A remark on the existence of slowly growing positive solutions to second order super-linear ordinary differential equations

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Abstract. In this paper second order super-linear ordinary differential equations are considered, and a sufficient condition for the existence of a slowly growing positive solution is given.

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1. Introduction

In this paper we consider the second order nonlinear ordinary differential equation

$$x'' + p(t)|x|^\gamma \operatorname{sgn} x = 0, \quad (1.1)$$

where γ is a positive constant and $p(t)$ is a positive and continuous function on an interval $[t_0, \infty)$. A solution $x(t)$ of (1.1) which exists on some interval of the form $[T, \infty)$, $T \geq t_0$, is said to be nonoscillatory if $x(t)$ is eventually positive or eventually negative. It is obvious that if $x(t)$ is an eventually negative solution of (1.1), then $-x(t)$ is an eventually positive solution of (1.1). Therefore there is no loss of generality in assuming that a nonoscillatory solution of (1.1) is eventually positive.

It is well known that an eventually positive solution $x(t)$ of (1.1) satisfies one and only one of the following three conditions:

$$\lim_{t \rightarrow \infty} x(t) \text{ exists and is a positive finite number;} \quad (1.2)$$

$$\lim_{t \rightarrow \infty} x(t) = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{x(t)}{t} = 0; \quad (1.3)$$

$$\lim_{t \rightarrow \infty} \frac{x(t)}{t} \text{ exists and is a positive finite number.} \quad (1.4)$$

A solution $x(t)$ of (1.1) which satisfies (1.2) [*resp.* (1.4)] is asymptotically equal to a positive constant function c [*resp.* a linear function ct] as $t \rightarrow \infty$ for some constant $c > 0$. The asymptotic growth of a solution $x(t)$ of (1.1) which satisfies (1.3) is asymptotically bigger than positive constant functions, and is asymptotically smaller than positive unbounded linear functions. In this paper we refer to eventually positive solutions $x(t)$ satisfying (1.3) as *slowly growing positive solutions*. Eventually positive solutions $x(t)$ satisfying (1.2), (1.3) and (1.4) are sometimes called *subdominant solutions*, *intermediate solutions* and *dominant solutions*, respectively ([1]).

The following results are classical and important.

- (A) Equation (1.1) has an eventually positive solution $x(t)$ satisfying (1.2) if and only if

$$\int_{t_0}^{\infty} tp(t)dt < \infty. \quad (1.5)$$

- (B) Equation (1.1) has an eventually positive solution $x(t)$ satisfying (1.4) if and only if

$$\int_{t_0}^{\infty} t^{\gamma} p(t)dt < \infty. \quad (1.6)$$

- (C) Let $\gamma > 1$. Equation (1.1) has an eventually positive solution if and only if (1.5) is satisfied.
 (D) Let $0 < \gamma < 1$. Equation (1.1) has an eventually positive solution if and only if (1.6) is satisfied.
 (E) Let $0 < \gamma < 1$. Equation (1.1) has a slowly growing positive solution [i.e., an eventually positive solution which satisfies (1.3)] if and only if

$$\int_{t_0}^{\infty} t^{\gamma} p(t)dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} tp(t)dt = \infty. \quad (1.7)$$

A characterization of the existence of slowly growing positive solutions of (1.1) for the case $\gamma > 1$ is a more difficult problem. In this paper we attempt to discuss the existence of slowly growing positive solutions of (1.1) for the case $\gamma > 1$. For some results related to this case, see Cecchi et al. [1], Naito [9] and the references therein.

Note that the above-mentioned results (A)–(E) for the Eq. (1.1) are extended to the more general equation

$$(|x'|^{\alpha} \operatorname{sgn} x')' + p(t)|x|^{\gamma} \operatorname{sgn} x = 0, \quad (1.8)$$

where α and γ are positive constants and $p(t)$ is a positive and continuous function on an interval $[t_0, \infty)$. See, for example, [2, 3, 6–8]. The problem of the existence of slowly growing positive solutions in the papers [1, 9] has been studied for the generalized Eq. (1.8).

