On the velocity of planar trajectories

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Abstract. Consider two trajectories each forming a Jordan curve with a finite length in the two dimensional plane. We show that the integral of the velocity of the portion of one curve that is included in the inside of the second curve, is bounded by half the length of the latter curve. We point out an application of the result to averaging.

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1. The main result

Consider a Jordan curve, say J , with a finite length L in the two-dimensional plane R^2 . It divides the plane into two regions, the *inside* of J , that we denote by $\text{In}(J)$, and the *outside* of J , that we denote by $\text{Out}(J)$. To avoid ambiguity we agree not to include in $\text{In}(J)$ the curve itself; thus the closure of $\text{In}(J)$ is $\text{In}(J)$ union J ; likewise with respect to the outside of J .

Consider another Jordan curve in the plane, given by $x(t) : [0, T] \rightarrow R^2$, namely $x(0) = x(T)$ and otherwise $x(t_1) \neq x(t_2)$ if $t_1 \neq t_2$. Suppose $x(\cdot)$ is absolutely continuous and let $u(t)$ denote its almost everywhere derivative. Denote by $|\cdot|$ the Euclidean norm on R^2 . Our main result is as follows.

Theorem A. *Let U denote the subset of $[0, T]$ for which $x(t) \in \text{In}(J) \cup J$. Then*

$$\left| \int_U u(t) dt \right| \leq \frac{L}{2} \quad (1.1)$$

Remark 1.1. Since $\int_0^T u(t) dt = 0$ and since the roles of inside and outside are interchangeable, it follows that (1.1) holds also when U is the subset of $[0, T]$ where $x(t)$ is in $\text{Out}(J)$, hence also when U is the subset of $[0, T]$ where

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$x(t)$ belongs to $\text{In}(J)$ and, likewise, when U is the subset of $[0, T]$ where $x(t)$ belongs to $\text{Out}(J) \cup J$.

The result is an extension of an observation that served us (in [1]) in establishing a Poincaré-Bendixson type result for infinite horizon optimal control problems, namely, verifying that if the state space is two-dimensional, it suffices to consider periodic solutions. We allude to this application in the closing section. In the next two sections we verify the result in two special cases, this in preparation to the completion of the proof of the theorem in Sect. 4, for which we establish some useful observations concerning approximations of Jordan curves. In the closing section we provide some examples and comments, showing, in particular, that the estimates are sharp, but can be improved when J is convex.

2. The case of a finite number of intersections

In the sequel, and without loss of generality, we assume that J is parameterized as $y(s) : [0, L] \rightarrow \mathbb{R}^2$, where $y(\cdot)$ is absolutely continuous, $y(0) = y(L)$ and otherwise $y(s_1) \neq y(s_2)$ if $s_1 \neq s_2$, and such that $|\frac{dy}{ds}| = 1$ for every s . (Here and in the sequel we reserve the variable s to indicate the parameterization of J .) This is the standard arc-length parameterization; it can be constructed by identifying one point on J as $y(0)$, prescribe an orientation on J and then define $y(s)$ to be the point on J with distance from $y(0)$, on J in the positive direction, of magnitude s . In particular, intervals on J correspond to intervals on $[0, L]$ when 0 is identified with L . We denote then $\frac{dy}{ds}(s) = v(s)$.

Before stating and verifying our result in a special case as follows, we display a claim that holds in general and, in fact, holds for periodic orbits in any dimension.

Lemma 2.1. *Let V be a Borel measurable subset of $[0, L]$. Then*

$$\left| \int_V v(s) ds \right| \leq \frac{L}{2}. \quad (2.1)$$

Proof. Since $\int_0^L v(s) ds = 0$ it follows that if $|\int_V v(s) ds| > \frac{L}{2}$ then the norm of the integral on the complement of V in $[0, L]$ is also greater than $\frac{L}{2}$. This would imply that $\int_0^L |v(s)| ds > L$, in contradiction with $\int_0^L |v(s)| ds$ being the length of J , namely equal to L . This completes the proof. $\square$

Here is the special case announced in the title of the section.

