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$W_0^{1,1}$-solutions for elliptic problems having gradient quadratic lower order terms

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Abstract. In this paper we deal with solutions of problems of the type

$$\begin{cases} -\operatorname{div}\left(\frac{a(x)Du}{(1+|u|)^2}\right) + u = \frac{b(x)|Du|^2}{(1+|u|)^3} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \alpha \leq a(x) \leq \beta, |b(x)| \leq \gamma, \gamma > 0, f \in L^2(\Omega)$ and Ω is a bounded subset of $\mathbb{R}^N$ with $N \geq 3$. We prove the existence of at least one solution for such a problem in the space $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ if the size of the lower order term satisfies a smallness condition when compared with the principal part of the operator. This kind of problems naturally appears when one looks for positive minima of a functional whose model is:

$$J(v) = \frac{\alpha}{2} \int_\Omega \frac{|Dv|^2}{(1+|v|)^2} + \frac{12}{\int_\Omega |v|^2} - \int_\Omega f\, v, \quad f \in L^2(\Omega)\,,$$

where in this case $a(x) \equiv b(x) = \alpha > 0$.

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1. Introduction

In this paper we prove existence results for a class of nonlinear elliptic problems with lower order term having quadratic (natural) growth with respect to the gradient and whose principal part is strongly degenerate. Our model problem is:

$$\begin{cases} -\operatorname{div}\left(\frac{\alpha Du}{(1+|u|)^2}\right) + u = \gamma\frac{|Du|^2}{(1+|u|)^3} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\alpha, \gamma > 0, f \in L^m(\Omega)$ with $m \geq 2$ and Ω is a bounded subset of $\mathbb{R}^N, N \geq 3$.

If $\alpha = \gamma$, such kind of problems naturally appear as the (formal) Euler–Lagrange equation satisfied by the positive minima of functionals J of the type

$$J(v) = \frac{1}{2}\int_\Omega \frac{\alpha|Dv|^2}{(1+|v|)^2} + \frac{1}{2}\int_\Omega |v|^2 - \int_\Omega f\, v.$$

Following the ideas of [7] the natural space to look for minima of J is the space $W_0^{1,1}(\Omega) \cap L^2(\Omega)$.

We deal with solutions of a class of problems that, in general, do not have a variational structure. Specifically, let $a(x)$ be a measurable function such that for some positive constants α and β it holds

$$0 < \alpha \le a(x) \le \beta. \tag{1.2}$$

Let also $H(x,s,\xi)$ be a Carathéodory function such that

$$|H(x,s,\xi)| \le \gamma \frac{|\xi|^2}{(1+|s|)^3}, \tag{1.3}$$

for some $\gamma > 0$. Suppose moreover that

$$f(x) \in L^m(\Omega), \quad m \ge 2. \tag{1.4}$$

Under the above assumptions, we study the existence of solutions u of the following boundary value problem:

$$\begin{cases} -\mathrm{div}\left(\frac{a(x)Du}{(1+|u|)^2}\right) + u = H(x,u,Du) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

in the following sense:

$$u \in W_0^{1,1}(\Omega) \cap L^2(\Omega), \text{ such that } \frac{|Du|^2}{(1+|u|)^3} \in L^1(\Omega),$$

and

$$\int_\Omega a(x)\frac{DuD\psi}{(1+|u|)^2} + \int_\Omega u\psi = \int_\Omega H(x,u,Du)\psi + \int_\Omega f\psi,$$

for every $\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Note that every term in the above integral identity makes sense. Indeed, by the assumptions (1.2) and (1.3), if $\frac{|Du|^2}{(1+|u|)^3} \in L^1(\Omega)$, then it follows that $\frac{Du}{(1+|u|)^2} \in L^2(\Omega)^N$ and $H(x,u,Du) \in L^1(\Omega)$.

We also observe that the presence of the zeroth order term in the left hand side in the equation is crucial. Indeed it gives a sort of coerciveness to the equation and it allows us to deduce that the differential operator is well defined in the space $W_0^{1,1}(\Omega) \cap L^2(\Omega)$. We basically recover the same type of results proved in [8] for $H(x,s,\xi) \equiv 0$. Equations with this type of degenerative coercivity have been introduced in [9] and also studied in [6].

Now we state our main results.

Theorem 1.1. *Assume that* (1.2), (1.3) *and* (1.4) *hold true. If*

$$\alpha\frac{m}{2}-\gamma>0, \tag{1.6}$$

then there exists a solution $u\in W_0^{1,1}(\Omega)\cap L^m(\Omega)$ *of* (1.5).
Moreover:

(1) *if* $2<m<4$, *then* $u\in W_0^{1,\frac{m}{2}}(\Omega)$;
(2) *if* $m\geq 4$, *then* $u\in W_0^{1,2}(\Omega)$;
(3) *if* $m\geq\max\{N,\frac{N}{2}(\frac{\gamma}{\alpha}+1)\}$, *then* $u\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

We first notice that assumption (1.6) relies to be a size condition on the right hand side in the equation of (1.5). Actually it means that the term $H(x,u,Du)$ cannot exceed the principal part of the operator, up to a coefficient $\frac{m}{2}$ that depends on the regularity of the datum f. This kind of condition is not completely new and naturally appears in this type of problems (see for instance [1] and [15]).

Note that if $f\in L^2(\Omega)$, the condition (1.6) implies that $\alpha>\gamma$ and thus Theorem 1.1 does not cover the case $m=2$ and $\alpha=\gamma$.

We show that the arguments used to prove the Theorem 1.1 can be adapted to the case $m=2$ and $\alpha=\gamma$ provided that f belongs to a slightly smaller space than $L^2(\Omega)$.