Consider the Eq. (1.8). In the case $0 < \alpha < \gamma$, assuming the existence of slowly growing positive solutions of (1.8), Kamo and Usami [4] have obtained the asymptotic forms as $t \rightarrow \infty$ of such solutions. Note however that the existence of slowly growing positive solutions of (1.8) is not proved. In the

case $0 < \gamma < \alpha$, the asymptotic forms as $t \rightarrow \infty$ of slowly growing positive solutions of (1.8) has been discussed by Naito [8]. See also [4, 5].

2. Results and examples

To discuss the existence of a slowly growing positive solution of (1.1) for the case $\gamma > 1$, let us first consider the particular equation

$$x'' + \kappa t^{-\mu} |x|^{\gamma} \operatorname{sgn} x = 0 \quad (\gamma > 1), \quad (2.1)$$

where κ is a positive constant and μ is a real constant. It is easy to see that (2.1) has a slowly growing positive solution of the form ct^{ν} ($c > 0, 0 < \nu < 1$) if and only if $2 < \mu < \gamma + 1$, and that this solution is uniquely determined by

$$x_0(t) = c_0 t^{\nu_0} \quad (2.2)$$

with

$$\nu_0 = \frac{\mu - 2}{\gamma - 1} \quad \text{and} \quad c_0 = \left[\frac{\nu_0(1 - \nu_0)}{\kappa} \right]^{1/(\gamma-1)}. \quad (2.3)$$

Observe here that $0 < \nu_0 < 1$ and $c_0 > 0$ under the conditions $\gamma > 1$ and $2 < \mu < \gamma + 1$. Then we may conjecture that if $p(t)$ is close to the function $\kappa t^{-\mu}$ ($\kappa > 0, 2 < \mu < \gamma + 1$) in some sense, then (1.1) has a slowly growing positive solution $x(t)$ satisfying

$$x(t) = x_0(t)(1 + o(1)) \quad (t \rightarrow \infty) \quad (2.4)$$

or

$$\begin{cases} x(t) = x_0(t)(1 + o(1)) & (t \rightarrow \infty), \\ x'(t) = x'_0(t)(1 + o(1)) & (t \rightarrow \infty), \end{cases} \quad (2.5)$$

where $x_0(t)$ is defined by (2.2) and (2.3). This conjecture is true to a certain extent. In fact, the following theorems can be proved. For convenience, we write the Eq. (1.1) in the form

$$x'' + \kappa t^{-\mu} (1 + \varepsilon(t)) |x|^{\gamma} \operatorname{sgn} x = 0, \quad (2.6)$$

where $\varepsilon(t)$ is a continuous function on $[t_0, \infty)$, $t_0 > 0$, such that $1 + \varepsilon(t) > 0$ for $t \geq t_0$.

Theorem 2.1. *Consider the Eq. (2.6) under the conditions*

$$\gamma > 1, \quad 2 < \mu < \gamma + 1 \quad \text{and} \quad \kappa > 0. \quad (2.7)$$

Set $x_0(t) = c_0 t^{\nu_0}$, where c_0 and ν_0 are constants given by (2.3). Suppose that there exists $\ell > 0$ such that

$$\left(\ell - \frac{\mu - [(\gamma + 3)/2]}{\gamma - 1} \right)^2 < \frac{1}{4} - \frac{\gamma(\gamma + 1 - \mu)(\mu - 2)}{(\gamma - 1)^2} \quad (2.8)$$

and

$$\lim_{t \rightarrow \infty} t^{\ell - [(\mu - 2)/(\gamma - 1)]} \int_{t_0}^t \left(\int_s^{\infty} r^{-(2\gamma - \mu)/(\gamma - 1)} |\varepsilon(r)| dr \right) ds = 0. \quad (2.9)$$

