Unconditional uniqueness in the charge class for the Dirac–Klein–Gordon equations in two space dimensions

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Abstract. Recently, Grünrock and Pecher proved global well-posedness of the 2d Dirac–Klein–Gordon equations given initial data for the spinor and scalar fields in H^s and $H^{s+1/2} \times H^{s-1/2}$, respectively, where $s \geq 0$, but uniqueness was only known in a contraction space of Bourgain type, strictly smaller than the natural solution space $C([0, T]; H^s \times H^{s+1/2} \times H^{s-1/2})$. Here we prove uniqueness in the latter space for $s \geq 0$. This improves a recent result of Pecher, where the range $s > 1/30$ was covered.

Mathematics Subject Classification (2010). 35Q40; 35L70.

Keywords. Dirac–Klein–Gordon equations, Uniqueness, Null structure, Bilinear space-time estimates.

1. Introduction

We consider the Dirac–Klein–Gordon system (DKG) in two space dimensions, which reads

$$\begin{cases} -i(\partial_t + \alpha \cdot \nabla)\psi = -M\beta\psi + \phi\beta\psi, \\ (-\square + m^2)\phi = \langle \beta\psi, \psi \rangle, \quad (\square = -\partial_t^2 + \Delta) \end{cases} \quad (1.1)$$

with initial data

$$\psi|_{t=0} = \psi_0 \in H^s, \quad \phi|_{t=0} = \phi_0 \in H^r, \quad \partial_t \phi|_{t=0} = \phi_1 \in H^{r-1}, \quad (1.2)$$

where $\psi(t, x)$ is the Dirac spinor, regarded as a column vector in $\mathbb{C}^2$, and $\phi(t, x)$ is real-valued. Here $t \in \mathbb{R}$, $x \in \mathbb{R}^2$; $M, m \geq 0$ are constants; $\nabla = (\partial_{x_1}, \partial_{x_2})$; $\langle u, v \rangle := \langle u, v \rangle_{\mathbb{C}^2} = v^\dagger u$ for column vectors $u, v \in \mathbb{C}^2$, where $v^\dagger$ is the complex conjugate transpose of v ; $H^s = (1 + \sqrt{-\Delta})^{-s} L^2(\mathbb{R}^2)$ is the standard Sobolev space of order s . The Dirac matrices α^j, β satisfy

$$\beta^\dagger = \beta, \quad (\alpha^j)^\dagger = \alpha^j, \quad \beta^2 = (\alpha^j)^2 = I, \quad \alpha^j \beta + \beta \alpha^j = 0. \quad (1.3)$$

A particular representation is

$$\alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Local well-posedness of this problem with low-regularity data was first studied in [1]. In [3], D’Ancona et al. proved the existence of a local solution

$$(\psi, \phi, \partial_t \phi) \in C([0, T]; H^s \times H^r \times H^{r-1})$$

for the Cauchy problem (1.1), (1.2) whenever

$$s > -\frac{1}{5}, \quad \max\left(\frac{1}{4} + \frac{|s|}{2}, s\right) < r < \min\left(\frac{3}{4} + 2s, 1 + s\right),$$

but uniqueness was only obtained in the contraction space, which is strictly smaller than the natural solution space $C([0, T]; H^s \times H^r \times H^{r-1})$. The question therefore arises whether uniqueness holds in the latter space; if this is the case, one says that *unconditional uniqueness* holds at the regularity (s, r) . The question of unconditional uniqueness is especially interesting for $s \geq 0$ and $r = s + 1/2$, since for that range it was proved by Grünrock and Pecher [5] that the local solution extends globally in time. Recently, Pecher [6] proved unconditional uniqueness for $s > 1/30, r = s + 1/2$. In this paper we improve this to $s \geq 0$.

Theorem 1.1. *If $s \geq 0$, uniqueness of the solution to (1.1), (1.2) holds in*

$$(\psi, \phi, \partial_t \phi) \in C([0, T]; H^s \times H^{s+1/2} \times H^{s-1/2}).$$

To prove this we rely on the null structure of DKG, found in [3], and on bilinear space-time estimates of Klainerman–Machedon type proved in [4]. Using also an idea of Zhou [7] we iteratively improve the known regularity of the solution, until we reach a space where uniqueness is known by the results in [3].