Theorem 1.2. *Assume that* (1.2) *and* (1.3) *hold with* $\alpha=\gamma$. *If*

$$\int_\Omega|f|^2\log(1+|f|)<\infty,$$

then there exists a solution of (1.5).

The technique we use to obtain the existence of a solution in Theorems 1.1 and 1.2 relies on approximate (1.5) by a sequence of nondegenerate problems with $L^\infty(\Omega)$—data for which we prove suitable a priori estimates and compactness results. Specifically, if $f_n(x)=\frac{f(x)}{1+\frac{1}{n}|f(x)|}$ and $H_n(x,s,\xi)=\frac{H(x,s,\xi)}{1+\frac{1}{n}|\xi|^2}$, we deal with a sequence $\{u_n\}\subset W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ such that:

$$\int_\Omega\frac{a(x)Du_nD\psi}{(1+T_n(|u_n|))^2}+\int_\Omega u_n\psi=\int_\Omega H_n(x,u_n,Du_n)\psi+\int_\Omega f_n\psi, \tag{1.7}$$

for any $\psi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, where for any $k\geq 0, T_k(s)$ is the classical truncation defined by $T_k(s)=\min\{k,\max\{-k,s\}\}$. Hereafter we also denote $G_k(s)=s-T_k(s)$.

We remark that the existence of such a sequence is a consequence of [12]. In order to prove that (1.5) is solvable, we pass to the limit in (1.7). The difficulty to make it is twofold. From one side we need to prove a strong L^1 compactness of the lower order term that has natural growth with respect to the gradient. On the other hand, the strong degeneracy of the principal part of the operator forces us to change framework. Indeed, we cannot expect the solutions to belong to $W_0^{1,2}(\Omega)$, but we look for them in a bigger space, namely $W_0^{1,1}(\Omega)\cap L^2(\Omega)$. It, consequently, implies that we need to prove a compactness result in such a space.

In addition, we prove that the condition on the integrability of $|f|^2\log(1+|f|)$ in Theorem 1.2 can be overcome in the case that (1.5) is variational. Specifically for $f\in L^2(\Omega)$, and $a(x)$ satisfying (1.2), we look for solutions of the equation

$$\int_\Omega \frac{a(x)DuD\varphi}{(1+|u|)^2}+\int_\Omega u\varphi=\int_\Omega f\varphi+\int_\Omega \frac{a(x)|Du|^2}{(1+|u|)^3}\text{sign}(u)\varphi, \tag{1.8}$$

for every $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Note that if $u\in W_0^{1,1}(\Omega)\cap L^2(\Omega)$ and $\frac{|Du|^2}{(1+|u|)^2}\in L^1(\Omega)$, then the functional J defined for $v\in W_0^{1,1}(\Omega)\cap L^2(\Omega)$ by

$$J(v)=\begin{cases}\dfrac12\displaystyle\int_\Omega \frac{a(x)|Dv|^2}{(1+|v|)^2}+\frac12\int_\Omega |v|^2-\int_\Omega f\,v, & \text{if } \int_\Omega \frac{|Dv|^2}{(1+|v|)^2}<+\infty,\\ +\infty, & \text{otherwise}\end{cases} \tag{1.9}$$

is differentiable at u along every direction $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ with derivative

$$\begin{aligned}&< J'(u),\varphi >\\ &\quad=\int_\Omega a(x)\frac{Du}{(1+|u|)^2}D\varphi+\int_\Omega u\varphi-\int_\Omega f\varphi-\int_\Omega a(x)\frac{|Du|^2}{(1+|u|)^3}\text{sign}(u)\varphi.\end{aligned}$$

Hence every minimum of J is a solution of (1.8).

In [7] it was proved the existence of a function $u\in W_0^{1,1}(\Omega)\cap L^2(\Omega)$ such that $J(u)\le J(v)$ for any $v\in W_0^{1,2}(\Omega)$. We slightly improve such a result (see Theorem 4.1 below) by showing that u is in fact a minimum for J in the whole $W_0^{1,1}(\Omega)\cap L^2(\Omega)$ and, therefore, a solution of (1.8).

Observe that in case that $f\ge 0$, it is not difficult to prove that the minimum u is positive ($J(u)\ge J(u^+)$). Hence, if furthermore the function $a(x)$ is constant, i.e. $a(x)\equiv\alpha>0$, then problem (1.8) relies to be in the limit case (condition (1.3) holds with $\alpha=\gamma$) and (1.8) becomes (1.1).

2. A priori estimates

In this section we assume (1.2), (1.3) and (1.4) and we suppose that

$$\alpha(m-1)-\gamma>0 \tag{2.1}$$

holds true. Notice that such an assumption is less restrictive than (1.6), since $m\ge 2$.

Lemma 2.1. *Assume that (1.2), (1.3), (1.4) and (2.1) hold true. If $u_n\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ is a solution of (1.7), then for any $k\ge 0$,*

$$\int_\Omega |G_k(u_n)|^m\le\int_{\{k\le|u_n|\}}|f|^m. \tag{2.2}$$

Moreover, there exist $R>0$ and C_0 depending on $\|f\|_{L^m(\Omega)},\alpha$ and γ, such that

$$\int_\Omega |Du_n|^2(1+|u_n|)^{m-4}\le R, \tag{2.3}$$

and, for any $k \geq 0$,

$$\int_\Omega |DT_k(u_n)|^2 \leq C_0\, k^3. \tag{2.4}$$

Proof. As we have already noticed, the existence of a solution $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ solving (1.7) is a consequence of [12].

