Saddle solutions for bistable symmetric semilinear elliptic equations

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Abstract. This paper concerns the existence and asymptotic characterization of saddle solutions in $\mathbb{R}^3$ for semilinear elliptic equations of the form

$$-\Delta u + W'(u) = 0, \quad (x, y, z) \in \mathbb{R}^3 \quad (0.1)$$

where $W \in C^3(\mathbb{R})$ is a double well symmetric potential, i.e. it satisfies $W(-s) = W(s)$ for $s \in \mathbb{R}$, $W(s) > 0$ for $s \in (-1, 1)$, $W(\pm 1) = 0$ and $W''(\pm 1) > 0$. Denoted with θ_2 the saddle planar solution of (0.1), we show the existence of a unique solution $\theta_3 \in C^2(\mathbb{R}^3)$ which is odd with respect to each variable, symmetric with respect to the diagonal planes, verifies $0 < \theta_3(x, y, z) < 1$ for $x, y, z > 0$ and $\theta_3(x, y, z) \rightarrow_{z \rightarrow +\infty} \theta_2(x, y)$ uniformly with respect to $(x, y) \in \mathbb{R}^2$.

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1. Introduction

We consider on $\mathbb{R}^3$, semilinear elliptic equations of the form

$$-\Delta u + W'(u) = 0 \quad (1.1)$$

assuming that W is a double well symmetric potential, i.e.

$$\begin{aligned} W \in C^3(\mathbb{R}) \text{ satisfies } W(-s) = W(s) \text{ for } s \in \mathbb{R}, W(s) > 0 \text{ for } s \in (-1, 1), \\ W(\pm 1) = 0 \text{ and } W''(\pm 1) > 0. \end{aligned} \quad (W)$$

The problem of looking for and classifying entire bounded solutions of this kind of equations in $\mathbb{R}^n$ has been extensively studied in the last years, providing a rich amount of differently shaped families of solutions. In particular, a

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long standing problem pointed out by De Giorgi in [16], is to characterize the set of the solutions $u \in C^2(\mathbb{R}^n)$ of (1.1), satisfying $|u(x)| \leq 1$ and $\partial_{x_1} u(x) > 0$ in $\mathbb{R}^n$. In [16], De Giorgi conjectured that if $n \leq 8$ and $W(s) = (s^2 - 1)^2$, any solutions of (1.1) of that kind, coincides up to space roto-translations with the unique solution θ_1 of the one dimensional heteroclinic problem

$$\begin{cases} -\ddot{q}(x) + W'(q(x)) = 0, & x \in \mathbb{R}, \\ q(0) = 0 \text{ and } q(\pm\infty) = \pm 1, \end{cases} \quad (1.3)$$

The conjecture has been proved for $n = 2$ by Ghoussoub and Gui in [29] and then by Ambrosio and Cabré in [8] (see also [2]) for $n = 3$, even for more general double well potentials W . A further step in the proof of the De Giorgi conjecture has been done by Savin in [39] where, when $n \leq 8$, the same one dimensional symmetry is obtained for solutions u such that $\partial_{x_1} u(x) > 0$ on $\mathbb{R}^n$ and $\lim_{x_1 \rightarrow \pm\infty} u(x) = \pm 1$ for all $(x_2, x_3, \dots, x_n) \in \mathbb{R}^{n-1}$ (see [9, 11, 24–27, 40] for related problems). That result is completed in [17, 18] where the existence of entire solutions without one dimensional symmetry which are increasing and asymptotic to ± 1 with respect to the first variable is proved in dimension $n > 8$.

Non monotonic solutions of equation (1.1) are the *saddle solutions*, introduced by Dang et al. in [15]. In that paper the Authors consider potentials W satisfying (W) together with the condition

$$W''(0) < 0 \text{ and the function } u \rightarrow W'(u)/u \text{ is strictly increasing on } (0, 1). \quad (W1)$$

Under these assumptions They prove that (1.1) has a unique solution $u \in C^2(\mathbb{R}^2)$ which is odd w.r.t. x and y and which is positive when $xy > 0$. In particular the saddle solution u has the same sign of the function xy . The monotonicity of the function $W'(u)/u$ allows the Authors to prove their result by the use of a supersolution–subsolution method.

Assuming (W)–(W1), Schatzman in [41] studied the stability properties of the saddle solution proving that its Morse index is at least equal to one and, more recently, Kowalczyk and Liu in [33] proved that the saddle solution is not degenerate in the sense that the corresponding linearized operator has zero kernel in $L^2(\mathbb{R}^2)$. Cabré and Terra in [13, 14] found in all even dimensions saddle type solutions (vanishing on the Simons' cones) characterizing their instability for dimensions $n = 4, 6$. A vectorial version of the result in [15] has been obtained by Alama et al. in [1] where systems of equations of the type (1.1) have been studied. We refer also to [30] where, among other results, existence, uniqueness and qualitative properties of the saddle solution of (1.1) has been studied by Gui assuming (W) and

$$W'(u) \neq 0 \quad \text{for } u \in (0, 1). \quad (W2)$$

The saddle solutions are examples of four ends solutions, which, roughly speaking, are planar solutions characterized by the fact that their nodal set is asymptotic at infinity to four half lines. This kind of solutions of (1.1) has been studied assuming (W) and (W2) in [20, 31, 34, 35] (see also [19]). In particular,

in [31] Gui proved that any four ends solution is, modulo roto-translations, even in its variables and monotonic in x and y on the first quadrant. In [35] it is shown that the set of even four end solutions constitutes a continuum containing the saddle solution.

The existence of more general $2k$ ends solutions with nearly parallel ends has been obtained in [20] for every $k \in \mathbb{N}$ in the case of the Allen Cahn potential. Other $2k$ ends solutions, built in [3] for potential satisfying (W) , are saddle type solutions with nodal set equal to the union of k lines through the origin forming angles equal to π/k .

All these solutions are characterized by the fact that along directions parallel to the end lines they are asymptotic to the one dimensional solution θ_1 (i.e. the saddle solution in dimension 1), suitably reflected and rotated, and along different directions they tend to ± 1 in an alternate way with respect to the end lines. In connection with the De Giorgi conjecture we remark that if $k > 1$ is odd, up to a rotation, the $2k$ ends solutions are example of non monotonic planar solution $u(x, y)$ of (1.1) which satisfy the asymptotic condition $\lim_{x \rightarrow \pm\infty} u(x, y) = \pm 1$ for every $y \in \mathbb{R}$.

In the present work we introduce a global variational approach suitable to study under the assumption (W) the problem of existence of a saddle type solutions of (1.1) in $\mathbb{R}^3$ providing directly a characterization of its asymptotic behaviour. Indeed, as showed in [3] and recalled in Sect. 3.1 below, the assumption (W) is sufficient to obtain existence and uniqueness of the planar saddle solution of (1.1) which we denote $\theta_2(x, y)$. The main Theorem of the paper is the following one

Theorem 1.1. *Assuming (W) there exists a unique solution $\theta_3 \in C^2(\mathbb{R}^3)$ of (1.1) verifying*

- i) $|\theta_3(x, y, z)| < 1$ on $\mathbb{R}^3$ and $\theta_3(x, y, z) > 0$ for $x, y, z > 0$,
- ii) θ_3 is odd with respect to each variable and symmetric with respect to the diagonal planes $x = y$, $x = z$ and $y = z$,
- iii) $\theta_3(x, y, z) \rightarrow \theta_2(x, y)$ as $z \rightarrow +\infty$ uniformly with respect to $(x, y) \in \mathbb{R}^2$.

Moreover $\theta_3(x, y, z)$ is increasing in each variable on the set $\{(x, y, z) / x, y, z > 0\}$.

We note that, since θ_3 is odd with respect to each of its variable and strictly positive on the set $\{(x, y, z) / x, y, z > 0\}$, there results that θ_3 has the same sign of the product function xyz and its nodal set coincides with the union of the three cartesian planes. We remark that the mere existence of solutions of that kind could be proved in $\mathbb{R}^n$ also for $n \geq 3$, at least in the case in which also $(W2)$ holds true, by using a suitable direct minimization argument on balls of radius R and then considering the limit as R goes to infinity.

The more tricky and interesting result in Theorem 1.1 is the asymptotic characterization of θ_3 given by property (iii) (see [32] for related results in the vectorial setting). Thanks to the symmetry properties of θ_3 , (iii) tells us, analogously to the planar case, that along directions parallel to the axes, θ_3 is uniformly asymptotic to the two dimensional saddle solution θ_2 , suitably

reflected and rotated. In this line one can try to generalize Theorem 1.1 in different ways. For example, analogous to the k -end planar solutions, one can try to find entire solutions on $\mathbb{R}^3$ which along different directions are asymptotic to θ_2 (reflected and rotated), or, one can ask whether an inductive statement of the same type holds true in any dimension: *if θ_{n-1} is a saddle solution in $\mathbb{R}^{n-1}$ then there exists a unique θ_n satisfying (i) and (ii) in $\mathbb{R}^n$ and such that $\theta_n(x_1, \dots, x_n) \rightarrow_{x_n \rightarrow +\infty} \theta_{n-1}(x_1, \dots, x_{n-1})$ for $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.*

We remark also that, thanks to (iii), θ_3 is asymptotic to ± 1 along directions parallel to the space diagonals and so, again in connection with the De Giorgi conjecture, rotating the saddle solution θ_3 we find an example in $\mathbb{R}^3$ of non monotonic entire solution $u(x, y, z)$ of (1.1) which satisfy the asymptotic condition $\lim_{x \rightarrow \pm\infty} u(x, y, z) = \pm 1$ for every $(y, z) \in \mathbb{R}^2$.

A last comment regards the uniqueness of θ_3 which is obviously linked again to property (iii). Indeed we note that since only (W) is assumed, up to dilation and the addition of a constant to W , Theorem 1.1 applies to general even potential $W \in C^3(R)$ having a local non degenerate minimum point $a > 0$ for which $W(s) > W(a)$ for $s \in (-a, a)$. In the case in which W has different non degenerate positive minimum points $a_i > 0$ satisfying that property, Theorem 1.1 gives different solutions $\theta_{3,i}$ of (1.1), all odd with respect to each variable and positive on the set $\{(x, y, z) / x, y, z > 0\}$, but clearly characterized by different asymptotic behaviours.

To prove Theorem 1.1 we develop a global variational procedure, inspired to the ones introduced in [3, 7], which allows us to recover the saddle type solution θ_3 from the minimum of a suitably renormalized action functional (for the use of renormalized functionals in different contexts we also refer to [4–6, 36] and to the comprehensive recent monograph [37]). We look for minima of the double renormalized functional

$$F_{\infty,3}(u) = \int_0^{+\infty} \left[\int_0^z \left[\int_{-y}^y \frac{1}{2} |\nabla u(x, y, z)|^2 + a(x)W(u(x, y, z)) dx - c_{1,y} \right] dy - c_{2,z} \right] dz$$

on the class $\Gamma_{\infty,3} = \{u \in H_{loc}^1(T_{\infty,3}) / u(x, y, z) = -u(-x, y, z) \text{ and } 0 \leq u(x, y, z) \leq 1 \text{ for } x > 0\}$, where $T_{\infty,3} = \{(x, y, z) \in \mathbb{R}^3 / 0 < |x| < y < z\}$. In the definition of $F_{\infty,3}$ enter the two renormalizing functions $c_{1,y}$ and $c_{2,z}$ which are related to minimal problems in dimension 1 and 2. Precisely

$$c_{1,y} = \min \left\{ \int_{-y}^y \frac{1}{2} |\dot{q}|^2 + W(q) dx / q \in \Gamma_{y,1} \right\}$$

where $\Gamma_{y,1} = \{q \in H^1(-y, y) / q(x) = -q(-x), 0 \leq q(x) \leq 1 \text{ for } x > 0\}$, and

$$c_{2,z} = \min \left\{ \int_0^z \int_{-y}^y \frac{1}{2} |\nabla u|^2 + W(u) dx - c_{1,y} dy / u \in \Gamma_{z,2} \right\}$$

where $\Gamma_{z,2} = \{u \in H^1(T_{z,2}) / u(x, y) = -u(-x, y), 0 \leq u(x, y) \leq 1 \text{ for } x > 0\}$ and $T_{z,2} = \{(x, y) \in \mathbb{R}^2 / 0 < |x| < y < z\}$.

This global variational approach allows us to directly control the asymptotes as $z \rightarrow +\infty$ of the minima of $F_{\infty,3}$. Indeed, if u is a minimum, then $\sup_{(x,y) \in T_{2,z}} |v(x, y, z) - \theta_2(x, y)| \rightarrow 0$ as $z \rightarrow +\infty$ (see Lemmas 4.2 and 4.3 below). Moreover a minimum of $F_{\infty,3}$ is in fact minimal with respect to $C_0^\infty(\mathbb{R}^3)$ perturbation (see Lemma 4.1 below) and the saddle solution θ_3 is recovered from it by recursive reflections with respect to the faces of $T_{\infty,3}$. The uniqueness and the monotonicity properties of θ_3 are obtained by standard elliptic techniques.

We remark that, even if obtained by recursive reflections of a minimal solution, the solution θ_3 is no more minimal for the action with respect to $C_0^\infty(\mathbb{R}^3)$ perturbations. In fact, since finite Morse index solutions are necessarily stable at infinity and since θ_2 has Morse index at least equal to 1 (see [41]), θ_3 has infinite Morse index. For finite Morse index solutions in $\mathbb{R}^3$ we refer to the papers by Del Pino et al. [22, 23], where solutions concentrating around complete embedded minimal surfaces of finite total curvature are constructed.

The paper is organized as follows. Sections 2 and 3 are devoted to recall and study a list of refined properties of the one dimensional and two dimensional problems, characterizing in particular the asymptotic properties of the minimal functions $c_{1,y}$ and $c_{2,z}$. The double renormalized functional $F_{\infty,3}$ is defined and studied in Sect. 4, where we complete the proof of Theorem 1.1.

