Lax–Hopf formula and Max-Plus properties of solutions to Hamilton–Jacobi equations

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Abstract. We state and prove a “Lax–Hopf formula” characterizing viable capture basins of targets investigated in viability theory and derive a “Max-Plus” morphism of capture basins with respect to the target. Capture basins are used to define “viability solutions” to Hamilton–Jacobi equations satisfying “trajectory conditions” (initial, boundary or Lagrangian conditions). The Max-Plus morphism property of Lax–Hopf formula implies the fact that the solution associated with inf-convolution of trajectory conditions is the inf-convolution of the solutions for each trajectory condition. For instance, Lipschitz regularization or decreasing envelopes of trajectory condition imply the Lipschitz regulation or decreasing envelopes of the solutions.

Keywords. Lax–Hopf, Viability, Capture Basins, Inf-convolution, Fenchel transform, Lagrangian, Hamiltonian.

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1. Introduction

Let $X := \mathbb{R}^n$ be a finite dimensional vector space. Let us consider

1. a concave upper semicontinuous Hamiltonian $\mathbf{h} : p \in X \mapsto \mathbf{h}(p) \in \mathbb{R}$;

2. an extended lower semicontinuous function $\mathbf{c} : (t, x) \in \mathbb{R}_+ \times \mapsto \mathbf{c}(t, x) \in \mathbb{R} \cup \{+\infty\}$, with which we associate the *domain tube* $t \leadsto \mathbf{C}(t)$ defined by

$$\mathbf{C}(t) := \{x \text{ such that } \mathbf{c}(t, x) < +\infty\} \quad (1)$$

which can be regarded as a thick trajectory.

An extended function $V : (t, x) \in \mathbb{R}_+ \times K \mapsto V(t, x) \in \mathbb{R}$ is a solution of the “trajectory-valued Hamilton–Jacobi problem” if:

$$\begin{cases} \text{(i)} & \forall t > 0, \forall x \notin \mathbf{C}(t), \quad \frac{\partial V(t, x)}{\partial t} = \mathbf{h}\left(-\frac{\partial V(t, x)}{\partial x}\right) \\ \text{(ii)} & \forall t \geq 0, \forall x \in \mathbf{C}(t), \quad V(t, x) \leq \mathbf{c}(t, x) \end{cases} \quad (2)$$

The second condition encompasses Cauchy, Dirichlet, Lagrangian conditions as well as many other “internal conditions”. We call it a “trajectory condition” (see [2] for examples and applications to traffic management).

The Lax–Hopf formula proved for Cauchy problems extend to general trajectory-valued problems: we associate with the Hamiltonian $\mathbf{h}$ its Fenchel transform

$$\forall u, \quad \mathbf{l}(u) := \sup_{p \in \text{Dom}(\mathbf{h})} [\mathbf{h}(p) - \langle p, u \rangle]$$

regarded as the Lagrangian. The Lax–Hopf formula reads

$$V(T, x) = \inf_{d \in [0, T]} \inf_{\xi \in \mathbf{C}(d)} \left[\mathbf{c}(d, \xi) + (T - d) \mathbf{l}\left(\frac{x - \xi}{T - d}\right) \right]$$

Furthermore, let us introduce the inf-convolution $\mathbf{v} \star \mathbf{c}$ of the trajectory condition $\mathbf{c}$ with any lower semicontinuous positively-homogeneous convex extended function $\mathbf{v}$ defined by

$$(\mathbf{v} \star \mathbf{c})(t, x) = \inf_y (\mathbf{v}(y) + \mathbf{c}(t, x - y))$$

We shall prove that the solution to the Hamilton–Jacobi equation associated with the inf-convoluted trajectory condition $(t, x) \mapsto (\mathbf{v} \star \mathbf{c})(t, x)$ by $\mathbf{v}$ is the inf-convolution $(t, x) \mapsto (\mathbf{v} \star V)(t, x)$ of the solution $(t, x) \mapsto V(t, x)$ to the Hamilton–Jacobi equation associated with trajectory condition $(t, x) \mapsto \mathbf{c}(t, x)$.

Among some consequences, we may single out:

1. if we regularize the trajectory condition by taking its “Lipschitz regularization” which is the largest smaller λ -Lipschitz function obtained by its Pasch–Hausdorff envelope, obtained as an inf-convolution product by the function $x \mapsto \lambda\|x\|$, we deduce that the solution is λ -Lipschitz, and, actually, the Pasch–Hausdorff envelope of the solution.
2. if we associate with the trajectory condition its “decreasing envelope”, which is the smallest larger decreasing function, obtained as an inf-convolution product by the indicator function of a cone, we deduce that the solution is decreasing, and, actually, the decreasing envelope of the solution.

The method used is derived from set-valued analysis and viability theory (see the books [1, 6, 7]), actually, from the concept of viable capture basin (see Definition 4.1, p. 191). We define Lax–Hopf formulas for computing them under adequate assumptions (Theorem 4.4, p. 193). We next define the epigraph of the “viability solution” to the trajectory-valued Hamilton–Jacobi problem as a viable capture basin of an auxiliary target under an auxiliary characteristic system. It is shown (see [4]) that viability solutions defined in this way are viscosity solutions in the weaker sense of Barron–Jensen/Frankowska. The Lax–Hopf formulas proved (in a very simple way) at the level of viable capture basins imply the announced results we were looking for.

After defining trajectory-valued Hamilton–Jacobi problems (Sect. 2) and introducing the Lagrangian (Sect. 3), we prove the Lax–Hopf formulas characterizing viable capture basins (Sect. 3) for computing the Lax–Hopf formula for the solution (Sect. 4). Section 5 is devoted to the “inf-convolution” morphism. The last section apply the above results to one-dimensional trajectory-valued Hamilton–Jacobi problems.

