Existence and multiplicity for positive solutions of a system of higher-order multi-point boundary value problems

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Abstract. We investigate the existence and multiplicity of positive solutions of multi-point boundary value problems for systems of nonlinear higher-order ordinary differential equations.

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1. Introduction

We consider the system of nonlinear higher-order ordinary differential equations

$$\begin{cases} u^{(n)}(t) + f(t, v(t)) = 0, & t \in (0, T), \quad n \in \mathbb{N}, \quad n \geq 2, \\ v^{(m)}(t) + g(t, u(t)) = 0, & t \in (0, T), \quad m \in \mathbb{N}, \quad m \geq 2, \end{cases} \quad (S)$$

with the multi-point boundary conditions

$$\begin{cases} u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i), \quad p \in \mathbb{N}, \quad p \geq 3, \\ v(0) = v'(0) = \dots = v^{(m-2)}(0) = 0, & v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i), \quad q \in \mathbb{N}, \quad q \geq 3. \end{cases} \quad (BC)$$

Under sufficient assumptions on f and g , we prove the existence and multiplicity of positive solutions of the above problem, by applying the fixed point index theory. By a positive solution of $(S) - (BC)$, we understand a pair of functions $(u, v) \in C^n([0, T]) \times C^m([0, T])$ satisfying (S) and (BC) with $u(t) \geq 0, v(t) \geq 0$ for all $t \in [0, T]$, and $\sup_{t \in [0, T]} u(t) > 0, \sup_{t \in [0, T]} v(t) > 0$. This problem is a generalization of the problem studied in [5], where $n = m = 2$. In [16], the authors investigated the existence and multiplicity of positive solutions for system (S) with $n = m = 2$ and $T = 1$ with the boundary conditions

$u(0) = 0, u(1) = \alpha u(\eta), v(0) = 0, v(1) = \alpha v(\eta), \eta \in (0, 1), 0 < \alpha\eta < 1$. We also mention the paper [14] where the authors used the fixed point index theory to prove the existence of positive solutions for the system (S) with $f(t, v(t))$ and $g(t, u(t))$ replaced by $c(t)f(u(t), v(t))$ and $d(t)\tilde{g}(u(t), v(t))$, respectively, and (BC) , where $\frac{1}{2} \leq \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1, \frac{1}{2} \leq \eta_1 < \eta_2 < \cdots < \eta_{q-2} < 1, (T = 1)$. Some multi-point boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were studied in recent years by Henderson, Luca, Ntouyas and Purnaras, by using the Guo–Krasnosel'skii fixed point theorem. In [3], the authors give sufficient conditions for λ, μ, f and g such that the system,

$$\begin{cases} u^{(n)}(t) + \lambda c(t)f(u(t), v(t)) = 0, & t \in (0, T), \quad n \in \mathbb{N}, \quad n \geq 2, \\ v^{(m)}(t) + \mu d(t)g(u(t), v(t)) = 0, & t \in (0, T), \quad m \in \mathbb{N}, \quad m \geq 2, \end{cases} \quad (S_1)$$

with the boundary conditions (BC) , has positive solutions $(u(t) \geq 0, v(t) \geq 0$ for all $t \in [0, T]$ and $\|u\| + \|v\| > 0$). The system (S_1) with $n = m = 2$ and the multi-point boundary conditions

$$\begin{cases} \alpha u(0) - \beta u'(0) = 0, & u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i), \quad p \in \mathbb{N}, \quad p \geq 3, \\ \gamma v(0) - \delta v'(0) = 0, & v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i), \quad q \in \mathbb{N}, \quad q \geq 3, \end{cases} \quad (BC_1)$$

has been investigated in [4]. Some particular cases of the above problems have been studied in [6–9, 12, 13, 15].

In recent years, the multi-point boundary value problems for second-order or higher-order differential or difference equations/systems have been investigated by many authors, by using different methods such as fixed point theorems in cones, the Leray–Schauder continuation theorem and its nonlinear alternatives and the coincidence degree theory.

In Sect. 2, we shall present some auxiliary results which investigate two boundary value problems for higher-order equations (the problems (1)–(2) and (10)–(11) below). In Sect. 3, inspired by the paper [16], we shall prove some existence and multiplicity results for positive solutions with respect to a cone for our problem $(S) - (BC)$, which are based on three fixed point index theorems. Finally, in Sect. 4, we shall present some examples which illustrate our main results.

2. Auxiliary results

2.1. Fixed point index theorems

First, we shall recall some theorems concerning the index fixed point theory. Let E be a real Banach space, $P \subset E$ a cone, “ $\leq$ ” the partial ordering defined by P and θ the zero element in E . For $\varrho > 0$, let $B_\varrho = \{u \in E, \|u\| < \varrho\}$. The proofs of our results are based on the following fixed point index theorems.

Theorem 2.1. [1] Let $A : \bar{B}_\varrho \cap P \rightarrow P$ be a completely continuous operator which has no fixed point on $\partial B_\varrho \cap P$. If $\|Au\| \leq \|u\|$ for all $u \in \partial B_\varrho \cap P$, then $i(A, B_\varrho \cap P, P) = 1$.

Theorem 2.2. [1] Let $A : \bar{B}_\varrho \cap P \rightarrow P$ be a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that $u - Au \neq \lambda u_0$, for all $\lambda \geq 0$ and $u \in \partial B_\varrho \cap P$, then $i(A, B_\varrho \cap P, P) = 0$.

Theorem 2.3. [16] Let $A : \bar{B}_\varrho \cap P \rightarrow P$ be a completely continuous operator which has no fixed point on $\partial B_\varrho \cap P$. If there exists a linear operator $L : P \rightarrow P$ and $u_0 \in P \setminus \{\theta\}$ such that

$$(i) \quad u_0 \leq Lu_0, \quad (ii) \quad Lu \leq Au, \quad \forall u \in \partial B_\varrho \cap P,$$

then $i(A, B_\varrho \cap P, P) = 0$.

2.2. Higher-order differential equations with multi-point boundary conditions

In this subsection, we shall present some auxiliary results (for the proofs see, for example, [11, 12]) related to the n th-order differential equation with p -point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \quad t \in (0, T), \quad (1)$$

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i). \quad (2)$$

Lemma 2.1. If $d = T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \neq 0$, $0 < \xi_1 < \dots < \xi_{p-2} < T$ and $y \in C([0, T])$, then the solution of (1)–(2) is given by

$$u(t) = \frac{t^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} y(s) ds - \frac{t^{n-1}}{d(n-1)!} \sum_{i=1}^{p-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s) ds \\ - \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s) ds, \quad 0 \leq t \leq T.$$

Lemma 2.2. Under the assumptions of Lemma 2.1, the Green's function for the boundary value problem (1)–(2) is given by

$$G_1(t, s) = \begin{cases} \frac{t^{n-1}}{d(n-1)!} \left[(T-s)^{n-1} - \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1} \right] - \frac{1}{(n-1)!} (t-s)^{n-1}, \\ \quad \text{if } \xi_j \leq s < \xi_{j+1}, \quad s \leq t, \\ \frac{t^{n-1}}{d(n-1)!} \left[(T-s)^{n-1} - \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1} \right], \\ \quad \text{if } \xi_j \leq s < \xi_{j+1}, \quad s \geq t, \quad j = 0, \dots, p-3, \\ \frac{t^{n-1}}{d(n-1)!} (T-s)^{n-1} - \frac{1}{(n-1)!} (t-s)^{n-1}, \text{ if } \xi_{p-2} \leq s \leq T, \quad s \leq t, \\ \frac{t^{n-1}}{d(n-1)!} (T-s)^{n-1}, \quad \text{if } \xi_{p-2} \leq s \leq T, \quad s \geq t, \quad (\xi_0 = 0), \end{cases}$$

for all $(t, s) \in [0, T] \times [0, T]$.