Let us choose $\psi = |G_k(u_n)|^{m-2} G_k(u_n)$ as test function in (1.7), and use (1.2) and (1.3) to deduce that

$$(m-1)\alpha \int_\Omega \frac{|Du_n|^2}{(1+T_n(|u_n|))^2} |G_k(u_n)|^{m-2} + \int_\Omega |u_n||G_k(u_n)|^{m-1}$$
$$\leq \int_\Omega |f_n||G_k(u_n)|^{m-1} + \gamma \int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^3} |G_k(u_n)|^{m-2}|G_k(u_n)|.$$

Thus, joining the terms involving the gradient, we get

$$((m-1)\alpha - \gamma) \int_{\{|u_n| \geq k+1\}} \frac{|Du_n|^2}{(1+|u_n|)^2} + \int_\Omega |G_k(u_n)|^m$$
$$\leq \int_\Omega \left[(m-1)\alpha - \gamma \frac{|G_k(u_n)|}{1+|u_n|} \right] \frac{|Du_n|^2}{(1+|u_n|)^2} |G_k(u_n)|^{m-2} + \int_\Omega |u_n||G_k(u_n)|^{m-1}$$
$$\leq \int_\Omega |f_n||G_k(u_n)|^{m-1}.$$

By (2.1) the first term is positive and consequently, by (1.4) and Young inequality, we derive that (2.2) holds true for any $k \geq 0$.

Let us choose now $\psi = [(1+|u_n|)^{m-1} - 1]\,\mathrm{sign}(u_n)$ as test function in (1.7). Again, using (1.2), (1.3) and (1.4), we have by Hölder inequality

$$((m-1)\alpha - \gamma) \int_\Omega \frac{|Du_n|^2}{(1+|T_n(u_n)|)^2} (1+|u_n|)^{m-2}$$
$$\leq \|f\|_{L^m(\Omega)} \left(\int_\Omega (1+|u_n|)^m \right)^{1-\frac{1}{m}}. \tag{2.5}$$

By (2.2) with $k=0$, (2.3) follows.

Finally we prove (2.4). If $m \geq 4$, as a consequence of (2.3), the whole sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and (2.4) holds.

In the case $2 \leq m < 4$, consider $\psi = T_k(u_n)$ as test function in (1.7). Using as usual (1.2) (1.3) and (1.4), we deduce that

$$\alpha \int_{\{|u_n| \leq k\}} \frac{|DT_k(u_n)|^2}{(1+|u_n|)^2} + \int_\Omega u_n T_k(u_n) \leq k(\|f_n\|_{L^1(\Omega)} + \|H_n(x,u_n,Du_n)\|_{L^1(\Omega)})$$
$$\leq k(\|f\|_{L^1(\Omega)} + \gamma\, R),$$

which also implies (2.4). □

Remark 2.2. Since $m \geq 2$, the growth assumption (1.3) on $H(x,s,\xi)$ and (2.3) imply that there exists $M = M(R)$ such that

$$\int_\Omega |H_n(x,u_n,Du_n)| \leq \gamma \int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^3} \leq M. \tag{2.6}$$

As a consequence of the estimates of Lemma 2.1 we obtain the following convergence results.

Proposition 2.3. *Assume that* (1.2), (1.3), (1.4) *and* (2.1) *hold true. If* $\{u_n\}$ *is a sequence of solutions of* (1.7), *then there exists* $u \in W_0^{1,1}(\Omega) \cap L^m(\Omega)$ *such that, up to a subsequence,*

$$u_n \longrightarrow u \quad \text{in } W_0^{1,1}(\Omega) \quad \text{and in} \quad L^m(\Omega).$$

In addition:

(i) *if* $m > 2$, *then*

$$u_n \longrightarrow u \quad \text{weakly in } W_0^{1,q}(\Omega) \text{ with } q = \min\{\frac{m}{2}, 2\};$$

(ii) *if* $m > \max\{N, \frac{N}{2}\left(1 + \frac{\gamma}{\alpha}\right)\}$, *then* $u \in L^\infty(\Omega)$.

Remark 2.4. A consequence of the strong compactness in $L^1(\Omega)^N$ of Du_n is that, up to a subsequence, $Du_n(x) \to Du(x)$ a.e. in Ω.

Remark 2.5. By the previous remark and the Fatou lemma we deduce from (2.3) that

$$\int_\Omega |Du|^2(1 + |u|)^{4-m} \le R.$$

We also have, once again by (2.3) and the a.e. convergence of Du_n, that

$$\frac{Du_n}{(1 + |u_n|)^{2-\frac{m}{2}}} \longrightarrow \frac{Du}{(1 + |u|)^{2-\frac{m}{2}}} \quad \text{weakly in } L^2(\Omega)^N.$$

Proof of Proposition 2.3.

Step 1. *Almost everywhere convergence of the sequence* $\{u_n\}$.
We observe that, since $m \ge 2$, a consequence of the estimate (2.3) is

$$\int_\Omega \frac{|Du_n|^2}{(1 + |u_n|)^2} \le R.$$

So that the sequence $\{\log(1+|u_n|)\}$ is bounded in $W_0^{1,2}(\Omega)$ and, by the Rellich–Kondrakov compact embedding, it is compact in $L^p(\Omega)$, for any $1 \le p < 2^*$. Hence there exists a function u such that, up to subsequences,

$$u_n(x) \to u(x) \text{ a.e. in } \Omega.$$

Step 2. *Weak convergence of truncations* $T_k(u_n)$ *in* $W_0^{1,2}(\Omega)$.
Thanks to Fatou lemma we deduce from (2.2) that u belongs to $L^m(\Omega)$. Moreover by (2.4) we have that, up to subsequences,

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,2}(\Omega). \tag{2.7}$$

Step 3. *Strong convergence of* u_n *in* $L^m(\Omega)$.
Notice that, by (2.2) with $k = 0$, we have, for any $j > 0$

$$\text{meas}\{j \le |u_n|\} \le \frac{\|f\|^m_{L^m(\Omega)}}{j^m}.$$

Thus, for any $\varepsilon > 0$, there exists $j^*(\varepsilon)$, independent from n, such that for any $j \geq j^*(\varepsilon)$, $2^{m-1}\int_{\{|u_n|\geq j\}}|f|^m \leq \varepsilon$, thanks to the absolute continuity of the integral.