Some notation: In estimates we shall use $X \lesssim Y$ as shorthand for $X \leq CY$, where $C \gg 1$ is a constant. We write $X \approx Y$ for $X \lesssim Y \lesssim X$. Throughout, ε is understood to be a sufficiently small positive number. The space-time Fourier transform of a function $u(t, x)$ is denoted $\mathcal{F}u(\tau, \xi) = \tilde{u}(\tau, \xi)$.

2. Null structure and bilinear estimates

To uncover the null structure of the system we follow [3] and use the Dirac projections

$$P_{\pm}(\xi) = \frac{1}{2} \left(I \pm \frac{\xi}{|\xi|} \cdot \alpha \right)$$

to split $\psi = \psi_+ + \psi_-$, where $\psi_{\pm} = P_{\pm}(D)\psi$. Here $D = -i\nabla$, which has Fourier symbol ξ . Applying $P_{\pm}(D)$ to the Dirac equation in (1.1), and using the identities $\alpha \cdot D = |D|P_+(D) - |D|P_-(D)$, $P_{\pm}^2(D) = P_{\pm}(D)$, $P_{\pm}(D)P_{\mp}(D) = 0$ and $P_{\pm}(D)\beta = \beta P_{\mp}(D)$, the Dirac equation becomes

$$(-i\partial_t \pm |D|)\psi_{\pm} = -M\beta\psi_{\mp} + P_{\pm}(D)(\phi\beta\psi).$$

On the other hand, splitting $\phi = \phi_+ + \phi_-$, where

$$\phi_{\pm} = \frac{1}{2} (\phi \pm i \langle D \rangle_m^{-1} \phi_t), \quad \langle D \rangle_m = \sqrt{m^2 + |D|^2},$$

the Klein–Gordon part of the system becomes

$$(-i\partial_t \pm \langle D \rangle_m) \phi_{\pm} = \mp (2\langle D \rangle_m)^{-1} \langle \beta \psi, \psi \rangle.$$

Thus, the DKG system has been rewritten as

$$\begin{cases} (-i\partial_t \pm |D|) \psi_{\pm} = -M\beta\psi_{\mp} + P_{\pm}(D)(\phi\beta\psi), \\ (-i\partial_t \pm \langle D \rangle_m) \phi_{\pm} = \mp (2\langle D \rangle_m)^{-1} \langle \beta \psi, \psi \rangle, \end{cases} \quad (2.1)$$

with initial data

$$\psi_{\pm}(0) = P_{\pm}(D)\psi_0 \in H^s, \quad \phi_{\pm}(0) = \frac{1}{2} (\phi_0 \pm i \langle D \rangle_m^{-1} \phi_1) \in H^{s+\frac{1}{2}}. \quad (2.2)$$

Theorem 1.1 then reduces to the following.

Theorem 2.1. *If $s \geq 0$, uniqueness of the solution to (2.1), (2.2) holds in*

$$(\psi_{\pm}, \phi_{\pm}) \in C\left([0, T]; H^s \times H^{s+1/2}\right).$$

The null structure in the second equation in (2.1) is quantified by the following estimate from [3]: For Schwartz functions $\psi, \psi' : \mathbb{R}^{1+2} \rightarrow \mathbb{C}^2$ and independent signs $\pm_1$ and $\pm_2$,

$$\begin{aligned} & |\mathcal{F} \langle \beta P_{\pm_1}(D)\psi, P_{\pm_2}(D)\psi' \rangle(\tau, \xi)| \\ & \lesssim \int_{\mathbb{R}^{1+2}} \theta(\pm_1\eta, \pm_2(\eta - \xi)) |\tilde{\psi}(\lambda, \eta)| |\tilde{\psi}'(\lambda - \tau, \eta - \xi)| d\lambda d\eta, \end{aligned} \quad (2.3)$$

where $\theta(\xi, \eta)$ denotes the angle between nonzero $\xi, \eta \in \mathbb{R}^2$. The null structure in the first equation in (2.1) is seen to be of the same type via a duality argument (see the next section). To exploit this structure we need the following estimate:

Lemma 2.2. *Let $a, b, c \in [0, \frac{1}{2}]$. For all signs $(\pm_1, \pm_2)$, all $\lambda, \mu \in \mathbb{R}$ and all nonzero $\eta, \zeta \in \mathbb{R}^2$,*

$$\theta(\pm_1\eta, \pm_2\zeta) \lesssim \left(\frac{|\langle \lambda - \mu | - |\eta - \zeta| \rangle|}{\min(\langle \eta \rangle, \langle \zeta \rangle)} \right)^a + \left(\frac{\langle \lambda \pm_1 | \eta \rangle}{\min(\langle \eta \rangle, \langle \zeta \rangle)} \right)^b + \left(\frac{\langle \mu \pm_2 | \zeta \rangle}{\min(\langle \eta \rangle, \langle \zeta \rangle)} \right)^c. \quad (2.4)$$

Proof. In [2, Section 5.1] it is shown that this holds when $a = b = c = 1/2$, and since $\theta(\pm_1\eta, \pm_2\zeta) \lesssim 1$ the lemma follows. $\square$

Next, we define some function spaces.

We remark that the symbols $\tau \pm |\xi|$ and $\tau \pm \langle \xi \rangle_m$ associated with the operators $-i\partial_t \pm |D|$ and $-i\partial_t \pm \langle D \rangle_m$ appearing in (2.1) are comparable, in the sense that $\langle \tau \pm |\xi| \rangle \approx \langle \tau \pm \langle \xi \rangle_m \rangle$, where $\langle \cdot \rangle_m = \sqrt{m^2 + |\cdot|^2}$. Hence, the $X^{s,b}$ spaces corresponding to the two operators are in fact equivalent. Specifically,

the space $X_{\pm}^{s,b}$ in question is defined (given $s, b \in \mathbb{R}$) as the completion of $\mathcal{S}(\mathbb{R}^{1+2})$ with respect to the norm

$$\|u\|_{X_{\pm}^{s,b}} = \|\langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b \widetilde{u}(\tau, \xi)\|_{L_{\tau, \xi}^2}.$$

We also need the wave-Sobolev space $H^{s,b}$ with norm

$$\|u\|_{H^{s,b}} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \widetilde{u}(\tau, \xi)\|_{L_{\tau, \xi}^2}.$$

For $T > 0$, let $X_{\pm}^{s,b}(S_T)$ and $H^{s,b}(S_T)$ be the respective restriction spaces to the slab $S_T = (0, T) \times \mathbb{R}^2$.

Clearly, for $b \geq 0$,

$$\|u\|_{H^{s,b}} \leq \|u\|_{X_{\pm}^{s,b}}. \quad (2.5)$$

We also recall that for $b > 1/2$, $H^{s,b}$ is a proper subspace of $C(\mathbb{R}; H^s)$, and the inclusion is continuous. Thus,

$$X_{\pm}^{s,b} \subset H^{s,b} \subset C(\mathbb{R}, H^s) \quad \text{for } b > 1/2.$$

Moreover, it is well known that the initial value problem

$$(-i\partial_t \pm |D|)u = F \in X_{\pm}^{s,b-1}, \quad u|_{t=0} = f \in H^s,$$

for any $s \in \mathbb{R}$ and $b > \frac{1}{2}$, has a unique solution $u \in C([0, T]; H^s)$, and u satisfies

$$\|u\|_{X_{\pm}^{s,b}(S_T)} \leq C \left(\|f\|_{H^s} + \|F\|_{X_{\pm}^{s,b-1}(S_T)} \right), \quad (2.6)$$

where C depends on b and T . See, e.g., [3].

The final key ingredient that we need for the proof of uniqueness is the following product estimate for the spaces $H^{s,b}$.

Theorem 2.3. *Assume*

$$b_0 \leq 0 < b_1, b_2 \quad (2.7)$$

$$b_0 + b_1 + b_2 > \frac{1}{2} \quad (2.8)$$

$$b_0 + b_1 > 0, \quad b_0 + b_2 > 0 \quad (2.9)$$

$$s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2) \quad (2.10)$$

$$s_0 + s_1 + s_2 > 1 - (b_0 + b_1) \quad (2.11)$$

$$s_0 + s_1 + s_2 > 1 - (b_0 + b_2) \quad (2.12)$$

$$s_0 + s_1 + s_2 > \frac{1}{2} - b_0 \quad (2.13)$$

$$s_0 + s_1 + s_2 > \frac{3}{4} \quad (2.14)$$

$$s_0 + b_0 + 2(s_1 + s_2) > 1 \quad (2.15)$$

$$s_1 + s_2 \geq -b_0, \quad s_0 + s_2 \geq 0, \quad s_0 + s_1 \geq 0. \quad (2.16)$$

Then the product estimate

$$\|uv\|_{H^{-s_0, -b_0}} \lesssim \|u\|_{H^{s_1, b_1}} \|v\|_{H^{s_2, b_2}} \quad (2.17)$$

holds for all $u, v \in \mathcal{S}(\mathbb{R}^{1+2})$.