Proposition 2.2. *Suppose that the curve $x(\cdot)$ intersects J in only a finite number of points. Then (1.1) holds.*

Proof. A key observation is that $\int_0^T u(t) dt = 0$, which follows from the periodicity. Hence the cases where $U = [0, T]$ or $U = \emptyset$ are trivial, and likewise (since the outside can play the role of inside) when $x(t)$ is always in $\text{Out}(J) \cup J$. Hence we have to consider only the case where the trajectory $x(\cdot)$ intersects both $\text{In}(J)$ and $\text{Out}(J)$ and, in particular, intersects J itself.

Among the finite number of points of intersection there are those where $x(\cdot)$ stays on the same side of J on short intervals before and after the intersection, and those for which the trajectory crosses the Jordan curve at the intersection, either from $\text{In}(J)$ to $\text{Out}(J)$ or from $\text{Out}(J)$ to $\text{In}(J)$. We consider now only the points of crossing. Let z_1 be one of these points. We choose an orientation on J , say clockwise, and denote by $z_2, z_3, \dots, z_n$ the other points at which $x(\cdot)$ crosses J , in the order they appear on J according to the chosen orientation (which may not necessarily be the order derived from $[0, T]$). We also identify $z_1 = z_{n+1}$. We shall need the following observation.

Observation. If z_i is a point where $x(\cdot)$ crosses J from inside to outside (or from outside to inside) then z_{i+1} is a point where $x(\cdot)$ crosses J from outside to inside (respectively, from inside to outside). In particular, n is even.

To verify the observation we use the two-dimensional topology. Let J_1 be the curve that consists of the range of $x(\cdot)$ starting at the time t_i where $x(t_i) = z_i$, until the time t_{i+1} where $x(t_{i+1}) = z_{i+1}$. Consider the concatenation of J_1 with the segment of J that joins z_i and z_{i+1} . If $x(\cdot)$ crosses J from inside to outside both at z_i and at z_{i+1} , then in order to reach z_i again, which it should due to periodicity, $x(\cdot)$ will have to cross the concatenated curve. This is not allowed since the curve induced by $x(\cdot)$ is Jordan and since there are no crossing points on J between z_i and z_{i+1} . The contradiction verifies the observation.

We continue with the proof of the proposition. We construct now a finite number of periodic trajectories $\theta_1, \dots, \theta_m$, each composed of portions of $x(\cdot)$ in $\text{In}(J)$ and portions of J itself, such that each θ_i intersects itself in at most a finite number of times and any two distinct θ_i intersect each other in at most a finite number of points (in fact, it is possible to show that they do not intersect but we shall not need it in the proof), and the union of $\theta_i, i = 1, \dots, m$, contains the range of $x(\cdot)$ within $\text{In}(J)$ and a portion of J (but may not contain the entire curve J).

Suppose that z_1 is a point of crossing from $\text{In}(J)$ to $\text{Out}(J)$. The construction of θ_1 starts at z_1 in the clockwise direction on J until z_2 is reached. By the displayed Observation, the point z_2 is a point of crossing from $\text{Out}(J)$ to $\text{In}(J)$. Now, θ_1 continues by following $x(\cdot)$ in $\text{In}(J)$ until a point of crossing from $\text{In}(J)$ to $\text{Out}(J)$ is reached; if this point is z_1 then θ_1 has been completed. If it is a different point, say z_j , we switch at z_j to a portion of J in the clockwise direction, until we reach z_{j+1} which, by the previous Observation, is a point of crossing from $\text{Out}(J)$ to $\text{In}(J)$. The latter point cannot be z_2 as z_2 has been reached by the clockwise arc starting at z_1 . We continue from z_{j+1} with $x(\cdot)$ inside J until a point of crossing outside is reached again; if this point is z_1 then θ_1 has been completed; otherwise we continue in the same manner as before. This process must end with reaching the original z_1 ; indeed, since $x(\cdot)$ forms a Jordan curve the first time the process reaches a point that is already in the constructed trajectory this point must be z_1 . The trajectory θ_1 may intersect itself but only on the portions of J contained in it, this since $x(\cdot)$ forms a Jordan curve; hence the self intersection of θ_1 may occur only at a finite number of points.