Then the Eq. (2.6) has a slowly growing positive solution $x(t)$ with the asymptotic property

$$x(t) = x_0(t)(1 + O(t^{-\ell})) \quad (t \rightarrow \infty). \quad (2.10)$$

If there is an $\ell > 0$ such that (2.8) holds, then

$$\frac{1}{4} - \frac{\gamma(\gamma + 1 - \mu)(\mu - 2)}{(\gamma - 1)^2} > 0. \quad (2.11)$$

This condition (2.11) can be regarded as a quadratic inequality with respect to μ . Then, under the conditions $\gamma > 1$ and $2 < \mu < \gamma + 1$, the condition (2.11) implies either

$$2 < \mu < \frac{\gamma + 3}{2} - \frac{\gamma - 1}{2} \sqrt{\frac{\gamma - 1}{\gamma}} \quad (2.12)$$

or

$$\frac{\gamma + 3}{2} + \frac{\gamma - 1}{2} \sqrt{\frac{\gamma - 1}{\gamma}} < \mu < \gamma + 1. \quad (2.13)$$

Note that, for the case (2.12), there is no $\ell > 0$ such that (2.8) holds. On the other hand, for the case (2.13), the condition (2.8) is satisfied by

$$\ell = \frac{\mu - [(\gamma + 3)/2]}{\gamma - 1} > 0, \quad (2.14)$$

in which the condition (2.9) reduces to

$$\lim_{t \rightarrow \infty} t^{-1/2} \int_{t_0}^t \left(\int_s^\infty r^{-(2\gamma - \mu)/(\gamma - 1)} |\varepsilon(r)| dr \right) ds = 0. \quad (2.15)$$

Thus we can conclude the following result as a corollary of Theorem 2.1.

Corollary 2.2. *Consider the Eq. (2.6) under the conditions (2.7) and (2.13). If (2.15) is satisfied, then (2.6) has a slowly growing positive solution $x(t)$ with the asymptotic property (2.4).*

Moreover, by the use of Theorem 2.1, we can prove the following theorem.

Theorem 2.3. *In the Eq. (2.6), assume that (2.7) holds and that there is an $\ell > 0$ such that, in addition to (2.8),*

$$0 < \ell < \frac{\mu - 2}{\gamma - 1} \quad (2.16)$$

and

$$\lim_{t \rightarrow \infty} t^{\ell + [(\gamma + 1 - \mu)/(\gamma - 1)]} \int_t^\infty s^{-(2\gamma - \mu)/(\gamma - 1)} |\varepsilon(s)| ds = 0. \quad (2.17)$$

Then (2.6) has a slowly growing positive solution $x(t)$ such that

$$\begin{cases} x(t) = x_0(t)(1 + O(t^{-\ell})) & (t \rightarrow \infty), \\ x'(t) = x'_0(t)(1 + O(t^{-\ell})) & (t \rightarrow \infty), \end{cases}$$

where $x_0(t) = c_0 t^{\nu_0}$, and c_0 and ν_0 are constants given by (2.3).

Notice that if $\varepsilon(t) \not\equiv 0$ and if $\ell > 0$ satisfies (2.9), then (2.16) holds. Notice moreover that (2.16) and (2.17) together imply (2.9).

In Theorem 2.3, if we take ℓ as in (2.14), then we obtain the next corollary.

Corollary 2.4. *Consider the Eq. (2.6) under the conditions (2.7) and (2.13). If*

$$\lim_{t \rightarrow \infty} t^{1/2} \int_t^\infty s^{-(2\gamma-\mu)/(\gamma-1)} |\varepsilon(s)| ds = 0 \quad (2.18)$$

is satisfied, then (2.6) has a slowly growing positive solution $x(t)$ with the asymptotic property (2.5).

We give an example illustrating our results.

