.

We introduce the following mappings acting on $(z_0, \xi) \in \mathbb{B}$: (as far as this is not clear, the facts that these mapping indeed are well defined and have the indicated target spaces will be proved below, together with additional mapping properties).

- $x_e : \mathbb{B} \longrightarrow X_c, \quad (z_0, \xi) \mapsto P^c z_0 + \int_0^\infty P^c \omega \xi(s) ds;$
- $z_e : \mathbb{B} \longrightarrow D(L), \quad z_e := (\text{id} + \psi) \circ x_e;$
- $Z_e : \mathbb{B} \longrightarrow \mathcal{E}, \quad Z_e := (\text{id} + \varphi) \circ z_e.$

For the sake of brevity we will write $J_e := J_e(z_0, \xi)$, $J \in \{x, z, Z\}$.

- $Z : \mathbb{B} \longrightarrow \mathbb{E}(\delta), \quad (z_0, \xi) \mapsto \zeta,$
where ζ ist the solution of

$$(\partial_t - \hat{L}, B)(\zeta) = (\omega \xi, 0), \quad \zeta(0) = z_0 - z_e. \quad (4.5)$$

- $M : \mathbb{B} \longrightarrow \mathbb{F}(\delta),$

$$M(z_0, \xi) := (\mathbf{F}, \mathbf{G})(Z_e + \zeta + \text{ext}_\delta(\varphi(z_0) - \varphi(z_e) - \xi(0)) + \xi) - (\mathbf{F}, \mathbf{G})(Z_e). \quad (4.6)$$

Finally,

- $\mathcal{V} : \mathbb{B} \longrightarrow \mathbb{E}(\delta), \quad (z_0, \xi) \mapsto \nu,$
where ν ist the solution of

$$(\partial_t + \omega - \hat{L}, B)(\nu) = M(z_0, \xi), \quad \nu(0) = \varphi(z_0) - \varphi(z_e). \quad (4.7)$$

These definitions are motivated by the following observation:

Remark 4.6. Let $(z_0, \xi) \in \mathbb{B}$ be such that $z_0 + \varphi(z_0) = \mu_0$ and

$$\xi = \nu(z_0, \xi). \quad (4.8)$$

Then

$$\mu = \xi + Z_e + \zeta$$

is a solution to (4.2).

By this observation, our problem is reduced to solving (4.8) in terms of ξ for given small $z_0 \in \mathcal{Z}$ which can be directly done via the Implicit Function theorem. For the following we observe that obviously

$$J_e \in C^1(\mathbb{B}, E_1), \quad J_e(0, 0) = 0, \quad J \in \{x, z, Z\}.$$

Lemma 4.7. *Let $(z_0, \xi) \in \mathbb{B}$ and $\omega > 0$. If $\delta > 0$ is small enough, there is a unique solution $\zeta \in \mathbb{E}(\delta)$ of (4.5). The solution map $Z := Z_\omega := (z_0, \xi) \mapsto \zeta$ satisfies*

$$Z \in C^1(\mathbb{B}, \mathbb{E}(\delta)), \quad Z(0) = 0.$$

Proof. To construct ζ we will use an ansatz based on the natural splitting

$$\zeta(t) + z_e = (x(t) + x_e) + (y(t) + \psi(x(t) + x_e)), \quad (4.9)$$

where x and y are (a priori unknown) functions valued in X_c and X_s , respectively. Observe that the terms in the first bracket on the right are in X_c while the terms in the second bracket are in X_s . We rewrite (4.9) as

$$\zeta(t) = x(t) + y(t) + \psi(x(t) + x_e) - \psi(x_e) \quad (4.10)$$

and apply $\partial_t - \hat{L}$ and the projections P^c and P^s . In this way we find that (4.5) is (via (4.10)) equivalent to the system