Hence, for every measurable subset $E \subset \Omega$ and any positive j we have, by (2.2), that

$$\int_E |u_n|^m \leq 2^{m-1}\int_E |T_j(u_n)|^m + 2^{m-1}\int_\Omega |G_j(u_n)|^m$$
$$\leq 2^{m-1}j^m \text{meas}(E) + 2^{m-1}\int_{\{|u_n|\geq j\}} |f|^m \leq 2^{m-1}j^m\text{meas}(E) + \varepsilon,$$

that implies the equi-integrability of the sequence $\{|u_n|^m\}$. The almost everywhere convergence of $u_n(x)$ to $u(x)$ and Vitali Theorem imply that u_n converges to u in $L^m(\Omega)$.

Step 4. *$Du_n \to Du$ weakly in $L^1(\Omega)^N$.*
We follow the ideas contained in [8]. For any measurable set $E \subset \Omega$ we have, using (2.3) with $m = 2$,

$$\int_E |Du_n| = \int_E \frac{|Du_n|}{1+|u_n|}(1+|u_n|)$$
$$\leq \left[\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^2}\right]^{\frac{1}{2}} \left[\int_E (1+|u_n|)^2\right]^{\frac{1}{2}} \leq R^{\frac{1}{2}}\left[\int_E (1+|u_n|)^2\right]^{\frac{1}{2}}.$$

Thanks to Vitali Theorem and by Step 3 we have that

$$\lim_{\text{meas}(E)\to 0}\int_E |Du_n| = 0, \quad \text{uniformly with respect to } n.$$

Thus by Dunford Pettis theorem, we deduce that Du exists in $L^1(\Omega)^N$ and that it is the weak-$L^1(\Omega)^N$ limit of Du_n.
Step 5. *Strong convergence of Du_n to Du in $L^1(\Omega)$.*
For the proof of this step, we have been inspired by [2] and [5]. For $k, h > 0$, let us choose $\psi = T_h[u_n - T_k(u)]$ as test function in (1.7) (this test function is admissible since $T_k(u) \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$) and we obtain

$$\int_\Omega \frac{a(x)Du_n}{(1+|T_n(u_n)|)^2}DT_h[u_n - T_k(u)] + \int_\Omega u_n T_h[u_n - T_k(u)] \leq hM, \qquad (2.8)$$

where $M = \|f\|_{L^1(\Omega)} + \gamma R$. By (2.7) we have

$$\lim_{n\to\infty}\int_\Omega \frac{a(x)DT_k(u)}{(1+|T_n(u_n)|)^2}DT_h[u_n - T_k(u)] = 0.$$

Moreover, thanks to the L^2 convergence of u_n, the second integral in (2.8) converges (as n diverges) to a positive number. Thus, it yields to

$$\alpha\int_\Omega \frac{|DT_h[u_n - T_k(u)]|^2}{(1+|u_n|)^2} \leq hM + \omega(n), \qquad (2.9)$$

where by $\omega(n)$ we denote any quantity that vanishes as n diverges. Hence, by Hölder inequality, we deduce that

$$\begin{aligned}\int_{\left\{\substack{|u_n-u|\le h\\ |u|\le k}\right\}} |D(u_n-u)| &= \int_\Omega |DT_h[u_n - T_k(u)]| \\ &\int_\Omega \frac{|DT_h[u_n-T_k(u)]|}{1+|u_n|}(1+|u_n|) \\ &\le \left[\int_\Omega \frac{|DT_h[u_n-T_k(u)]|^2}{(1+|u_n|)^2}\right]^{\frac{1}{2}} \left[\int_\Omega (1+|u_n|)^2\right]^{\frac{1}{2}} \le \sqrt{\frac{hM+\omega(n)}{\alpha}}C_f, \quad (2.10)\end{aligned}$$

where in the last inequality we have used (2.9) and (2.2) with $m = 2$ and $k = 0$.

Fix, now, $\epsilon > 0$ and $h > 0$ such that $C_f\sqrt{\frac{hM}{\alpha}} \le \varepsilon$. Thanks to Step 4 and the absolute continuity of the integral, there exists k^* (independent from n) such that, for $k > k^*$, we have

$$\int_{\{|u|>k\}} |Du_n| + \int_{\{|u|>k\}} |Du| \le \epsilon. \quad (2.11)$$

In addition, by the convergence in measure of u_n to u, Step 4 and once again by Dunford Pettis Theorem, we deduce that there exists $n(h, \epsilon)$ such that, for $n > n(h, \epsilon)$, we have

$$\int_{\{|u_n-u|>h\}} |D(u_n - u)| \le \epsilon. \quad (2.12)$$

As a consequence of (2.10), (2.11), (2.12), we conclude

$$\begin{aligned}&\int_\Omega |D(u_n-u)| \\ &= \int_{\left\{\substack{|u_n-u|\le h\\ |u|\le k}\right\}} |D(u_n-u)| + \int_{\left\{\substack{|u_n-u|\le h\\ |u|>k}\right\}} |D(u_n - u)| + \int_{\{|u_n-u|>h\}} |D(u_n - u)| \\ &\le 3\epsilon + \omega(n), \quad \forall\, n > n(h, \epsilon).\end{aligned}$$

This proves the strong convergence of Du_n to Du in $L^1(\Omega)^N$.

Step 6. *We prove, following the idea of [10], that (i) holds true.*

It is clear that if $m \ge 4$, then (2.3) directly implies that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$.