Remark 1.1. We precise now some basic consequences of the assumption (W) fixing some constants which will remain unchanged in the rest of the paper. First, since $W \in C^3(\mathbb{R})$ and $W''(\pm 1) > 0$, there exists $\bar{\delta} \in (0, \frac{1}{4})$ such that setting $\underline{w} = W''(\pm 1)/2$ and $\bar{w} = 3\underline{w}$ we have

$$\bar{w} \geq W''(s) \geq \underline{w} \quad \text{for any } |s| \in [1 - 2\bar{\delta}, 1 + 2\bar{\delta}]. \quad (1.6)$$

In particular, since $W(\pm 1) = W'(\pm 1) = 0$, setting $\chi(s) = \min\{|1-s|, |1+s|\}$, we have that

$$\frac{w}{2} \chi(s)^2 \leq W(s) \leq \frac{\bar{w}}{2} \chi(s)^2 \quad \text{and} \quad |W'(s)| \leq \bar{w} \chi(s), \quad \forall |s| \in [1 - 2\bar{\delta}, 1 + 2\bar{\delta}]. \quad (1.7)$$

Finally, since we are looking for solution u such that $|u(x, y, z)| \leq 1$, by (W) it is not restrictive to assume, and in fact we will do it below, that $W(s) \geq 0$ for $s \in \mathbb{R}$.

2. The one dimensional problem

In this section we describe some results concerning the one dimensional ODE associated to (1.1).

2.1. The heteroclinic problem.

We consider the heteroclinic ODE problem associated to (1.1)

$$\begin{cases} \ddot{q}(t) = W'(q(t)) & \text{for any } t \in \mathbb{R} \\ \lim_{t \rightarrow \pm\infty} q(t) = \pm 1 \end{cases} \quad (P_{\infty,1})$$

It is well known, that the problem admits a unique solution modulo time translation and we denote θ_1 the solution which satisfies $\theta_1(0) = 0$. We recall here below some relevant and known properties of the solution θ_1 .

Remark 2.1. (*Symmetry, monotonicity and exponential decaying of θ_1*). The function θ_1 is odd and increasing:

$$\theta_1(x) = -\theta_1(-x) \quad \text{and} \quad \dot{\theta}_1(x) > 0 \quad \text{for every } x \in \mathbb{R}.$$

In particular $|\theta_1(x)| < 1$. Moreover θ_1 behaves exponentially at infinity verifying that there exists a constant $C > 0$ such that

$$0 \leq 1 - \theta_1(x) \leq Ce^{-\sqrt{\frac{w}{2}}x} \quad \text{and} \quad 0 < \dot{\theta}_1(x) \leq Ce^{-\sqrt{\frac{w}{2}}x} \quad \text{for all } x > 0.$$

Remark 2.2. (*Non Degeneracy property of θ_1*) The set of bounded solutions of the linearized equation about θ_1 , $-\ddot{h}(x) + W''(\theta_1(x))h(x) = 0$, $x \in \mathbb{R}$, is one dimensional and contains the function θ_1 .

Remark 2.3. (*Minimality of θ_1*). We consider the action functional

$$F_{\infty,1}(q) = \int_{\mathbb{R}} \frac{1}{2} |\dot{q}(x)|^2 + W(q(x)) \, dx.$$

on the space

$$\mathcal{H} = z_0 + H^1(\mathbb{R}),$$

where $z_0 \in C^\infty(\mathbb{R})$ is a fixed odd increasing function such that $z_0(x) \rightarrow \pm 1$, as $x \rightarrow \pm\infty$, and $|z_0(x)| = 1$ for any $|x| \geq 1$. Setting

$$c_{\infty,1} = \inf_{\mathcal{H}} F_{\infty,1}$$

we have $\theta_1 \in \mathcal{H}$ and $F_{\infty,1}(\theta_1) = c_{\infty,1}$, i.e., θ_1 is a minimum of $F_{\infty,1}$ on $\mathcal{H}$. Note that $\mathcal{H}$ is an affine Hilbertian Manifold if endowed with the $H^1(\mathbb{R})$ scalar product and $F \in C^2(\mathcal{H})$ with

$$\begin{aligned} F'_{\infty,1}(\theta_1)h &= \int_{\mathbb{R}} \dot{\theta}_1 \dot{h} + W'(\theta_1)h \, dx \quad \text{and} \\ F''_{\infty,1}(\theta_1)hk &= \int_{\mathbb{R}} \dot{h} \dot{k} + W''(\theta_1)hk \, dx \quad \text{for any } h, k \in H^1(\mathbb{R}). \end{aligned}$$

Since θ_1 is a minimum of $F_{\infty,1}$ on $\mathcal{H}$, we have in particular that $F'_{\infty,1}(\theta_1) = 0$ and that $F''_{\infty,1}(\theta_1)hh \geq 0$ for any $h \in H^1(\mathbb{R})$.

Lemma 2.1. (Stability of θ_1). *There exists $\lambda_1 \in (0, 2\underline{w}]$ such that for all odd functions $h \in H^1(\mathbb{R})$ there results*

$$F''_{\infty,1}(\theta_1)hh = \int_{\mathbb{R}} |\dot{h}|^2 + W''(\theta_1)hh \, dx \geq \lambda_1 \|h\|_{L^2(\mathbb{R})}^2.$$

Proof. Indeed, recalling that $2\underline{w} = W''(\pm 1)$, since $\theta_1(x) \rightarrow \pm 1$ as $x \rightarrow \pm\infty$, we recover that the selfadjoint operator $\mathcal{L}_{\infty,1} : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$,

$$\mathcal{L}_{\infty,1}(h) = -\ddot{h} + W''(\theta_1)h = -\Delta h + 2\underline{w}h + (W''(\theta_1) - 2\underline{w})h$$

is a compact perturbation of the operator $\mathring{A} : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $\mathring{A}h = -\ddot{h} + 2\underline{w}h$. Then, the essential spectrum of $\mathcal{L}_{\infty,1}$ coincides with the one of $\mathring{A}$, i.e., $\sigma_{ess}(\mathcal{L}_{\infty,1}) = \sigma_{ess}(\mathring{A}) = [2\underline{w}, +\infty)$. By the minimality property of θ_1 (Remark 2.3) we know that $\langle \mathcal{L}_{\infty,1}(h), h \rangle_{L^2(\mathbb{R})} = F''_{\infty,1}(\theta_1)hh \geq 0$ for any $h \in H^2(\mathbb{R})$ and by the non degeneracy property (Remark 2.2) we conclude that $\lambda_0 = 0$ is a simple eigenvalue whose eigenspace is spanned by $\dot{\theta}_1$.

To characterize the remaining part of $\sigma(\mathcal{L}_{\infty,1})$ we can use the min-max principle ([38], Theorem XIII.1) to conclude that

$$\sigma(\mathcal{L}_{\infty,1}) \cap (-\infty, 2\underline{w}] = \{\lambda_0 \leq \lambda_1 \leq \dots\}$$

where

$$\lambda_j = \sup_{V \subset H^2(\mathbb{R}), \dim V = j} \inf_{\psi \perp V, \|\psi\|=1} \langle \mathcal{L}_{\infty,1}\psi, \psi \rangle_{L^2(\mathbb{R})}$$

We have that λ_1 is either equal to $2\underline{w}$ or strictly less than $2\underline{w}$ and so an eigenvalue of $\mathcal{L}_{\infty,1}$. Since λ_0 is simple, we obtain that in any case $\lambda_1 > \lambda_0 = 0$. Now, note that if $h \in H^2(\mathbb{R})$ is odd then $h \perp \dot{\theta}_1$ and hence, using e.g. the resolution of the identity relative to $\mathcal{L}_{\infty,1}$, we obtain

$$F''_{\infty,1}(\theta_1)hh = \langle \mathcal{L}_{\infty,1}h, h \rangle_{L^2(\mathbb{R})} \geq \lambda_1 \|h\|_{L^2(\mathbb{R})}^2$$

Then, by density, the Lemma follows. $\square$

A last consideration concerning the heteroclinic problem is about the compactness of the minimizing sequences of $F_{\infty,1}$ with respect to the $H^1(\mathbb{R})$ topology. We remark that since W is an even function we can restrict ourself to look for the minimizing sequences of $F_{\infty,1}$ constituted by odd functions which are not negative for $x > 0$, i.e. belonging to the space

$$\Gamma_{\infty,1} = \{q \in \mathcal{H} / q(x) = -q(-x) \text{ and } q(x) \geq 0 \text{ for } x > 0\}.$$

Then it can be proved (see e.g. Lemma 2.4 in [3]) the following result.

Lemma 2.2. (Compactness property) *If $(q_n) \subset \Gamma_{\infty,1}$ verifies $F_{\infty,1}(q_n) \rightarrow c_{\infty,1}$ then $\|q_n - \theta_1\|_{H^1(\mathbb{R})} \rightarrow 0$. In particular, for all $r > 0$ there exists $\mu_r > 0$ such that*

$$\text{if } q \in \Gamma_{\infty,1} \text{ verifies } \|q_n - \theta_1\|_{H^1(\mathbb{R})} \geq r \text{ then } F_{\infty,1}(q) \geq c_{\infty,1} + \mu_r. \quad (2.1)$$

2.2. The approximating one dimensional problems

Given $L > 0$, denoted $I_L = (-L, L)$, we consider the functional

$$F_{L,1}(q) = \int_{I_L} \frac{1}{2} |\dot{q}(x)|^2 + W(q(x)) dx \quad \text{on} \quad \Gamma_{L,1} = \{q|_{I_L} / q \in \Gamma_{\infty,1}\}.$$

It is standard to show that the minima of $F_{L,1}$ on $\Gamma_{L,1}$ are solutions to

$$\begin{cases} -\ddot{q} + W'(q) = 0, & x \in I_L, \\ q(x) = -q(-x), & x \in I_L, \\ \dot{q}(\pm L) = 0. \end{cases} \quad (P_{L,1})$$

We set

$$c_{L,1} = \inf_{\Gamma_{L,1}} F_{L,1} \quad \text{and} \quad \mathcal{K}_{L,1} = \{q \in \Gamma_{L,1} / F_{L,1}(q) = c_{L,1}\}.$$

The application of the direct method of the calculus of variation allows to show that $\mathcal{K}_{L,1}$ is not empty for any $L > 0$. We note that since any $q \in \mathcal{K}_{L,1}$ is a solution to $(P_{L,1})$, it can be reflected about L and then continued by periodicity obtaining in fact a $4L$ -periodic solution of $-\ddot{q} + W'(q) = 0$ on $\mathbb{R}$.

As described in the following Lemma the functions $q \in \mathcal{K}_{L,1}$ are uniformly exponentially close to 1 when x is large. This allows also to get informations on the asymptotic behaviour of the map $c_{L,1}$ (see Proposition 2.2 and Lemma 2.5 in [3] for the proofs).

Lemma 2.3. (Asymptotic behaviour of $c_{L,1}$). *There results*

i) *there exists a constant $C > 0$ such that*

$$0 \leq 1 - q(x) \leq Ce^{-\sqrt{w/2}x} \text{ for any } x > 0, L > 0 \text{ and } q \in \mathcal{K}_{L,1};$$

ii) *the map $L > 0 \rightarrow c_{L,1}$ is monotone increasing and satisfies*

$$0 \leq c_{\infty,1} - c_{L,1} \leq Ce^{-\sqrt{2w}L}, \quad \forall L > 0. \quad (2.2)$$

The asymptotic behaviour of $c_{L,1}$ described in (2.2) expresses a particular relation of $F_{L,1}$ with $F_{\infty,1}$ when $L \rightarrow +\infty$. Another related property, linked with Lemma 2.2, is the following one (see Lemma 2.6 in [3]).

Lemma 2.4. *Let $L_n \rightarrow +\infty$ and $(q_n) \subset \Gamma_{L_n,1}$ be such that $F_{L_n,1}(q_n) - c_{L_n,1} \rightarrow 0$ as $n \rightarrow +\infty$. Then,*

$$\|q_n - \theta_1\|_{H^1(I_{L_n})} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

By Lemma 2.4 we recover an analogous of (2.1), i.e., we have that for every $r \in (0, 1)$ there exists $\mu_r > 0$ and $\ell_r > 0$ such that

$$\text{if } L \geq \ell_r, \quad q \in \Gamma_{L,1} \text{ verify } \|q - \theta_1\|_{H^1(I_L)} \geq r \text{ then } F_{L,1}(q) - c_L \geq \mu_r. \quad (2.3)$$

Consider now the space

$$\mathcal{H}_L = \{q|_{I_L} / q \in \mathcal{H}\}$$

which is a Hilbert space if endowed with the $H^1(I_L)$ scalar product. It is classical to show that the functional $F_{L,1}$ is in $C^2(\mathcal{H}_L)$ and that for any $q, h, k \in \mathcal{H}_L$ there results

$$F'_{L,1}(q)h = \int_{-L}^L q\dot{h} + W'(q)h \, dx \quad \text{and} \quad F''_{L,1}(q)hk = \int_{-L}^L \dot{h}\dot{k} + W''(q)hk \, dx.$$

For our study, it is useful to study the behaviour with respect to $L > 0$ of $F_{L,1}(\theta_1)$, $F'_{L,1}(\theta_1)$ and $F''_{L,1}(\theta_1)$.