$$\left. \begin{aligned} \dot{x} &= P^c \omega \xi, & t > 0; & \quad x(0) = x_0 - x_e; \\ \dot{y} - L^s y &= S(x, x_e, \xi), & t > 0; & \quad y(0) = y_0; \\ By(t) &= 0, & t > 0, & \end{aligned} \right\} \quad (4.11)$$

where

$$S(x, x_e, \xi) = -\psi'(x + x_e)(P^c \omega \xi) + L^s \psi(x + x_e) - L^s \psi(x_e) + P^s \omega \xi, \quad (4.12)$$

$$x_0 = P^c z_0, \quad (4.13)$$

$$y_0 = z_0 - x_0 - \psi(x_0). \quad (4.14)$$

We observe that

$$z_0 \mapsto (x_0, y_0) \in C^1(\mathbb{B}_{\mathcal{Z}}(0, \gamma), X_c \times (X_s \cap \mathcal{Z})),$$

provided $\gamma > 0$ is small enough, and that x is determined by

$$x(t) := - \int_t^\infty P^c \omega \xi(s) ds.$$

Moreover, first just formally, we have

$$y(t) := (\partial_t - L^s, \text{tr})^{-1}(S(x, x_e, \xi), y_0).$$

To make this observation rigorous, we introduce the spaces

$$\mathbb{X}_c(\delta) := W_p^1(\mathbb{R}^+, \delta, X_c), \quad \mathbb{X}_s(\delta) := W_p^1(\mathbb{R}^+, \delta, X_s) \cap L_p(\mathbb{R}^+, \delta, D(L^s)).$$

From the decomposition

$$e^{\delta t} x(t) = - \int_t^\infty e^{\delta(t-s)} e^{\delta s} P^c \omega \xi(s) ds$$

and well-known facts about convolutions one gets that

$$\xi \mapsto x(\xi) \in C^1(\mathbb{E}(\delta), \mathbb{X}_c(\delta)).$$

As ψ, ψ', ψ'' are bounded functions near zero and $\psi(0) = 0, \psi'(0) = 0$, we can straightforwardly check that the Nemytskii operator induced by $\psi \circ (\cdot + \mathbf{x}_e) - \psi(\mathbf{x}_e)$ is a C^1 -mapping from the space $\mathbb{X}_c(\delta)$ to $L_p(\mathbb{R}^+, \delta, X_s \cap D(L)) \cap \mathbb{X}(\delta)$. Thus

$$(z_0, \xi) \mapsto S(\mathbf{x}, \mathbf{x}_e, \xi) \in C^1(\mathbb{B}, L_p(\mathbb{R}^+, \delta, X_s)).$$

Due to Corollary 3.2, we can use arguments parallel to those in the proof of [1], Theorem III.4.10.7 to conclude that

$$(\partial_t - L^s, \text{tr})^{-1} \in \mathcal{L}(L_p(\mathbb{R}^+, \delta, X_s) \times \mathcal{Z} \cap X_s, \mathbb{X}_s(\delta)),$$

provided $\delta > 0$ is small enough. Therefore,

$$(z_0, \xi) \mapsto \mathbf{y}(z_0, \xi) \in C^1(\mathbb{B}, \mathbb{X}_s(\delta)),$$

and thus $(z_0, \xi) \mapsto \zeta \in C^1(\mathbb{B}, \mathbb{X}(\delta))$. It remains to prove some additional regularity: If $(z_0, \xi) := (z_0, (\xi_\pm, \sigma_\xi)) \in \mathbb{B}$ is given, let $\zeta = \zeta(z_0, \xi) := (w_\pm, \rho)$. Then

$$\rho_t = -\alpha_+ \partial_n w_+ + \omega \sigma_\xi.$$

As $w \mapsto \partial_n w$ is a bounded linear operator from $W_p^{2,1}(\mathbb{R}^+, \delta, \Omega_+)$ to $W_p^{1/2-1/(2p)}(\mathbb{R}^+, \delta, L_p(S))$ we have

$$(z_0, \xi) \mapsto \zeta \in C^1(\mathbb{B}, \mathbb{E}(\delta)),$$

provided, $\delta, \gamma > 0$ are small enough. The statement $Z(0) = 0$ is clear. $\square$