Using the Heaviside function on $\mathbb{R}$, $H(x) = 1$ for $x \geq 0$, and $H(x) = 0$ for $x < 0$, the above Green's function can be written in a compact form

$$G_1(t, s) = \frac{t^{n-1}}{d(n-1)!} \left[(T-s)^{n-1} - \sum_{i=1}^{p-2} a_i (\xi_i - s)^{n-1} H(\xi_i - s) \right] - \frac{1}{(n-1)!} (t-s)^{n-1} H(t-s), \quad (t, s) \in [0, T] \times [0, T].$$

By using the above Green's function the solution of problem (1)–(2) is expressed as $u(t) = \int_0^T G_1(t, s)y(s) ds$.

Lemma 2.3. *Under the assumptions of Lemma 2.1, the Green's function for the boundary value problem (1)–(2) can be expressed as*

$$G_1(t, s) = g_1(t, s) + \frac{t^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s), \quad (3)$$

where

$$g_1(t, s) = \frac{1}{(n-1)!T^{n-1}} \begin{cases} t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}, & 0 \leq s \leq t \leq T, \\ t^{n-1}(T-s)^{n-1}, & 0 \leq t \leq s \leq T. \end{cases} \quad (4)$$

Proof. If $s \leq t$ and $\xi_j \leq s < \xi_{j+1}$ for $j \in \{0, 1, \dots, p-3\}$, we have

$$\begin{aligned} G_1(t, s) &= \frac{1}{d(n-1)!} \left[t^{n-1}(T-s)^{n-1} - t^{n-1} \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1} \right. \\ &\quad \left. - (t-s)^{n-1} T^{n-1} + (t-s)^{n-1} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \right] \\ &= \frac{1}{d(n-1)!} \left[t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1} + (t-s)^{n-1} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \right. \\ &\quad \left. - \frac{t^{n-1}(T-s)^{n-1}}{T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} + \frac{t^{n-1}(T-s)^{n-1}}{T^{n-1}} \sum_{i=1}^j a_i \xi_i^{n-1} \right. \\ &\quad \left. + \frac{t^{n-1}(T-s)^{n-1}}{T^{n-1}} \sum_{i=j+1}^{p-2} a_i \xi_i^{n-1} - t^{n-1} \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1} \right] \\ &= \frac{1}{d(n-1)!T^{n-1}} \left\{ \left(T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \right) [t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}] \right. \\ &\quad \left. + t^{n-1}(T-s)^{n-1} \sum_{i=1}^j a_i \xi_i^{n-1} + t^{n-1} \sum_{i=j+1}^{p-2} a_i [\xi_i^{n-1}(T-s)^{n-1} \right. \\ &\quad \left. - T^{n-1}(\xi_i - s)^{n-1}] \right\} \\ &= \frac{1}{(n-1)!T^{n-1}} [t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}] \end{aligned}$$

$$\begin{aligned}
& + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^j a_i \xi_i^{n-1} (T-s)^{n-1} \\
& + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=j+1}^{p-2} a_i \left[\xi_i^{n-1} (T-s)^{n-1} - T^{n-1} (\xi_i - s)^{n-1} \right] \\
& = g_1(t, s) + \frac{t^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s).
\end{aligned}$$

We use the convention that $\sum_{i=1}^0 \alpha_i = 0$.

If $s \geq t, \xi_j \leq s < \xi_{j+1}$ for $j \in \{0, 1, \dots, p-3\}$, we obtain

$$\begin{aligned}
G_1(t, s) &= \frac{1}{d(n-1)!} \left[t^{n-1} (T-s)^{n-1} - t^{n-1} \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1} \right] \\
&= \frac{1}{d(n-1)!} \left[t^{n-1} (T-s)^{n-1} - \frac{t^{n-1} (T-s)^{n-1}}{T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \right. \\
&\quad + \frac{t^{n-1} (T-s)^{n-1}}{T^{n-1}} \sum_{i=1}^j a_i \xi_i^{n-1} + \frac{t^{n-1} (T-s)^{n-1}}{T^{n-1}} \\
&\quad \times \left. \sum_{i=j+1}^{p-2} a_i \xi_i^{n-1} - t^{n-1} \sum_{i=j+1}^{p-2} a_i (\xi_i - s)^{n-1} \right] = \frac{t^{n-1} (T-s)^{n-1}}{d(n-1)!T^{n-1}} \\
&\quad \times \left(T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \right) + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^j a_i \xi_i^{n-1} (T-s)^{n-1} \\
&\quad + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=j+1}^{p-2} a_i \left[\xi_i^{n-1} (T-s)^{n-1} - T^{n-1} (\xi_i - s)^{n-1} \right] \\
&= \frac{1}{(n-1)!T^{n-1}} t^{n-1} (T-s)^{n-1} + \frac{t^{n-1}}{d(n-1)!} \sum_{i=1}^j a_i \xi_i^{n-1} (T-s)^{n-1} \\
&\quad + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=j+1}^{p-2} a_i \left[\xi_i^{n-1} (T-s)^{n-1} - T^{n-1} (\xi_i - s)^{n-1} \right] \\
&= g_1(t, s) + \frac{t^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s).
\end{aligned}$$

If $s \leq t$ and $\xi_{p-2} \leq s \leq T$, we have

$$\begin{aligned}
G_1(t, s) &= \frac{t^{n-1}}{d(n-1)!} (T-s)^{n-1} - \frac{1}{(n-1)!} (t-s)^{n-1} \\
&= \frac{t^{n-1}}{d(n-1)!} (T-s)^{n-1} - \frac{1}{d(n-1)!T^{n-1}} t^{n-1} (T-s)^{n-1} \sum_{i=1}^{p-2} a_i \xi_i^{n-1}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{(n-1)!}(t-s)^{n-1} + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} (T-s)^{n-1} \\
& = \left(T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \right) \frac{1}{d(n-1)!T^{n-1}} t^{n-1} (T-s)^{n-1} \\
& \quad - \frac{1}{(n-1)!}(t-s)^{n-1} + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} (T-s)^{n-1} \\
& = \frac{1}{(n-1)!T^{n-1}} [t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}] \\
& \quad + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} (T-s)^{n-1} \\
& = g_1(t, s) + \frac{t^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s).
\end{aligned}$$