First note that, thanks to the exponential asymptotic behaviour of θ_1 (Remark 2.1) and by (1.7) we deduce the existence of a constant $C > 0$ such that

$$0 < F_{\infty,1}(\theta_1) - F_{L,1}(\theta_1) \leq Ce^{-\sqrt{2w}L} \quad \text{for all } L > 0. \quad (2.4)$$

Concerning $F'_{L,1}(\theta_1)$, we note that since $\ddot{\theta}_1 = W'(\theta_1)$, integrating by parts we obtain $|F'_{L,1}(\theta_1)h| = 2|\dot{\theta}_1(L)h(L)|$ for all $h \in \Gamma_{L,1}$. Then, by Remark 2.1, we recover that there exists a constant $C > 0$ such that

$$|F'_{L,1}(\theta_1)h| = 2|\dot{\theta}_1(L)h(L)| \leq Ce^{-\sqrt{\frac{w}{2}}L} \|h\|_{H^1(I_L)} \quad \forall L \geq 1, h \in \Gamma_{L,1}. \quad (2.5)$$

Finally, considering $F''_{L,1}(\theta_1)$ we shall show that thanks to the stability property of θ_1 (Lemma 2.1), if L is sufficiently large then, $F''_{L,1}(\theta_1)$ is a positive definite bilinear form on the space of odd functions in $H^1(I_L)$ uniformly with respect to L . Indeed there results

Lemma 2.5. *There exists $\ell_0 > 0$ such that if $L \geq \ell_0$, $h \in H^1(I_L)$ and h is odd, then*

$$F''_{L,1}(\theta_1)hh \geq \frac{\lambda_1}{8} \|h\|_{L^2(I_L)}^2.$$

Proof. Set $\bar{W} = \max_{[-1,1]} |W''(s)|$ and let $h \in H^1(I_L)$ odd. If $\|\dot{h}\|_{L^2(I_L)}^2 \geq (\bar{W} + \frac{\lambda_1}{8}) \|h\|_{L^2(I_L)}^2$ we recover that

$$F''_{L,1}(\theta_1)hh \geq \|\dot{h}\|_{L^2(I_L)}^2 - \bar{W} \|h\|_{L^2(I_L)}^2 \geq \frac{\lambda_1}{8} \|h\|_{L^2(I_L)}^2$$

and we have nothing more to show. So let us assume

$$\|\dot{h}\|_{L^2(I_L)}^2 < \left(\bar{W} + \frac{\lambda_1}{8} \right) \|h\|_{L^2(I_L)}^2. \quad (2.6)$$

Observe that if $x_h \in [L/2, L-1]$ is such that

$$\|h\|_{H^1(x_h, x_h+1)}^2 = \min_{x \in [L/2, L-1]} \|h\|_{H^1(x, x+1)}^2,$$

then (denoting $[L/2]$ the entire part of $L/2$) there results

$$\|h\|_{H^1(I_L)}^2 \geq \sum_{i=0}^{[L/2]-1} \|h\|_{H^1(L/2+i, L/2+i+1)}^2 \geq [L/2] \|h\|_{H^1(x_h, x_h+1)}^2. \quad (2.7)$$

Consider the cutoff function β_h equal to 1 for $|x| \leq x_h$, equal to 0 for $|x| \geq x_h + 1$ and such that $\beta_h(x) = 1 - (|x| - x_h)$ for $0 < |x| - x_h < 1$. We have

$$\begin{aligned}
F''_{L,1}(\theta_1)hh &= F''_{L,1}(\theta_1)(\beta_h h)(\beta_h h) + 2F''_{L,1}(\theta_1)((1 - \beta_h)h)(\beta_h h) \\
&\quad + F''_{L,1}(\theta_1)((1 - \beta_h)h)((1 - \beta_h)h).
\end{aligned} \tag{2.8}$$

By the stability property of θ_1 (Lemma 2.1) we can estimate the first addendum as

$$F''_{L,1}(\theta_1)(\beta_h h)(\beta_h h) \geq \lambda_1 \|\beta_h h\|_{L^2(I_L)}^2.$$

For the third addendum, observe that, since $\theta_1(x) \rightarrow 1$ as $x \rightarrow +\infty$ and since $W''(1) \geq \lambda_1$, there exists $\ell_0 > 0$ such that $W''(\theta_1(x)) \geq \lambda_1/2$ for any $x \geq \ell_0/2$. Then if $L > \ell_0$, since $x_h > L/2$, we recover

$$\begin{aligned}
F''_{L,1}(\theta_1)((1 - \beta_h)h)((1 - \beta_h)h) &\geq \frac{\lambda_1}{2} \|(1 - \beta_h)h\|_{L^2(L/2 < |x| < L)}^2 \\
&= \frac{\lambda_1}{2} \|(1 - \beta_h)h\|_{L^2(I_L)}^2.
\end{aligned}$$

Finally if $L > \ell_0$, setting $\mathcal{N}_h = (x_h, x_h + 1)$ and using (2.7) and (2.6), we can bound from below the second addendum as follows

$$\begin{aligned}
&F''_{L,1}(\theta_1)((1 - \beta_h)h)(\beta_h h) \\
&= 2 \left[2 \int_{\mathcal{N}_h} \beta_h \dot{h} h \, dx - \|h\|_{L^2(\mathcal{N}_h)}^2 - \int_{\mathcal{N}_h} \dot{h} h \, dx \right. \\
&\quad \left. + \int_{\mathcal{N}_h} (1 - \beta_h) \beta_h \dot{h}^2 \, dx + \int_{\mathcal{N}_h} (1 - \beta_h) \beta_h W''(\theta_1) h^2 \, dx \right] \\
&\geq 4 \int_{\mathcal{N}_h} \beta_h \dot{h} h \, dx - 2 \|h\|_{L^2(\mathcal{N}_h)}^2 - 2 \int_{\mathcal{N}_h} \dot{h} h \, dx \geq -5 \|h\|_{H^1(\mathcal{N}_h)}^2 \\
&\geq -\frac{5}{[L/2]} \|h\|_{H^1(I_L)}^2 \geq -\frac{5}{[L/2]} \left(1 + \bar{W} + \frac{\lambda_1}{8} \right) \|h\|_{L^2(I_L)}^2.
\end{aligned}$$

Taking ℓ_0 bigger if necessary, we can assume that $\frac{5}{[L/2]} (1 + \bar{W} + \frac{\lambda_1}{8}) \leq \frac{\lambda_1}{8}$ for any $L \geq \ell_0$ and so, gathering the estimates above in (2.8) and using (2.6), we conclude that if $L \geq \ell_0$ then

$$\begin{aligned}
F''_{L,1}(\theta_1)hh &\geq \frac{\lambda_1}{2} \left(\|\beta_h h\|_{L^2(I_L)}^2 + \|(1 - \beta_h)h\|_{L^2(I_L)}^2 \right) - \frac{\lambda_1}{8} \|h\|_{L^2(I_L)}^2 \\
&\geq \frac{\lambda_1}{2} \left(\|h\|_{L^2(I_L)}^2 - 4 \|h\|_{L^2(\mathcal{N}_h)}^2 \right) - \frac{\lambda_1}{8} \|h\|_{L^2(I_L)}^2 \\
&\geq \frac{\lambda_1}{4} \|h\|_{L^2(I_L)}^2 - \frac{\lambda_1}{8} \|h\|_{L^2(I_L)}^2 = \frac{\lambda_1}{8} \|h\|_{L^2(I_L)}^2
\end{aligned}$$

and the Lemma follows. $\square$

Remark 2.4. By (2.3), taking the constant ℓ_0 in Lemma 2.5 bigger if necessary, we can assume that setting $\tilde{W} = \frac{1}{6} \max_{s \in [-2,2]} |W'''(s)|$, there exists $\mu_0 > 0$ such that if $L \geq \ell_0$, $q \in \Gamma_{L,1}$ and $F_{L,1}(q) - c_{L,1} \leq \mu_0$ then

$$\|q - \theta_1\|_{H^1(I_L)} \leq \frac{\lambda_1}{8\tilde{W}c_0}, \tag{2.9}$$

where c_0 is the immersion constant $H^1(I_1) \rightarrow L^\infty(I_1)$.

As last result in this section we give the following Lemma which expresses a uniform coerciveness property of the functionals F_L .

Lemma 2.6. *There exist a constant $C > 0$, such that if $L \geq \ell_0$, $q \in \Gamma_{L,1}$, $\|q\|_{L^\infty(I_L)} \leq 1$ are such that $F_{L,1}(q) - c_{L,1} \leq \mu_0$ then*

$$F_{L,1}(q) - c_{L,1} \geq \frac{\lambda_1}{16} \|q - \theta_1\|_{L^2(I_L)}^2 - Ce^{-\sqrt{\frac{w}{2}}L}.$$

Proof. Let $L \geq \ell_0$ and $q \in \Gamma_{L,1}$ such that $\|q\|_{L^\infty(I_L)} \leq 1$. We note that since $\|q\|_{L^\infty(I_L)} \leq 1$, by (W) we recover that on I_L there results

$$|W(q) - W(\theta_1) - W'(\theta_1)(q - \theta_1) - \frac{1}{2}W''(\theta_1(x))(q - \theta_1)^2| \leq \frac{1}{2}\tilde{W}|q - \theta_1|^3. \quad (2.10)$$

Since we have the identity

$$\begin{aligned} F_{L,1}(q) - F_{L,1}(\theta_1) &= F'_{L,1}(\theta_1)(q - \theta_1) + \frac{1}{2}F''_{L,1}(\theta_1)(q - \theta_1)(q - \theta_1) + \int_{-L}^L W(q) \\ &\quad - W(\theta_1) - W'(\theta_1)(q - \theta_1) - \frac{1}{2}W''(\theta_1)(q - \theta_1)^2 dx, \end{aligned}$$

by (2.5), Lemma 2.5 and (2.10), we obtain that

$$\begin{aligned} F_{L,1}(q) - F_{L,1}(\theta_1) &\geq \frac{\lambda_1}{8} \|q - \theta_1\|_{L^2(I_L)}^2 - Ce^{-\sqrt{\frac{w}{2}}L} \|q - \theta_1\|_{H^1(I_L)} \\ &\quad - \frac{\tilde{W}}{2} \|q - \theta_1\|_{L^3(I_L)}^3 \geq \left(\frac{\lambda_1}{8} - \frac{\tilde{W}}{2} c_0 \|q - \theta_1\|_{H^1(I_L)} \right) \|q - \theta_1\|_{L^2(I_L)}^2 \\ &\quad - Ce^{-\sqrt{\frac{w}{2}}L} \|q - \theta_1\|_{H^1(I_L)}. \end{aligned} \quad (2.11)$$

Since by (2.4) and (2.2) we have that for a constant $C > 0$

$$\begin{aligned} F_{L,1}(q) - c_{L,1} &= F_{L,1}(q) - F_{L,1}(\theta_1) + F_{L,1}(\theta_1) - F_{\infty,1}(\theta_1) + c_{\infty,1} - c_{L,1} \\ &\geq F_{L,1}(q) - F_{L,1}(\theta_1) - Ce^{-\sqrt{2w}L} \quad \text{for any } L > \ell_0 \text{ and } q \in \Gamma_{L,1}, \end{aligned}$$

by (2.11) we deduce

$$\begin{aligned} F_{L,1}(q) - c_{L,1} &\geq \left(\frac{\lambda_1}{8} - \frac{\tilde{W}}{2} c_0 \|q - \theta_1\|_{H^1(I_L)} \right) \|q - \theta_1\|_{L^2(I_L)}^2 + \\ &\quad - Ce^{-\sqrt{\frac{w}{2}}L} \|q - \theta_1\|_{H^1(I_L)} - Ce^{-\sqrt{2w}L}. \end{aligned}$$

Then the lemma follows using (2.9). $\square$

3. The two dimensional problem

In this section we study and characterize some variational properties of certain classes of planar entire solutions of (1.1) which are odd in each variable. Precisely in the Sect. 3.1 we recall the main steps in the construction of saddle solutions given in [3], pointing out some further variational properties. In Sect. 3.2, analogously to the study made in Sect. 2 in the one dimensional case, we consider related approximating problems studying the relationship between this class of approximating (periodic) solutions with the saddle solution.

3.1. Saddle Planar solutions.

As in [3] the existence of a saddle solution of (1.1) is studied by considering the triangular set

$$T_{\infty,2} = \{(x, y) \in \mathbb{R}^2 / x \in I_y, y > 0\},$$

and looking for minima of the renormalized functional

$$F_{\infty,2}(u) = \int_0^{+\infty} \frac{1}{2} \|\partial_y u(\cdot, y)\|_{L^2(I_y)}^2 + (F_{y,1}(u(\cdot, y)) - c_{y,1}) dy$$

on the space

$$\Gamma_{\infty,2} = \{u \in H_{loc}^1(T_{\infty,2}) / u(\cdot, y) \in \Gamma_{y,1} \text{ for a.e. } y > 0\}.$$

Note that if $u \in \Gamma_{\infty,2}$ then $u(\cdot, y) \in \Gamma_{y,1}$ for a.e. $y > 0$ and so $F_{y,1}(u(\cdot, y)) - c_{y,1} \geq 0$ for a.e. $y > 0$. Hence $F_{\infty,2}$ is well defined on $\Gamma_{\infty,2}$ with values in $[0, +\infty]$. The asymptotic property of the function $y \rightarrow c_{y,1}$ described in (2.2) allows to simply show that $F_{\infty,2}(\theta_1) < +\infty$ (considering θ_1 as a function in $\Gamma_{\infty,2}$, see Lemma 3.1 in [3]). Then we have in particular that

$$c_{\infty,2} \equiv \inf_{\Gamma_{\infty,2}} F_{\infty,2} < +\infty.$$

The functional $F_{\infty,2}$ is weakly lower semicontinuous with respect to the $H_{loc}^1(T_{\infty,2})$ topology and we are interested in finding minima of $F_{\infty,2}$ on $\Gamma_{\infty,2}$. We denote

$$\mathcal{K}_{\infty,2} = \{u \in \Gamma_{\infty,2} \mid F_{\infty,2}(u) = c_{\infty,2}\}.$$

Remark 3.1. From the minima of $F_{\infty,2}$ on $\Gamma_{\infty,2}$ we recover entire planar solutions of (1.1) recursively reflecting them with respect to the edges of $T_{\infty,2}$. Indeed, as in Lemma 3.3 of [3], if $\bar{u} \in \mathcal{K}_{\infty,2}$ we have

$$\int_{T_{\infty,2}} \nabla \bar{u} \nabla \psi + W'(\bar{u}) \psi dx dy = 0 \text{ for any } \psi \in C_0^\infty(\mathbb{R}^2). \quad (3.1)$$