On the other hand, if $2 < m < 4$, we use Hölder inequality with exponents $\frac{4}{m}$ and $\frac{4}{4-m}$, to obtain

$$\begin{aligned}\int_\Omega |Du_n|^{\frac{m}{2}} &= \int_\Omega \frac{|Du_n|^{\frac{m}{2}}}{(1+|u_n|)^{\frac{m}{4}(4-m)}}(1+|u_n|)^{\frac{m}{4}(4-m)} \\ &\le \left[\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{4-m}}\right]^{\frac{4}{m}} \left[\int_\Omega (1+|u_n|)^m\right]^{1-\frac{4}{m}}.\end{aligned}$$

Hence, in this case, we conclude by using (2.2) and (2.3) that the sequence $\{u_n\}$ is bounded in $W_0^{1,\frac{m}{2}}(\Omega)$.

Therefore, in both cases, $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$, with $q = \min\{\frac{m}{2}, 2\}$, and consequently (i) is easily deduced.

Step 7. *Finally, we prove that (ii) holds.*
Since $m > \frac{N}{2}(1 + \frac{\gamma}{\alpha})$, we have $\frac{1}{2}\big(\frac{\gamma}{\alpha} - 1\big) < \frac{m}{N} - 1$. Let us choose $\sigma > 0$ such that

$$\frac{1}{2}\left(\frac{\gamma}{\alpha} - 1\right) < \sigma < \frac{m}{N} - 1.$$

By (1.2) and (1.3), if we take $[(1 + |u_n|)^{2\sigma+1} - (1 + k)^{2\sigma+1}]^{+}\text{sign}(u_n)$ as test function in (1.7), we get

$$\begin{aligned}(2\sigma+1)\alpha \int_\Omega |DG_k(u_n)|^2(1+|u_n|)^{2\sigma-2} + \int_\Omega |u_n|[(1 + |u_n|)^{2\sigma+1} - (1 + k)^{2\sigma+1}]^{+} \\ \leq \gamma \int_\Omega \frac{|DG_k(u_n)|^2}{(1+|u_n|)^3}(1 + |u_n|)^{2\sigma+1} + \int_\Omega |f_n|[(1+|u_n|)^{2\sigma+1} - (1 + k)^{2\sigma+1}]^{+}.\end{aligned}$$

By Sobolev and Young inequalities, we deduce that:

$$\begin{aligned}[(2\sigma + 1)\alpha - \gamma]\frac{\mathcal{S}^2}{\sigma^2}\left(\int_{\{k\leq|u_n|\}} [(1 + |u_n|)^\sigma - (1 + k)^\sigma]^{2^*}\right)^{\frac{2}{2^*}} \\ \leq \int_{\{k\leq|u_n|\}} C_1 + \int_{\{k\leq|u_n|\}} C_1\,|f|^{2\sigma+2} \leq C_m \text{meas}\,\{k \leq |u_n|\}^{1-\frac{2\sigma+2}{m}},\end{aligned}$$

where $\mathcal{S}$ denotes the best constant in Sobolev inequality. Defining $z_n = (1 + |u_n|)^\sigma$ and $h = (1 + k)^\sigma$, the above inequality becomes

$$[(2\sigma + 1)\alpha - \gamma]\frac{\mathcal{S}^2}{\sigma^2}\left(\int_{\{h\leq z_n\}} (z_n - h)^{2^*}\right)^{\frac{2}{2^*}} \leq C_m \text{meas}\{h \leq z_n\}^{1-\frac{2\sigma+2}{m}}.$$

Since the assumption $\sigma < \frac{m}{N} - 1$ implies that $\frac{2}{2^*} > 1 - \frac{2\sigma+2}{m}$, the boundedness of the sequence $\{z_n\}$ (and then of $\{u_n\}$) follows by Lemma 3.1 in [16]. □

Remark 2.6. Observe that if $2 < m < 4$, then it is easy to deduce the strong convergence of u_n to u in $W_0^{1,q}(\Omega)$ with $q = m/2$. Indeed, since we know that u_n is strongly compact in $L^m(\Omega)$ and its gradient a.e. converges in Ω, for any $E \subset \Omega$ measurable, we have by (2.3):

$$\int_E |Du_n|^{\frac{m}{2}} \leq \left[\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^{4-m}}\right]^{\frac{4}{m}} \left[\int_E (1+|u_n|)^m\right]^{1-\frac{4}{m}} \leq R^{\frac{4}{m}} C \left[\int_E |u_n|^m\right]^{1-\frac{4}{m}}$$

for a positive constant C. Hence by Vitali Theorem u_n is strongly compact in $W_0^{1,q}(\Omega)$.

3. Passing to the limit

This section is devoted to the proof of Theorem 1.1, which follows the ideas of [1], [4], [11], [13] and [14].

Proof of Theorem 1.1. We first observe that since $m \geq 2$, condition (1.6) implies (2.1). Hence the results of the previous section hold true. Let

$\alpha\frac{m}{2}-\gamma>0$ and let us define the following function:

$$g(s)=\begin{cases}\frac{1+s}{\alpha\frac{m}{2}-\gamma}, & \text{if } s\geq 0,\\ \frac{1}{(1-s)(\alpha\frac{m}{2}-\gamma)}, & \text{is } s<0.\end{cases} \tag{3.1}$$

Observe that $g\in C^1(\mathbb{R})$ verifies

$$\begin{cases}\alpha\frac{m}{2}g'(s)-\gamma\frac{g(s)}{1+|s|}>0, & \text{in } \mathbb{R},\\ g(s)>0, & \text{in } \mathbb{R}.\end{cases} \tag{3.2}$$

Moreover, we define the following family of cut-off functions:

$$R_k(s)=1-|T_1(G_k(s))|, \quad \forall k>0. \tag{3.3}$$

Clearly, $R_k\geq 0$, supp $R_k(s)\subset[-k-1,k+1]$ and

$$R'(s)=\begin{cases}1 & \text{if } -k-1\leq s\leq -k,\\ -1 & \text{if } \quad k\leq s\leq k+1,\\ 0 & \text{otherwise.}\end{cases} \tag{3.4}$$

To show that u is a solution of (1.5), the first part is the proof of the inequality

$$\int_\Omega \frac{a(x)DuD\phi}{(1+|u|)^2}+\int_\Omega u\phi\leq\int_\Omega H(x,u,Du)\phi+\int_\Omega f\phi, \tag{3.5}$$

for any $\phi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega), \phi\geq 0$.