If $s \geq t$ and $\xi_{p-2} \leq s \leq T$, we obtain

$$\begin{aligned}
G_1(t, s) &= \frac{t^{n-1}(T-s)^{n-1}}{d(n-1)!} = \frac{t^{n-1}(T-s)^{n-1}}{d(n-1)!} - \frac{t^{n-1}(T-s)^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \\
& \quad + \frac{t^{n-1}(T-s)^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \\
& = \frac{t^{n-1}(T-s)^{n-1}}{d(n-1)!T^{n-1}} \left(T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \right) \\
& \quad + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} (T-s)^{n-1} = \frac{1}{(n-1)!T^{n-1}} t^{n-1} (T-s)^{n-1} \\
& \quad + \frac{t^{n-1}}{d(n-1)!T^{n-1}} \sum_{i=1}^{p-2} a_i \xi_i^{n-1} (T-s)^{n-1} \\
& = g_1(t, s) + \frac{t^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s).
\end{aligned}$$

Therefore, in every case from above, we obtain the relation (3). $\square$

If $a_i = 0$ for $i = 1, \dots, p-2$, then $G_1(t, s) = g_1(t, s)$ is the Green's function for the problem

$$w^{(n)}(t) + y_0(t) = 0, \quad t \in (0, T), \quad (5)$$

$$w(0) = w'(0) = \dots = w^{(n-2)}(0) = 0, \quad w(T) = 0. \quad (6)$$

The solution of the above problem is

$$\begin{aligned} w(t) &= \frac{t^{n-1}}{(n-1)!T^{n-1}} \int_0^T (T-s)^{n-1} y_0(s) ds \\ &\quad - \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y_0(s) ds, \quad t \in [0, T] \\ \Leftrightarrow \quad w(t) &= \int_0^T g_1(t, s) y_0(s) ds, \quad t \in [0, T], \end{aligned}$$

where g_1 is given in (4).

Therefore, the function g_1 satisfies the conditions $g_1(0, s) = 0$, $\frac{\partial g_1}{\partial t}(0, s) = 0, \dots, \frac{\partial^{n-2} g_1}{\partial t^{n-2}}(0, s) = 0$ for all $s \in [0, T]$. Moreover, we have $g_1(t, 0) = 0$ and $g_1(t, T) = 0$ for all $t \in [0, T]$. We also observe that g_1 is a continuous function on $[0, T] \times [0, T]$ and $g_1(t, s) \geq 0$ for all $(t, s) \in [0, T] \times [0, T]$. Indeed, if $0 \leq t \leq s \leq T$, it is obvious that $g_1(t, s) \geq 0$. For $0 \leq s \leq t \leq T$, we have

$$g_1(t, s) = \frac{1}{(n-1)!T^{n-1}} [(tT - ts)^{n-1} - (tT - sT)^{n-1}] \geq 0,$$

because $tT - ts \geq tT - sT \Leftrightarrow t \leq T$.

By using the above properties of g_1 and similar arguments as those used in the proof of Lemma 3.3 from [10], we obtain the following result.

Lemma 2.4. *The Green's function g_1 of problem (5)–(6) satisfies the inequalities*

- (a) $g_1(t, s) \leq g_1(\theta_1(s), s)$, for all $(t, s) \in [0, T] \times [0, T]$,
- (b) For any $c \in (0, \frac{T}{2})$,

$$\min_{t \in [c, T-c]} g_1(t, s) \geq \frac{c^{n-1}}{T^{n-1}} g_1(\theta_1(s), s), \quad \text{for all } s \in [0, T]$$

$$\text{where } \theta_1(s) = s \text{ if } n = 2 \text{ and } \theta_1(s) = \begin{cases} \frac{s}{1 - (1 - \frac{s}{T})^{\frac{n-1}{n-2}}}, & s \in (0, T], \\ \frac{T(n-2)}{n-1}, & s = 0, \end{cases} \quad \text{if } n \geq 3.$$

In the case $n \geq 3$, we choose the values of θ_1 in $s = 0$ and $s = T$ such that θ_1 be a continuous function on $[0, T]$.

Lemma 2.5. *Assume that $d > 0, 0 < \xi_1 < \dots < \xi_{p-2} < T, a_i \geq 0$ for all $i = 1, \dots, p-2$. Then the Green's function G_1 of the problem (1)–(2) is continuous on $[0, T] \times [0, T]$ and satisfies $G_1(t, s) \geq 0$ for all $(t, s) \in [0, T] \times [0, T]$. Moreover, if $y \in C([0, T])$ satisfies $y(t) \geq 0$ for all $t \in [0, T]$, then the unique solution u of problem (1)–(2) satisfies $u(t) \geq 0$ for all $t \in [0, T]$.*

The proof of Lemma 2.5 is immediate by using the nonnegativity property of g_1 and the expressions for G_1 and u .

Lemma 2.6. *Assume that $d > 0, 0 < \xi_1 < \dots < \xi_{p-2} < T, a_i \geq 0$ for all $i = 1, \dots, p-2$. Then the Green's function G_1 of problem (1)–(2) satisfies the*

inequalities

$$(a) \quad G_1(t, s) \leq I_1(s), \quad \forall (t, s) \in [0, T] \times [0, T],$$

$$\text{where } I_1(s) = g_1(\theta_1(s), s) + \frac{T^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s), \quad \forall s \in [0, T]; \quad (7)$$

(b) For every $c \in (0, \frac{T}{2})$,

$$\min_{t \in [c, T-c]} G_1(t, s) \geq \frac{c^{n-1}}{T^{n-1}} I_1(s), \quad \forall s \in [0, T]. \quad (8)$$

Proof. From (3) and Lemma 2.4 we obtain the inequality (a), and for any $c \in (0, \frac{T}{2})$, we have

$$\begin{aligned} G_1(t, s) &\geq \frac{c^{n-1}}{T^{n-1}} g_1(\theta_1(s), s) + \frac{c^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s) \\ &= \frac{c^{n-1}}{T^{n-1}} \left[g_1(\theta_1(s), s) + \frac{T^{n-1}}{d} \sum_{i=1}^{p-2} a_i g_1(\xi_i, s) \right] \\ &= \frac{c^{n-1}}{T^{n-1}} I_1(s), \quad \forall s \in [0, T], \quad t \in [c, T-c], \end{aligned}$$

and then $\min_{t \in [c, T-c]} G_1(t, s) \geq \frac{c^{n-1}}{T^{n-1}} I_1(s)$ for all $s \in [0, T]$. $\square$

Lemma 2.7. Assume that $d > 0$, $0 < \xi_1 < \dots < \xi_{p-2} < T$, $a_i \geq 0$ for all $i = 1, \dots, p-2$, $c \in (0, \frac{T}{2})$ and $y \in C([0, T])$ satisfies $y(t) \geq 0$ for all $t \in [0, T]$. Then the solution $u(t)$, $t \in [0, T]$ of problem (1)–(2) satisfies the inequality

$$\min_{t \in [c, T-c]} u(t) \geq \frac{c^{n-1}}{T^{n-1}} \max_{t' \in [0, T]} u(t'). \quad (9)$$

Proof. By using the inequalities (7) and (8), we obtain

$$\begin{aligned} u(t) &= \int_0^T G_1(t, s) y(s) ds \geq \frac{c^{n-1}}{T^{n-1}} \int_0^T I_1(s) y(s) ds \\ &\geq \frac{c^{n-1}}{T^{n-1}} \int_0^T G_1(t', s) y(s) ds = \frac{c^{n-1}}{T^{n-1}} u(t'), \quad \forall t \in [c, T-c], \quad t' \in [0, T]. \end{aligned}$$

Therefore, we conclude

$$\min_{t \in [c, T-c]} u(t) \geq \frac{c^{n-1}}{T^{n-1}} \max_{t' \in [0, T]} u(t'),$$

that is the inequality (9). $\square$

We can also formulate similar results as Lemmas 2.1–2.7 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \quad t \in (0, T), \quad (10)$$

$$v(0) = v'(0) = \dots = v^{(m-2)}(0) = 0, \quad v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i), \quad (11)$$

where $0 < \eta_1 < \dots < \eta_{q-2} < T$, $b_i \geq 0$ for $i = 1, \dots, q-2$ and $h \in C([0, T])$. If $e = T^{m-1} - \sum_{i=1}^{q-2} b_i \eta_i^{m-1} \neq 0$, we denote by G_2 the Green's function associated to problem (10)–(11) and defined in a similar manner as G_1 . We also denote by g_2, θ_2 and I_2 the corresponding functions for (10)–(11) defined in a similar manner as g_1, θ_1 and I_1 , respectively.