Then, denoting the $\pi/2$ planar rotation matrix with R and setting $T^k = R^k T_{\infty,2}$ for $k = 0, \dots, 3$, we have $\mathbb{R}^2 = \cup_{k=0}^3 T^k$, and defining

$$\theta_2(x, y) = (-1)^k \bar{u}(R^{-k}(x, y)), \quad \forall (x, y) \in T^k,$$

we obtain $\theta_2 \in H_{loc}^1(\mathbb{R}^2)$ and, by the change of variables theorem and (3.1),

$$\int_{T^k} \nabla \theta_2 \cdot \nabla \psi + W'(\theta_2) \psi dx dy = 0 \quad \text{for any } \psi \in C_0^\infty(\mathbb{R}^2) \text{ and } k = 0, \dots, 3.$$

Then, for any $\psi \in C_0^\infty(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} \nabla \theta_2 \cdot \nabla \psi + W'(\theta_2) \psi dx dy = \sum_{k=0}^3 \int_{T^k} \nabla \theta_2 \cdot \nabla \psi + W'(\theta_2) \psi dx dy = 0,$$

i.e. θ_2 is a weak and so, by standard arguments, a classical $C^2(\mathbb{R}^2)$ solution of (1.1). We finally remark that if $\bar{u} \in \mathcal{K}_{\infty,2}$ then $\|\bar{u}\|_{L^\infty(T_{\infty,2})} \leq 1$ (if not, defining $\bar{v} = \min\{1, \max\{-1, \bar{u}\}\}$ we have $\bar{v} \in \Gamma_{\infty,2}$ and $F_{\infty,2}(\bar{v}) < F_{\infty,2}(\bar{u})$)

and so $\|\theta_2\|_{L^\infty(\mathbb{R}^2)} \leq 1$. By Schauder estimates we then obtain the existence of a constant $\bar{C} > 0$ such that

$$\|\bar{u}\|_{C^2(T_{\infty,2})} = \|\theta_2\|_{C^2(\mathbb{R}^2)} \leq \bar{C} \quad \text{for any } \bar{u} \in \mathcal{K}_{\infty,2}. \quad (3.2)$$

Given an interval $I \subset \mathbb{R}^+$, it is useful to consider the functional

$$F_{I,2}(u) = \int_I \frac{1}{2} \|\partial_y u(\cdot, y)\|_{L^2(I_y)}^2 + (F_{y,1}(u(\cdot, y)) - c_{y,1}) dy,$$

which is well defined on $\Gamma_{\infty,2}$, positive and weakly lower semicontinuous with respect to the weak $H_{loc}^1(T_{\infty,2})$ topology. The study of the existence of minima of $F_{\infty,2}$ on $\Gamma_{\infty,2}$ is based on the following two estimates:

i) Continuity estimate. If $u \in \Gamma_{\infty,2}$ is such that $F_{(\sigma,\tau),2} < +\infty$ for a certain $(\sigma, \tau) \subset (0, +\infty)$, we have that the map $y \in (\sigma, \tau) \mapsto u(\cdot, y) \in L^2(I_\sigma)$ is continuous. Indeed, if $\sigma \leq y_1 < y_2 \leq \tau$ then, by the Hölder inequality,

$$\begin{aligned} \|u(\cdot, y_2) - u(\cdot, y_1)\|_{L^2(I_\sigma)}^2 &\leq (y_2 - y_1) \int_{I_\sigma} \int_{y_1}^{y_2} |\partial_y u(x, y)|^2 dy dx \\ &\leq (y_2 - y_1) 2F_{(\sigma,\tau),2}(u). \end{aligned} \quad (3.3)$$

ii) Coerciveness estimate. If $u \in \Gamma_{\infty,2}$ is such that $F_{y,1}(u(\cdot, y)) - c_{y,1} \geq \mu > 0$ for a.e. $y \in (\sigma, \tau) \subset \mathbb{R}^+$ then

$$\begin{aligned} F_{(\sigma,\tau),2}(u) &\geq \frac{1}{2(\tau - \sigma)} \|u(\cdot, \tau) - u(\cdot, \sigma)\|_{L^2(I_\sigma)}^2 + \mu(\tau - \sigma) \\ &\geq \sqrt{2\mu} \|u(\cdot, \tau) - u(\cdot, \sigma)\|_{L^2(I_\sigma)}. \end{aligned} \quad (3.4)$$

Estimates (3.3) and (3.4) can be used together with (2.3) to obtain the following characterizations of asymptotic behaviour of the functions $u \in \Gamma_{\infty,2}$ with $F_{\infty,2}(u) < +\infty$ (see Lemma 3.2 in [3]).

Lemma 3.1. *For any $r \in (0, 1]$ there exists $\eta_r > 0$ and $L_r > 0$ such that for all $L \geq L_r$*

$$\text{if } u \in \Gamma_{\infty,2} \text{ and } F_{(L,+\infty),2}(u) \leq \eta_r \text{ then } \|u(\cdot, y) - \theta_1\|_{L^2(I_y)} \leq r \quad \forall y \geq L. \quad (3.5)$$

Note that, by Lemma 3.1, we recover that if $u \in \Gamma_{\infty,2}$ and $F_{\infty,2}(u) < +\infty$ then

$$\|u(\cdot, y) - \theta_1\|_{L^2(I_y)} \rightarrow 0 \quad \text{as } y \rightarrow +\infty. \quad (3.6)$$

Proposition 3.1. *There results $\mathcal{K}_{\infty,2} \neq \emptyset$. Moreover, every $\bar{u} \in \mathcal{K}_{\infty,2}$ satisfies $1 > \bar{u}(x, y) > 0$ for $(x, y) \in T_{\infty,2}$, $x > 0$ and $\|\bar{u}(\cdot, y) - \theta_1\|_{L^\infty(I_y)} \rightarrow 0$ as $y \rightarrow +\infty$.*

Proof. It is standard to show that $\mathcal{K}_{\infty,2} \neq \emptyset$. Indeed, letting $(u_n) \subset \Gamma_{2,\infty}$ be such that $F_{2,\infty}(u_n) \rightarrow c_{\infty,2}$, by (W), we can assume without restrictions that $\|u_n\|_{L^\infty(T_{\infty,2})} \leq 1$ for any $n \in \mathbb{N}$. Since for any $L > 0$ there results

$$\int_0^L \int_{-y}^y |\nabla u_n(x, y)|^2 dx dy \leq 2(F_{\infty,2}(u_n) + c_{L,1}L)$$

we obtain that (u_n) is a bounded sequence in $H^1_{loc}(T_{\infty,2})$. Hence there exists $\bar{u} \in H^1_{loc}(T_{\infty,2})$ such that, along a subsequence, $u_n - \bar{u} \rightarrow 0$ weakly in $H^1_{loc}(T_{\infty,2})$ and for a.e. $(x, y) \in T_{\infty,2}$. By pointwise convergence, we recover that $\bar{u} \geq 0$ for $x > 0$, that $\bar{u}(x, y) = -\bar{u}(-x, y)$ for a.e. $(x, y) \in T_{\infty,2}$ and that $\|\bar{u}\|_{L^\infty(T_{\infty,2})} \leq 1$. Then $\bar{u} \in \Gamma_{\infty,2}$ and, by semicontinuity, $F_{\infty,2}(\bar{u}) = c_{\infty,2}$.

To show that $\bar{u}(x, y) > 0$ for $x > 0$, $(x, y) \in T_{\infty,2}$, we note that by Remark 3.1 it holds that $-\Delta\bar{u} + W'(\bar{u}) = 0$ on $T_{\infty,2}$. Hence, since $\bar{u}(x, y) \geq 0$ for $x > 0$, $(x, y) \in T_{\infty,2}$, it satisfies the differential inequality $-\Delta\bar{u} + \max\{a(x, y), 0\}\bar{u} \geq 0$ on $T_{\infty,2}$ when $x > 0$ where

$$a(x, y) = \begin{cases} W'(\bar{u})/\bar{u} & \text{if } \bar{u} \neq 0 \\ W''(0) & \text{if } \bar{u} = 0 \end{cases}$$

is continuous on $T_{\infty,2}$. Then the maximum principle tells us that either $\bar{u} \equiv 0$ or $\bar{u} > 0$ on $T_{\infty,2}$ when $x > 0$. By (3.6) we conclude that $\bar{u} > 0$ on $T_{\infty,2}$ when $x > 0$. Analogously one proves that $\bar{u} < 1$ on $T_{\infty,2}$ when $x > 0$.

We now finally show that $\|\bar{u}(\cdot, y) - \theta_1\|_{L^\infty(I_y)} \rightarrow 0$ as $y \rightarrow +\infty$. For that, assume by contradiction that there exists a sequence $(x_n, y_n) \in T_{\infty,2}$ with $y_n \rightarrow +\infty$ such that $|\bar{u}(x_n, y_n) - \theta_1(x_n)| \geq \rho_0 > 0$ for every $n \in \mathbb{N}$. By (3.2) we have $\|\bar{u}\|_{C^2(T_{\infty,2})} < +\infty$ and so there exists $r_0 > 0$ such that $|\bar{u}(x, y_n) - \theta_1(x)| \geq \rho_0/2$ whenever $|x - x_n| \leq r_0$, $(x, y_n) \in T_{\infty,2}$ and $n \in \mathbb{N}$. Then $\|\bar{u}(\cdot, y_n) - \theta_1\|_{L^2(I_{y_n})}^2 \geq r_0 \rho_0^2/4$ for all $n \in \mathbb{N}$, in contradiction with (3.6). $\square$

As stated in the following Lemma the saddle solution is unique.

Lemma 3.2. *There exists a unique $\bar{u} \in \mathcal{K}_{\infty,2}$. Denoted with θ_2 the corresponding saddle solution we have moreover $\partial_x \theta_2(x, y) > 0$ and $\partial_y \theta_2(x, y) > 0$ for $(x, y) \in Q_1 = \{(x, y) \in \mathbb{R}^2 / x > 0, y > 0\}$.*

Proof. Let $u, v \in \mathcal{K}_{\infty,2}$. To prove that $u = v$ we use a moving plane like argument.

As in Remark 3.1 we extend u and v as saddle solution on $\mathbb{R}^2$ of (1.1). By Proposition 3.1 we derive that $0 < u(x, y), v(x, y) < 1$ on Q_1 and that $u(x, y), v(x, y) \rightarrow \theta_1(x)$ as $y \rightarrow +\infty$ unif. w.r.t. $x \in \mathbb{R}$, $u(x, y), v(x, y) \rightarrow \theta_1(y)$ as $x \rightarrow +\infty$ unif. w.r.t. $y \in \mathbb{R}$. Then, denoting $Q_{1,r} = \{(x+r, y+r) / (x, y) \in Q_1\}$, we can fix $L > 0$ such that $1 > u(x, y), v(x, y) > 1 - \bar{\delta}$ on $Q_{1,L}$. Relatively to L there exists $\delta_L > 0$ such that $0 < u(x, y), v(x, y) < 1 - \delta_L$ on $Q_1 \setminus Q_{1,L}$. For $r > 0$ we define $v_r(x, y) = v(x - r, y - r)$ for $(x, y) \in \overline{Q_{1,r}}$. By the asymptotic properties of u we can fix $r_0 > L$ such that $u > v_{r_0}$ on $\overline{Q_{1,r_0}} \setminus Q_{1,r_0+L}$. Moreover since $1 > u(x, y), v_{r_0}(x, y) > 1 - \bar{\delta}$ on Q_{1,r_0+L} by (1.6) we deduce that

$$\Delta(u(x, y) - v_{r_0}(x, y))^2 \geq 2w(u(x, y) - v_{r_0}(x, y))^2 \quad \text{for } (x, y) \in Q_{1,r_0+L}.$$

This implies that the function $u - v_{r_0}$ cannot have internal negative relative minima on Q_{1,r_0+L} and so, since $u(x, y) - v_{r_0}(x, y) > 0$ on $\partial Q_{1,r_0+L}$ and $\liminf u(x, y) - v_{r_0}(x, y) \geq 0$ as $\|(x, y)\| \rightarrow +\infty$, $(x, y) \in Q_{1,r_0+L}$, we deduce that $u(x, y) - v_{r_0}(x, y) \geq 0$ on Q_{1,r_0+L} and so that $u(x, y) - v_{r_0}(x, y) \geq 0$ on Q_{1,r_0} . This last inequality is in fact strict since defining $a(x, y) = (W'(u) - W'(v_{r_0})) / (u - v_{r_0})$ if $u > v_{r_0}$ and $a(x, y) = W''(0)$ if $u = v_{r_0}$, we have that a is

continuous on Q_{1,r_0} and $-\Delta(u - v_{r_0}) + \max\{a(x, y), 0\}(u - v_{r_0}) \geq 0$ on Q_{1,r_0} . Since $u > v_{r_0}$ on $\overline{Q_{1,r_0}} \setminus Q_{1,r_0+L}$ by the maximum principle we conclude that $u > v_{r_0}$ on $\overline{Q_{1,r_0}}$.