2. Trajectory-valued Hamilton–Jacobi problems

Definition 2.1. (*Hamiltonian and trajectory conditions*) Let $X := \mathbb{R}^n$ be a finite dimensional vector space. We associate with a Hamiltonian $\mathbf{h} : \mathbb{R}_+ \times X \times \mathbb{R}_+ \mapsto \mathbb{R}$ the Hamilton–Jacobi partial differential equation

$$\frac{\partial V(t, x)}{\partial t} = \mathbf{h} \left(-\frac{\partial V(t, x)}{\partial x} \right)$$

A *trajectory condition* is described by a lower semicontinuous function $\mathbf{c} : (t, x) \in \mathbb{R}_+ \times X \mapsto \mathbf{c}(t, x) \in \mathbb{R} \cup \{+\infty\}$, with which we associate the *domain tube* $t \rightsquigarrow \mathbf{C}(t)$ defined by

$$\mathbf{C}(t) := \{x \text{ such that } \mathbf{c}(t, x) < +\infty\} \quad (3)$$

These general trajectory conditions cover the following classical and less classical examples:

1. *Cauchy initial conditions* $x \mapsto \gamma_0(x)$. In this case, we set $\mathbf{c}_0(0, x) := \gamma_0(x)$ and $\mathbf{c}_0(t, x) = +\infty$ whenever $t > 0$. Then $\mathbf{C}(0) = \text{Dom}(\gamma_0)$ and $\mathbf{C}(t) = \emptyset$;
2. *Dirichlet boundary conditions* $x \in \partial K \mapsto \delta(t, x)$ defined on the boundary of the domain, with which we associate $\mathbf{c}_\delta(t, x) := \delta(t, x)$ if $x \in \partial K$ and $\mathbf{c}_\delta(t, x) = +\infty$ whenever $x \in \overset{\circ}{K}$;
3. *Lagrangian “mobile” conditions* $t \mapsto \gamma_i(t)$ during some time intervals $[\underline{\tau}_i, \bar{\tau}_i]$, with which we associate $\mathbf{c}_{\gamma_i}(t, \gamma_i(t)) := \delta_i(t)$ and $\mathbf{c}_{\gamma_i}(t, x) := +\infty$ whenever $x \neq \gamma_i(t)$;
4. *prescribed upper bounds* on solutions to Hamilton–Jacobi equations;
5. *Combined Lagrangian conditions* $\gamma_i(t), i = 1, \dots, p, \mathbf{C}(t) = \bigcup_i \text{such that } \underline{\tau}_i \leq t \leq \bar{\tau}_i \{ \gamma_i(t) \}$ and the trajectory condition $\mathbf{c}(t, \gamma_i(t))$ is defined on the graphs $(t, \gamma_i(t))_{t \geq 0}$ of the evolution of each vehicle i .

Definition 2.2. (*The trajectory-valued Hamilton–Jacobi problem*) The trajectory-valued Hamilton–Jacobi problem takes into account the two above requirements on its solution $V : (t, x) \in \mathbb{R}_+ \times K \mapsto V(t, x) \in \mathbb{R}$:

$$\begin{cases} \text{(i)} & \forall t > 0, \forall x \notin \mathbf{C}(t), \quad \frac{\partial V(t, x)}{\partial t} = \mathbf{h}\left(-\frac{\partial V(t, x)}{\partial x}\right) \\ \text{(ii)} & \forall t \geq 0, \forall x \in \mathbf{C}(t), \quad V(t, x) \leq \mathbf{c}(t, x) \end{cases} \quad (4)$$

Since the Lagrangian conditions may and do take infinite values and are assumed only to be semicontinuous, partial differential equation techniques (see for instance [11]) may not work. The lack of regularity is not only motivated by mathematical search of generality, but here, by very concrete applications.

3. Hamiltonians and Lagrangians

It is “mathematically” natural in the framework of duality theory in mechanics, in economics and in convex analysis to associate with a (concave) Hamiltonian $\mathbf{h} : X \mapsto \mathbb{R}$ a (convex) Lagrangian $\mathbf{l} : X^* := X \mapsto \mathbb{R}$. Since *Werner Fenchel*, we know that there exists a bijective correspondence between lower semicontinuous convex functions defined on a vector space and their conjugate functions defined on the dual. Here, there is a slight adaptation to perform, since the Hamiltonian $\mathbf{h}$ is concave and the Lagrangian $\mathbf{l}$ is convex. The adaptation to this situation, tedious as it is, poses no problem, and the results we shall use are summarized in the following Lemma:

Lemma 3.1. (Hamiltonians and Lagrangians) *Assume that the Hamiltonian $\mathbf{h}$ is concave and upper semicontinuous. Then its Lagrangian $\mathbf{l}$ defined by*

$$\forall u, \quad \mathbf{l}(u) := \sup_{p \in \text{Dom}(\mathbf{h})} [\mathbf{h}(p) - \langle p, u \rangle] \quad (5)$$

is lower semicontinuous and convex. The Hamiltonian is related to the Lagrangian by the relation

$$\forall p, \quad \mathbf{h}(p) = \inf_{u \in \text{Dom}(\mathbf{l})} [\mathbf{l}(u) + \langle p, u \rangle] \quad (6)$$

so that the Fenchel inequality

$$\forall u, \forall p, \quad \langle p, u \rangle \leq \mathbf{l}(u) - \mathbf{h}(-p)$$

always holds true. Furthermore, $\bar{u}$ achieves the minimum in the minimization problem $\mathbf{h}(p) = \mathbf{l}(\bar{u}) + \langle p, \bar{u} \rangle$ if and only if $\bar{u} \in \partial_+ \mathbf{h}(-p)$ and $\bar{p}$ achieves the maximum in $\mathbf{l}(u) := \mathbf{h}(\bar{p}) - \langle \bar{p}, u \rangle$ if and only if $\bar{p} \in \partial_- \mathbf{l}(u)$. The three following statements are thus equivalent:

$$\begin{cases} \text{(i)} & \langle \bar{p}, \bar{u} \rangle \geq \mathbf{l}(\bar{u}) - \mathbf{h}(-\bar{p}) \\ \text{(ii)} & \bar{u} \in \partial_+ \mathbf{h}(-\bar{p}) \\ \text{(iii)} & \bar{p} \in \partial_- \mathbf{l}(\bar{u}) \end{cases} \quad (7)$$

Moreover, whenever $\mathbf{h}_i$ is upper semicontinuous and concave (or $\mathbf{l}_i$ is lower semicontinuous and convex), $i = 1, 2$, then

$$\mathbf{h}_1 \leq \mathbf{h}_2 \quad \text{if and only if } \mathbf{l}_1 \leq \mathbf{l}_2$$

(See [3] or [14] for convex analysis among many other books).

Using extended functions (taking infinite values) is mandatory because solutions of Hamilton–Jacobi partial differential equation may take infinite values and may be non differentiable. However, they are viscosity solutions in the weak sense of Barron–Jensen/Frankowska solutions (see [8, 12, 13], which are lower semicontinuous “viscosity solutions” (see [9, 10]). More details can be found in [5] and the book [6].