First of all, note that the a.e. convergence of Du_n (see Remark 2.4) and its L^2 boundedness (see (2.5)) imply both

$$\frac{Du_n}{(1+T_n(|u_n|))^2}g^{\frac{m}{2}}(u_n)\longrightarrow\frac{Du}{(1+|u|)^2}g^{\frac{m}{2}}(u) \quad \text{weakly in } L^2(\Omega)^N, \tag{3.6}$$

and

$$\frac{Du_n}{(1+T_n(|u_n|))^2}\frac{1}{g^{\frac{m}{2}}(u_n)}\longrightarrow\frac{Du}{(1+|u|)^2}\frac{1}{g^{\frac{m}{2}}(u)} \quad \text{weakly in } L^2(\Omega)^N, \tag{3.7}$$

where g is the function defined in (3.1).

Recalling also the definition of R_k given (3.3) we use

$$\psi=\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k(u)\phi$$

as test function in (1.7) and we deduce that

$$\int_\Omega \frac{a(x)Du_nD\phi}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k(u) - \frac{m}{2}\int_\Omega \frac{a(x)Du_nDu}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}+1}(u)}g'(u)R_k(u)\phi$$
$$+\int_\Omega \frac{a(x)Du_nDu}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k'(u)\phi$$
$$+\frac{m}{2}\int_\Omega \frac{a(x)Du_nDu_n}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}-1}(u_n)}{g^{\frac{m}{2}}(u)}g'(u_n)R_k(u)\phi$$
$$-\int_\Omega H_n(x,u_n,Du_n)\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k(u)\phi$$
$$+\int_\Omega u_n\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k(u)\phi = \int_\Omega f_n\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k(u)\phi. \tag{3.8}$$

Using (1.2), (1.3), (1.6) and (3.2) we have

$$\left[\frac{m}{2}\frac{a(x)Du_nDu_n}{(1+T_n(|u_n|))^2}g'(u_n) - H_n(x,u_n,Du_n)g(u_n)\right]$$
$$\geq \left[\alpha\frac{m}{2}g'(u_n) - \gamma\frac{g(u_n)}{1+|u_n|}\right]\frac{|Du_n|^2}{(1+|u_n|)^2} \geq 0. \tag{3.9}$$

Hence, by Fatou lemma, we obtain

$$\liminf_{n\to\infty}\frac{m}{2}\int_\Omega \frac{a(x)Du_nDu_n}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}-1}(u_n)}{g^{\frac{m}{2}}(u)}g'(u_n)\phi - \int_\Omega H_n(x,u_n,Du_n)\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}\phi$$
$$\geq \frac{m}{2}\int_\Omega \frac{a(x)DuDu}{(1+|u|)^2}\frac{g'(u)}{g(u)}\phi - \int_\Omega H(x,u,Du)\phi. \tag{3.10}$$

In addition, since $\phi \in W_0^{1,2}(\Omega)$ and $\dfrac{R_k(u)}{g^{\frac{m}{2}}(u)}$ is bounded, by (3.6) we have

$$\lim_{n\to\infty}\int_\Omega \frac{a(x)Du_nD\phi}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k(u) = \int_\Omega \frac{a(x)DuD\phi}{(1+|u|)^2}R_k(u).$$

Similarly, using the strong $L^m(\Omega)$ compactness of $(u_n - f_n)$, we also deduce that

$$\lim_{n\to\infty}\int_\Omega (u_n - f_n)\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k(u)\phi = \int_\Omega (u-f)R_k(u)\phi. \tag{3.11}$$

On the other hand, it is not hard to see that

$$\frac{Du}{g^{\frac{m}{2}+1}(u)}g'(u)R_k(u) \in L^2(\Omega)^N,$$

so that, again by (3.6), we have:

$$\lim_{n\to\infty} -\frac{m}{2}\int_\Omega \frac{a(x)Du_nDu}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}+1}(u)}g'(u)R_k(u)\phi$$
$$= -\frac{m}{2}\int_\Omega \frac{a(x)DuDu}{(1+|u|)^2}\frac{g'(u)}{g(u)}R_k(u)\phi.$$

Moreover, we claim that

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_\Omega \frac{a(x)Du_nDu}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k'(u)\phi = 0. \tag{3.12}$$

Indeed, recalling (3.4) we deduce that

$$\lim_{n\to\infty}\int_\Omega \frac{a(x)Du_nDu}{(1+T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u_n)}{g^{\frac{m}{2}}(u)}R_k'(u)\phi = \int_\Omega \frac{a(x)DuDu}{(1+|u|)^2}R_k'(u)\phi.$$

Choosing $\psi = T_1(G_k(u_n))$ as test function in (1.7), we obtain

$$\int_{\{k\le|u_n|\le k+1\}}\frac{a(x)Du_nDu_n}{(1+T_n(|u_n|))^2}\le\int_{\{k\le|u_n|\}}|f_n|+\int_\Omega|H(x,u_n,Du_n)||T_1(G_k(u_n))|$$
$$\le\int_{\{k\le|u_n|\}}|f_n|+\int_{\{k\le|u_n|\}}\gamma\frac{|Du_n|^2}{(1+|u_n|)^3}\le\int_{\{k\le|u_n|\}}|f_n|+\int_{\{k\le|u_n|\}}\gamma\frac{|Du_n|^2}{(1+|u_n|)^2}.$$

Using Fatou lemma and Remark 2.5 we deduce that

$$\lim_{k\to\infty}\int_{\{k\le|u|\le k+1\}}\frac{a(x)DuDu}{(1+|u|)^2}=0,$$

and (3.12) follows.