We can now translate back the function v considering the functions $v_r(x, y) = v(x - r, y - r)$ on $Q_{1,r}$ for $r \in (0, r_0]$ showing by using again the maximum principle that $\inf\{r > 0 / u(x, y) > v_r(x, y) \text{ on } Q_{1,r}\} = 0$ and so that $u(x, y) \geq v(x, y)$ on Q_1 . That conclude the proof of uniqueness. For the monotonicity it is possible to use a similar argument. $\square$

Remark 3.2. The saddle solution θ_2 enjoys the properties:

$$\begin{aligned} \theta_2(x, -y) &= -\theta_2(x, y), \quad \theta_2(x, y) = \theta_2(y, x) \\ &\text{on } \mathbb{R}^2 \text{ and } \theta_2(x, y) = \bar{u}(x, y) \text{ on } T_{\infty,2}. \end{aligned}$$

Since $\bar{u}(x, y) = -\bar{u}(-x, y)$ on $T_{\infty,2}$ we then deduce that also $\theta_2(-x, y) = -\theta_2(x, y)$. Then, since $\bar{u}(x, y) > 0$ if $x > 0$ we derive that $\theta_2(x, y)$ has the same sign of xy . Moreover, since $\|\bar{u}(\cdot, y) - \theta_1\|_{L^\infty(I_y)} \rightarrow 0$ as $y \rightarrow +\infty$, we finally recognize that

$$\begin{aligned} \lim_{y \rightarrow \pm\infty} \theta_2(x, y) &= \pm \theta_1(x) \quad \text{uniformly with respect to } x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} \theta_2(x, y) &= \pm \theta_1(y) \quad \text{uniformly with respect to } y \in \mathbb{R}. \end{aligned}$$

In the following Lemma we finally better characterize the convergence properties of the minimizing sequences of $F_{\infty,2}$ on $\Gamma_{\infty,2}$.

Lemma 3.3. *If $(u_n) \subset \Gamma_{\infty,2}$ verifies $\|u_n\|_{L^\infty(T_{\infty,2})} \leq 1$ and $F_{\infty,2}(u_n) \rightarrow c_{\infty,2}$ then $u_n - \theta_2 \rightarrow 0$ in $L^2(T_{\infty,2})$.*

Proof. As in the proof of Proposition 3.1, if $(u_n) \subset \Gamma_{\infty,2}$ is a minimizing sequence for $F_{\infty,2}$ such that $|u_n(x, y)| \leq 1$ for all $(x, y) \in T_{\infty,2}$, we can extract a subsequence (denoted again (u_n)) such that $u_n \rightarrow \theta_2$ weakly in $H^1(T_{L,2})$ and strongly in $L^2(T_{L,2})$ for all $L > 0$, where $T_{L,2} = \{(x, y) \in T_{\infty,2} / y \in (0, L)\}$. The Lemma will follow once we show that

$$\forall \varepsilon > 0 \exists L_\varepsilon > 0, \bar{n} \in \mathbb{N} \text{ such that } \|u_n - \theta_2\|_{L^2(T_{\infty,2} \setminus T_{L,2})} \leq \varepsilon, \quad \forall n \geq \bar{n}. \quad (3.7)$$

To this aim, we first note that in general, if φ, φ_1 and φ_2 are lower semi-continuous functionals such that $\varphi = \varphi_1 + \varphi_2$ then if $u_n \rightarrow u$ and $\varphi(u_n) \rightarrow \varphi(u)$ then also $\varphi_1(u_n) \rightarrow \varphi_1(u)$ and $\varphi_2(u_n) \rightarrow \varphi_2(u)$.

Then setting $\varphi_1(u) = \frac{1}{2} \int_{T_{\infty,2}} |\partial_y u|^2 dx dy + \int_0^L F_{y,1}(u_n(\cdot, y)) - c_{y,1} dy$ and $\varphi_2(u) = \int_L^{+\infty} F_{y,1}(u_n(\cdot, y)) - c_{y,1} dy$, the above remark applies to the functional $\varphi(u) = \varphi_1(u) + \varphi_2(u) = F_{\infty,2}(u)$ and we deduce that for every $L \geq 0$

$$\int_L^{+\infty} F_{y,1}(u_n(\cdot, y)) - c_{y,1} dy \rightarrow \int_L^{+\infty} F_{y,1}(\theta_2(\cdot, y)) - c_{y,1} dy \text{ as } n \rightarrow +\infty. \quad (3.8)$$

Let $\eta_1 > 0$ and $L_1 > 0$ be given by Lemma 3.1 in correspondence of $r = 1$, let ℓ_0 as defined in Lemma 2.5) and $\mu_0 > 0$ be given by Remark 2.4.

Given $\varepsilon \in (0, \eta_1)$, we fix $L_\varepsilon \geq \max\{\ell_0, L_1\}$ such that $\int_{L_\varepsilon}^{+\infty} F(\theta_2(\cdot, y)) - m \, dy < \frac{\varepsilon}{2}$ and we note that, by (3.8), there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\int_{L_\varepsilon}^{+\infty} F_{y,1}(u_n(\cdot, y)) - c_{y,1} \, dy < \varepsilon \quad \text{for all } n \geq n_\varepsilon. \quad (3.9)$$

We define

$$\mathring{A}_n = \{y > L_\varepsilon / F_{y,1}(u_n(\cdot, y)) - c_{y,1} > \mu_0\},$$

noting that, by (3.9), $|\mathring{A}_n| \leq \frac{\varepsilon}{\mu_0}$ for all $n \geq n_\varepsilon$. Moreover, since $\varepsilon < \eta_1$ and $L_\varepsilon > L_1$ by (3.9) and Lemma 2.2 we recover that $\|u_n(\cdot, y) - \theta_1\|_{L^2(I_y)} \leq 1$ for any $y \geq L_\varepsilon$, from which we deduce

$$\int_{\mathring{A}_n} \|u_n(\cdot, y) - \theta_1\|_{L^2(I_y)}^2 dy \leq \frac{\varepsilon}{\mu_0} \quad \text{for any } n \geq n_\varepsilon. \quad (3.10)$$

On the other hand, if $n \geq n_\varepsilon$ and $y \in (L_\varepsilon, +\infty) \setminus \mathring{A}_n$, by definition of $\mathring{A}_n$ we have $F_{y,1}(u_n(\cdot, y)) - c_{y,1} \leq \mu_0$. Then, since $L_\varepsilon \geq \ell_0$, by Lemma 2.6 we recover that there exists a constant $C > 0$, such that for any $y \geq L_\varepsilon$, $y \notin \mathring{A}_n$ and $n \geq n_\varepsilon$

$$\|u_n(\cdot, y) - \theta_1\|_{L^2(I_y)}^2 \leq \frac{16}{\lambda_1} (F_{y,1}(u_n(\cdot, y)) - c_{y,1}) + \frac{16}{\lambda_1} C e^{-\sqrt{\frac{w}{2}}y}$$

and so, integrating on $(L_\varepsilon, +\infty) \setminus \mathring{A}_n$, by (3.9) we conclude that for any $n \geq n_\varepsilon$ there results

$$\int_{(L_\varepsilon, +\infty) \setminus \mathring{A}_n} \|u_n(\cdot, y) - \theta_1\|_{L^2(I_y)}^2 dy \leq \frac{16}{\lambda_1} \varepsilon + \frac{16}{\lambda_1} C \int_{L_\varepsilon}^{+\infty} e^{-\sqrt{\frac{w}{2}}y} dy. \quad (3.11)$$

By (3.10) and (3.11) we recover (3.7) and the Lemma follows. $\square$

3.2. Approximating planar problem. Periodic saddle type planar solutions.

Given $L > 0$ and defined $T_{L,2} = \{(x, y) \in T_{\infty,2} / y < L\}$, we consider on the space $\Gamma_{L,2} = \{u|_{T_{L,2}} / u \in \Gamma_{\infty,2}\}$ the functional

$$F_{L,2}(u) = F_{(0,L),2}(u) = \int_0^L \frac{1}{2} \|\partial_y u(\cdot, y)\|_{L^2(I_y)}^2 + (F_{y,1}(u(\cdot, y)) - c_{y,1}) \, dy.$$

We set

$$c_{L,2} = \inf_{\Gamma_{L,2}} F_{L,2} \quad \text{and} \quad \mathcal{K}_{L,2} = \{u \in \Gamma_{L,2} / F_{L,2}(u) = c_{L,2}\}.$$

Remark 3.3. One plainly recognizes that the function $L \mapsto c_{L,2}$ is monotone increasing for $L > 0$, i.e.,

$$\text{if } 0 < L_1 < L_2 \quad \text{then} \quad c_{L_1,2} \leq c_{L_2,2} < c_{\infty,2}.$$

Moreover, as in the proof of Proposition 3.1, the application of the direct method of the calculus of variation allows to show that $\mathcal{K}_{L,2}$ is not empty for any $L > 0$ and by (W) that if $u \in \mathcal{K}_{L,2}$ then $\|u\|_{L^\infty(T_{L,2})} \leq 1$.

Remark 3.4. Arguing as in Remark 3.1, from the minima of $F_{L,2}$ on $\Gamma_{L,2}$ we recover by reflections entire periodic planar solutions of (1.1). Indeed, one proves that if $u \in \mathcal{K}_{L,2}$ then

$$\int_{T_{L,2}} \nabla u \nabla \psi + W'(u) \psi \, dx \, dy = 0 \quad \text{for any } \psi \in C_0^\infty(\mathbb{R}^2). \quad (3.12)$$

If R is the $\pi/2$ rotation matrix already introduced in Remark 3.1, we consider the function

$$u_p(x, y) = (-1)^k u(R^{-k}(x, y)), \quad \forall (x, y) \in T_{L,2}^k,$$

where $T_{L,2}^k = R^k T_{L,2}$. Setting $Q_L = \cup_{k=0}^3 T_{L,2}^k = (-L, L)^2$, by (3.12) we recognize that $\int_{Q_L} \nabla u \nabla \psi + W'(u) \psi \, dx \, dy = 0$ for any $\psi \in C_0^\infty(\mathbb{R}^2)$ and so that u_p is a solution to (1.1) on $Q_L = (-L, L)^2$ satisfying the Neumann boundary condition $\partial_\nu u_p(x, y) = 0$ for $(x, y) \in \partial Q_L$. Then we can reflect u_p with respect to the axes $x = L$, $y = L$ and then continue it by periodicity in the x and y -direction. We then obtain a periodic entire solution of period $4L$ both in the x and y variable. With abuse of notation we denote by u_p that periodic solution. Since $\|u_p\|_{L^\infty(\mathbb{R}^2)} \leq 1$ for any $u \in \mathcal{K}_{L,2}$ and $L > 0$, by Schauder estimates there exists $C > 0$ such that

$$\|u_p\|_{C^2(\mathbb{R}^2)} \leq \bar{C} \quad \text{for any } u \in \mathcal{K}_{L,2} \text{ and } L > 0. \quad (3.13)$$

We have now a first concentration property of the elements of $\mathcal{K}_{L,2}$. The proof can be done adapting the argument already used in the proof of Lemma 4.2 in [7].

Lemma 3.4. *For all $\varepsilon > 0$ there exists $L_\varepsilon \geq \ell_0$ such that*

$$\text{if } L > L_\varepsilon \text{ and } u \in \mathcal{K}_{L,2} \text{ then } \|u(\cdot, y) - \theta_1\|_{L^\infty(I_y)} < \varepsilon \quad \forall y \in (L_\varepsilon, L]. \quad (3.14)$$

Remark 3.5. Since $u \in \mathcal{K}_{L,2}$ and θ_1 solve (1.1) on $T_{L,2}$ we can use Lemma 3.4 to derive via the Schauder estimates that for every $\varepsilon > 0$ there exists $L_\varepsilon \geq \ell_0$ such that

$$\text{if } L > L_\varepsilon \text{ and } u \in \mathcal{K}_{L,2} \text{ then } \|u(\cdot, y) - \theta_1\|_{C^2(T_{L,2} \setminus T_{L_\varepsilon,2})} < \varepsilon \quad \forall y \in (L_\varepsilon, L]. \quad (3.15)$$

Lemma 3.5. *There exist $L_0 \geq \ell_0$ and a constant $C > 0$ such that for all $L \geq L_0$, if $u \in \mathcal{K}_{L,2}$ then*

$$0 \leq 1 - u(x, y) \leq C(e^{\sqrt{2\overline{w}}(L_0-x)} + e^{\sqrt{2\overline{w}}(L_0-y)})^{1/2} \quad (3.16)$$

and

$$|\partial_x u(x, y)| \leq C(e^{\sqrt{2\overline{w}}(L_0-x)} + e^{\sqrt{2\overline{w}}(L_0-y)})^{1/2}. \quad (3.17)$$

for every $(x, y) \in T_{L,2}$ with $x \in (L_0, L)$. Moreover,

$$\forall \varepsilon > 0, \exists \bar{L}_\varepsilon > L_0 / \text{ if } u \in \mathcal{K}_{L,2}, y \in (\bar{L}_\varepsilon, L) \text{ then } \|u(\cdot, y) - \theta_1\|_{H^1(I_y)} < \varepsilon. \quad (3.18)$$

Proof. By Remark 2.1 and (3.14), we can fix $L_0 \geq \ell_0$ such that if $L > L_0$ and $u \in \mathcal{K}_{L,2}$, then the corresponding periodic solution u_p (see Remark 3.4) satisfies

$$0 \leq 1 - u_p(x, y) \leq \bar{\delta} \quad \text{for any } (x, y) \in \tilde{Q}_{L, L_0} \quad (3.19)$$

where $\tilde{Q}_{L, L_0}$ is the square centered at the vertex (L, L) of the triangle $T_{L,2}$ and of side length $2(L - L_0)$.

Setting $\phi(x, y) = (1 - u_p(x, y))^2$ we have $0 \leq \phi(x, y) \leq \bar{\delta}^2$ on $\tilde{Q}_{L, L_0}$ and by (1.6)

$$\Delta \phi(x, y) \geq 2\underline{w} \phi(x, y), \quad \text{for all } (x, y) \in \tilde{Q}_{L, L_0}$$

Then, by the maximum principle we obtain

$$0 \leq 1 - u_p(x, y) \leq \bar{\delta} \left(\frac{\cosh(\sqrt{2\underline{w}}(x - L)) + \cosh(\sqrt{2\underline{w}}(y - L))}{\cosh(\sqrt{2\underline{w}}(L - L_0))} \right)^{1/2},$$

from which (3.16) is plainly derived.