4. A Lax–Hopf formula for viable capture basins

We shall define the epigraph of viability solution to a trajectory-valued Hamilton–Jacobi problem as a viable capture basin, one of the main concepts studied in viability theory:

Definition 4.1. (*Capture basin of a target viable in an environment*) Let $K \subset X$ be an environment, $C \subset K$ a target in this environment and differential inclusion $x'(t) \in F(x(t))$. The capture basin $\text{Capt}_F(K, C)$ is the subset of initial states $x \in K$ from which starts at least one evolution $x(\cdot)$ to differential inclusion $x'(t) \in F(x(t))$ such that, for some finite time $t^* \geq 0$, this evolution

1. reaches the target C at time t^* at $x(t^*) \in C$;
2. is viable in K on the interval $[0, t^*]$ ($\forall t \in [0, t^*], x(t) \in K$).

For constant differential inclusions $x'(t) \in G$, we can obtain simple formulas of the capture basins $\text{Capt}_G(K, C)$ when G is a closed convex subset, and even for non convex right hand sides G for which there exists a closed convex cone P such that $G - P$ is closed and convex.

If H is a closed subset, we denote by

$$\mathbb{R}_+H = \bigcup_{\lambda \geq 0} \lambda H \text{ the cone spanned by the subset } H$$

We shall need the following

Lemma 4.2. Let H be a compact subset and P a closed convex cone. If $\text{co}(H) \cap P \neq \emptyset$, then

$$\mathbb{R}_+\text{co}(H) - P = \mathbb{R}_+(\text{co}(H - P)) \text{ is closed} \quad (8)$$

Consequently, if $G := H - P$ is closed and convex, then

$$\mathbb{R}_+\text{co}(H) - P = \mathbb{R}_+G \quad (9)$$

Proof. First, we observe that $\mathbb{R}_+\text{co}(H) - P = \mathbb{R}_+\text{co}(H - P)$ because P is a closed convex cone. The subset H being assumed to be compact, its convex hull $\text{co}(H) = \overline{\text{co}(H)}$ is also compact. We have to prove that if $\text{co}(H) \cap P = \emptyset$, then $\mathbb{R}_+(\text{co}(H) - P)$ is closed. For that purpose, let us take a sequence $\lambda_n > 0, x_n \in \text{co}(H)$ such that the sequence $y_n := \lambda_n x_n - p_n \in \text{co}(H) - P$

converges to some y . Since x_n ranges over a compact subset, a subsequence (again denoted by) x_n converges to some $x \in \text{co}(H)$. Next, let us prove that the sequence λ_n is bounded. If not, there would exist a subsequence (again denoted by) λ_n going to $+\infty$. Dividing by λ_n , we infer that $x_n = \frac{y_n}{\lambda_n} + q_n$ where $q_n := \frac{p_n}{\lambda_n} \in P$. Since $\frac{y_n}{\lambda_n}$ converges to 0, we infer that q_n converges to $q = x \in P$, because P is closed. Hence $x \in \text{co}(H) \cap P$, which is impossible. Therefore, the sequence λ_n being bounded, there exists a subsequence (again denoted by) λ_n converging to some $\lambda \geq 0$. Consequently, $p_n = \lambda_n x_n - y_n$ converges to some $p = \lambda x - y \in P$, so that y belongs to $\text{co}(H) - P$, which is then closed. $\square$

Remark. By taking $P := \{0\}$ and assuming that H is a compact convex subset which does not contain $\{0\}$, the cone $\mathbb{R}_+ H$ spanned by H is a closed convex cone. Such a subset H is called a *sole* of the closed convex cone $\mathbb{R}_+ H$. $\square$

We now state and prove the Lax–Hopf formula for capture basins:

Theorem 4.3. (Lax–Hopf formula for capture basins) *Assume that the target C is contained in the environment K . The capture basin satisfies the inclusion*

$$\text{Capt}_F(K, C) \subset K \cap (C - \mathbb{R}_+ \overline{\text{co}}(\text{Im}(F))) \quad (10)$$

If the set-valued map $F(x) := G$ is constant, then

$$K \cap (C - \mathbb{R}_+ G) \subset \text{Capt}_G(\overline{\text{co}}(K), C) \quad (11)$$

Consequently, if K is a closed convex subset, $C \subset K$ is closed and G is a constant set-valued map with a closed convex image G , then the capture basin enjoys the Lax–Hopf formula

$$\text{Capt}_G(K, C) = K \cap (C - \mathbb{R}_+ G) \quad (12)$$

Proof. First, let us consider an element $x \in \text{Capt}_F(K, C)$. Then x belongs to K and there exist a solution $x(\cdot)$ to the differential inclusion $x'(t) \in F(x(t))$ and $T \geq 0$ such that $x(T) \in C$ and $\forall t \in [0, T]$, $x(t) \in K$.

Hence

$$\frac{x(T) - x}{T} \in \frac{1}{T} \int_0^T F(x(t)) dt \subset \frac{1}{T} \int_0^T \text{Im}(F) dt = \overline{\text{co}}(\text{Im}(F)) \quad (13)$$

This implies that

$$x = x(T) - T \frac{x(T) - x}{T} \in C - T \overline{\text{co}}(\text{Im}(F)) \subset C - \mathbb{R}_+ \overline{\text{co}}(\text{Im}(F))$$

On the other hand, if the set-valued map $F(x) \equiv G$ is constant, let us take $x \in \text{co}(K) \cap (C - \mathbb{R}_+ G)$. Hence, there exist $g \in G, T \geq 0$ and $\xi \in C$ such that $x = \xi - Tg$. The evolution $x(\cdot) : t \mapsto x(t) := x + tg$ is a solution to differential equation $x'(t) = g \in G$ starting at $x \in K$ and satisfying $x(T) := x + Tg = \xi \in C$. It is viable in $\overline{\text{co}}(K)$. Indeed, since $x \in K$ and since $\xi = x + Tg \in C \subset K$, then $x(t) := x + tg = (1 - \frac{t}{T})x + \frac{t}{T}(x + Tg) \in \overline{\text{co}}(K)$. This means that x belongs to the capture basin $\text{Capt}_F(\overline{\text{co}}(K), C)$.