Example. Consider the equation

$$x'' + \kappa t^{-13/5} (1 + \varepsilon(t)) |x|^{5/3} \operatorname{sgn} x = 0, \quad \kappa > 0, \quad (2.19)$$

where $\varepsilon(t)$ is a continuous function on $[1, \infty)$ such that $1 + \varepsilon(t) > 0$ for $t \geq 1$. For this equation, $\gamma = 5/3$, $\mu = 13/5$; and so $\nu_0 = 9/10$, $c_0 = (3^3/10^3)\kappa^{-3/2}$ and $x_0(t) = (3^3/10^3)\kappa^{-3/2}t^{9/10}$. The conditions (2.7) and (2.13) are satisfied. Further, (2.15) and (2.18) reduce to

$$\lim_{t \rightarrow \infty} t^{-1/2} \int_1^t \left(\int_s^\infty r^{-11/10} |\varepsilon(r)| dr \right) ds = 0 \quad (2.20)$$

and

$$\lim_{t \rightarrow \infty} t^{1/2} \int_t^\infty s^{-11/10} |\varepsilon(s)| ds = 0, \quad (2.21)$$

respectively. Therefore, by Corollary 2.2, we find that if (2.20) is satisfied, then (2.19) has a slowly growing positive solution $x(t)$ such that

$$x(t) = (3^3/10^3)\kappa^{-3/2}t^{9/10}(1 + o(1)) \quad (t \rightarrow \infty).$$

Moreover, by Corollary 2.4, we can conclude that if (2.21) is satisfied, then (2.19) has a slowly growing positive solution $x(t)$ such that

$$\begin{cases} x(t) = (3^3/10^3)\kappa^{-3/2}t^{9/10}(1 + o(1)) & (t \rightarrow \infty), \\ x'(t) = (3^5/10^4)\kappa^{-3/2}t^{-1/10}(1 + o(1)) & (t \rightarrow \infty). \end{cases} \quad (2.22)$$

It is worth noting that the condition (2.21) is stronger than (2.20).

In (2.19), let $\varepsilon(t) = \lambda t^{-1/2}$ (λ is a nonzero constant). Then (2.21) is satisfied, and so we see that (2.19) with $\varepsilon(t) = \lambda t^{-1/2}$ ($\lambda \neq 0$) has a slowly growing positive solution $x(t)$ which satisfies (2.22).

Next consider the case $\varepsilon(t) = \lambda t^{-1/2}$ ($\lambda \neq 0$) in the general Eq. (2.6). Then (2.18) is always satisfied under the assumptions $\gamma > 1$ and $\mu < \gamma + 1$. Therefore, by Corollary 2.4, we find that if (2.7) and (2.13) hold, then the Eq. (2.6) with $\varepsilon(t) = \lambda t^{-1/2}$ ($\lambda \neq 0$) has a slowly growing positive solution $x(t)$ satisfying (2.5). For a given $\gamma > 1$, the range of values of the parameter μ which is determined by (2.13) is the open interval

$$I_\gamma = (\gamma^*, \gamma + 1),$$

where

$$\gamma^* = \frac{\gamma + 3}{2} + \frac{\gamma - 1}{2} \sqrt{\frac{\gamma - 1}{\gamma}}.$$

The length $|I_\gamma|$ of the interval I_γ is an increasing function of $\gamma > 1$, and $|I_\gamma| \rightarrow 0$ as $\gamma \rightarrow 1 + 0$ and $|I_\gamma| \rightarrow 1/4$ as $\gamma \rightarrow \infty$.

In the paper [9] the author has discussed the existence of slowly growing positive solutions of (1.8) for the case $\alpha < \gamma$. Applying Theorem 1.1 in [9] to the Eq. (2.19), we see that if there is a ρ such that $\rho > (\sqrt{22} - 4)/10$ and

$$\lim_{t \rightarrow \infty} t^\rho \int_t^\infty s^{-1/5} |\varepsilon(s)| ds = 0, \quad (2.23)$$

then (2.19) has a slowly growing positive solution $x(t)$ which satisfies (2.22).