3. Main results

In this section, we shall investigate the existence and multiplicity of positive solutions for our problem $(S) - (BC)$, under various assumptions on f and g .

We present the assumptions that we shall use in the sequel

(H1) $0 < \xi_1 < \dots < \xi_{p-2} < T, a_i \geq 0, i = 1, \dots, p-2, d = T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} > 0, 0 < \eta_1 < \dots < \eta_{q-2} < T, b_i \geq 0, i = 1, \dots, q-2, e = T^{m-1} - \sum_{i=1}^{q-2} b_i \eta_i^{m-1} > 0$.

(H2) The functions $f, g \in C([0, T] \times [0, \infty), [0, \infty))$ and $f(t, 0) = 0, g(t, 0) = 0$ for all $t \in [0, T]$.

(H3) There exists a positive constant $p_1 \in (0, 1]$ such that

(i) $f_\infty^i = \liminf_{u \rightarrow \infty} \inf_{t \in [0, T]} \frac{f(t, u)}{u^{p_1}} \in (0, \infty]$; (ii) $g_\infty^i = \liminf_{u \rightarrow \infty} \inf_{t \in [0, T]} \frac{g(t, u)}{u^{1/p_1}} = \infty$.

(H4) There exists a positive constant $q_1 \in (0, \infty)$ such that

(i) $f_0^s = \limsup_{u \rightarrow 0^+} \sup_{t \in [0, T]} \frac{f(t, u)}{u^{q_1}} \in [0, \infty)$; (ii) $g_0^s = \limsup_{u \rightarrow 0^+} \sup_{t \in [0, T]} \frac{g(t, u)}{u^{1/q_1}} = 0$.

(H5) There exists a positive constant $r \in (0, \infty)$ such that

(i) $f_\infty^s = \limsup_{u \rightarrow \infty} \sup_{t \in [0, T]} \frac{f(t, u)}{u^r} \in [0, \infty)$; $g_\infty^s = \limsup_{u \rightarrow \infty} \sup_{t \in [0, T]} \frac{g(t, u)}{u^{1/r}} = 0$.

(H6) The following conditions are satisfied

$f_0^i = \liminf_{u \rightarrow 0^+} \inf_{t \in [0, T]} \frac{f(t, u)}{u} \in (0, \infty]$; $g_0^i = \liminf_{u \rightarrow 0^+} \inf_{t \in [0, T]} \frac{g(t, u)}{u} = \infty$.

(H7) For each $t \in [0, T]$, $f(t, u)$ and $g(t, u)$ are nondecreasing with respect to u , and there exists a constant $N > 0$ such that

$$f \left(t, m_0 \int_0^T g(s, N) ds \right) < \frac{N}{m_0}, \quad \forall t \in [0, T],$$

where $m_0 = \max\{K_1 T, K_2\}$, $K_1 = \max_{s \in [0, T]} I_1(s)$, $K_2 = \max_{s \in [0, T]} I_2(s)$ and I_1, I_2 are defined in Sect. 2.

The pair of functions $(u, v) \in C^n([0, T]) \times C^m([0, T])$ is a solution for our problem $(S) - (BC)$ if and only if $(u, v) \in C([0, T]) \times C([0, T])$ is a solution for the nonlinear integral system

$$\begin{cases} u(t) = \int_0^T G_1(t, s) f(s, v(s)) ds, & t \in [0, T], \\ v(t) = \int_0^T G_2(t, s) g(s, u(s)) ds, & t \in [0, T]. \end{cases} \quad (12)$$

Besides, the system (12) can be written as the nonlinear integral system

$$\begin{cases} u(t) = \int_0^T G_1(t, s) f \left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right) ds, & t \in [0, T], \\ v(t) = \int_0^T G_2(t, s) g(s, u(s)) ds, & t \in [0, T]. \end{cases}$$

We consider the Banach space $X = C([0, T])$ with supremum norm $\|\cdot\|$ and define the cone $P \subset X$ by $P = \{u \in X, u(t) \geq 0, \forall t \in [0, T]\}$.

We also define the operators $A : P \rightarrow X$ by

$$(Au)(t) = \int_0^T G_1(t, s) f \left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right) ds, \quad t \in [0, T],$$

and $B : P \rightarrow X, C : P \rightarrow X$ by

$$(Bu)(t) = \int_0^T G_1(t, s) u(s) ds, \quad (Cu)(t) = \int_0^T G_2(t, s) u(s) ds, \quad t \in [0, T].$$

Under the assumptions (H1) and (H2), using also Lemma 2.5, it is easy to see that A, B and C are completely continuous from P to P . Thus the existence and multiplicity of positive solutions of the system $(S) - (BC)$ are equivalent to the existence and multiplicity of fixed points of the operator A .

Theorem 3.1. *Assume that (H1)–(H4) hold. Then the problem $(S) - (BC)$ has at least one positive solution $(u(t), v(t)), t \in [0, T]$.*

Proof. From assumption (i) of (H3), we conclude that there exist $C_1, C_2 > 0$ such that

$$f(t, u) \geq C_1 u^{p_1} - C_2, \quad \forall (t, u) \in [0, T] \times [0, \infty). \quad (13)$$

Then for $u \in P$, by using (13), the reverse form of Holder's inequality and Lemma 2.6, we have for $p_1 \in (0, 1)$

$$\begin{aligned} (Au)(t) &= \int_0^T G_1(t, s) f \left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right) ds \\ &\geq \int_0^T G_1(t, s) \left[C_1 \left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right)^{p_1} - C_2 \right] ds \\ &\geq \int_0^T G_1(t, s) \left[C_1 \int_0^T (G_2(s, \tau) g(\tau, u(\tau)))^{p_1} d\tau \left(\int_0^T d\tau \right)^{p_1/q_0} \right] ds \\ &\quad - C_2 \int_0^T I_1(s) ds = C_1 T^{p_1/q_0} \int_0^T G_1(t, s) \\ &\quad \times \left(\int_0^T (G_2(s, \tau))^{p_1} (g(\tau, u(\tau)))^{p_1} d\tau \right) ds - C_3, \quad \forall t \in [0, T], \end{aligned}$$

where $q_0 = p_1/(p_1 - 1)$ and $C_3 = C_2 \int_0^T I_1(s) ds$.