The last statement follows from inclusions (10), p. 192 and (11), p. 192 when K is assumed convex and the constant set-valued map G is closed and convex. $\square$

We infer from these definitions the following consequence of Lax–Hopf formula (12), p. 192:

Theorem 4.4. (Lax–Hopf formula for the sum of two targets) *Let P be a closed convex cone contained in K . Assume that H is a compact subset which does not contain 0 and satisfies $\text{co}(H) \cap P \neq \emptyset$. Then, for any target $C \subset K$,*

$$\text{Capt}_H(K, C) + P \subset \text{Capt}_{\text{co}(H-P)}(K + P, C + P) \subset \text{Capt}_H(K + P, C + P)$$

Consequently, if $G := H - P$ is closed and convex, if the closed convex environment K is equal to $K + P$ and if the closed target C is equal to $C + P$, then

$$\text{Capt}_H(K, C) = \text{Capt}_{\text{co}(H-P)}(K + P, C + P) = \text{Capt}_H(K, C) + P \quad (14)$$

Proof. Formula (10), p. 192 of Theorem 4.3, p. 192 implies that

$$\begin{cases} \text{Capt}_H(K, C) + P \subset K \cap (C - \mathbb{R}_+ \overline{\text{co}}(H)) + P \\ \subset (K + P) \cap (C - (\mathbb{R}_+ \overline{\text{co}}(H) - P)) \end{cases}$$

By formula (8), p. 191 of Lemma 4.2, p. 191, we infer that $\mathbb{R}_+ \overline{\text{co}}(H) - P = \mathbb{R}_+ \text{co}(H - P)$, and thus, by formula (12), p. 192, that

$$\text{Capt}_H(K, C) + P \subset (K + P) \cap (C - \mathbb{R}_+ \text{co}(H - P)) = \text{Capt}_{\text{co}(H-P)}(K + P, C)$$

On the other hand, by formula (11), p. 192, we obtain

$$\begin{cases} \text{Capt}_{\text{co}(H-P)}(K + P, C) = (K + P) \cap (C - \mathbb{R}_+(H - P)) \\ = (K + P) \cap (C + P - \mathbb{R}_+(H)) \subset \text{Capt}_H(K + P, C + P) \end{cases}$$

We thus derived inclusions

$$\text{Capt}_H(K, C) + P \subset \text{Capt}_{\text{co}(H-P)}(K + P, C) \subset \text{Capt}_H(K + P, C + P)$$

Consequently, if $K = K + P$ and $C = C + P$ are convex, then

$$\text{Capt}_H(K, C) \subset \text{Capt}_H(K, C) + P \subset \text{Capt}_{\text{co}(H-P)}(K, C) \subset \text{Capt}_H(K, C) \quad (15)$$

which implies that equation (15), p. 193 holds true. $\square$

We deduce that the map $A \mapsto \text{Capt}_H(K, A)$ is a Max-Plus morphism:

Theorem 4.5. (Max-Plus morphism) *Let P be a closed convex cone contained in K and H a constant subset. Assume that $H - P$ is closed and convex and that $K = K + P$ and $C = C + P$ (where $C \subset K$). Then the map $A \mapsto \text{Capt}_H(K, A)$ satisfies*

$$\begin{cases} \text{(i)} & \text{Capt}_H(K, A \cup B) = \text{Capt}_H(K, A) \cup \text{Capt}_H(K, B) \\ \text{(ii)} & \text{Capt}_H(K, C + A) = \text{Capt}_H(K, C) + A \end{cases} \quad (16)$$

Proof. The first statement is obvious. The second one follows from the Lax–Hopf formula. Observe first that if $C = C + P$, then, for any subset A , $C + A = C + A + P$ because $(C + A) + P = (C + P) + A = C + A$. Therefore, formula (15), p. 193 of Theorem 4.4, p. 193 implies that

$$\text{Capt}_H(K, C + A) = \text{Capt}_{H-P}(K, C + A)$$

Since $H - P$ is closed and convex by assumption, formula (12), p. 192 implies that

$$\begin{cases} \text{Capt}_{H-P}(K, C + A) = (C + A) - \mathbb{R}_+(H - P) \\ = (C - \mathbb{R}_+(H - P)) + A = \text{Capt}_{H-P}(K, C) + A \end{cases}$$

and thus, by (15), p. 193 again, that

$$\text{Capt}_H(K, C + A) = \text{Capt}_H(K, C) + A$$

□

5. The viability solution

We associate with partial differential equation (4)(i), p. 190 and the Lagrangian **1** the *characteristic system*

$$\begin{cases} \text{(i)} & \tau'(t) = -1 \\ \text{(ii)} & x'(t) = -u(t) \\ \text{(iii)} & y'(t) = -\mathbf{l}(u(t)) \\ & \text{where } u(t) \in \text{Dom}(\mathbf{l}) \end{cases} \quad (17)$$

controlled by celerities $u(\cdot)$.

Set $H := \{-1\} \times (-\text{Graph}(\mathbf{l})) \subset \mathbb{R}_+ \times X \times \mathbb{R}$. The characteristic system is the differential inclusion $(\tau'(t), x'(t), y'(t)) \in H$ with constant right hand side. This will allow us to use the Lax–Hopf formula for capture basins and its consequences exposed in Sect. 4, p. 191 because we shall define the solution of the trajectory-valued problem as a capture basin.

The epigraph $\mathcal{E}p(\mathbf{c}) \subset \mathbb{R}_+ \times X \times \mathbb{R}$ of an extended function $\mathbf{c} : (t, x) \in \mathbb{R}_+ \times X \mapsto \mathbf{c}(t, x) \in \mathbb{R} \cup \{+\infty\}$ is the subset of triples (t, x, y) such that $\mathbf{c}(t, x) \leq y$. Epigraphs of functions are thus subsets such that

$$\mathcal{E}p(\mathbf{c}) = \mathcal{E}p(\mathbf{c}) + \{0\} \times \{0\} \times \mathbb{R}_+ = \mathcal{E}p(\mathbf{c}) + \mathcal{P} \text{ where } \mathcal{P} := \{0\} \times \{0\} \times \mathbb{R}_+$$

Recall that an extended function is convex (resp. lower semicontinuous, positively homogeneous) if and only if its epigraph is convex (resp. closed, a cone).

We take for environment $K := \mathbb{R}_+ \times X \times \mathbb{R}$ and for target $C := \mathcal{E}p(\mathbf{c})$.