This follows from Theorems 4.1 and 6.1 in [4].

3. Proof of uniqueness

We now prove Theorem 2.1. Without loss of generality take $s = 0$. From [3] it is known that the solution is unique in the iteration space

$$(\psi_{\pm}, \phi_{\pm}) \in X_{\pm}^{-\frac{5}{32}-, \frac{1}{2}+}(S_T) \times X_{\pm}^{\frac{11}{32}-, \frac{1}{2}+}(S_T),$$

where we use the notation $a_{\pm} := a \pm \delta$ for sufficiently small $\delta > 0$. Thus, it suffices to show that if

$$(\psi_{\pm}, \phi_{\pm}) \in C\left([0, T], L^2 \times H^{\frac{1}{2}}\right), \quad (3.1)$$

is a solution of (2.1), (2.2), then

$$(\psi_{\pm}, \phi_{\pm}) \in X_{\pm}^{-\frac{5}{32}-, \frac{1}{2}+}(S_T) \times X_{\pm}^{\frac{11}{32}-, \frac{1}{2}+}(S_T). \quad (3.2)$$

To this end, we split $\psi_{\pm}$ as

$$\psi_{\pm} = \psi_{\pm}^h + \psi_{\pm}^l + \Psi_{\pm},$$

where $\psi_{\pm}^h$ is the homogeneous part while $\psi_{\pm}^l$ and $\Psi_{\pm}$ are the inhomogeneous parts corresponding to the linear and bilinear terms, respectively, in the right-hand side of the first equation in (2.1). Similarly, we split $\phi_{\pm}$ as

$$\phi_{\pm} = \phi_{\pm}^h + \Phi_{\pm},$$

where $\phi_{\pm}^h$ and $\Phi_{\pm}$ are the homogeneous and inhomogeneous parts of $\phi_{\pm}$.

First note that by (2.6) and assumption (3.1),

$$\psi_{\pm}^h \in X_{\pm}^{0, \frac{1}{2}+}(S_T) \quad \text{and} \quad \phi_{\pm}^h \in X_{\pm}^{\frac{1}{2}, \frac{1}{2}+}(S_T).$$

Moreover, (3.1) implies

$$\psi_{\pm} \in X_{\pm}^{0,0}(S_T),$$

so (2.6) gives

$$\psi_{\pm}^l \in X_{\pm}^{0,1}(S_T).$$

Thus it only remains to show that the pair $(\Psi_{\pm}, \Phi_{\pm})$ satisfies (3.2). To this end we start with (3.1) and use the null structure and product estimates to successively improve the regularity.

Note that by the above, $\Psi_{\pm} = \psi_{\pm} - \psi_{\pm}^h - \psi_{\pm}^l \in X_{\pm}^{0,0}(S_T)$, and a similar argument shows that $\Phi_{\pm} = \phi_{\pm} - \phi_{\pm}^h \in X_{\pm}^{\frac{1}{2},0}$.

3.1. First estimate for $\Psi_{\pm}$

We first claim

$$\Psi_{\pm} \in X_{\pm}^{-\frac{1}{2}\alpha, \alpha}(S_T) \quad \text{for } \alpha \in [0, 1]. \quad (3.3)$$

Indeed, using (2.6),

$$\begin{aligned} \|\Psi_{\pm}\|_{X_{\pm}^{-\frac{1}{2}, 1}(S_T)} &\lesssim \|\phi\beta\psi\|_{X_{\pm}^{-\frac{1}{2}, 0}(S_T)} = \|\phi\beta\psi\|_{L_t^2 H_x^{-\frac{1}{2}}(S_T)} \\ &\lesssim T^{\frac{1}{2}} \|\phi\|_{L_t^{\infty} H_x^{\frac{1}{2}}(S_T)} \|\psi\|_{L_t^{\infty} L_x^2(S_T)}, \end{aligned}$$

where in the last line we used the product Sobolev inequality in the x -variable (see, e.g., the introduction in [4]). Interpolating $\Psi_{\pm} \in X_{\pm}^{-\frac{1}{2}, 1}(S_T)$ with $\Psi_{\pm} \in X_{\pm}^{0, 0}(S_T)$, we get (3.3).