If the periodic trajectory constructed in the first stage has consumed all the crossing points we are done, in one step. Otherwise, there are crossing points not on the constructed trajectory. Consider a point, say z_i , that is on θ_1 but such that z_{i+1} is not in θ_1 . Then z_i is a point of crossing inside J , otherwise the arc joining z_i and z_{i+1} would have been in θ_1 . By the Observation, the point z_{i+1} crossing from $\text{In}(J)$ to $\text{Out}(J)$. We start now θ_2 with z_{i+1} following the process described in the construction of θ_1 . (It is not difficult to argue that all the crossing points on θ_2 are on a segment of J with interior that does not contain crossing points of θ_1 , but we do not need it in the proof.) The curve θ_2 does not cross θ_1 in the inside of J , and if it crosses it on J (it is not difficult to show that this cannot happen, but we do not need it in the proof) it is only in a finite number of points. Therefore, the construction of θ_2 ends when z_{i+1} is reached again, where θ_2 is completed.

If needed, we continue successively. After a finite number of iterations the construction of $\theta_1, \dots, \theta_m$ is complete with the desired properties being satisfied.

With the construction of the curves $\theta_1, \dots, \theta_m$ the completion of the proof of the proposition is simple. Indeed, the periodic curves we constructed are the images of intervals in $[0, T]$ whose union is U , and intervals in $[0, L]$, whose union is, say, V . The periodicity of the curves θ_i implies that the sum of the integrals of $u(\cdot)$ and $v(\cdot)$ on, respectively, U and V , results with the vector 0. In view of Lemma 2.1 the integral of $v(\cdot)$ over V results in a vector of norm less than or equal to $\frac{L}{2}$. Hence the norm of the integral of $u(\cdot)$ on U is also less than or equal to $\frac{L}{2}$ which is what is claimed in (1.1). This completes the proof of Proposition 2.2. $\square$

3. The smooth and the convex cases

We now state and verify the result under an additional condition as follows. Let D_ε denote the ε -neighborhood of $\text{In}(J) \cup J$. Denote the boundary of D_ε by J_ε .

Proposition 3.1. *Assume that for ε small enough the curves J_ε are Jordan curves. Then (1.1) holds.*

Proof. We shall employ the following observation.

Auxiliary Proposition. For a set of ε of full Lebesgue measure around 0 the range of $x(\cdot)$ intersects J_ε in only a finite number of points.

To verify the auxiliary proposition consider a re-parameterization of $x(\cdot)$, say $x_1(\tau) : [0, T_1] \rightarrow \mathbb{R}^2$, such that $x_1(\tau)$ is Lipschitz in the variable τ . Such a parameterization clearly exists (compare with the first paragraph of the previous section). Since the new parameterization does not change the number of intersections with each J_ε , it is enough to verify the claim with respect to this new parameterization. Consider now the mapping $r(\tau) = \varepsilon$ when $x_1(\tau) \in J_\varepsilon$. It is well defined and its range includes an open interval around 0. This mapping is Lipschitz. Now the result follows from the extension of Sard's theorem for Lipschitz maps, see Wells [7].

With the previous auxiliary proposition the conclusion of the proof of the proposition is easy. First, recall Steiner's theorem (see, e.g., Guggenheimer [5, page 81]) which implies that the length L_ε of J_ε satisfies

$$L_\varepsilon \leq L + 2\pi\varepsilon. \quad (3.1)$$

Denote by U_ε the set of points on $[0, T]$ where $x(t)$ is inside or on J_ε . By Proposition 2.2 for a set of ε of full measure around 0, the estimate (1.1) holds when we replace U by U_ε . The intersection of the closed sets U_ε for ε positive is U . Hence (1.1) holds for the original U and Proposition 3.1 is verified. $\square$

Corollary 3.2. *Suppose that J is either twice continuously differentiable or that $\text{In}(J)$ is a convex set. Then (1.1) holds.*

Proof. In both cases it is easy to see that the condition of Proposition 3.1 is met. $\square$

4. The general case

In this section we prove Theorem A. We adopt the setting displayed at the beginning of Sect. 2. Our first aim is a construction of an appropriate Jordan curve approximation, as follows.