For the case $\varepsilon(t) = \lambda t^{-1/2} (\lambda \neq 0)$, the infinite integral in (2.23) does not converge, and so we cannot apply the above result which is derived from [9, Theorem 1.1] to (2.19) with $\varepsilon(t) = \lambda t^{-1/2} (\lambda \neq 0)$.

Actually the results in [9] and the results in the present paper are independent as the following example shows.

Example. Consider the equation

$$x'' + \kappa t^{-3} (1 + \varepsilon(t)) |x|^3 \operatorname{sgn} x = 0, \quad \kappa > 0, \quad (2.24)$$

where $\varepsilon(t)$ is a continuous function on $[1, \infty)$ such that $1 + \varepsilon(t) > 0$ for $t \geq 1$. It is seen ([9, Example 1.1]) that if there is ρ satisfying

$$\rho > \frac{\sqrt{2}}{2} \quad \text{and} \quad \lim_{t \rightarrow \infty} t^\rho \int_t^\infty \frac{|\varepsilon(s)|}{s} ds = 0,$$

then (2.24) has a slowly growing positive solution $x(t)$ such that

$$\begin{cases} x(t) = \frac{1}{2\sqrt{\kappa}} t^{1/2} (1 + O(t^{-\rho})) & (t \rightarrow \infty), \\ x'(t) = \frac{1}{4\sqrt{\kappa}} t^{-1/2} (1 + O(t^{-\rho})) & (t \rightarrow \infty). \end{cases}$$

Equation (2.24) is the case $\gamma = 3$ and $\mu = 3$, and in this case the condition (2.13) is not satisfied. Therefore we cannot apply the results in the present paper to (2.24).

3. Proofs of main results

In this section we give the proofs of Theorems 2.1 and 2.3. The idea of the proof of Theorem 2.1 is as follows. It is easy to see that $x = x(t)$ is a positive solution of (1.1) on an interval $[T, \infty)$, $T \geq t_0$, if and only if it satisfies

$$x(t) = x(T) + \int_T^t \left(\int_s^\infty p(r) x(r)^\gamma dr \right) ds, \quad t \geq T. \quad (3.1)$$

The condition (2.10) is equivalent to the condition

$$(1 - Kt^{-\ell})x_0(t) \leq x(t) \leq (1 + Kt^{-\ell})x_0(t), \quad t \geq T, \quad (3.2)$$

where K is a positive constant and $T(\geq t_0)$ is taken sufficiently large. Therefore we seek a positive solution $x(t)$ of (3.1) which satisfies (3.2). For this purpose we use the Schauder–Tychonoff fixed point theorem. Theorem 2.3 may be easily derived from Theorem 2.1.

Proof of Theorem 2.1. Let ℓ be a positive number satisfying (2.8) and (2.9). First we note that the condition (2.8) is rewritten in the form

$$(\ell + \mu - \nu_0\gamma - 2)(\ell + \mu - \nu_0\gamma - 1) < -\frac{\gamma(\gamma + 1 - \mu)(\mu - 2)}{(\gamma - 1)^2}, \quad (3.3)$$

where ν_0 is a constant given by (2.3). By (3.3) we see that

$$\ell + \mu - \nu_0\gamma - 2 < 0 \quad \text{and} \quad \ell + \mu - \nu_0\gamma - 1 > 0. \quad (3.4)$$

From (3.3) it is possible to take a sufficiently small positive number h such that

$$(\ell + \mu - \nu_0\gamma - 2)(\ell + \mu - \nu_0\gamma - 1) < -\frac{\gamma(\gamma + 1 - \mu)(\mu - 2)}{(\gamma - 1)^2} (1 + h)^2. \quad (3.5)$$