3.2. First estimate for $\Phi_{\pm}$

We show that

$$\Phi_{\pm} \in X_{\pm}^{\frac{1}{4}-\varepsilon, \frac{1}{2}+\varepsilon}(S_T). \quad (3.4)$$

Using (3.3) with $\alpha = \frac{1}{4} + 2\varepsilon$ and (2.6), the claim reduces to the bilinear estimate

$$\|\langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle\|_{X_{\pm 0}^{-\frac{3}{4}-\varepsilon, -\frac{1}{2}+\varepsilon}} \lesssim \|\psi\|_{X_{\pm 1}^{-\frac{1}{8}-\varepsilon, \frac{1}{4}+2\varepsilon}} \|\psi'\|_{X_{\pm 2}^{-\frac{1}{8}-\varepsilon, \frac{1}{4}+2\varepsilon}}.$$

Without loss of generality, take $\pm_1 = +, \pm_2 = \pm$. By Plancherel and using (2.3) we reduce to

$$I \lesssim \|u\|_{X_+^{-\frac{1}{8}-\varepsilon, \frac{1}{4}+2\varepsilon}} \|v\|_{X_{\pm}^{-\frac{1}{8}-\varepsilon, \frac{1}{4}+2\varepsilon}},$$

where

$$I = \left\| \iint \frac{\theta(\eta, \pm(\eta - \xi))}{\langle \xi \rangle^{\frac{3}{4}+\varepsilon} \langle |\tau| - |\xi| \rangle^{\frac{1}{2}-\varepsilon}} \widehat{u}(\lambda, \eta) \widehat{v}(\lambda - \tau, \eta - \xi) \, d\lambda \, d\eta \right\|_{L_{\tau, \xi}^2}.$$

Applying Lemma 2.2 with $a = \frac{1}{2} - \varepsilon$ and $b = c = \frac{1}{4} + 2\varepsilon$, and using also (2.5), the estimate for I reduces to the following three estimates:

$$\begin{cases} \|uv\|_{H^{-\frac{3}{4}-\varepsilon, 0}} \lesssim \|u\|_{H^{\frac{3}{8}-2\varepsilon, \frac{1}{4}+2\varepsilon}} \|v\|_{H^{-\frac{1}{8}-\varepsilon, \frac{1}{4}+2\varepsilon}} \\ \|uv\|_{H^{-\frac{3}{4}-\varepsilon, -\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{H^{\frac{1}{8}+\varepsilon, 0}} \|v\|_{H^{-\frac{1}{8}-\varepsilon, \frac{1}{4}+2\varepsilon}} \\ \|uv\|_{H^{-\frac{3}{4}-\varepsilon, -\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{H^{-\frac{1}{8}-\varepsilon, 0}} \|v\|_{H^{\frac{1}{8}+\varepsilon, \frac{1}{4}+2\varepsilon}}. \end{cases} \quad (3.5)$$

All these hold by Theorem 2.3 (via duality for the last two), proving (3.4). Interpolating with $\Phi_{\pm} \in X_{\pm}^{\frac{1}{2}, 0}$ we get moreover

$$\Phi_{\pm} \in X_{\pm}^{\frac{7}{16}-\frac{\varepsilon}{4}, \frac{1}{8}+\frac{\varepsilon}{4}}(S_T), \quad (3.6)$$

which we now use to improve (3.3).