Definition 5.1. (*Viability solution*) Let us consider the epigraph $\mathcal{E}p(\mathbf{c})$ of the function $\mathbf{c}$. The *viability solution (associated with the trajectory condition $\mathbf{c}$)* V (or $V_{\mathbf{c}}$ when the reference to the trajectory condition is useful) to problem (4), p. 190 is defined by the following formula

$$V(T, x) := \inf_{(T, x, y) \in \text{Capt}_{(17)}(\mathbb{R}_+ \times X \times \mathbb{R}, \mathcal{E}p(\mathbf{c}))} y \quad (18)$$

We recall that the viability solution, when it is differentiable, is a solution to the Hamilton–Jacobi equation satisfying the trajectory conditions. Otherwise, when it is not differentiable, but only lower semicontinuous, we can give a meaning to a solution as a solution in the Barron–Jensen/Frankoska sense, using for that purpose subdifferential of lower semicontinuous functions defined in non-smooth analysis [5, 6]. This is not that important for two reasons: all other properties of viability solutions that are proven in this paper

FIGURE 1. Viability solution. The figure illustrates Definition 5.1, p. 194 of the viability solution for a Lagrangian condition $t \mapsto \mathbf{c}(t, \gamma(t))$ defined on a nominal $t \mapsto \gamma(t)$ in the case when $\nu^\# = 0$. Its epigraph is the capture basin of the epigraph of the Lagrangian condition under system (17), p. 194, which is mentioned in the figure. The point $(T, x, y) + (-1, -u, -l(u))$ is represented as well as the element $(T - t^*, x - t^*u, y - t^*l(u)) \in \mathcal{E}p(c)$ where an evolution reaches the epigraph of the Lagrangian condition

are derived directly from the properties of capture basins without using the concept of derivatives, usual or generalized. In particular, the fact that the viability solution is a solution to the Hamilton–Jacobi equation derives from the tangential conditions characterization of viable-capture basins provided by the Viability Theorem (see [4] for details which are not duplicated in this study, for instance).

For proving that the viability solution is lower semicontinuous, i.e., that the capture basin $\text{Capt}_{(17)}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathcal{E}p(\mathbf{c}))$ characterizing it is closed. This will be easier to prove thanks to the Lax–Hopf formula, which also provides a simpler formula of the viability solution.

The independence of the Hamiltonian $\mathbf{h}(p)$ on (t, x) and the convexity of the Lagrangian imply the *Lax–Hopf formula* for partial differential equations (Fig. 1).

Theorem 5.2. (The Lax–Hopf formula for the viability solution) *Assume that the Hamiltonian h is concave and upper semicontinuous. Then the viability solution is lower semicontinuous and is given by the following formula:*

$$V(T, x) = \inf_{d \in [0, T]} \inf_{\xi \in \mathbf{C}(d)} \left[\mathbf{c}(d, \xi) + (T - d) \mathbf{l} \left(\frac{x - \xi}{T - d} \right) \right] \quad (19)$$

which can also be written in the following form:

$$V(T, x) = \inf_{s \in [0, T]} \inf_{u \in \text{Dom}(\mathbf{l})} [\mathbf{c}(T - s, x - su) + s\mathbf{l}(u)] \quad (20)$$

Proof. We apply Lax–Hopf formula (see Theorem 4.3, p. 192) in the following context. The state space of variables (t, x, y) is $\mathcal{X} := \mathbb{R}_+ \times X \times \mathbb{R}$, the subset H is equal to $\{-1\} \times (-\times \text{Graph}(\mathbf{l}))$ since the characteristic system (17), p. 194 can be written in the form

$$(\tau'(t), x'(t), y'(t)) \in H$$

The subset H is compact, thanks to Proposition 7.3, p.202, but not convex: we cannot use the Viability Theorem. However, setting $P := \{0\} \times \{0\} \times \mathbb{R}_+$ (which is a closed convex cone),

$$G := H - P = \{-1\} \times (-\mathcal{E}p(\mathbf{l})) = \{-1\} \times (-\mathcal{E}p(\mathbf{l})) - \{0\} \times \{0\} \times \mathbb{R}_+$$

is a closed and convex, since the graph of $\mathbf{l}$ is compact and the function $\mathbf{l}$ is convex.

We deduce that

$$G := H - P = \{-1\} \times (-\mathcal{E}p(\mathbf{l})) \text{ is closed and convex}$$

We exploit the fact that, setting $P := \{0\} \times \{0\} \times \mathbb{R}_+$, closed epigraphs $\mathcal{E}p(V)$ of extended functions satisfy $\mathcal{E}p(V) = \mathcal{E}p(V) + \mathbb{R}_+ \times X \times \mathbb{R}_+$.

Formula (15), p. 193 of Theorem 4.4, p. 193 stating that

$$\text{Capt}_H(K, C) = \text{Capt}_{H-P}(K, C) = \text{Capt}_H(K, C) + P$$

implies that the epigraph of the viability epigraphical solution satisfies

$$\begin{cases} \mathcal{E}p(V) := \text{Capt}_{\{-1\} \times (-\text{Graph}(\mathbf{l}))}(\mathcal{E}p(\mathbf{c})) \\ = \text{Capt}_{\{-1\} \times (-\mathcal{E}p(\mathbf{l}))}(\mathcal{E}p(\mathbf{c})) = \text{Capt}_{\{-1\} \times (-\text{Graph}(\mathbf{l}))}(\mathcal{E}p(\mathbf{c})) + \{0\} \\ \times \{0\} \times \mathbb{R}_+ = \mathcal{E}p(\mathbf{c}) + \mathbb{R}_+(\{1\} \times \mathcal{E}p(\mathbf{l})) \end{cases}$$

Consequently, for any $(T, x, y) \in \mathcal{E}p(V)$, there exist $(\tau, \xi, \eta) \in \mathcal{E}p(\mathbf{c})$, $u \in \text{Dom}(\mathbf{l})$ and $s \geq 0$ such that

$$(T, x, y) = (\tau, \xi, \eta) - s(-1, -u, -\mathbf{l}(u) - \pi)$$

is such that

$$y = \eta + s\mathbf{l}(u) + \pi \geq \mathbf{c}(\tau, \xi) + s\mathbf{l}(u) = \mathbf{c}(T - s, x - su) + s\mathbf{l}(u)$$

By taking the infimum over $y \geq V(T, x)$, $s \in [0, T]$ and $u \in \text{Dom}(\mathbf{l})$, we deduce that

$$V(T, x) \geq U(T, x) := \inf_{s \in [0, T]} \inf_{u \in \text{Dom}(\mathbf{l})} [\mathbf{c}(T - s, x - su) + s\mathbf{l}(u)]$$