Gathering together (3.10), (3.11), (3.12) and passing to the limit in (3.8) first with respect to n and after as k tends to infinity, we deduce that (3.5) holds true.

We want, now, to show that the reverse inequality holds true, namely that

$$\int_\Omega\frac{a(x)}{(1+u)^2}DuD\phi+\int_\Omega u\phi\ge\int_\Omega H(x,u,Du)\phi+\int_\Omega f\phi, \tag{3.13}$$

for any $\phi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega), \phi\ge 0$. In order prove such inequality we choose

$$\psi=\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}}(u_n)}R_k(u)\phi$$

as test function in (1.7) and we get

$$\int_\Omega\frac{a(x)Du_nD\phi}{(1{+}T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}}(u_n)}R_k(u)+\int_\Omega\frac{a(x)Du_nDu}{(1{+}T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}}(u_n)}R_k'(u)\phi$$
$$+\frac{m}{2}\int_\Omega\frac{a(x)Du_nDu}{(1{+}T_n(|u_n|))^2}\frac{g^{\frac{m}{2}-1}(u)}{g^{\frac{m}{2}}(u_n)}g'(u)R_k(u)\phi$$
$$+\int_\Omega u_n\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}}(u_n)}R_k(u)\phi-\int_\Omega f_n\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}}(u_n)}R_k(u)\phi$$
$$=\frac{m}{2}\int_\Omega\frac{a(x)Du_nDu_n}{(1{+}T_n(|u_n|))^2}\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}+1}(u_n)}g'(u_n)R_k(u)\phi$$
$$+\int_\Omega H_n(x,u_n,Du_n)\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}}(u_n)}R_k(u)\phi. \tag{3.14}$$

We follow, step by step, the proof of inequality (3.5) with just a small change. Indeed we can pass to the limit in the left hand side above exploiting that (3.11) holds and using (3.7).

In order to get rid of the right hand side of (3.14) we use Fatou lemma, using that

$$\begin{aligned}&\left[\frac{m}{2}\frac{a(x)Du_nDu_n}{(1+T_n(|u_n|))^2}g'(u_n)+H_n(x,u_n,Du_n)g(u_n)\right]R_k(u)\phi\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}+1}(u_n)}\\&\quad\geq\left[\alpha\frac{m}{2}g'(u_n)R_k(u)-\gamma g(u_n)\right]R_k(u)\phi\frac{g^{\frac{m}{2}}(u)}{g^{\frac{m}{2}+1}(u_n)}\frac{|Du_n|^2}{(1+|u_n|)^2}\geq 0,\end{aligned}$$

where last inequality holds since g verifies (3.2). By applying Fatou lemma we deduce the convergence of the right hand side of (3.14) and consequently (3.13) follows.

Gathering together (3.5) and (3.13) we deduce that $u\in W_0^{1,1}(\Omega)\cap L^m(\Omega)$ solves (1.5) for any test function $\phi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ positive. In order to generalize it to any test function (without any restriction on its sign) we just have to deal first with its positive part and then with its negative part.

Finally, the regularity results (1)–(3) of Theorem 1.1 are a direct consequence of Proposition 2.3. □

4. The limit case

This section is devoted to prove the results in the limit case. We start by proving Theorem 1.2, that is, the existence of a solution in the case $\alpha=\gamma$ if the datum belongs to a smaller space than $L^2(\Omega)$.

Proof of Theorem 1.2. The proof follows the steps of Theorem 1.1.

Step 1. u_n *a.e. converges to* $u\in L^2(\Omega)$.

Let us consider the real function η given by

$$\eta(t)=(1+|t|)\log(1+|t|)\text{sign}(t),\quad t\in\mathbb{R}.$$

Choosing $\psi=\eta(u_n)$ as test function in (1.7), we deduce, by (1.2) and (1.3) with $\alpha=\gamma$, that

$$\alpha\int_\Omega\frac{|Du_n|^2}{(1+|u_n|)^2}\eta'(u_n)+\int_\Omega|u_n||\eta(u_n)|\leq\int_\Omega|f_n||\eta(u_n)|+\alpha\int_\Omega\frac{|Du_n|^2}{(1+|u_n|)^3}\eta(u_n).$$

Since $\eta'(t)-\dfrac{|\eta(t)|}{1+|t|}=1$ for every $t\in\mathbb{R}$, we derive that

$$\begin{aligned}&\alpha\int_\Omega\frac{|Du_n|^2}{(1+|u_n|)^2}+\frac{1}{2}\int_\Omega|u_n||\eta(u_n)|\\&\quad\leq\int_{\{\frac12|u_n|\leq|f|\}}|f||\eta(u_n)|+\int_{\{\frac12|u_n|\geq|f|\}}|f||\eta(u_n)|\\&\quad\leq\int_{\{\frac12|u_n|\leq|f|\}}|f||\eta(2|f|)|+\frac{1}{2}\int_{\{\frac12|u_n|\geq|f|\}}|u_n||\eta(u_n)|\end{aligned}$$

and, consequently, there exists a constant $C_1 > 0$ such that

$$\alpha \int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^2} + \frac{1}{2}\int_\Omega |u_n||\eta(u_n)| \leq C_1\Big(1 + \int_\Omega |f|^2 \log(1+|f|)\Big).$$

This implies that $\log(1+|u_n|)$ is bounded in $W_0^{1,2}(\Omega)$ and Step 1 is concluded.