Therefore, for $u \in P$ and $p_1 \in (0, 1]$, we have

$$(Au)(t) \geq \tilde{C}_1 \int_0^T G_1(t, s) \left(\int_0^T (G_2(s, \tau))^{p_1} (g(\tau, u(\tau)))^{p_1} d\tau \right) ds - C_3, \quad \forall t \in [0, T], \quad (14)$$

where $\tilde{C}_1 = C_1 T^{p_1/q_0}$ for $p_1 \in (0, 1)$ and $\tilde{C}_1 = C_1$ for $p_1 = 1$.

For $c \in (0, \frac{T}{2})$, we define the cone

$$P_0 = \left\{ u \in P; \inf_{t \in [c, T-c]} u(t) \geq \gamma \|u\| \right\},$$

where $\gamma = \min \{c^{n-1}/T^{n-1}, c^{m-1}/T^{m-1}\}$.

From our assumptions and Lemma 2.7, it can be shown that for any $y \in P$ the functions $u(t) = (By)(t)$ and $v(t) = (Cy)(t)$ satisfy the inequalities

$$\inf_{t \in [c, T-c]} u(t) \geq \frac{c^{n-1}}{T^{n-1}} \|u\| \geq \gamma \|u\|, \quad \inf_{t \in [c, T-c]} v(t) \geq \frac{c^{m-1}}{T^{m-1}} \|v\| \geq \gamma \|v\|.$$

So, $u = By \in P_0, v = Cy \in P_0$. Therefore, we deduce that $B(P) \subset P_0, C(P) \subset P_0$.

Now we consider the function $u_0(t), t \in [0, T]$, the solution of problem (1)–(2) with $y = y_0$, where $y_0(t) = 1$ for all $t \in [0, T]$. Then $u_0(t) = \int_0^T G_1(t, s) ds = (By_0)(t), t \in [0, T]$. Obviously, we have $u_0(t) \geq 0$ for all $t \in [0, T]$. We also consider the set

$$M = \{u \in P; \text{there exists } \lambda \geq 0 \text{ such that } u = Au + \lambda u_0\}.$$

We will show that $M \subset P_0$ and M is a bounded subset of X . If $u \in M$, then there exists $\lambda \geq 0$ such that $u(t) = (Au)(t) + \lambda u_0(t), t \in [0, T]$. From definition of u_0 , we have

$$u(t) = (Au)(t) + \lambda (By_0)(t) = B(Fu)(t) + \lambda (By_0)(t) = B(Fu(t) + \lambda y_0(t)) \in P_0,$$

where $F : P \rightarrow P$ is defined by $(Fu)(t) = f\left(t, \int_0^T G_2(t, s)g(s, u(s)) ds\right)$. Therefore, $M \subset P_0$, and from the definition of P_0 , we have

$$\|u\| \leq \frac{1}{\gamma} \inf_{t \in [c, T-c]} u(t), \quad \forall u \in M. \quad (15)$$

From (ii) of assumption (H3), we conclude that for $\varepsilon_0 = \left(\frac{2T^{p_1(m-1)+n-1}}{\tilde{C}_1 m_1 m_2 c^{p_1(m-1)+n-1}}\right)^{1/p_1} > 0$ there exists $C_4 > 0$ such that

$$(g(t, u))^{p_1} \geq \varepsilon_0^{p_1} u - C_4, \quad \forall (t, u) \in [0, T] \times [0, \infty), \quad (16)$$

where $m_1 = \int_c^{T-c} I_1(\tau) d\tau > 0, m_2 = \int_c^{T-c} (I_2(\tau))^{p_1} d\tau > 0$.

For $u \in M$ and $t \in [c, T - c]$, by using Lemma 2.6 and the relations (14), (16), it follows that

$$\begin{aligned}
 u(t) &= (Au)(t) + \lambda u_0(t) \geq (Au)(t) \\
 &\geq \tilde{C}_1 \int_0^T G_1(t, s) \left[\int_0^T (G_2(s, \tau))^{p_1} (g(\tau, u(\tau)))^{p_1} d\tau \right] ds - C_3 \\
 &\geq \tilde{C}_1 \int_c^{T-c} G_1(t, s) \left[\int_c^{T-c} (G_2(s, \tau))^{p_1} (g(\tau, u(\tau)))^{p_1} d\tau \right] ds - C_3 \\
 &\geq \frac{\tilde{C}_1 c^{p_1(m-1)+n-1}}{T^{p_1(m-1)+n-1}} \int_c^{T-c} I_1(s) \left(\int_c^{T-c} (I_2(\tau))^{p_1} (g(\tau, u(\tau)))^{p_1} d\tau \right) ds - C_3 \\
 &\geq \frac{\tilde{C}_1 c^{p_1(m-1)+n-1}}{T^{p_1(m-1)+n-1}} \left(\int_c^{T-c} I_1(s) ds \right) \left(\int_c^{T-c} (I_2(\tau))^{p_1} (\varepsilon_0^{p_1} u(\tau) - C_4) d\tau \right) - C_3 \\
 &= \frac{\tilde{C}_1 \varepsilon_0^{p_1} c^{p_1(m-1)+n-1}}{T^{p_1(m-1)+n-1}} \left(\int_c^{T-c} I_1(s) ds \right) \left(\int_c^{T-c} (I_2(\tau))^{p_1} u(\tau) d\tau \right) - C_5 \\
 &\geq \frac{\tilde{C}_1 \varepsilon_0^{p_1} c^{p_1(m-1)+n-1}}{T^{p_1(m-1)+n-1}} \left(\int_c^{T-c} I_1(s) ds \right) \left(\int_c^{T-c} (I_2(\tau))^{p_1} d\tau \right) \inf_{\tau \in [c, T-c]} u(\tau) - C_5 \\
 &= 2 \inf_{\tau \in [c, T-c]} u(\tau) - C_5,
 \end{aligned}$$

where $C_5 = C_3 + \frac{C_4 \tilde{C}_1 m_1 m_2 c^{p_1(m-1)+n-1}}{T^{p_1(m-1)+n-1}} > 0$.

Hence, $\inf_{t \in [c, T-c]} u(t) \geq 2 \inf_{t \in [c, T-c]} u(t) - C_5$, and so

$$\inf_{t \in [c, T-c]} u(t) \leq C_5, \quad \forall u \in M. \quad (17)$$

Now from relations (15) and (17), it can be shown that $\|u\| \leq \frac{1}{\gamma} \inf_{t \in [c, T-c]} u(t) \leq \frac{C_5}{\gamma}$, for all $u \in M$, that is, M is a bounded subset of X .