3.3. Second estimate for $\Psi_{\pm}$

We show that

$$\Psi_{\pm} \in X_{\pm}^{-\frac{7}{32}, \frac{1}{2} + \varepsilon}(S_T). \quad (3.7)$$

Applying (2.6) and using (3.3) (with $\alpha = \frac{1}{2} - 2\varepsilon$) and (3.6), the claim reduces to the bilinear estimate

$$\|P_{\pm 2}(D)(\phi \beta P_{\pm 1}(D)\psi)\|_{X_{\pm 2}^{-\frac{7}{32}, -\frac{1}{2} + \varepsilon}} \lesssim \|\phi\|_{X_{\pm 0}^{\frac{7}{16} - \frac{\varepsilon}{4}, \frac{1}{8} + \frac{\varepsilon}{4}}} \|\psi\|_{X_{\pm 1}^{-\frac{1}{4} + \varepsilon, \frac{1}{2} - 2\varepsilon}}.$$

Duality reduces this to

$$\|\langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle\|_{X_{\pm 0}^{-\frac{7}{16} + \frac{\varepsilon}{4}, -\frac{1}{8} - \frac{\varepsilon}{4}}} \lesssim \|\psi'\|_{X_{\pm 1}^{\frac{7}{32}, \frac{1}{2} - \varepsilon}} \|\psi\|_{X_{\pm 2}^{-\frac{1}{4} + \varepsilon, \frac{1}{2} - 2\varepsilon}}.$$

Taking again $\pm_1 = +, \pm_2 = \pm$ without loss of generality, applying Plancherel and using (2.3), we reduce further to

$$J \lesssim \|u\|_{X_{+}^{\frac{7}{32}, \frac{1}{2} - \varepsilon}} \|v\|_{X_{+}^{-\frac{1}{4} + \varepsilon, \frac{1}{2} - 2\varepsilon}},$$

where

$$J = \left\| \iint \frac{\theta(\eta, \pm(\eta - \xi))}{\langle \xi \rangle^{\frac{7}{16} - \frac{\varepsilon}{4}} \langle |\tau| - |\xi| \rangle^{\frac{1}{8} + \frac{\varepsilon}{4}}} \widehat{u}(\lambda, \eta) \widehat{v}(\lambda - \tau, \eta - \xi) d\lambda d\eta \right\|_{L_{\tau, \xi}^2},$$

Applying Lemma 2.2 with $a = \frac{1}{2}, b = \frac{1}{2} - \varepsilon$ and $c = \frac{1}{2} - 2\varepsilon$, and using (3.5), we reduce to the following six estimates:

$$\left\{ \begin{array}{l} \|uv\|_{H^{-\frac{7}{16} + \frac{\varepsilon}{4}, \frac{3}{8} - \frac{\varepsilon}{4}}} \lesssim \|u\|_{H^{\frac{23}{32}, \frac{1}{2} - \varepsilon}} \|v\|_{H^{-\frac{1}{4} + \varepsilon, \frac{1}{2} - 2\varepsilon}} \\ \|uv\|_{H^{-\frac{7}{16} + \frac{\varepsilon}{4}, \frac{3}{8} - \frac{\varepsilon}{4}}} \lesssim \|u\|_{H^{\frac{7}{32}, \frac{1}{2} - \varepsilon}} \|v\|_{H^{\frac{1}{4} + \varepsilon, \frac{1}{2} - 2\varepsilon}} \\ \|uv\|_{H^{-\frac{7}{16} + \frac{\varepsilon}{4}, -\frac{1}{8} - \frac{\varepsilon}{4}}} \lesssim \|u\|_{H^{\frac{23}{32} - \varepsilon, 0}} \|v\|_{H^{-\frac{1}{4} + \varepsilon, \frac{1}{2} - 2\varepsilon}} \\ \|uv\|_{H^{-\frac{7}{16} + \frac{\varepsilon}{4}, -\frac{1}{8} - \frac{\varepsilon}{4}}} \lesssim \|u\|_{H^{\frac{7}{32}, 0}} \|v\|_{H^{\frac{1}{4}, \frac{1}{2} - 2\varepsilon}} \\ \|uv\|_{H^{-\frac{7}{16} + \frac{\varepsilon}{4}, -\frac{1}{8} - \frac{\varepsilon}{4}}} \lesssim \|u\|_{H^{\frac{23}{32} - 2\varepsilon, \frac{1}{2} - \varepsilon}} \|v\|_{H^{-\frac{1}{4} + \varepsilon, 0}} \\ \|uv\|_{H^{-\frac{7}{16} + \frac{\varepsilon}{4}, -\frac{1}{8} - \frac{\varepsilon}{4}}} \lesssim \|u\|_{H^{\frac{7}{32}, \frac{1}{2} - \varepsilon}} \|v\|_{H^{\frac{1}{4} - \varepsilon, 0}} \end{array} \right. \quad (3.8)$$

All these hold by Theorem 2.3 (via duality for the last four), proving (3.7). Interpolation with $\Psi_{\pm} \in X_{\pm}^{0,0}$ yields

$$\Psi_{\pm} \in X_{\pm}^{-\frac{7}{64}, \frac{1}{4} + \frac{\varepsilon}{2}}(S_T). \quad (3.9)$$

We now use this to improve (3.4).