Proposition 4.1. *For every $\eta > 0$ there exists a Jordan curve $J_{\eta,0}$, with distance η from J , with length $L_{\eta,0}$ less than or equal to $L + 2\pi\eta$, such that the intersection of the insides of $J_{\eta,0}$ is $\text{In}(J) \cup J$.*

Proof. For $\eta > 0$ let D_η be the closed η -neighborhood of $\text{In}(J) \cup J$, namely, the set of points in $\mathbb{R}^2$ whose Euclidean distance from $\text{In}(J) \cup J$ is less than or equal to η . Then the interior of D_η is arc-wise connected. Indeed, the interior of D_η is the union of $\text{In}(J)$ with the open η -neighborhood of the Jordan curve J .

Now consider the boundary ∂D_η of D_η . By Steiner's theorem, the boundary ∂D_η , also called the parallel curve, is closed and has length, say L_η , satisfying

$$L_\eta \leq L + 2\pi\eta. \quad (4.1)$$

See Guggenheimer [5, page 81].

The boundary curve ∂D_η may not be Jordan (if it is, the proof is finished). Consider the complement of D_η . It is an open set. It may have many connected components.

Claim 1. *In the one-point compactification of $\mathbb{R}^2$ (namely considering it on the two dimensional sphere in three space) any component of the complement of D_η is simply connected.*

To verify the claim notice that otherwise the component would include a Jordan curve with points of D_η inside and outside it, in contradiction to D_η being arc-wise connected.

Consider now one component, say O_η of the complement of D_η , and denote its boundary by J_η .

Claim 2. *The boundary J_η of a component O_η of the complement of D_η is a Jordan curve with length less than or equal to $L + 2\pi\eta$.*

Since J_η is part of ∂D_η , each point on the curve J_η is a boundary point of the interior of D_η . If the complement of J_η has more than two connected components, then two distinct such components will contain points in the interior of D_η . This contradicts Claim 1; indeed, any arc that connects two points in distinct components must intersect J_η , hence intersects ∂D_η and cannot be included in the interior of D_η . This proves that J_η is Jordan. Being also a closed subset of ∂D_η implies, in view of (4.1), the estimate on its length.

The previous two claims apply to any component of the complement of D_η . To prove the proposition we choose one of these components. To this end we identify a point x_0 in $\text{Out}(J)$. This point will be fixed for every small enough η . Suppose that η is smaller than the distance between x_0 and J . Among the open components of the complement of D_η we choose the one that contains x_0 , say $O_{\eta,0}$. Its boundary is the Jordan curve $J_{\eta,0}$. We call $O_{\eta,0}$ the outside of $J_{\eta,0}$. We claim that the family $J_{\eta,0}$ satisfies the requirements of the proposition.

Being part of ∂D_η the requirements on the distance from J and the length of $J_{\eta,0}$ (see Claim 2) are met. It remains to prove the statement concerning the intersection of the insides of $J_{\eta,0}$.

Claim 3. *For every point x in $\text{Out}(J)$, if η is small enough, then x belongs to the outside of $J_{\eta,0}$.*

To verify the claim we employ the fact that J is Jordan, hence we can connect x_0 and x with a closed curve, say C , within $\text{Out}(J)$. Let δ be the least distance between points of C and of J , namely, for each $y \in J$ and $z \in C$ we have $|y - z| > \delta$. Now, if $\eta < \delta$, the curve $J_{\eta,0}$ does not intersect with C , hence x_0 and x are in the same side of $J_{\eta,0}$. This verifies the claim.