Besides, there exists a sufficiently large $L > 0$ such that

$$u \neq Au + \lambda u_0, \quad \forall u \in \partial B_L \cap P, \quad \forall \lambda \geq 0.$$

From Theorem 2.2, we deduce that

$$i(A, B_L \cap P, P) = 0. \quad (18)$$

Next, from (i) of assumption (H4), we conclude that there exists $M_0 > 0$ such that

$$f(t, u) \leq M_0 u^q, \quad \forall (t, u) \in [0, T] \times [0, 1]. \quad (19)$$

From (ii) of assumption (H4) and (H2), it can be shown that for $\varepsilon_1 = \min \left\{ \frac{1}{M_2}, \left(\frac{1}{2M_0 M_1 M_2^{q_1}} \right)^{1/q_1} \right\} > 0$, there exists $\delta_1 \in (0, 1)$ such that $g(t, u) \leq \varepsilon_1 u^{1/q_1}$ for all $(t, u) \in [0, T] \times [0, \delta_1]$, where $M_1 = \int_0^T I_1(s) ds > 0$, $M_2 = \int_0^T I_2(s) ds > 0$. Hence, we obtain

$$\begin{aligned}
 \int_0^T G_2(t, s) g(s, u(s)) ds &\leq \varepsilon_1 \int_0^T I_2(s) (u(s))^{1/q_1} ds \\
 &\leq \varepsilon_1 M_2 \|u\|^{1/q_1} \leq 1, \quad \forall u \in \bar{B}_{\delta_1} \cap P, \quad \forall t \in [0, T]. \quad (20)
 \end{aligned}$$

Therefore, by (19) and (20) we deduce

$$\begin{aligned}(Au)(t) &= \int_0^T G_1(t, s) f\left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\leq M_0 \int_0^T G_1(t, s) \left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau\right)^{q_1} ds \\ &\leq M_0 \varepsilon_1^{q_1} M_2^{q_1} \|u\| \int_0^T I_1(s) ds = M_0 \varepsilon_1^{q_1} M_1 M_2^{q_1} \|u\| \leq \frac{1}{2} \|u\|, \\ &\quad \forall u \in \bar{B}_{\delta_1} \cap P, \quad t \in [0, T].\end{aligned}$$

This implies that $\|Au\| \leq \frac{1}{2} \|u\|$, $\forall u \in \partial B_{\delta_1} \cap P$. From Theorem 2.1, we conclude

$$i(A, B_{\delta_1} \cap P, P) = 1. \quad (21)$$

Combining (18) and (21), we obtain

$$i(A, (B_L \setminus \bar{B}_{\delta_1}) \cap P, P) = i(A, B_L \cap P, P) - i(A, B_{\delta_1} \cap P, P) = -1.$$

We conclude that A has at least one fixed point $u_1 \in (B_L \setminus \bar{B}_{\delta_1}) \cap P$, that is $\delta_1 < \|u_1\| < L$.

Let $v_1(t) = \int_0^T G_2(t, s) g(s, u_1(s)) ds$. Then $(u_1, v_1) \in P \times P$ is a solution of $(S) - (BC)$. In addition $\|v_1\| > 0$. Indeed, if we suppose that $v_1(t) = 0$, for all $t \in [0, T]$, then by using $(H2)$ we have $f(s, v_1(s)) = f(s, 0) = 0$, for all $s \in [0, T]$. This implies $u_1(t) = \int_0^T G_1(t, s) f(s, v_1(s)) ds = 0$, for all $t \in [0, T]$, which contradicts $\|u_1\| > 0$. By using Theorem 1.1 from [11] (see [2]), we obtain $u_1(t) > 0$ and $v_1(t) > 0$ for all $t \in (0, T - c]$. The proof of Theorem 3.1 is completed. $\square$

Theorem 3.2. Assume that $(H1)$, $(H2)$, $(H5)$ and $(H6)$ hold. Then the problem $(S) - (BC)$ has at least one positive solution $(u(t), v(t))$, $t \in [0, T]$.

Proof. From the assumption $(H5)(i)$, we deduce that there exist $C_6, C_7 > 0$ such that

$$f(t, u) \leq C_6 u^r + C_7, \quad \forall (t, u) \in [0, T] \times [0, \infty). \quad (22)$$

From $(H5)(ii)$, we conclude that for $\varepsilon_2 = \left(\frac{1}{2C_6 M_1 M_2^r}\right)^{1/r}$ there exists $C_8 > 0$ such that

$$g(t, u) \leq \varepsilon_2 u^{1/r} + C_8, \quad \forall (t, u) \in [0, T] \times [0, \infty). \quad (23)$$

Hence, for $u \in P$, by using (22) and (23), we obtain

$$\begin{aligned}
 (Au)(t) &= \int_0^T G_1(t, s) f \left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right) ds \\
 &\leq \int_0^T G_1(t, s) \left[C_6 \left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right)^r + C_7 \right] ds \\
 &\leq C_6 \int_0^T G_1(t, s) \left[\int_0^T G_2(s, \tau) \left(\varepsilon_2 (u(\tau))^{1/r} + C_8 \right) d\tau \right]^r ds + M_1 C_7 \\
 &\leq C_6 \int_0^T G_1(t, s) \left[\int_0^T G_2(s, \tau) \left(\varepsilon_2 \|u\|^{1/r} + C_8 \right) d\tau \right]^r ds + M_1 C_7 \\
 &= C_6 \left(\varepsilon_2 \|u\|^{1/r} + C_8 \right)^r \left(\int_0^T I_1(s) ds \right) \left(\int_0^T I_2(\tau) d\tau \right)^r + M_1 C_7, \\
 &\quad \forall t \in [0, T].
 \end{aligned}$$

Therefore, we have

$$(Au)(t) \leq C_6 M_1 M_2^r \left(\varepsilon_2 \|u\|^{1/r} + C_8 \right)^r + M_1 C_7, \quad \forall t \in [0, T]. \quad (24)$$

After some computations, it can be shown that

$$\lim_{\|u\| \rightarrow \infty} \left[C_6 M_1 M_2^r \left(\varepsilon_2 \|u\|^{1/r} + C_8 \right)^r + M_1 C_7 \right] / \|u\| = \frac{1}{2},$$

and so, there exists a sufficiently large $R > 0$ such that

$$C_6 M_1 M_2^r \left(\varepsilon_2 \|u\|^{1/r} + C_8 \right)^r + M_1 C_7 \leq \frac{3}{4} \|u\|, \quad \forall u \in P, \quad \|u\| \geq R. \quad (25)$$

Hence, from (24) and (25), we obtain $\|Au\| \leq \frac{3}{4} \|u\| < \|u\|$, for all $u \in \partial B_R \cap P$, and from Theorem 2.1, we have

$$i(A, B_R \cap P, P) = 1. \quad (26)$$

On the other hand, from (H6)(i) we deduce that there exist positive constants $C_9 > 0$ and $\tilde{u}_1 > 0$ such that $f(t, u) \geq C_9 u, \forall (t, u) \in [0, T] \times [0, \tilde{u}_1]$. From (H6)(ii), for $\varepsilon = C_0/C_9 > 0$ with $C_0 = \frac{T^{n+m-2}}{c^{n+m-2} m_1 m_3} > 0$ and $m_3 = \int_c^{T-c} I_2(\tau) d\tau > 0$, we conclude that there exists $\tilde{\tilde{u}}_1 > 0$ such that $g(t, u) \geq \frac{C_0}{C_9} u$ for all $(t, u) \in [0, T] \times [0, \tilde{\tilde{u}}_1]$. We consider $u_1 = \min\{\tilde{u}_1, \tilde{\tilde{u}}_1\}$ and then we obtain

$$f(t, u) \geq C_9 u, \quad g(t, u) \geq \frac{C_0}{C_9} u, \quad \forall (t, u) \in [0, T] \times [0, u_1]. \quad (27)$$