3.4. Second estimate for $\Phi_{\pm}$

We show that

$$\Phi_{\pm} \in X_{\pm}^{\frac{9}{32} - \varepsilon, \frac{1}{2} + \varepsilon}(S_T). \quad (3.10)$$

Applying (2.6) and using (3.9) we reduce to

$$\|\langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle\|_{X_{\pm 0}^{-\frac{23}{32} - \varepsilon, -\frac{1}{2} + \varepsilon}} \lesssim \|\psi\|_{X_{\pm 1}^{-\frac{7}{64}, \frac{1}{4} + \frac{\varepsilon}{2}}} \|\psi'\|_{X_{\pm 2}^{-\frac{7}{64}, \frac{1}{4} + \frac{\varepsilon}{2}}}.$$

Proceeding as in Sect. 3.2, applying Lemma 2.2 with $a = \frac{1}{2} - \varepsilon$ and $b = c = \frac{1}{4} + \frac{\varepsilon}{2}$, and using (2.5), we further reduce to the following three estimates:

$$\begin{cases} \|uv\|_{H^{-\frac{23}{32}-\varepsilon,0}} \lesssim \|u\|_{H^{\frac{25}{64}-\varepsilon,\frac{1}{4}+\frac{\varepsilon}{2}}} \|v\|_{H^{-\frac{7}{64},\frac{1}{4}+\frac{\varepsilon}{2}}} \\ \|uv\|_{H^{-\frac{23}{32}-\varepsilon,-\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{H^{\frac{9}{64}+\frac{\varepsilon}{2},0}} \|v\|_{H^{-\frac{7}{64},\frac{1}{4}+\frac{\varepsilon}{2}}} \\ \|uv\|_{H^{-\frac{23}{32}-\varepsilon,-\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{H^{-\frac{7}{64},0}} \|v\|_{H^{\frac{9}{64}+\frac{\varepsilon}{2},\frac{1}{4}+\frac{\varepsilon}{2}}} \end{cases} \quad (3.11)$$

All these are seen to hold by Theorem 2.3 (via duality for the last two).

Interpolating (3.10) with $\Phi_{\pm} \in X_{\pm}^{\frac{1}{2},0}$, we further get

$$\Phi_{\pm} \in X_{\pm}^{\frac{25}{64}-\frac{\varepsilon}{2},\frac{1}{4}+\frac{\varepsilon}{2}}(S_T). \quad (3.12)$$

3.5. Third estimate for $\Psi_{\pm}$

We prove

$$\Psi_{\pm} \in X_{\pm}^{-\frac{5}{32}-3\varepsilon,\frac{1}{2}+\varepsilon}(S_T). \quad (3.13)$$

Applying (2.6) and using (3.7) and (3.12), the claim reduces (after duality) to

$$\|\langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle\|_{X_{\pm 0}^{-\frac{25}{64}+\frac{\varepsilon}{2},-\frac{1}{4}-\frac{\varepsilon}{2}}} \lesssim \|\psi'\|_{X_{\pm 1}^{\frac{5}{32}+3\varepsilon,\frac{1}{2}-\varepsilon}} \|\psi\|_{X_{\pm 2}^{-\frac{7}{32},\frac{1}{2}+\varepsilon}}.$$