Step 2. A priori estimates.

Choosing now, for any $k \geq 0$, $\psi = \eta(G_k(u_n))$ as test function in (1.7) and arguing as in the above Step 1, we deduce that

$$\int_\Omega \frac{|DG_k(u_n)|^2}{(1+|u_n|)^2} + \frac{1}{2}\int_\Omega |u_n||\eta(G_k(u_n))| \leq C_1\Big(1 + \int_{\{k\leq|u_n|\}} |f|^2 \log(1+|f|)\Big).$$

Step 3. *Du_n weakly converges to Du in $L^1(\Omega)^N$.*

Arguing as in Steps 3 and 4 of the proof of Proposition 2.3, we get the strong compactness of the sequence $\{u_n\}$ in $L^2(\Omega)$ and the weak $L^1(\Omega)^N$ convergence of Du_n is deduced.

Step 4. *u is a solution of* (1.5).

Following the ideas of Step 5 of Proposition 2.3, it is easy to show the strong compactness of u_n in $W_0^{1,1}(\Omega)$.

Thus, in order to prove that u turns out to be a solution of (3.13), we first show that it is a subsolution of (1.5), i.e. u satisfies (3.5). To make it, we follow the ideas of the proof in Theorem 1.1 by choosing in this case, for any positive k, the test function

$$\psi = \frac{h(u_n)}{h(u)} R_k(u)\phi$$

in (1.7), where $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \phi \geq 0$, $R_k(s)$ is the function defined in (3.3) and $h(s)$ is given by

$$h(s) = \begin{cases} 1+s & \text{if } s \geq 0 \\ \dfrac{1}{1-s} & \text{if } s < 0. \end{cases} \tag{4.1}$$

Notice that h is positive, increasing and it verifies

$$h'(s) - \frac{h(s)}{1+|s|} \geq 0 \qquad \text{in } \mathbb{R}.$$

Hence, we perform the same proof of of the first part of Theorem 1.1 and we get (3.5).

Analogously, in order to prove that u is supersolution of (1.5) [i.e., u verifies (3.13)] we deal with the following test function:

$$\psi = \frac{h(u)}{h(u_n)} R_k(u)\phi$$

in (1.7), where $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \phi \geq 0, R_k(s)$ is defined in (3.3) and $h(s)$ in (4.1). Following the second part of the proof of Theorem 1.1, we obtain (3.13). Thus we deduce the existence of a solution for (1.5) just by gathering together (3.5) and (3.13) and dealing, as before, with the positive and negative part of ϕ. □

Finally, we prove the existence of a minimum in $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ of the functional J given by (1.9). As it has been mentioned it implies the existence of a solution for (1.8). The proof is essentially contained in [7].

Theorem 4.1. *If $f \in L^2(\Omega)$ and $a(x)$ satisfies (1.2), then there exists a solution of (1.8) which is a minimum in $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ of the functional J defined in* (1.9).

Proof. In [7] it is proved the existence of a function $u \in W_0^{1,1}(\Omega) \cap L^2(\Omega)$ such that

$$J(u) \leq J(v), \quad \forall v \in W_0^{1,2}(\Omega), \tag{4.2}$$

where J is the functional given by (1.9).

We claim that u is a minimum in the whole $W_0^{1,1}(\Omega) \cap L^2(\Omega)$. Indeed, assume by contradiction that there exists $w \in W_0^{1,1}(\Omega) \cap L^2(\Omega)$ such that $J(w) < J(u) < +\infty$. This implies that

$$\int_\Omega \frac{a(x)|Dw|^2}{(1+|w|)^2} < +\infty, \tag{4.3}$$

and consequently

$$\limsup_{k\to\infty} J(T_k(w)) \leq J(w) + \lim_{k\to\infty} \int_\Omega f G_k(w) = J(w) < J(u).$$

In particular, if we fix k large enough, the function $z := T_k(w) \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$ satisfies

$$J(z) < J(u). \tag{4.4}$$

Since $z \in L^\infty(\Omega)$ and (4.3) holds true, we get from (1.2) that

$$\frac{\alpha}{(1+\|z\|_\infty)^2} \int_\Omega |Dz|^2 \leq \int_\Omega \frac{a(x)|Dz|^2}{(1+|z|)^2} < +\infty,$$

i.e., $z \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Therefore (4.4) contradicts (4.2) proving that necessarily

$$J(u) \leq J(v), \quad \forall v \in W_0^{1,1}(\Omega) \cap L^2(\Omega)$$

and, thus, u satisfies (1.8). □

Remark 4.2. Observe that, if the solution is bounded, then the principal part of the operator is not anymore degenerate and the existence of a solution of (1.1) becomes trivial. Hence, we want to stress that if the datum on the right hand side is not sufficiently regular, then unbounded solutions may exist. Indeed assume that $f(x) = \frac{\alpha(N-1)+1-\gamma}{|x|} - 1$, then a positive solution of

$$-\text{div}\left(\alpha \frac{\nabla u}{(1+u)^2}\right) + u - \gamma \frac{|\nabla u|^2}{(1+u)^3} = f(x) \quad \text{in } B(0,1),$$

is $u(|x|) = \frac{1}{|x|} - 1$. Since f belongs to $L^m(B(0,1))$, $\forall 1 < m < N$, according to the hypotheses of Theorem 1.1, we can only deduce that u belongs to $W_0^{1,2}(\Omega) \cap L^m(B(0,1))$, but not to $L^\infty(B(0,1))$.

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