Because $g(t, 0) = 0$ for all $t \in [0, T]$, and g is continuous, it can be shown that there exists a sufficiently small $\delta_2 \in (0, u_1)$ such that $g(t, u) \leq \frac{u_1}{M_2}$ for all $(t, u) \in [0, T] \times [0, \delta_2]$. Hence,

$$\begin{aligned}
 \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau &\leq \int_0^T I_2(\tau) g(\tau, u(\tau)) d\tau \leq u_1, \quad \forall u \in \bar{B}_{\delta_2} \cap P, \\
 s &\in [0, T].
 \end{aligned} \quad (28)$$

From (27), (28) and Lemma 2.6, we deduce that for any $u \in \bar{B}_{\delta_2} \cap P$, we have

$$\begin{aligned} (Au)(t) &= \int_0^T G_1(t, s) f \left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right) ds \\ &\geq C_9 \int_0^T G_1(t, s) \left(\int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right) ds \\ &\geq C_0 \int_0^T G_1(t, s) \left(\int_0^T G_2(s, \tau) u(\tau) d\tau \right) ds \\ &\geq C_0 \int_c^{T-c} G_1(t, s) \left(\int_0^T G_2(s, \tau) u(\tau) d\tau \right) ds \\ &\geq \frac{C_0 c^{m-1}}{T^{m-1}} \int_c^{T-c} G_1(t, s) \left(\int_0^T I_2(\tau) u(\tau) d\tau \right) ds = (Lu)(t), \quad \forall t \in [0, T], \end{aligned}$$

where the linear operator $L : P \rightarrow P$ is defined by

$$(Lu)(t) = \frac{C_0 c^{m-1}}{T^{m-1}} \left(\int_0^T I_2(\tau) u(\tau) d\tau \right) \left(\int_c^{T-c} G_1(t, s) ds \right), \quad t \in [0, T].$$

Hence, we obtain

$$Au \geq Lu, \quad \forall u \in \partial B_{\delta_2} \cap P. \quad (29)$$

For $w_0(t) = \int_c^{T-c} G_1(t, s) ds, t \in [0, T]$, we have $w_0 \in P \setminus \{\theta_0\}$ and

$$\begin{aligned} (Lw_0)(t) &= \frac{C_0 c^{m-1}}{T^{m-1}} \left[\int_0^T I_2(\tau) \left(\int_c^{T-c} G_1(\tau, s) ds \right) d\tau \right] \left(\int_c^{T-c} G_1(t, s) ds \right) \\ &\geq \frac{C_0 c^{n+m-2}}{T^{n+m-2}} \left(\int_c^{T-c} I_2(\tau) d\tau \right) \left(\int_c^{T-c} I_1(\tau) d\tau \right) \left(\int_c^{T-c} G_1(t, s) ds \right) \\ &= \frac{C_0 c^{n+m-2} m_1 m_3}{T^{n+m-2}} \int_c^{T-c} G_1(t, s) ds = \int_c^{T-c} G_1(t, s) ds = w_0(t), \quad t \in [0, T]. \end{aligned}$$

Therefore,

$$Lw_0 \geq w_0. \quad (30)$$

We may suppose that A has no fixed point on $\partial B_{\delta_2} \cap P$ (otherwise the proof is finished). From (29), (30) and Theorem 2.3 (with $u_0 = w_0$), we conclude that

$$i(A, B_{\delta_2} \cap P, P) = 0. \quad (31)$$

Therefore, from (26) and (31), we have

$$i(A, (B_R \setminus \bar{B}_{\delta_2}) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_{\delta_2} \cap P, P) = 1.$$

Then A has at least one fixed point in $(B_R \setminus \bar{B}_{\delta_2}) \cap P$. Thus, the problem $(S) - (BC)$ has at least one positive solution $(u, v) \in P \times P$ with $u(t) \geq 0, v(t) \geq 0$ for all $t \in [0, T]$ and $\|u\| > 0, \|v\| > 0$. This completes the proof of Theorem 3.2. $\square$

Theorem 3.3. *Assume that (H1)–(H3), (H6) and (H7) hold. Then the problem (S) – (BC) has at least two positive solutions $(u_1(t), v_1(t)), (u_2(t), v_2(t)), t \in [0, T]$.*

Proof. From Sect. 2, we have $0 \leq G_1(t, s) \leq I_1(s) \leq K_1$ and $G_2(t, s) \leq I_2(s) \leq K_2$, for all $(t, s) \in [0, T] \times [0, T]$. By using (H7), for any $u \in \partial B_N \cap P$, we obtain

$$\begin{aligned} (Au)(t) &= \int_0^T G_1(t, s) f \left(s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right) ds \\ &\leq \int_0^T G_1(t, s) f \left(s, K_2 \int_0^T g(\tau, u(\tau)) d\tau \right) ds \\ &\leq \int_0^T G_1(t, s) f \left(s, m_0 \int_0^T g(\tau, N) d\tau \right) ds \\ &< \frac{N}{m_0} \int_0^T G_1(t, s) ds \leq \frac{N}{m_0} K_1 T \leq N, \quad \forall t \in [0, T]. \end{aligned}$$

So, $\|Au\| < \|u\|$ for all $u \in \partial B_N \cap P$.

By Theorem 2.1, we conclude that

$$i(A, B_N \cap P, P) = 1. \quad (32)$$

On the other hand, from (H3), (H6) and the proofs of Theorems 3.1 and 3.2, we know that there exists a sufficiently large $L > N$ and a sufficiently small δ_2 with $0 < \delta_2 < N$ such that

$$i(A, B_L \cap P, P) = 0, \quad i(A, B_{\delta_2} \cap P, P) = 0. \quad (33)$$

From the relations (32) and (33), we obtain

$$\begin{aligned} i(A, (B_L \setminus \bar{B}_N) \cap P, P) &= i(A, B_L \cap P, P) - i(A, B_N \cap P, P) = -1, \\ i(A, (B_N \setminus \bar{B}_{\delta_2}) \cap P, P) &= i(A, B_N \cap P, P) - i(A, B_{\delta_2} \cap P, P) = 1. \end{aligned}$$

Then A has at least one fixed point u_1 in $(B_L \setminus \bar{B}_N) \cap P$ and has one fixed point u_2 in $(B_N \setminus \bar{B}_{\delta_2}) \cap P$, respectively. Therefore, the problem (S) – (BC) has two distinct positive solutions $(u_1, v_1), (u_2, v_2) \in P \times P$ with $u_i(t) \geq 0, v_i(t) \geq 0$ for all $t \in [0, T]$ and $\|u_i\| > 0, \|v_i\| > 0, i = 1, 2$. The proof of Theorem 3.3 is completed. $\square$

4. Examples

In this section, we shall present some examples which illustrate our results.

Example 1. Let $f(t, u) = (at^\alpha + b)u^{1/2}(2 + \sin u)$ and $g(t, u) = (ct^\beta + d)u^3(3 + \cos u)$ with $a, c \geq 0, b, d > 0, \alpha, \beta > 0$ and $p_1 = 1/2, q_1 = 1/2$. Then the

assumptions (H3) and (H4) are satisfied; indeed, we have

$$\begin{aligned} f_{\infty}^i &= \liminf_{u \rightarrow \infty} \inf_{t \in [0, T]} \frac{(at^{\alpha} + b)u^{1/2}(2 + \sin u)}{u^{1/2}} = b, \\ g_{\infty}^i &= \liminf_{u \rightarrow \infty} \inf_{t \in [0, T]} \frac{(ct^{\beta} + d)u^3(3 + \cos u)}{u^2} = \infty, \\ f_0^s &= \limsup_{u \rightarrow 0^+} \sup_{t \in [0, T]} \frac{(at^{\alpha} + b)u^{1/2}(2 + \sin u)}{u^{1/2}} = 2(aT^{\alpha} + b), \\ g_0^s &= \limsup_{u \rightarrow 0^+} \sup_{t \in [0, T]} \frac{(ct^{\beta} + d)u^3(3 + \cos u)}{u^2} = 0. \end{aligned}$$

Under the assumption (H1), by Theorem 3.1, we deduce that the problem (S) – (BC) has at least one positive solution.