Since

$$\lim_{a \rightarrow +0} \frac{(1+a)^\gamma - 1}{a} = \gamma \quad \text{and} \quad \lim_{a \rightarrow +0} \frac{1 - (1-a)^\gamma}{a} = \gamma,$$

there is $a_0, 0 < a_0 < 1$, such that

$$(1+a)^\gamma \leq 1 + \gamma(1+h)a \quad \text{and} \quad (1-a)^\gamma \geq 1 - \gamma(1+h)a \quad (3.6)$$

for $0 \leq a < a_0$. Now, let K be an arbitrary positive number. We take a positive number $T(\geq t_0)$ such that $Kt^{-\ell} < a_0 (< 1)$ for $t \geq T$ and

$$\begin{aligned} & t^{\ell - [(\mu - 2)/(\gamma - 1)]} \int_T^t \left(\int_s^\infty r^{-(2\gamma - \mu)/(\gamma - 1)} |\varepsilon(r)| dr \right) ds \\ & \leq \frac{hK}{(1+h)[1 + \gamma(1+h)a_0]\kappa c_0^{\gamma-1}}, \quad t \geq T. \end{aligned} \quad (3.7)$$

The above inequality (3.7) can be deduced from the assumption (2.9). Then, by (3.6),

$$(1 + Kt^{-\ell})^\gamma \leq 1 + \gamma(1+h)Kt^{-\ell}, \quad t \geq T, \quad \text{and} \quad (3.8)$$

$$(1 - Kt^{-\ell})^\gamma \geq 1 - \gamma(1+h)Kt^{-\ell}, \quad t \geq T. \quad (3.9)$$

In particular, (3.8) gives

$$(1 + Kt^{-\ell})^\gamma \leq 1 + \gamma(1+h)a_0, \quad t \geq T. \quad (3.10)$$

We denote by X the set of all functions $x \in C[T, \infty)$ such that (3.2) holds. The space $C[T, \infty)$ is regarded as the Fréchet space consisting of all continuous functions on $[T, \infty)$ with the topology of uniform convergence on compact subintervals of $[T, \infty)$. Observe that if $x \in X$, then $x(t) \geq (1 - a_0)x_0(t) > 0$ for $t \geq T$.

Define the operator $\Phi : X \rightarrow C[T, \infty)$ by

$$(\Phi x)(t) = x_0(T) + \kappa \int_T^t \left(\int_s^\infty r^{-\mu} (1 + \varepsilon(r)) x(r)^\gamma dr \right) ds, \quad t \geq T. \quad (3.11)$$

It will be shown by the aid of the Schauder–Tychonoff theorem that Φ has a fixed point x in $X(\subset C[T, \infty))$. For brevity, we set $p_0(t) = \kappa t^{-\mu}$. Then, $x = x_0(t)$ is a slowly growing positive solution of the equation

$$x'' + p_0(t)|x|^\gamma \operatorname{sgn} x = 0, \quad (3.12)$$

and the above operator Φ is rewritten in the form

$$(\Phi x)(t) = x_0(T) + \int_T^t \left(\int_s^\infty p_0(r)(1 + \varepsilon(r))x(r)^\gamma dr \right) ds, \quad t \geq T. \quad (3.13)$$

(i) *The operator Φ is well defined on X and maps X into X .*

Let $x \in X$. Then, by (3.2), (3.8) and (3.10),

$$\begin{aligned} & p_0(r)(1 + |\varepsilon(r)|)x(r)^\gamma \\ & \leq p_0(r)(1 + Kr^{-\ell})^\gamma x_0(r)^\gamma + p_0(r)|\varepsilon(r)|(1 + Kr^{-\ell})^\gamma x_0(r)^\gamma \\ & \leq p_0(r)x_0(r)^\gamma + \gamma(1 + h)Kp_0(r)r^{-\ell}x_0(r)^\gamma \\ & \quad + [1 + \gamma(1 + h)a_0]p_0(r)|\varepsilon(r)|x_0(r)^\gamma \end{aligned}$$

for $r \geq T$. Denote the first, second and third term of the last side in the above inequality by $\varphi_1(r)$, $\varphi_2(r)$ and $\varphi_3(r)$, respectively:

$$\begin{aligned} \varphi_1(r) &= p_0(r)x_0(r)^\gamma, \\ \varphi_2(r) &= \gamma(1 + h)Kp_0(r)r^{-\ell}x_0(r)^\gamma, \\ \varphi_3(r) &= [1 + \gamma(1 + h)a_0]p_0(r)|\varepsilon(r)|x_0(r)^\gamma. \end{aligned}$$