Consider now the function $\partial_x u_p$. Since it verifies $-\Delta \partial_x u_p + W''(u_p) \partial_x u_p = 0$, we have $\Delta |\partial_x u_p|^2 \geq W''(u_p) |\partial_x u_p|^2$ on $\mathbb{R}^2$. Then, by (1.6) and (3.19) we have

$$\Delta |\partial_x u_p(x, y)|^2 \geq 2\underline{w} |\partial_x u_p(x, y)|^2 \quad \text{for } (x, y) \in \tilde{Q}_{L, L_0}.$$

By (3.13) and the maximum principle, we then obtain

$$|\partial_x u_p(x, y)| \leq C \left(\frac{\cosh(\sqrt{2\underline{w}}(x - L)) + \cosh(\sqrt{2\underline{w}}(y - L))}{\cosh(\sqrt{2\underline{w}}(L - L_0))} \right)^{1/2}$$

for $(x, y) \in \tilde{Q}_{L, L_0}$

for a constant $C > 0$ (independent of L) and (3.17) follows.

Finally (3.18) follows using the exponential decay property of θ_1 described in Remark 2.1, the concentration property of u displayed by (3.15), the estimates (3.16), (3.17) and the oddness with respect to the variable x of u and θ_1 . $\square$

Here below, using the stability property of θ_1 given in Lemma 2.5 we give a refined version of (3.18).

Lemma 3.6. *Setting $m_0 = \frac{1}{2}(\frac{\lambda_1}{16})^{1/2}$, there exist $\bar{L} \geq L_0$ and a constant $C > 0$ such that if $L > \bar{L}$, $u \in \mathcal{K}_{L,2}$ then*

$$\|u(\cdot, y) - \theta_1\|_{L^2(I_y)} \leq C e^{-m_0 y} \quad \text{for all } y \in [\bar{L}, L] \quad (3.20)$$

and

$$\|\partial_x(u(\cdot, y) - \theta_1)\|_{L^2(I_y)} \leq C e^{-m_0 y} \quad \text{for all } y \in [\bar{L}, L]. \quad (3.21)$$

Proof. By (W) and the L^∞ -concentration estimate (3.14) we can choose $\bar{L} \geq L_0$ such that given $L > \bar{L}$, $u \in \mathcal{K}_{L,2}$ there results that for any $y \geq \bar{L}$ and $x \in I_y$

$$|W'(u(x, y)) - W'(\theta_1(x)) - W''(\theta_1(x))(u(x, y) - \theta_1(x))| \leq \frac{\lambda_1}{16} |u(x, y) - \theta_1(x)| \quad (3.22)$$

By (3.18) we can also assume that $\bar{L}$ is such that if $L > \bar{L}$, $u \in \mathcal{K}_{L,2}$ then

$$\|u(\cdot, y) - \theta_1\|_{H^1(I_y)} \leq 1 \quad \text{for every } y \in (\bar{L}, L). \quad (3.23)$$

Given $L \geq \bar{L}$ and $u \in \mathcal{K}_{L,2}$, letting u_p be defined as in Remark 3.4, we define

$$\phi(y) = \begin{cases} \|u_p(\cdot, y) - \theta_1\|_{L^2(I_y)}^2 & \text{if } y \in (0, L] \\ \|u_p(\cdot, 2L - y) - \theta_1\|_{L^2(I_{2L-y})}^2 & \text{if } y \in (L, 2L) \end{cases}$$

observing that, since u_p is regular and symmetric with respect to $y = L$ we have that $\phi \in C^2((0, 2L))$ and $\phi(y) = \phi(2L - y)$ on $(0, 2L)$. Denoted by $\langle \cdot, \cdot \rangle_y$ the bracket symbol for the scalar product in $L^2(I_y)$, since the functions u , $\partial_y u$ and θ_1 are odd in the x variable, we have that for any $y \in (0, L)$

$$\frac{d}{dy}\phi(y) = 2\langle u(\cdot, y) - \theta_1, \partial_y u(\cdot, y) \rangle_y = 2 \int_{-y}^y (u(x, y) - \theta_1(x)) \partial_y u(x, y) dx.$$

and then

$$\ddot{\phi}(y) \geq 2\langle u(\cdot, y) - \theta_1, \partial_{yy} u(\cdot, y) \rangle_y \quad \text{for every } y \in (0, L). \quad (3.24)$$

Since u and θ_1 solve (1.1), by (3.22) we deduce

$$\begin{aligned} \langle u(\cdot, y) - \theta_1, \partial_{yy} u(\cdot, y) \rangle_y &\geq F''_{y,1}(\theta_1)(u(\cdot, y) - \theta_1)(u(\cdot, y) - \theta_1) + \\ &\quad - 2(u(y, y) - \theta_1(y))\partial_x(u(y, y) - \theta_1(y)) - \frac{\lambda_1}{16}\|u(\cdot, y) - \theta_1\|_{L^2(I_y)}^2 \end{aligned}$$

for every $y \in (\bar{L}, L)$. Hence, recalling the definition of $\phi(y)$, by (3.24) and Lemma 2.5, we conclude

$$\ddot{\phi}(y) \geq \frac{\lambda_1}{8}\phi(y) - 4(u(y, y) - \theta_1(y))\partial_x(u(y, y) - \theta_1(y)) \quad \text{for every } y \in (\bar{L}, L). \quad (3.25)$$

Note now that by the exponential decaying of θ_1 (Remark 2.1) and by (3.16)–(3.17), we obtain that there exists $C > 0$ independent of L such that for $y \in (\bar{L}, L)$

$$4|u(y, y) - \theta_1(y)| |\partial_x(u(y, y) - \theta_1(y))| \leq C e^{\sqrt{2w}(L_0 - y)}.$$

Then, since $e^{\sqrt{2w}(L_0 - y)} \leq \frac{\cosh(\sqrt{2w}(L - y))}{\cosh(\sqrt{2w}(L - L_0))}$ on (L_0, L) , by symmetry, (3.25) and (3.23) we deduce that ϕ verifies

$$\phi(y) \leq 1 \text{ and } \ddot{\phi}(y) \geq \frac{\lambda_1}{8}\phi(y) - C \frac{\cosh(\sqrt{2w}(y - L))}{\cosh(\sqrt{2w}(L - L_0))} \text{ for any } y \in (\bar{L}, 2L - \bar{L}). \quad (3.26)$$

Setting $\psi(y) = \frac{-C}{2w - \lambda_1/8} \frac{\cosh(\sqrt{2w}(y - L))}{\cosh(\sqrt{2w}(L - L_0))}$ (by Lemma 2.1 $\frac{\lambda_1}{8} < 2w$) and

$$\eta_\alpha(y) = \alpha \cosh(\sqrt{\lambda_1/8}(y - L)) + \psi(y) - \phi(y), \quad \alpha > 0, \quad y \in [\bar{L}, 2L - \bar{L}],$$

by (3.26) we recover that for any $\alpha > 0$ there results

$$\ddot{\eta}_\alpha(y) - \frac{\lambda_1}{8}\eta_\alpha(y) \leq 0 \quad \text{on } (\bar{L}, 2L - \bar{L}).$$

Setting $\alpha_0 = \frac{2}{\cosh(\sqrt{\lambda_1/8}(\bar{L}-L))}(\frac{C}{2\underline{w}-\lambda_1/8}+1)$ we have $\eta_{\alpha_0}(\bar{L}) = \eta_{\alpha_0}(2L-\bar{L}) > 0$ and by the maximum principle we derive that for any $y \in (\bar{L}, 2L-\bar{L})$

$$\begin{aligned}\phi(y) &\leq \alpha_0 \cosh(\sqrt{\lambda_1/8}(L-y)) + \psi(y) \\ &\leq 2 \left(\frac{C}{2\underline{w}-\lambda_1/8} + 1 \right) \frac{\cosh(\sqrt{\lambda_1/8}(y-L))}{\cosh(\sqrt{\lambda_1/8}(L-\bar{L}))}.\end{aligned}\quad (3.27)$$

Since on $(\bar{L}, L)$ we have $\frac{\cosh(\sqrt{\lambda_1/8}(y-L))}{\cosh(\sqrt{\lambda_1/8}(L-\bar{L}))} \leq 2e^{\sqrt{\lambda_1/8}(\bar{L}-y)}$, (3.20) follows.

To recover (3.21), we set $\psi = u(\cdot, y) - \theta_1$ and integrating by parts we get

$$\|\partial_x \psi\|_{L^2(I_y)}^2 \leq 2|\psi(y)| |\partial_x \psi(y)| + (2y)^{1/2} \|\partial_x^2 \psi\|_{L^\infty(I_y)} \|\psi\|_{L^2(I_y)} \quad (3.28)$$

By (3.13) we know $\|\partial_x^2 \psi\|_{L^\infty(I_y)} \leq \bar{C}$ for $y \in (\bar{L}, L)$. Using the exponential decaying of θ_1 (Remark 2.1), (3.16), (3.17) and (3.27) we obtain

$$2|\psi(y)| |\partial_x \psi(y)| \leq Ce^{-\sqrt{2\underline{w}}y} \text{ and } \|\psi\|_{L^2(I_y)} \leq Ce^{-\frac{1}{2}\sqrt{\lambda_1/8}y} \text{ on } (\bar{L}, L),$$

where C is a positive constant independent on L . Then, taking C bigger if necessary, by (3.28) we conclude

$$\|\partial_x(u(\cdot, y) - \theta_1)\|_{L^2(I_y)}^2 \leq C(e^{-\sqrt{2\underline{w}}y} + y^{1/2}e^{-\frac{1}{2}\sqrt{\lambda_1/8}y}) \quad \text{on } (\bar{L}, L),$$

from which (3.21) follows recalling that $m_0 = \frac{1}{2}\sqrt{\lambda_1/16}$. $\square$

Thanks to (3.21) we are now able to estimate the asymptotic behaviour of the function $c_{\infty,2} - c_{L,2}$ as $L \rightarrow +\infty$.

Lemma 3.7. *There exists $C > 0$ such that*

$$c_{\infty,2} - c_{L,2} \leq Ce^{-m_0L} \quad \text{for all } L \geq \bar{L}. \quad (3.29)$$

Proof. In the proof we will denote by C a positive constant independent of L whose value can change line to line. Letting $u \in \mathcal{K}_{L,2}$ We consider the function $\tilde{u}(x, y)$ defined on $T_{\infty,2}$ to be equal to $\theta_1(x)$ for $y \geq L+1$, equal to $u(x, y)$ for $y \leq L$ and such that for $y \in (L, L+1)$

$$\tilde{u}(x, y) = \begin{cases} (L+1-y)u(x, L) + (y-L)\theta_1(x) & -L \leq x \leq L, \\ (L+1-y)u(L, L) + (y-L)\theta_1(x) & L < x < y \\ (L+1-y)u(-L, L) + (y-L)\theta_1(x) & -y < x < -L. \end{cases}$$

We recognize that $\tilde{u} \in \Gamma_{\infty,2}$ and so $F_{\infty,2}(\tilde{u}) \geq c_{\infty,2}$. Then

$$F_{\infty,2}(\tilde{u}) = c_{L,2} + F_{(L,L+1),2}(\tilde{u}) + (F_{\infty,2}(\theta_1) - F_{L+1,2}(\theta_1)) \geq c_{\infty,2}$$

and so

$$c_{\infty,2} - c_{L,2} \leq F_{(L,L+1),2}(\tilde{u}) + (F_{\infty,2}(\theta_1) - F_{L+1,2}(\theta_1)). \quad (3.30)$$

Recall that by (2.4)

$$F_{\infty,2}(\theta_1) - F_{L+1,2}(\theta_1) \leq Ce^{-\sqrt{2\underline{w}}L} \quad \text{for all } L > 0. \quad (3.31)$$

To evaluate $F_{(L,L+1),2}(\tilde{u})$ we recall that $F_{(L,L+1),2}(\tilde{u}) = \int_L^{L+1} \frac{1}{2} \|\partial_y \tilde{u}(\cdot, y)\|_{L^2(I_y)}^2 + (F_{y,1}(\tilde{u}(\cdot, y)) - c_{y,1}) dy$, and we proceed estimating the two terms $\|\partial_y \tilde{u}(\cdot, y)\|_{L^2(I_y)}^2$ and $(F_{y,1}(\tilde{u}(\cdot, y)) - c_{y,1})$.