Lemma 4.8. *There are $\omega, \delta > 0$ such that*

$\mathbb{L}_\omega := (\partial_t + \omega - \hat{L}, B, \text{tr}) \in \mathcal{L}_{is}(\mathbb{E}(\delta), \mathbb{F}_0(\delta))$, *where*

$$\mathbb{F}_0(\delta) := \{(f_\pm, h, g_1, g_2; \mu) \in \mathbb{F}(\delta) \times E; B(\mu) = (g_1, g_2)(0)\}.$$

Proof. If $\omega > 0$ is large enough, then, in the autonomous case (i.e. the case of operators having time independent coefficients) Theorem 2.1 in [2] holds still true, if the finite interval $[0, T]$ is replaced by $\mathbb{R}^+$, provided the operator ∂_t is replaced by $\partial_t + \omega$. We apply this modified theorem to the problem that arises when $(u, \rho)(t)$ is replaced by $e^{\delta t}(u, \rho)(t)$. Choosing $\omega - \delta$ large enough, this yields the assertion. $\square$

Lemma 4.9. *For the mapping M , defined in (4.6), we have that*

- $M \in C^1(\mathbb{B}, \mathbb{F}(\delta));$
- $(M, D_\xi M)(0, 0) = (0, 0);$
- $(M(z_0, \xi), \varphi(z_0) - \varphi(z_e)) \in \mathbb{F}_0(\delta).$

Proof. The first statement is a consequence of the regularity properties of the mappings $\mathbf{F}, \mathbf{G}, Z, Z_e, z_e$. The second statement follows straightforwardly from the fact that

$$(\mathbf{F}, \mathbf{G}, Z, Z_e, z_e)(0) = 0, \quad (\mathbf{F}, \mathbf{G})'(0) = 0.$$

The third statement is obtained from the observations

- $M(z_0, \xi)(0) = (\mathbf{F}, \mathbf{G})(z_0 + \varphi(z_0)) - (\mathbf{F}, \mathbf{G})(z_e + \varphi(z_e));$
- $B(\varphi(z_0) - \varphi(z_e)) = \mathbf{G}(z_0 + \varphi(z_0)) - \mathbf{G}(z_e + \varphi(z_e)).$

$\square$

Fix $\omega > 0$ large enough. By Lemmas 4.8, 4.9, the following mapping is well defined:

$$K : \mathbb{B} \rightarrow \mathbb{E}(\delta), \quad (z_0, \xi) \mapsto \mathbb{L}_\omega^{-1}(M(z_0, \xi), \varphi(z_0) - \varphi(z_e)) - \xi.$$

Theorem 4.10. *There exist $\delta, \gamma > 0$ such that, if $\|z_0\|_{\mathcal{Z}} < \gamma$, then there is unique $\xi \in \mathbb{E}(\delta)$ such that $K(z_0, \xi) = 0$. The mapping*

$$z_0 \mapsto \xi(z_0) : \mathbb{B}_{\mathcal{Z}}(0, \gamma) \rightarrow \mathbb{E}(\delta)$$

is C^1 .

Proof. By the results of this section we have that $K \in C^1(\mathbb{B}, \mathbb{E}(\delta))$, $K(0, 0) = 0$, $\partial_\xi K(0, 0) = -\text{id}$. Thus, the assertion follows from the Implicit Function theorem. $\square$

Using Lemma 4.4 (i), from Theorem 4.10, Remark 4.6 and the regularity of Z and Z_e we easily derive the statements of Theorem 4.2 except uniqueness. The uniqueness of global strong solutions follows by adjusting the results in Sect. 4 of [9] to the two-phase case in an obvious way.

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