Proceeding as in Sect. 3.3, applying Lemma 2.2 with $a = \frac{1}{2}, b = \frac{1}{2} - \varepsilon$ and $b = \frac{1}{2}$, and using (2.5), we reduce to

$$\begin{cases} \|uv\|_{H^{-\frac{25}{64}+\frac{\varepsilon}{2},\frac{1}{4}-\frac{\varepsilon}{2}}} \lesssim \|u\|_{H^{\frac{21}{32}+3\varepsilon,\frac{1}{2}-\varepsilon}} \|v\|_{H^{-\frac{7}{32},\frac{1}{2}+\varepsilon}} \\ \|uv\|_{H^{-\frac{25}{64}+\frac{\varepsilon}{2},\frac{1}{4}-\frac{\varepsilon}{2}}} \lesssim \|u\|_{H^{\frac{5}{32}+3\varepsilon,\frac{1}{2}-\varepsilon}} \|v\|_{H^{\frac{9}{32},\frac{1}{2}+\varepsilon}} \\ \|uv\|_{H^{-\frac{25}{64}+\frac{\varepsilon}{2},-\frac{1}{4}-\frac{\varepsilon}{2}}} \lesssim \|u\|_{H^{\frac{21}{32}+2\varepsilon,0}} \|v\|_{H^{-\frac{7}{32},\frac{1}{2}+\varepsilon}} \\ \|uv\|_{H^{-\frac{25}{64}+\frac{\varepsilon}{2},-\frac{1}{4}-\frac{\varepsilon}{2}}} \lesssim \|u\|_{H^{\frac{5}{32}+3\varepsilon,0}} \|v\|_{H^{\frac{9}{32}-\varepsilon,\frac{1}{2}+\varepsilon}} \\ \|uv\|_{H^{-\frac{25}{64}+\frac{\varepsilon}{2},-\frac{1}{4}-\frac{\varepsilon}{2}}} \lesssim \|u\|_{H^{\frac{21}{32}+3\varepsilon,\frac{1}{2}-\varepsilon}} \|v\|_{H^{-\frac{7}{32},0}} \\ \|uv\|_{H^{-\frac{25}{64}+\frac{\varepsilon}{2},-\frac{1}{4}-\frac{\varepsilon}{2}}} \lesssim \|u\|_{H^{\frac{5}{32}+3\varepsilon,\frac{1}{2}-\varepsilon}} \|v\|_{H^{\frac{9}{32},0}} \end{cases} \quad (3.14)$$

all of which hold by Theorem 2.3 (via duality for the last four).

3.6. Third estimate for $\Phi_{\pm}$

We finally prove that

$$\Phi_{\pm} \in X_{\pm}^{\frac{11}{32}-3\varepsilon,\frac{1}{2}+\varepsilon}(S_T). \quad (3.15)$$

By (2.6) and (3.13) we reduce to

$$\|\langle \beta P_{\pm 1}(D)\psi, P_{\pm 2}(D)\psi' \rangle\|_{X_{\pm 0}^{-\frac{21}{32}-3\varepsilon,-\frac{1}{2}+\varepsilon}} \lesssim \|\psi\|_{X_{\pm 1}^{-\frac{5}{32}-3\varepsilon,\frac{1}{2}+\varepsilon}} \|\psi'\|_{X_{\pm 2}^{-\frac{5}{32}-3\varepsilon,\frac{1}{2}+\varepsilon}}.$$

Proceeding as in Sect. 3.2, applying Lemma 2.2 with $a = \frac{1}{2} - \varepsilon$ and $b = c = \frac{1}{2}$, the estimate reduces to

$$\begin{cases} \|uv\|_{H^{-\frac{21}{32}-3\varepsilon,0}} \lesssim \|u\|_{H^{\frac{11}{32}-4\varepsilon,\frac{1}{2}+\varepsilon}} \|v\|_{H^{-\frac{5}{32}-3\varepsilon,\frac{1}{2}+\varepsilon}} \\ \|uv\|_{H^{-\frac{21}{32}-3\varepsilon,-\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{H^{\frac{11}{32}-3\varepsilon,0}} \|v\|_{H^{-\frac{5}{32}-3\varepsilon,\frac{1}{2}+\varepsilon}} \\ \|uv\|_{H^{-\frac{21}{32}-3\varepsilon,-\frac{1}{2}+\varepsilon}} \lesssim \|u\|_{H^{-\frac{5}{32}-3\varepsilon,0}} \|v\|_{H^{\frac{11}{32}-3\varepsilon,\frac{1}{2}+\varepsilon}} \end{cases} \quad (3.16)$$

All these are true by Theorem 2.3.

This finishes the proof of Theorem 2.1.

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Received: 19 October 2011.

Accepted: 5 August 2012.