We have

$$p_0(r)(1 + |\varepsilon(r)|)x(r)^\gamma \leq \varphi_1(r) + \varphi_2(r) + \varphi_3(r), \quad r \geq T. \quad (3.14)$$

Likewise, using (3.2), (3.8), (3.9) and (3.10), we have

$$p_0(r)(1 - |\varepsilon(r)|)x(r)^\gamma \geq \varphi_1(r) - \varphi_2(r) - \varphi_3(r), \quad r \geq T. \quad (3.15)$$

Since $\varphi_1(r)$, $\varphi_2(r)$ and $\varphi_3(r)$ are integrable on $[T, \infty)$, the inequality (3.14) implies that $p_0(r)(1 + |\varepsilon(r)|)x(r)^\gamma$ is also integrable on $[T, \infty)$ for each $x \in X$. Therefore the operator Φ is well defined on X .

By the fact that $x_0(t)$ is a slowly growing positive solution of (3.12) (or, by a direct computation), we have

$$\int_t^\infty \varphi_1(r)dr = \int_t^\infty p_0(r)x_0(r)^\gamma dr = x_0'(t). \quad (3.16)$$

Moreover, by a direct computation, we find

$$\int_t^\infty \varphi_2(r)dr = \frac{\gamma(1 + h)\kappa c_0^\gamma K}{\ell + \mu - \nu_0\gamma - 1} t^{-\ell - \mu + \nu_0\gamma + 1} \quad (3.17)$$

for $t \geq T$. Furthermore we have

$$\int_t^\infty \varphi_3(r)dr = [1 + \gamma(1 + h)a_0]\kappa c_0^\gamma \int_t^\infty r^{-(2\gamma - \mu)/(\gamma - 1)} |\varepsilon(r)|dr \quad (3.18)$$

for $t \geq T$. By (3.16), it is clear that

$$\int_T^t \left(\int_s^\infty \varphi_1(r)dr \right) ds = x_0(t) - x_0(T), \quad t \geq T.$$

By virtue of (3.17), (3.5) and (2.3), we find that

$$\int_T^t \left(\int_s^\infty \varphi_2(r) dr \right) ds \leq \frac{K}{1+h} t^{-\ell} x_0(t), \quad t \geq T.$$

The verification is left to the readers. Finally, from (3.18) and (3.7) we obtain

$$\int_T^t \left(\int_s^\infty \varphi_3(r) dr \right) ds \leq \frac{hK}{1+h} t^{-\ell} x_0(t), \quad t \geq T.$$

Thus, by (3.13) and (3.14),

$$(\Phi x)(t) \leq x_0(t) + K t^{-\ell} x_0(t), \quad t \geq T,$$

for each $x \in X$. Similarly, from (3.13) and (3.15),

$$(\Phi x)(t) \geq x_0(t) - K t^{-\ell} x_0(t), \quad t \geq T,$$

for each $x \in X$. These show that if $x \in X$, then $\Phi x \in X$, which proves the claim (i).

(ii) *The operator Φ is continuous on X .*

Assume that $x_\infty \in X, x_n \in X (n = 1, 2, 3, \dots)$, and that

$$x_n(t) \rightarrow x_\infty(t) \quad \text{as } n \rightarrow \infty$$

uniformly on any compact subinterval $[T, S]$ of $[T, \infty)$. The inequality (3.14) implies that, for every $x \in X$, the function $|p_0(r)(1 + \varepsilon(r))x(r)^\gamma|$ is bounded by the integrable function $\varphi_1(r) + \varphi_2(r) + \varphi_3(r)$ on $[T, \infty)$. Therefore, by the Lebesgue dominated convergence theorem,

$$(\Phi x_n)(t) \rightarrow (\Phi x_\infty)(t) \quad \text{as } n \rightarrow \infty$$

uniformly on any compact subinterval $[T, S]$ of $[T, \infty)$. This proves the claim (ii).