Employing Claim 3 we conclude that the union of the outsides of $J_{\eta,0}$ is $\text{Out}(J)$. This completes the proof of Proposition 4.1. $\square$

Corollary 4.2. *For every $\eta > 0$ and $\varepsilon > 0$ there exists a twice continuously differentiable Jordan curve $J_{\eta,\varepsilon}$, each point of which is of distance between $\eta - \varepsilon$ and $\eta + \varepsilon$ from J , with length $L_{\eta,\varepsilon}$ less than or equal to $L + 2\pi\eta + \varepsilon$ and such that the intersection of the insides of $J_{\eta,\varepsilon}$ is $\text{In}(J) \cup J$.*

Proof. It is well known that a Jordan curve with a finite length can be approximated by a twice continuously differentiable Jordan curve with location and length arbitrarily close to that of the original one. (A direct proof is easy to construct based on a piecewise constant approximation in measure of the derivative, then smoothening, say a convolution, procedure. The approximation step in Tverberg [6] follows a similar argument. Alternatively, one can construct an approximation relying on the Whitney's extension theorem, see Federer [4, Theorem 3.1.16], and then apply a smoothening procedure.) $\square$

Proof of Theorem A. Consider the twice continuously differentiable curves $J_{\eta,\varepsilon}$ established in Corollary 4.2. Denote by $U_{\eta,\varepsilon}$ the set of points in

$[0, T]$ such that $x(t)$ is in the inside of $J_{\eta, \varepsilon}$ or on $J_{\eta, \varepsilon}$. By Corollary 3.2

$$\left| \int_{U_{\eta, \varepsilon}} u(t) dt \right| \leq \frac{L}{2} + 2\pi\eta + \varepsilon. \quad (4.2)$$

The proof of the theorem will be concluded by taking in (4.2) the limit as η and ε go to 0. (An alternative proof, avoiding the smooth approximation, would be to use the Auxiliary Proposition of Proposition 3.1 to show that for a set of full measure the intersection of $x(\cdot)$ with ∂D_η is finite. Then use Proposition 2.2 and Claims 1 and 2 above to argue that the integral of $u(t)$ over the sets of points where $x(t)$ is in any component of the complement of D_η , is bounded by half the length of the boundary of that component. Now, it is easy to see that the boundaries of distinct components intersect in only a finite number of points, hence the sum of the lengths of these boundaries does not exceed $L + 2\pi\eta$. Taking now the limit as $\eta \rightarrow 0$ completes the proof.)

5. Examples, comments

It is clear that the consequence of Theorem A does not hold for curves in more than two dimensional space, nor for curves that are not Jordan curves. The estimate $\frac{L}{2}$ is sharp in general, but can be improved when J is convex. Here are the statement and two examples.

Remark 5.1. The estimate $\frac{L}{2}$ in Theorem A can be improved in case J is convex to be equal to the diameter D of J in R^2 . Indeed, when J is convex the estimate in Lemma 2.1 (that served us all along) can be set to be D ; just follow the proof of Lemma 2.1 when $v(s)$ is replaced by the projection of $v(s)$ on a line where the diameter is attained.

Example 5.2. Let J be the curve in R^2 connecting $(0, 0)$, $(1, 0)$, $(1, 2h)$ and $(0, 2h)$ and let J_1 be the curve connecting $(0, h)$, $(1, h)$, $(1, 3h)$ and $(0, 3h)$ (i.e., J_1 is the shift of J upward by h). The value of the integral in (1.1) for this case is equal to 1, and for h small this is close to half the circumference, namely, half the length of J , and also close to the diameter of the convex set determined by J . This shows that the estimate D is sharp.

Example 5.3. We show now that, in general, the estimate $\frac{L}{2}$ cannot be replaced by D . Consider a comb-like figure, say C , determined by the segment joining $(0, 0)$ and $(\Delta, 0)$, and the vertical lines joining $(\frac{j}{N}\Delta, 0)$ and $(\frac{j}{N}\Delta, 1)$, for $j = 0, 1, \dots, N$. Let J be the boundary of all the points that are of distance δ from C , with δ small relative to $\frac{\Delta}{N}$. Let J_1 be the curve obtained by shifting J , say to the right, by $\frac{\delta}{2}$. When Δ is small and N is large, then L is large and the integral in (1.1) converges to $\frac{L}{2}$, while the diameter of J converges to 1.