By direct computation we recognize that

$$\partial_y \tilde{u}(x, y) = \begin{cases} u(L, L) - \theta_1(x), & \text{for } y \in (L, L+1) \text{ and } |x| \in (L, y) \\ u(x, L) - \theta_1(x), & \text{for } y \in (L, L+1) \text{ and } x \in (-L, L) \\ u(-L, L) - \theta_1(x), & \text{for } y \in (L, L+1) \text{ and } x \in (-y, -L) \end{cases}$$

and so, since u and θ_1 are odd in the variable x , using Remark (2.1, (3.16) and (3.20) we obtain that for any $L \geq \bar{L}$ and $y \in (L, L+1)$,

$$\|\partial_y \tilde{u}(\cdot, y)\|_{L^2(I_y)}^2 = 2\|u(L, L) - \theta_1\|_{L^2(L, y)}^2 + \|u(\cdot, L) - \theta_1\|_{L^2(I_L)}^2 \leq Ce^{-m_0 L}. \quad (3.32)$$

Considering the term $(F_{y,1}(\tilde{u}(\cdot, y)) - c_{y,1})$ we first note that by (2.2)

$$F_{y,1}(\tilde{u}(\cdot, y)) - c_{y,1} \leq F_{y,1}(\tilde{u}(\cdot, y)) - F_{y,1}(\theta_1) + Ce^{-\sqrt{2\bar{w}}L}. \quad (3.33)$$

Secondly we write

$$F_{y,1}(\tilde{u}(\cdot, y)) - F_{y,1}(\theta_1) = 2(F_{(L,y),1}(\tilde{u}(\cdot, y)) - F_{(L,y),1}(\theta_1)) + F_{L,1}(\tilde{u}) - F_{L,1}(\theta_1),$$

and we note that for any $y \in (L, L+1)$

$$\begin{aligned} F_{(L,y),1}(\tilde{u}(\cdot, y)) - F_{(L,y),1}(\theta_1) &= +\frac{1}{2}(\|(L-y)\dot{\theta}_1\|_{L^2(L,y)}^2 - \|\dot{\theta}_1\|_{L^2(L,y)}^2) \\ &\quad + \int_L^y W(\theta_1 + (L+1-y)(u(L, L) - \theta_1)) \\ &\quad - W(\theta_1) dx \leq \|\dot{\theta}_1\|_{L^2(L,L+1)}^2 + C\|u(L, L) - \theta_1\|_{L^2(L,L+1)}^2. \end{aligned}$$

by Remark 2.1 and (3.16) we recover $\|\dot{\theta}_1\|_{L^2(L,L+1)}^2 \leq Ce^{-\sqrt{2\bar{w}}L}$ and $\|u(L, L) - \theta_1\|_{L^2(L,L+1)}^2 \leq Ce^{-\sqrt{2\bar{w}}L}$ and so

$$F_{(L,y),1}(\tilde{u}(\cdot, y)) - F_{(L,y),1}(\theta_1) \leq Ce^{-\sqrt{2\bar{w}}L}. \quad (3.34)$$

Finally, since $|W(\theta_1(x) + (L+1-y)(u(x, L) - \theta_1(x))) - W(\theta_1(x))| \leq |W'(\theta_1)||u(x, L) - \theta_1(x)| + C|u(x, L) - \theta_1(x)|^2$ for $x \in I_L$ and since $\|W'(\theta_1)\|_{L^2(I_L)} \leq C$ we obtain that for any $y \in (L, L+1)$

$$\begin{aligned} F_{L,1}(\tilde{u}(\cdot, y)) - F_{L,1}(\theta_1) &\leq \frac{1}{2}\|\partial_x(u(\cdot, L) - \theta_1)\|_{L^2(I_L)}^2 + |\langle \dot{\theta}_1, \partial_x(u(\cdot, L) - \theta_1) \rangle_L| \\ &\quad + \int_{-L}^L W(\theta_1(x) + (L+1-y)(u(x, L) - \theta_1(x))) \\ &\leq -W(\theta_1(x)) dx C(\|\partial_x(u(\cdot, L) - \theta_1)\|_{L^2(I_L)} \\ &\quad + \|u(\cdot, L) - \theta_1\|_{L^2(I_L)}) \end{aligned}$$

and by (3.20) and (3.21) we recover that for any $y \in (L, L+1)$

$$F_{L,1}(\tilde{u}(\cdot, y)) - F_{L,1}(\theta_1) \leq Ce^{-m_0 L}. \quad (3.35)$$

By (3.32), (3.34) and (3.35) we obtain that $F_{(L,L+1),2}(\tilde{u}) \leq Ce^{-m_0 L}$, which, together with (3.30) and (3.31), implies (3.29). $\square$

Finally, by Lemma 2.4, adapting the argument used in [7], Lemma 4.5, we have

Lemma 3.8. *Let $L_n \rightarrow +\infty$, $u_n \in \Gamma_{L_n,2}$ such that $|u_n(x, y)| \leq 1$ on $T_{L_n,2}$ and $F_{L_n,2}(u_n) - c_{L_n,2} \rightarrow 0$ as $n \rightarrow \infty$. Then, for any $L > 0$ there results*

$$\|u_n - \theta_2\|_{L^2(T_{L,2})} \rightarrow 0.$$

Moreover, if $\bar{L}_n \in (0, L_n)$ is such that $\bar{L}_n \rightarrow +\infty$ then $F_{(\bar{L}_n, L_n),2}(u_n) \rightarrow 0$.

4. The saddle solution on $\mathbb{R}^3$

In this last section we collect the results obtained above to prove Theorem 1.1. The proof iterates the variational argument already used in Sect. 2 constructing the two dimensional saddle solution θ_2 . In fact we inductively define the simplicial set

$$T_{\infty,3} = \{(x, y, z) \in \mathbb{R}^3 / z \geq 0, (x, y) \in T_{z,2}\}$$

on which we look for a solution of (1.1) verifying the Neumann boundary condition $\partial_\nu u|_{\partial T_{\infty,3}} = 0$ and the asymptotic condition $\lim_{z \rightarrow +\infty} u(x, y, z) - \theta_2(x, y) = 0$ for every $(x, y) \in T_{\infty,2}$.

We define the space

$$\Gamma_{\infty,3} = \{u \in H_{loc}^1(T_{\infty,3}) / u(\cdot, \cdot, z) \in \Gamma_{z,2} \text{ for a.e. } z > 0\}$$

on which we consider the functional

$$F_{\infty,3}(u) = \int_0^{+\infty} \frac{1}{2} \|\partial_z u(\cdot, \cdot, z)\|_{L^2(T_{z,2})}^2 + (F_{y,2}(u(\cdot, \cdot, z)) - c_{z,2}) dz.$$

Note that if $u \in \Gamma_{\infty,3}$ then $u(\cdot, z) \in \Gamma_{z,2}$ for a.e. $z > 0$ and so $F_{y,2}(u(\cdot, z)) - c_{z,2} \geq 0$ for a.e. $z > 0$. Hence $F_{\infty,3}$ is non negative on $\Gamma_{\infty,3}$ and, as one can simply recognize, $F_{\infty,3}$ is weakly lower semicontinuous with respect to the $H_{loc}^1(T_{\infty,3})$ topology. We look for a minimum of $F_{\infty,3}$ on $\Gamma_{\infty,3}$ and we set

$$c_{\infty,3} = \inf_{\Gamma_{\infty,3}} F_{\infty,3} \quad \text{and} \quad \mathcal{K}_{\infty,3} = \{u \in \Gamma_{\infty,3} / F_{\infty,2}(u) = c_{\infty,3}\}.$$

The key Lemma to recover entire solutions of (1.1) from elements of $\mathcal{K}_{\infty,3}$ is the following

Lemma 4.1. *If $\bar{u} \in \mathcal{K}_{\infty,3}$ then $F_{\infty,3}(\bar{u} + \psi) \geq F_{\infty,3}(\bar{u})$ for any $\psi \in C_0^\infty(\mathbb{R}^3)$. Hence*

$$\int_{T_{\infty,3}} \nabla \bar{u} \cdot \nabla \psi + W'(\bar{u})\psi \, dx \, dy \, dz = 0 \text{ for any } \psi \in C_0^\infty(\mathbb{R}^3).$$

Proof. Since W is even, if $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfies $\psi(x, y, z) = -\psi(-x, y, z)$ then defined

$$\phi(x, y, z) = \begin{cases} \bar{u}(x, y, z) + \psi(x, y, z) & \text{if } x > 0 \text{ and } \bar{u}(x, y, z) + \psi(x, y, z) \geq 0 \\ -\bar{u}(x, y, z) - \psi(x, y, z) & \text{if } x > 0 \text{ and } \bar{u}(x, y, z) + \psi(x, y, z) \leq 0 \\ -\phi(-x, y, z) & \text{if } x < 0 \end{cases}$$

we have $\phi \in \Gamma_{\infty,3}$ and $F_{\infty,3}(\phi) = F_{\infty,3}(\bar{u} + \psi)$. Since $\bar{u}$ is a minimum of $F_{\infty,3}$ on $\Gamma_{\infty,3}$, we recover that if $\psi \in C_0^\infty(\mathbb{R}^3)$ is such that $\psi(x, y, z) = -\psi(-x, y, z)$ then

$$F_{\infty,3}(\bar{u} + \psi) = F_{\infty,3}(\phi) \geq F_{\infty,3}(\bar{u}). \quad (4.1)$$

Given now any $\psi \in C_0^\infty(\mathbb{R}^2)$, we set $\psi_o(x, y, z) = \frac{1}{2}(\psi(x, y, z) - \psi(-x, y, z))$, $\psi_e(x, y, z) = \frac{1}{2}(\psi(x, y, z) + \psi(-x, y, z))$, and observing that the functions $\nabla \bar{u} \cdot \nabla \psi_e$ and $\nabla \psi_o \cdot \nabla \psi_e$ are odd in the variable x we recover

$$\begin{aligned} F_{\infty,3}(\bar{u} + t\psi) - F_{\infty,3}(\bar{u} + t\psi_o) &= \int_{T_{\infty,3}} \frac{t^2}{2} |\nabla \psi_e|^2 + W(\bar{u} + t\psi) \\ &\quad - W(\bar{u} + t\psi_o) dx dy dz. \end{aligned}$$

By (4.1) and since the function $W'(\bar{u})\psi_e$ is odd in the variable x , we conclude

$$\begin{aligned} &\int_{T_{\infty,3}} \nabla \bar{u} \cdot \nabla \psi + W'(\bar{u})\psi dx dy dz \\ &= \lim_{t \rightarrow 0^+} \frac{1}{t} (F_{\infty,3}(\bar{u} + t\psi) - F_{\infty,3}(\bar{u})) \\ &\geq \lim_{t \rightarrow 0^+} \int_{T_{\infty,3}} \frac{W(\bar{u} + t\psi) - W(\bar{u})}{t} + \frac{W(\bar{u}) - W(\bar{u} + t\psi_o)}{t} dx dy dz \\ &= \int_{T_{\infty,3}} W'(\bar{u})\psi_e dx dy dz = 0. \end{aligned}$$

Then $\int_{T_{\infty,3}} \nabla \bar{u} \cdot \nabla \psi + W'(\bar{u})\psi dx dy dz \geq 0$ for any $\psi \in C_0^\infty(\mathbb{R}^3)$, and the Lemma follows. $\square$

By Lemma 4.1, to construct a saddle entire solutions on $\mathbb{R}^3$ we can proceed recursively reflecting a function $\bar{u} \in \mathcal{K}_{\infty,3}$. Precisely, we consider the operations of exchange of coordinates

$$\gamma_{2,1}(x, y, z) = (y, x, z), \quad \gamma_{3,1}(x, y, z) = (z, y, x), \quad \gamma_{3,2}(x, y, z) = (x, z, y),$$

to reflect with respect to the diagonal planes $\{y = x\}$, $\{z = x\}$, $\{z = y\}$, and the operations of changing sign of the coordinates

$$r_2(x, y, z) = (x, -y, z), \quad r_3(x, y, z) = (x, y, -z),$$

to reflect with respect to the plane $\{y = 0\}$ and $\{z = 0\}$. Denoting

$$\gamma_{3,0}(x, y, z) = \gamma_{2,0}(x, y, z) = (x, y, z),$$

this operations can be composed to generate the 24 transformations

$$M_{(k_3, k_2); (i_3, i_2)} = \gamma_{3, k_3} r_3^{i_3} \gamma_{2, k_2} r_2^{i_2}, \quad k_3 \in \{0, 1, 2\}, \quad k_2 \in \{0, 1\} \quad (i_3, i_2) \in \{0, 1\}^2.$$

Denoting the corresponding rigid movements of $T_{\infty,3}$

$$T_{(k_3, k_2); (i_3, i_2)} = M_{(k_3, k_2); (i_3, i_2)} T_{\infty,3}, \quad k_3 \in \{0, 1, 2\}, \quad k_2 \in \{0, 1\} \quad (i_3, i_2) \in \{0, 1\}^2.$$

we have $\mathbb{R}^3 = \cup_{k_3=0}^2 \cup_{k_2=0}^1 \cup_{(i_3, i_2) \in \{0,1\}^2} T(k_3, k_2); (i_3, i_2)$. We define

$$\theta_3(x, y, z) = (-1)^{i_2+i_3} \bar{u}(M_{(k_3, k_2); (i_3, i_2)}^{-1}(x, y, z)), \quad \forall (x, y, z) \in T(k_3, k_2); (i_3, i_2)$$

noting that, since $\bar{u} \in H_{loc}^1(T_{\infty, 3})$, by the properties of the reflections operators (see e.g. [12] Lemma 9.2.) $\theta_3 \in H_{loc}^1(\mathbb{R}^3)$. Then, by Lemma 4.1 and by the change of variables Theorem,

$$\begin{aligned} & \int_{T(k_3, k_2); (i_3, i_2)} \nabla \theta_3 \cdot \nabla \psi + W'(\theta_3) \psi \, dx \, dy \, dz \\ &= (-1)^{i_2+i_3} \int_{T_{\infty, 3}} \nabla \bar{u} \cdot \nabla \psi \circ M_{(k_3, k_2); (i_3, i_2)} \\ & \quad + W'(\bar{u}) \psi \circ M_{(k_3, k_2); (i_3, i_2)} \, dx \, dy \, dz = 0 \end{aligned}$$

for any $\psi \in C_0^\infty(\mathbb{R}^3)$, $k_3 \in \{0, 1, 2\}$, $k_2 \in \{0, 1\}$ and $(i_3, i_2) \in \{0, 1\}^2$. Then

$$\int_{\mathbb{R}^3} \nabla \theta_3 \cdot \nabla \psi + W'(\theta_3) \psi \, dx \, dy \, dz = 0 \text{ for any } \psi \in C_0^\infty(\mathbb{R}^3),$$

i.e. θ_3 is a weak and so, by standard bootstrap arguments, a classical $C^2(\mathbb{R}^3)$ solution of equation (1.1).

After these preliminary considerations we now proceed proving the existence of a minimum of $F_{\infty, 3}$ on $\Gamma_{\infty, 3}$.