Example 2. Let $f(t, u) = u^{1/2}$, $g(t, u) = u^{1/3}$ and $r = 1/2$. Then the assumptions (H5) and (H6) are satisfied; indeed, we have

$$\begin{aligned} f_{\infty}^s &= \lim_{u \rightarrow \infty} \frac{u^{1/2}}{u^{1/2}} = 1, \quad g_{\infty}^s = \lim_{u \rightarrow \infty} \frac{u^{1/3}}{u^2} = 0, \quad f_0^i = \lim_{u \rightarrow 0^+} \frac{u^{1/2}}{u} = \infty, \\ g_0^i &= \lim_{u \rightarrow 0^+} \frac{u^{1/3}}{u} = \infty. \end{aligned}$$

Under the assumption (H1), by Theorem 3.2, we conclude that the problem (S) – (BC) has at least one positive solution.

Example 3. We consider the following problem

$$\begin{cases} u^{(3)}(t) + a(v^{\alpha} + v^{\beta}) = 0, & t \in (0, 1), \\ v^{(4)}(t) + b(u^{\gamma} + u^{\delta}) = 0, & t \in (0, 1), \end{cases} \quad (S_0)$$

with the multi-point boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, & u(1) = 2u(\frac{1}{4}) + u(\frac{1}{2}) + \frac{1}{2}u(\frac{3}{4}), \\ v(0) = v'(0) = v''(0) = 0, & v(1) = v(\frac{1}{3}) + 3v(\frac{2}{3}), \end{cases} \quad (BC_0)$$

where $\alpha > 1, \beta < 1, \gamma > 2, \delta < 1, a, b > 0$. Here $T = 1, n = 3, m = 4, p = 5, q = 4, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}, \xi_3 = \frac{3}{4}, a_1 = 2, a_2 = 1, a_3 = \frac{1}{2}, \eta_1 = \frac{1}{3}, \eta_2 = \frac{2}{3}, b_1 = 1, b_2 = 3$. Then, we have $d = 1 - \sum_{i=1}^3 a_i \xi_i^2 = \frac{11}{32} > 0, e = 1 - \sum_{i=1}^3 b_i \eta_i^3 = \frac{2}{27} > 0$. Here,

$$\theta_1(s) = \frac{1}{2-s}, \quad s \in [0, 1]; \quad \theta_2(s) = \begin{cases} \frac{s}{1-(1-s)^{3/2}}, & s \in (0, 1], \\ \frac{2}{3}, & s = 0. \end{cases}$$

For the functions I_1 and I_2 , we obtain

$$I_1(s) = g_1(\theta_1(s), s) + \frac{1}{d} \left[2g_1\left(\frac{1}{4}, s\right) + g_1\left(\frac{1}{2}, s\right) + \frac{1}{2}g_1\left(\frac{3}{4}, s\right) \right]$$

$$= \begin{cases} \frac{2s - 5s^2 + 4s^3 - s^4}{2(2-s)^2} + \frac{1}{22}(-91s^2 + 46s), & 0 \leq s < \frac{1}{4}, \\ \frac{2s - 5s^2 + 4s^3 - s^4}{2(2-s)^2} + \frac{1}{22}(-27s^2 + 14s + 4), & \frac{1}{4} \leq s < \frac{1}{2}, \\ \frac{2s - 5s^2 + 4s^3 - s^4}{2(2-s)^2} + \frac{1}{22}(5s^2 - 18s + 12), & \frac{1}{2} \leq s < \frac{3}{4}, \\ \frac{2s - 5s^2 + 4s^3 - s^4}{2(2-s)^2} + \frac{21}{22}(1 - 2s + s^2), & \frac{3}{4} \leq s \leq 1, \end{cases}$$

$$I_2(s) = g_2(\theta_2(s), s) + \frac{1}{e} \left[g_2\left(\frac{1}{3}, s\right) + 3g_2\left(\frac{2}{3}, s\right) \right]$$

$$= \begin{cases} 0, & s = 0, \\ \frac{s^3(1-s)^3 - s^3(1-s)^{9/2}}{6[1 - (1-s)^{3/2}]^3} + \frac{1}{12}(83s^3 - 114s^2 + 42s), & 0 < s < \frac{1}{3}, \\ \frac{s^3(1-s)^3 - s^3(1-s)^{9/2}}{6[1 - (1-s)^{3/2}]^3} + \frac{1}{12}(56s^3 - 87s^2 + 33s + 1), & \frac{1}{3} \leq s < \frac{2}{3}, \\ \frac{s^3(1-s)^3 - s^3(1-s)^{9/2}}{6[1 - (1-s)^{3/2}]^3} + \frac{25}{12}(1 - 3s + 3s^2 - s^3), & \frac{2}{3} \leq s \leq 1. \end{cases}$$

We have $K_1 = \max_{s \in [0,1]} I_1(s) \approx 0.306$, $K_2 = \max_{s \in [0,1]} I_2(s) \approx 0.398$. Then $m_0 = \max\{K_1 T, K_2\} = K_2$. The functions $f(t, u)$ and $g(t, u)$ are nondecreasing with respect to u , for any $t \in [0, 1]$, and for $p_1 = 1/2$ the assumptions (H3) and (H6) are satisfied; indeed we obtain

$$f_\infty^i = \lim_{u \rightarrow \infty} \frac{a(u^\alpha + u^\beta)}{u^{1/2}} = \infty, \quad g_\infty^i = \lim_{u \rightarrow \infty} \frac{b(u^\gamma + u^\delta)}{u^2} = \infty,$$

$$f_0^i = \lim_{u \rightarrow 0^+} \frac{a(u^\alpha + u^\beta)}{u} = \infty, \quad g_0^i = \lim_{u \rightarrow 0^+} \frac{b(u^\gamma + u^\delta)}{u} = \infty.$$

We take $N = 1$ and then $\int_0^1 g(s, 1) ds = 2b$ and $f(t, 2bm_0) = a[(2bm_0)^\alpha + (2bm_0)^\beta]$. If $a[(2bm_0)^\alpha + (2bm_0)^\beta] < \frac{1}{m_0} \Leftrightarrow a \left[m_0^{\alpha+1} (2b)^\alpha + m_0^{\beta+1} (2b)^\beta \right] < 1$, then the assumption (H7) is satisfied. For example, if $\alpha = 2, \beta = 1/2, b = 1/2$ and $a < \frac{1}{m_0^3 + m_0^{3/2}}$ (e.g. $a \leq 3$), then the above inequality is satisfied. By Theorem 3.3, we deduce that the problem $(S_0) - (BC_0)$ has at least two positive solutions.

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