Remark 5.4. The results of this paper extend the weaker (both the estimate and the assumptions) result employed in [1] in order to establish the existence of optimal periodic solutions to infinite horizon optimal control problem with planar state space. The reasoning in Sects. 2 and 3 follows that of [1], with the needed adaptation. However, the reasoning has been used in the proofs in [1]

without displaying the result. Likewise with the consequence of our analysis, which we display next.

Let J be a Jordan curve in R^2 , generated by the absolutely continuous mapping $x(t) : [0, T] \rightarrow R^2$. Let $u(\cdot)$ be the almost everywhere derivative of $x(\cdot)$. Along with J we consider the state-velocity *occupational measure*, say $\mu(J)$, induced by $x(\cdot)$ on $R^2 \times R^2$, namely, the probability measure on $R^2 \times R^2$ defined by

$$\mu(J)(E) = \frac{1}{T} \lambda(\{t \in [0, T] : (x(t), u(t)) \in E\}) \quad (5.1)$$

where λ is the Lebesgue measure, namely, $\mu(J)(E)$ is the proportion of time in $[0, T]$ that the trajectory $(x(\cdot), u(\cdot))$ stays in E (thus, $\mu(J)$ depends on the parameterization).

Consider now a family J_i of Jordan curves in R^2 . Let L_i be the length of J_i . Pick a parameterization $x_i(\cdot)$ of J_i and suppose that $\mu(J_i)$ converges, in the weak convergence of measures, to μ_0 (for the terminology and results that we use in regard to probability measures see, e.g., Ash [2], Billingsley [3]). Let $\mathbf{p}$ be the marginal of μ_0 on the state space R^2 and let $\nu_0(x)$ be the disintegration of μ_0 with respect to $\mathbf{p}$ (notice that for $\mathbf{p}$ -almost every x the measure $\nu_0(x)$ is a probability measure on the u -variable).

Proposition 5.5. *If $L_i \rightarrow \infty$ as $i \rightarrow \infty$ then*

$$\int_{R^2} u \nu_0(x)(du) = 0 \quad (5.2)$$

for $\mathbf{p}$ -almost every x .

Proof. Let x be in the support of $\mathbf{p}$. Consider a small ball, say $B_{x,r}$, of radius r around x . Then, applying (1.1) for J being the circumference of $B_{x,r}$ and for $x(\cdot)$ generating the Jordan curve J_i implies that

$$\int_{B_{x,r} \times R^2} u \mu(J_i)(dx \times du) \leq \frac{1}{L_i} 2r. \quad (5.3)$$

Taking the limit as $i \rightarrow \infty$ yields

$$\int_{B_{x,r} \times R^2} u \mu_0(dx \times du) = 0. \quad (5.4)$$

Since (5.4) holds for every ball, hence for every Borel set, it clearly implies (5.2). This completes the proof. $\square$

The interpretation of the previous result is the following two-dimensional phenomenon. If the density of a sequence of longer and longer Jordan curves in R^2 converges, then, in the limit, the velocities average out locally, namely, the density of velocities tends weakly to zero. Clearly, the consequence may not be true if the curves are not Jordan.

References

- [1] Artstein, Z., Bright, I.: Periodic optimization suffices for infinite horizon planar optimal control. *SIAM J. Control Optim.* **48**, 4963–4986 (2010)
- [2] Ash, R.B.: *Real Analysis and Probability*. Academic Press, New York (1972)
- [3] Billingsley, P.: *Convergence of Probability Measures*, 2nd edn. Wiley, New York (1999)
- [4] Federer, H.: *Geometric Measure Theory*. Springer, Berlin (1969)
- [5] Guggenheimer, H.W.: *Differential Geometry*. McGraw-Hill Book Co., Inc., New York (1963)
- [6] Tverberg, H.: A proof of the Jordan Curve Theorem. *Bull. Lond. Math. Soc.* **12**, 34–38 (1980)
- [7] Wells, J.C.: A note on a corollary of Sard's theorem. *Proc. Am. Math. Soc.* **48**, 513–514 (1975)

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