(iii) *$\Phi(X)$ is relatively compact.*

To prove the relative compactness of $\Phi(X)$, it is enough to show that $\Phi(X)$ is uniformly bounded and equicontinuous on any compact subinterval $[T, S]$ of $[T, \infty)$. The former follows from the fact that, for any $x \in X$, the function $|(\Phi x)(t)|$ is bounded by $(1 + K t^{-\ell})x_0(t)$ on $[T, \infty)$. The latter follows from the equality

$$(\Phi x)'(t) = \int_t^\infty p_0(r)(1 + \varepsilon(r))x(r)^\gamma dr, \quad t \geq T,$$

and the fact that, for any $x \in X$, the function $|p_0(r)(1 + \varepsilon(r))x(r)^\gamma|$ is bounded by the integrable function $\varphi_1(r) + \varphi_2(r) + \varphi_3(r)$ on $[T, \infty)$. Thus the claim (iii) is also true.

In view of (i)–(iii), the Schauder–Tychonoff theorem shows that Φ has a fixed point x in X . This fixed point $x = x(t) (\in X)$ is a solution of

$$x(t) = x_0(T) + \kappa \int_T^t \left(\int_s^\infty r^{-\mu}(1 + \varepsilon(r))x(r)^\gamma dr \right) ds, \quad t \geq T,$$

and satisfies (3.2). Consequently, $x(t)$ is a solution of (2.6) which satisfies (2.10). This completes the proof of Theorem 2.1.

Proof of Theorem 2.3. As stated before, (2.16) and (2.17) together imply (2.9). Therefore, as shown in the proof of Theorem 2.1, the Eq. (2.6) has a slowly growing positive solution $x(t)$ which satisfies (3.2). In (3.2), K is an arbitrary positive number and T is taken as in the proof of Theorem 2.1. To complete the proof of Theorem 2.3, we will prove that such a solution $x(t)$ satisfies

$$(1 - Mt^{-\ell})x'_0(t) \leq x'(t) \leq (1 + Mt^{-\ell})x'_0(t), \quad t \geq T, \quad (3.19)$$

for some positive constant M .

We have, in addition to (3.2),

$$x'(t) = \int_t^\infty p_0(r)(1 + \varepsilon(r))x(r)^\gamma dr, \quad t \geq T, \quad (3.20)$$

where $p_0(t) = \kappa t^{-\mu}$. From the proof of Theorem 2.1, the inequalities (3.14) and (3.15) hold, and the integral estimates (3.16), (3.17) and (3.18) are valid. By the condition (2.17) we may suppose

$$\int_t^\infty r^{-(2\gamma-\mu)/(\gamma-1)} |\varepsilon(r)| dr \leq t^{-\ell - [(\gamma+1-\mu)/(\gamma-1)]} = t^{-\ell - \mu + \nu_0\gamma + 1} \quad (3.21)$$

for $t \geq T$. Then, by (3.14), (3.15), (3.16), (3.17), (3.18), (3.20) and (3.21), we obtain the following estimates for $x'(t)$:

$$\begin{aligned} x'(t) &\leq x'_0(t) + Mt^{-\ell}x'_0(t), \quad t \geq T, \quad \text{and} \\ x'(t) &\geq x'_0(t) - Mt^{-\ell}x'_0(t), \quad t \geq T, \end{aligned}$$

where

$$M = \left[\frac{\gamma(1+h)\kappa c_0^\gamma K}{\ell + \mu - \nu_0\gamma - 1} + [1 + \gamma(1+h)a_0] \kappa c_0^\gamma \right] \frac{1}{c_0\nu_0} > 0.$$

This proves (3.19), and the proof of Theorem 2.3 is complete.

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