Conversely, for any $\varepsilon > 0$, there exist $s_\varepsilon \in [0, T]$ and $u_\varepsilon \in \text{Dom}(\mathbf{l})$,

$$U(T, x) \leq \mathbf{c}(T - s_\varepsilon, x - s_\varepsilon u_\varepsilon) + s_\varepsilon \mathbf{l}(u_\varepsilon) + \varepsilon \leq V(T, x) + \varepsilon$$

so that, letting ε converge to 0, we derive that $V(T, x) = U(T, x)$, i.e., formula (20), p. 196. By making the change of variables $d = T - s \in [0, T]$, where d can be regarded as a departure date whenever $s \in [0, T]$ is regarded as a travel

time, and $\xi = x - (T - d)u$, regarded as a departure position, the Lax–Hopf formula reads

$$V(T, x) = \inf_{d \in [0, T]} \inf_{\xi \in \mathbf{C}(d)} \left[\mathbf{c}(d, \xi) + (T - d)\mathbf{l} \left(\frac{x - \xi}{T - d} \right) \right] \quad (21)$$

Since the capture basin is closed, we deduce from Definition 5.1, p. 194 that it is equal to the epigraph $\mathcal{E}p(V) := \text{Capt}_H(\mathcal{E}p(\mathbf{c}))$ of the *viability solution* V to Hamilton–Jacobi problem (4), p. 190. $\square$

We shall use this Lax–Hopf formula for deriving in Sect. 7, p. 201 explicit formulas and estimates of the viability solution for Cauchy and Lagrangian trajectory conditions.

The question arises to know precisely the domains $\text{Dom}(V(t, \cdot))$ of the viability solution, i.e., the set of states x such that $V(t, x) < +\infty$: we shall prove that it is couched in terms of the set-valued map $\mathbf{C}$:

Theorem 5.3. (Domain of the viability solution) *Assume that the Hamiltonian $\mathbf{h}$ is a Hamiltonian defined in Definition 7.1, p. 201. For any $t \geq 0$, the domains of the viability solutions $V(t, \cdot)$ associated with the trajectory condition $\mathbf{c}$ are equal to*

$$\text{Dom}(V(t, \cdot)) = \bigcup_{s \in [0, t], u \in \text{Dom}(\mathbf{l})} (\mathbf{C}(t - s) + su) \quad (22)$$

If we assume furthermore that

$$\forall t \geq 0, \quad \mathbf{C}(t) \subset \mathbf{C}(0) + t\text{Dom}(\mathbf{l}) \quad (23)$$

then

$$\text{Dom}(V(t, \cdot)) = \mathbf{C}(0) + t\text{Dom}(\mathbf{l}) \quad (24)$$

Proof. For any $x \in \text{Dom}(V(t, \cdot))$ and any $\varepsilon > 0$, there exist $s \in [0, t], u \in \text{Dom}(\mathbf{l})$ such that

$$\mathbf{c}(t - s, x - su) + s\mathbf{l}(u) \leq V(t, x) + \varepsilon < +\infty$$

so that $x - su \in \mathbf{C}(t - s)$, and thus,

$$\text{Dom}(V(t, \cdot)) \subset \mathbf{C}(t - s) + su \subset \bigcup_{s \in [0, t], u \in \text{Dom}(\mathbf{l})} (\mathbf{C}(t - s) + su)$$

Conversely, let us take $x \in \bigcup_{s \in [0, t], u \in \text{Dom}(\mathbf{l})} (\mathbf{C}(t - s) + su)$, and thus, take $s \in [0, t]$ and $u \in \text{Dom}(\mathbf{l})$ such that $x \in \mathbf{C}(t - s) + su$. Therefore, $\mathbf{c}(t - s, x - su) < +\infty$ is finite and

$$V(t, x) \leq \mathbf{c}(t - s, x - su) + s\mathbf{l}(u) < +\infty$$

so that $x \in \text{Dom}(V)(t, \cdot)$.

As a particular case,

$$\mathbf{C}(0) + t\text{Dom}(\mathbf{l}) \subset \bigcup_{s \in [0, t], u \in \text{Dom}(\mathbf{l})} (\mathbf{C}(t - s) + su) = \text{Dom}(V(t, \cdot))$$

Assume now that the tube $\mathbf{C}$ satisfies (23), p. 197. Take any $x \in \text{Dom}(V(t, \cdot))$, with which we associate $s \in [0, t]$, $\xi \in \mathbf{C}(t - s)$ and $u \in \text{Dom}(\mathbf{I})$ such that $x = \xi + su$ thanks to (22), p. 197. Property 23) implies that

$$\xi \in \mathbf{C}(t - s) \subset \mathbf{C}(0) + (t - s)\text{Dom}(\mathbf{I})$$

so that there exists $v \in \text{Dom}(\mathbf{I})$ such that $\xi \in \mathbf{C}(0) + (t - s)v$. Hence $x \in \mathbf{C}(0) + su + (t - s)v$. Since $\text{Dom}(\mathbf{I})$ is convex and since u and v belong to it and $t - s \geq 0$, then $su + (t - s)u = tw$ where $w \in \text{Dom}(\mathbf{I})$. Hence x belongs to $\mathbf{C}(0) + t\text{Dom}(\mathbf{I})$ and thus,

$$\text{Dom}(V(t, \cdot)) \subset \mathbf{C}(0) + t\text{Dom}(\mathbf{I})$$

This completes the characterization of the domain of the viability solution. $\square$

Lemma 5.4. (Properties of the domain tube) *Let us assume that property (23) holds true. Then, for any $u \in \text{Dom}(\mathbf{I})$ such that $(\mathbf{C}(0) + tu) \setminus \mathbf{C}(t) \neq \emptyset$ and for any $x \in (\mathbf{C}(0) + tu) \cap \overline{\mathbf{C}(t)}$, there exists $\bar{s} \in [0, t]$ such that $x \in \partial\mathbf{C}(t - \bar{s}) + \bar{s}u$.*

Proof. Let $u \in \text{Dom}(\mathbf{I})$ and $t \geq 0$ such that there exists $x \in (\mathbf{C}(0) + tu) \setminus \mathbf{C}(t)$. We introduce the subsets $E(u)^- := \{s \in [0, t] \text{ such that } x - su \in \mathbf{C}(t - s)\}$ and $E(u)^+ := \{s \in [0, t] \text{ such that } x - su \in \overline{\mathbf{C}(t - s)}\}$.