Remark 4.1. It is simple to recognize that $c_{\infty, 3} < +\infty$ and so the minimum problem is meaningful. Indeed, considering the function $v(x, y, z) = \theta_2(x, y)$, we have $v \in \Gamma_{\infty, 3}$ and, by (3.29),

$$F_{\infty, 3}(v) = \int_0^{+\infty} F_{z, 2}(\theta_2) - c_{z, 2} \, dz \leq \int_0^{+\infty} c_{\infty, 2} - c_{z, 2} \, dz \leq C \int_0^{+\infty} e^{-m_0 z} \, dz.$$

In the construction an important role is played by estimates concerning the functional $F_{\infty,3}$ which are analogous to the continuity estimate (3.3) and the coerciveness estimate (3.4) given for the functional $F_{\infty,2}$.

First we note that if $u \in \Gamma_{\infty,3}$ then $F_{z,2}(u(\cdot, \cdot, z)) \geq c_{z,2}$ for a.e. $z > 0$ and so

$$\|\partial_z u\|_{L^2(T_{\infty,3})}^2 \leq 2F_{\infty,3}(u) \quad \forall u \in \Gamma_{\infty,3}. \quad (4.2)$$

Then we observe that if $u \in \Gamma_{\infty,3}$ we have $u(x, y, \cdot) \in H_{loc}^1(z, +\infty)$ for a.e. $(x, y) \in T_{z,2}$, $z > 0$. Hence, if $0 < z_1 < z_2$, then $u(x, y, z_2) - u(x, y, z_1) = \int_{z_1}^{z_2} \partial_z u(x, y, z) dz$ holds for all $u \in \Gamma_{\infty,3}$ and a.e. $(x, y) \in T_{z_1,2}$. So, if $u \in \Gamma_{\infty,3}$ and $z_1 < z_2 \in \mathbb{R}^+$, by (4.2) we have

$$\begin{aligned} & \|u(\cdot, \cdot, z_2) - u(\cdot, \cdot, z_1)\|_{L^2(T_{z_1,2})}^2 \\ &= \int_{T_{z_1,2}} \left| \int_{z_1}^{z_2} \partial_z u(x, y, z) dz \right|^2 dx dy \\ &\leq |z_2 - z_1| \int_{T_{z_1,2}} \int_{\mathbb{R}^+} |\partial_z u(x, y, z)|^2 dz dx dy \leq 2F_{\infty,3}(u) |z_2 - z_1|, \end{aligned} \quad (4.3)$$

which is the analogous of the continuity estimate (3.3). To recover the analogous of (3.4), given an interval $I \subset \mathbb{R}_+$ and $u \in \Gamma_{\infty,3}$ we set

$$F_{I,3}(u) = \int_I \frac{1}{2} \|\partial_z u(\cdot, \cdot, z)\|_{L^2(T_{z,2})}^2 + (F_{z,2}(u(\cdot, \cdot, z)) - c_{z,2}) dz.$$

and we observe that if $u \in \Gamma_{\infty,3}$ is such that $F_{z,2}(u(\cdot, \cdot, z)) - c_{z,2} \geq \nu > 0$ for a.e. $z \in (\sigma, \tau) \subset \mathbb{R}^+$, then

$$\begin{aligned} F_{(\sigma,\tau),3}(u) &\geq \frac{1}{2(\tau - \sigma)} \|u(\cdot, \cdot, \tau) - u(\cdot, \cdot, \sigma)\|_{L^2(T_{2,\sigma})}^2 + \nu(\tau - \sigma) \\ &\geq \sqrt{2\nu} \|u(\cdot, \cdot, \tau) - u(\cdot, \cdot, \sigma)\|_{L^2(T_{2,\sigma})}. \end{aligned} \quad (4.4)$$

The estimates (4.3), (4.4) and Lemma 3.8, can be used to characterize the asymptotic behaviour, as $z \rightarrow +\infty$, of the functions $u \in \Gamma_{\infty,3}$ such that $F_{\infty,3}(u) < +\infty$.

Lemma 4.2. *If $u \in \Gamma_{\infty,3}$ is such that $|u(x, y, z)| \leq 1$ for a.e. $(x, y, z) \in T_{\infty,3}$ and $F_{\infty,3}(u) < +\infty$ then, fixed any $L > 0$, we have*

$$\|u(\cdot, \cdot, z) - \theta_2\|_{L^2(T_{L,2})} \rightarrow 0 \quad \text{as } z \rightarrow +\infty.$$

Proof. Let $u \in \Gamma_{\infty,3}$ and $F_{\infty,3}(u) < +\infty$. Since $F_{z,2}(u(\cdot, \cdot, z)) - c_{z,2} \geq 0$ for a.e. $z > 0$, there exists a sequence $z_n \rightarrow +\infty$ such that $F_{z_n,2}(u(\cdot, \cdot, z_n)) - c_{z_n,2} \rightarrow 0$. Fixed $L > 0$, by Lemma 3.8, there exists an increasing subsequence $(z_{n_k}) \subset (z_n)$ with $z_{n_1} \geq L$ such that $\|u(\cdot, \cdot, z_{n_k}) - \theta_2\|_{L^2(T_{L,2})} \rightarrow 0$ as $k \rightarrow +\infty$.

Assuming by contradiction that $\limsup_{z \rightarrow +\infty} \|u(\cdot, \cdot, z) - \theta_2\|_{L^2(T_{L,2})} = 4\delta > 0$, using (4.3), we obtain the existence of a sequence of intervals $(\sigma_k, \tau_k) \subset \mathbb{R}^+$ such that

- (i) $\sigma_{k+1} > \tau_k \rightarrow +\infty$,
- (ii) $\|u(\cdot, \cdot, \tau_k) - u(\cdot, \cdot, \sigma_k)\|_{L^2(T_{L,2})} = \delta$,
- (iii) $2\delta \geq \|u(\cdot, \cdot, z) - \theta_2\|_{L^2(T_{L,2})} \geq \delta$, for every $z \in (\sigma_k, \tau_k)$.

By Lemma 3.8, and (iii) there exist $\bar{k} \in \mathbb{N}$ and $\nu > 0$ such that if $z \in \cup_{k \geq \bar{k}} (\sigma_k, \tau_k)$ then $F_{z,2}(u(\cdot, \cdot, z)) - c_{z,2} \geq \nu$. By (4.4) and (ii) we obtain $F_{(\sigma_k, \tau_k),3}(u) \geq \sqrt{2\nu}\delta > 0$ for all $k \geq \bar{k}$ and so, by (i), $F_{\infty,3}(u) \geq \sum_{k \geq \bar{k}} F_{(\sigma_k, \tau_k),3}(u) = +\infty$, a contradiction. $\square$

We are now able to prove the existence of a minimum of $F_{\infty,3}$ on $\Gamma_{\infty,3}$.

Proposition 4.1. *There exists $\bar{u} \in \Gamma_{\infty,3}$ such that $F_{\infty,3}(\bar{u}) = c_{\infty,3}$. Moreover for every $L > 0$*

$$\|\bar{u}(\cdot, \cdot, z) - \theta_2\|_{L^2(T_{L,2})} \rightarrow 0 \quad \text{as } z \rightarrow +\infty.$$

Proof. Let $(u_n)_{n \in \mathbb{N}} \subset \Gamma_{\infty,3}$ be a minimizing sequence for $F_{\infty,3}$ which we can assume such that $|u_n(x, y, z)| \leq 1$ for a.e. $(x, y, z) \in T_{\infty,3}$, $n \in \mathbb{N}$.

It is not difficult to recognize that, fixed any $r > 0$, setting $T_{r,3} = T_{\infty,3} \cap \{z < r\}$, then u_n is a bounded sequence on $H^1(T_{r,3})$. Indeed, since $|u_n(x, y, z)| \leq 1$ a.e. in $T_{\infty,3}$ we have $\|u_n\|_{L^2(T_{r,3})} < +\infty$ for any $n \in \mathbb{N}$. Moreover

$$\begin{aligned} & \|\nabla u_n\|_{L^2(T_{r,3})}^2 \\ &= \int_0^r \int_0^z \int_{-y}^y |\nabla u_n(x, y, z)|^2 dx dy dz \\ &\leq 2F_{\infty,3}(u_n) + 2 \int_0^r c_{z,2} dz + 2 \int_0^r \int_0^z c_{y,1} dy dz \\ &\leq 2F_{\infty,3}(u_n) + 2r\bar{c}_{\infty,2} + r^2 c_{\infty,1} = 2c_{\infty,3} + 2rc_{\infty,2} + r^2 c_{\infty,1} + o(1) \end{aligned}$$

and our claim follows.

Thus, by a classical diagonal argument, there exists $\bar{u} \in \cap_{r>0} H^1(T_{r,2})$ and a subsequence of (u_n) , still denoted (u_n) , such that $u_n - \bar{u} \rightarrow 0$ weakly in $H^1(T_{r,2})$ for any $r > 0$ and for a.e. $(x, y, z) \in T_{\infty,2}$.

By pointwise convergence, we have $|\bar{u}(x, y, z)| \leq 1$ and since $u_n \in \Gamma_{\infty,3}$ we derive also $\bar{u}(x, y, z) = -\bar{u}(-x, y, z)$ a.e. on $T_{\infty,3}$ and $\bar{u}(x, y, z) \geq 0$ whenever $x \geq 0$. This proves that $\bar{u} \in \Gamma_{\infty,3}$ and by semicontinuity we recover that $F_{\infty,3}(\bar{u}) = c_{\infty,3}$. Finally, by Lemma 4.2, fixed any $L > 0$ we have $\|u(\cdot, \cdot, z) - \theta_2\|_{L^2(T_{L,2})} \rightarrow 0$ as $z \rightarrow +\infty$ and the Lemma follows. $\square$

As explained above, by Proposition 4.1 and Lemma 4.1, from $\bar{u}$ we can construct a classical entire solution $\theta_3 \in C^2(\mathbb{R}^3)$ of (1.1). Since $|\bar{u}(x, y, z)| \leq 1$ on $T_{\infty,3}$ we have $|\theta_3(x, y, z)| \leq 1$ on $\mathbb{R}^3$. Then, by Schauder estimates, there exists a constant $\bar{C} > 0$ such that

$$\|\theta_3\|_{C^2(T_{\infty,3})} \leq \bar{C} \quad (4.5)$$

This is used to finally show that θ_3 satisfied property (iii) of Theorem 1.1.

Lemma 4.3. *There results $\|\theta_3(\cdot, \cdot, z) - \theta_2\|_{L^\infty(T_{z,2})} \rightarrow 0$ as $z \rightarrow +\infty$.*

Proof. Assume by contradiction that there exists $\rho_0 > 0$ and $(x_n, y_n, z_n) \in T_{\infty,3}$ with $z_n \rightarrow +\infty$ such that $|\theta_3(x_n, y_n, z_n) - \theta_2(x_n, y_n)| \geq 4\rho_0$. By (4.5) there exists $r_0 > 0$ such that if $n \in \mathbb{N}$ and $(x, y, z) \in T_{\infty,3}$ is such that $|x - x_n|, |y - y_n|, |z - z_n| \leq r_0$ then $|\theta_3(x, y, z) - \theta_2(x, y)| \geq 2\rho_0$. Since by Proposition 4.1 we have $\|\theta_3(\cdot, \cdot, z) - \theta_2\|_{L^2(T_{L,2})} \rightarrow 0$ as $z \rightarrow +\infty$ for every

$L > 0$, we deduce that $y_n \rightarrow +\infty$. Since $\|\theta_2(\cdot, y) - \theta_1\|_{L^\infty(I_y)} \rightarrow 0$ as $y \rightarrow +\infty$, we recover that there exists $\bar{n} > 0$ such that if $n \geq \bar{n}$ and $(x, y, z) \in T_{\infty,3}$ is such that $|x - x_n|, |y - y_n|, |z - z_n| \leq r_0$ then $|\theta_3(x, y, z) - \theta_1(x)| \geq \rho_0$. In particular, if $n \geq \bar{n}$, $|z - z_n| \leq r_0$, $y \in [y_n - r_0, y_n]$, then

$$\|\theta_3(\cdot, y, z) - \theta_1\|_{L^2(I_y)} \geq \sqrt{2r_0\rho_0}. \quad (4.6)$$

By (4.6) and (2.3) we derive that there exists $\mu > 0$ such that if $n \geq \bar{n}$ (with $\bar{n}$ eventually larger), $|z - z_n| \leq r_0$, $y \in [y_n - r_0, y_n]$, then

$$F_{y,1}(\theta_3(\cdot, y, z)) \geq c_{y,1} + \mu$$

and so in particular if $n \geq \bar{n}$, $|z - z_n| \leq r_0$ there results

$$F_{(y_n-r_0, y_n),2}(\theta_3(\cdot, \cdot, z)) \geq \int_{y_n-r_0}^{y_n} F_{y,1}(\theta_3(\cdot, y, z)) - c_{y,1} dy \geq \mu r_0 \quad (4.7)$$

Denoting $A = \cup_{n \geq \bar{n}} [z_n - r_0, z_n + r_0]$, we have $\int_A F_{z,2}(\theta_3(\cdot, \cdot, z)) - c_{z,2} dz \leq F_{\infty,3}(\theta_3) < +\infty$ and so there exists a sequence $\zeta_n \rightarrow +\infty$ with $\zeta_n \in [z_n - r_0, z_n + r_0]$, such that $F_{\zeta_n,2}(\theta_3(\cdot, \cdot, \zeta_n)) - c_{\zeta_n,2} \rightarrow 0$. By Lemma 3.8, since $y_n - r_0 \rightarrow +\infty$ we deduce that

$$F_{(y_n-r_0, y_n),2}(\theta_3(\cdot, \cdot, \zeta_n)) \leq F_{(y_n-r_0, \zeta_n),2}(\theta_3(\cdot, \cdot, \zeta_n)) \rightarrow 0,$$

in contradiction with (4.7). $\square$

Remark 4.2. The symmetry properties of θ_3 and Lemma 4.3 ensure that property (iii) of Theorem 1.1 is satisfied. The uniqueness of θ_3 , its positivity and monotonicity properties on the first octant are obtained again as in the proof of Lemma 3.2 by moving plane like techniques. Then Theorem 1.1 is completely proved.

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