Since the domain of the function $\mathbf{c}$ is assumed to be closed, i.e., that the graph of the tube $\mathbf{C}$ is closed, so are the subsets $E(u)^-$ and $E(u)^+$. By definition, $[0, t] = E(u)^- \cup E(u)^+$. We observe that 0 belongs to $E(u)^+$ since we have assumed that $x \in \overline{\mathbf{C}(t)}$. On the hand, since $x \in \mathbf{C}(0) + tu$, we deduce that $t \in E(u)^-$. The interval $[0, t]$ being connected and covered by the union of the two non empty closed subsets $E(u)^-$ and $E(u)^+$, we infer that there exists some $\bar{s} \in E(u)^- \cap E(u)^+$, i.e., such that $x - \bar{s}u \in \partial\mathbf{C}(t - \bar{s})$. This completes the proof of Lemma 5.4, p. 198. $\square$

6. Inf-convolution

We begin by recalling few words about inf-convolutions:

Definition 6.1. (*Inf-Convolution and decreasing envelopes of functions*) The inf-convolution $\star_i \mathbf{u}_i$ of functions is defined by

$$\mathcal{E}p(\star_i \mathbf{u}_i) := \sum_i \mathcal{E}p(\mathbf{u}_i)$$

One observe immediately that

$$(\star_i \mathbf{u}_i)(x) \geq \inf_{x = \sum_i x_i} \sum_i \mathbf{u}_i(x_i)$$

Equality in this formula holds true whenever the functions $\mathbf{u}_i$ are lower semi-continuous and inf-compact. This implies that the function $\star_i(\mathbf{c}_i)$ is lower semicontinuous and that

$$\star_i \mathbf{u}_i(x) := \inf_{\sum_i x_i = x} \sum_i \mathbf{u}_i(x_i)$$

We see at once that the inf-convolution of convex functions is convex. The inf-convolution of lower semicontinuous is lower semicontinuous only under adequate conditions.

Lemma 6.2. (Lower semicontinuity of inf-convolutions) *Setting*

$$D := \{(p)_{j=1,\dots,J}\}_{p \in X^*} \subset \prod_{j=1}^J X^*$$

the inf-convolution $\star_{j=1}^J \mathbf{v}_j$ of lower semicontinuous $\mathbf{v}_j$ functions is lower semicontinuous whenever assumption

$$0 \in \text{Int} \left(D + \prod_{j=1}^J \text{Dom}(\mathbf{v}_j^*) \right) \quad (25)$$

holds true. Furthermore, if the functions $\mathbf{v}_j$ are inf-compact, so is their weighted inf-convolution $\star_{j=1}^J \mathbf{v}_j$ and there exist J elements $\bar{x}_j \in X_j$ such that

$$\sum_{j=1}^J \bar{x}_j = x \text{ and } \mathbf{v}(x) = \sum_{j=1}^J \mathbf{v}_j(\bar{x}_j)$$

In the case of two functions, we obtain

$$(\mathbf{u} \star \mathbf{v})(x) = \inf_y (\mathbf{u}(y) + \mathbf{v}(x - y))$$

from which the name of the operation is derived (when $\inf_y$ is replaced by $\int_y$ for the usual convolution in analysis). We refer to monographs on convex analysis (for instance, [3] or [14]) for more details.

There are many examples of “inf-convolution” operators $\mathbf{u} \mapsto \mathbf{v} \star \mathbf{u}$ by a function $\mathbf{v}$.

1. **Pasch–Hausdorff envelopes** (or Lipschitz regularization). This example is provided by the function $x \mapsto \lambda \| \cdot \| (x) := \lambda \|x\|$, the epigraph of which is a closed convex cone. We observe that a function $\mathbf{u}$ is λ -Lipschitz if and only if $\mathbf{u} = \lambda \| \cdot \| \star \mathbf{u}$. The function $\lambda \| \cdot \| \star \mathbf{u}$ defined by

$$(\lambda \| \cdot \| \star \mathbf{u})(x) := \inf_y (\mathbf{u}(y) + \lambda \|x - y\|)$$

is called the λ -Pasch–Hausdorff envelope of $\mathbf{u}$ and can be regarded as the λ -Lipschitz regularization of $\mathbf{u}$. Indeed,

$$\begin{cases} (\lambda \| \cdot \| \star \mathbf{u})(x_1) := \inf_y (\mathbf{u}(y) + \lambda \|x_1 - x_2 + x_2 - y\|) \\ \leq \inf_y (\mathbf{u}(y) + \lambda \|x_1 - x_2 + x_2 - y\|) + \lambda \|x_1 - x_2\| \\ = (\lambda \| \cdot \| \star \mathbf{u})(x_2) + \lambda \|x_1 - x_2\| \end{cases}$$

is Lipschitz with constant λ ;

2. **Decreasing envelopes** Let us consider a closed convex cone $P \subset X$. It defines an order relation by setting $y \geq x$ if and only if $y \in x + P$.

On the other hand, let us associate with the cone P its indicator function ψ_P defined by $\psi_P(x) = 0$ if $x \in P$ and $\psi_P(x) = +\infty$ otherwise. Therefore, $\mathcal{E}p(\psi_P) = P \times \mathbb{R}_+$. Consequently,

$$\mathcal{E}p(\psi_P \star \mathbf{u}) = \mathcal{E}p(\mathbf{u}) + P \times \mathbb{R}_+$$

We observe that the two following statements are equivalent:

- $\mathbf{u}$ is *decreasing* in the sense that for any $y \in x + P$, then $\mathbf{u}(y) \leq \mathbf{u}(x)$
- The epigraph of the function $\mathbf{u}$ satisfies

$$\mathcal{E}p(\mathbf{u}) + P \times \mathbb{R}_+ = \mathcal{E}p(\mathbf{u})$$

In this case, $\text{Dom}(\mathbf{u}) = \text{Dom}(\mathbf{u}) + P$. The function $\mathbf{u}_{\searrow}$ defined by

$$\mathcal{E}p(\mathbf{u}_{\searrow}) := \mathcal{E}p(\mathbf{u}) + P \times \mathbb{R}_+$$

is called the *P-decreasing envelope* of the function $\mathbf{u}$ (or simply *decreasing envelope* if there is no ambiguity). The decreasing envelope $\mathbf{u}_{\nearrow}$ of $\mathbf{u}$ is larger than or equal to

$$\inf_{p \in P} \mathbf{u}(x - p) \leq \mathbf{u}_{\searrow}(x) \text{ and } \text{Dom}(\mathbf{u}_{\searrow}) \subset \text{Dom}(\mathbf{u}) + P$$

Equality

$$\inf_{p \in P} \mathbf{u}(x - p) = \mathbf{u}_{\searrow}(x) \text{ and } \text{Dom}(\mathbf{u}_{\searrow}) = \text{Dom}(\mathbf{u}) + P \quad (26)$$

holds true whenever $\mathbf{u}$ is lower semicontinuous and inf-compact.

For any family $\{\mathbf{u}_i\}_{i \in I}$ of extended functions, the decreasing envelope of the infimum is the infimum of their decreasing envelopes:

$$(\inf_{i \in I} \mathbf{u}_i)_{\searrow} = \inf_{i \in I} \mathbf{u}_{i_{\searrow}} \quad (27)$$

Theorem 4.5, p. 193 using the Lax–Hopf formula, once translated in terms of epigraphs, implies very useful properties on viability solutions. The epigraph of the minimum $\mathbf{u} := \min_{i \in I} \mathbf{u}_i$ being obviously the union $\mathcal{E}p(\mathbf{u}) = \bigcup_{i \in I} \mathcal{E}p(\mathbf{u}_i)$ of their epigraphs and the epigraph of their inf-convolution $\mathbf{u} := \star_{i \in I} \mathbf{u}_i$ being the sum $\mathcal{E}p(\mathbf{u}) = \sum_{i \in I} \mathcal{E}p(\mathbf{u}_i)$ of their epigraphs by Definition 6.1, p. 198, we obtain

Theorem 6.3. (The min inf-convolution morphism) *Assume that the Hamiltonian $\mathbf{h}$ is a Hamiltonian defined in Definition 7.1, p. 201. Let us consider a finite family of trajectory conditions $\mathbf{c}_i, i \in I$. We denote their viability solutions by $V_{\mathbf{c}_i}$.*

1. *The viability solution associated to the infimum $\min_i(\mathbf{c}_i)$ is the infimum of the viability solutions associated with trajectory condition $\mathbf{c}_i$:*

$$V_{\min_i(\mathbf{c}_i)} = \min_i(V_{\mathbf{c}_i})$$

This allows us to study the contribution of each trajectory condition whenever the trajectory condition is their infimum.

2. *The viability solution associated with the inf-convolution $\star_i \mathbf{c}_i$ of trajectory conditions defined by*

$$\mathcal{E}p(\star_i \mathbf{c}_i) := \sum_i \mathcal{E}p(\mathbf{c}_i)$$

is the inf-convolution of the viability solutions associated with trajectory condition $\mathbf{c}_i$:

$$V_{\star_i(\mathbf{c}_i)} = \star_i(V_{\mathbf{c}_i})$$

Proof. These formulas are consequences of Theorem 4.5, p. 193 stating that the capture basin is a Max Plus morphism.

1. We recall that

$$\mathcal{E}p(\mathbf{c}_{\min_I}) = \bigcup_{i \in I} \mathcal{E}p(\mathbf{c}_i)$$

Since the union of the epigraphs of functions $\mathbf{u}_i$ is the epigraph of their infimum, we deduce that if a finite number of trajectory conditions $\mathbf{c}_i$ are given and denoting their viability solutions by $V_{\mathbf{c}_i}$, we infer that $V_{\min_i(\mathbf{c}_i)} = \min_i(V_{\mathbf{c}_i})$.

2. Since the sum of the epigraphs of functions $\mathbf{u}_i$ is the epigraph of their inf-convolution $\star_i \mathbf{u}_i$ defined by

$$\star_i \mathbf{u}_i(x) := \inf_{\sum_i x_i = x} \sum_i \mathbf{u}_i(x_i)$$

we deduce that if a finite number of trajectory conditions $\mathbf{c}_i$ are given and denoting their viability solutions by $V_{\mathbf{c}_i}$, we infer that $V_{\star_i(\mathbf{c}_i)} = \star_i(V_{\mathbf{c}_i})$.

This completes the proof. $\square$

7. One-dimensional example

We choose $X := \mathbb{R}$ and a concave Hamiltonians $\mathbf{h} : \mathbb{R} \mapsto \mathbb{R}$. We shall classify them according to characteristic parameters $(\nu^b, \nu^\sharp, \omega, \delta)$ where

- at $p = 0$, $\mathbf{h}(0) = 0$ and $\mathbf{h}'(0) := \nu^b \geq 0$;
- at $p = \omega$, $\mathbf{h}(\omega) = 0$ and $\mathbf{h}'(\omega) := -\nu^\sharp$ where $\nu^\sharp \geq 0$ (when $\omega = +\infty$, we assume that it is concave increasing and that $\lim_{p \rightarrow +\infty} h(p) = \delta < +\infty$ and set $\nu^\sharp = 0$);
- we set $\delta := \max_{p \in [0, \omega]} \mathbf{h}(p)$ and denote by $[\beta^b, \beta^\sharp]$ the interval on which $\mathbf{h}$ achieves its maximum.

Definition 7.1. (*Hamiltonian*) We associate with scalars $\nu^b \geq 0$, $\nu^\sharp \geq 0$, $\omega > 0$ and $\delta > 0$ the class of *Hamiltonians* $\mathbf{h} : \mathbb{R} \mapsto \mathbb{R}$ associated with those parameters which is any concave function $\mathbf{h}$ satisfying

$$\begin{cases} \mathbf{h}(p) := \nu^b p & \text{if } p \in]-\infty, 0] \\ \mathbf{h}(p) \in [0, \delta] & \text{if } p \in [0, \omega] \\ \mathbf{h}(p) \in \nu^\sharp(\omega - p) & \text{if } p \in [\omega, +\infty[\end{cases} \quad (28)$$

and

$$\begin{cases} \text{(i)} & \mathbf{h}(0) = 0 \quad \text{and} \quad \mathbf{h}'(0) = \nu^b \\ \text{(ii)} & \mathbf{h}(\omega) = 0 \quad \text{and} \quad \mathbf{h}'(\omega) = -\nu^\sharp \end{cases} \quad (29)$$

FIGURE 2. *Left* This diagram displays the graphs of several Hamiltonians $\mathbf{h}_{(\alpha^\#, \alpha^b, \omega, \delta)}$. *Right* This diagram displays the graphs of the associated Lagrangians $\mathbf{l}_{(\nu^b, \nu^\#, \omega, \delta)}$

Proposition 7.2. (Examples of Hamiltonians and Lagrangians)

Let us consider the trapezoidal Hamiltonian $\mathbf{h}_{(\nu^b, \nu^\#, \omega, \delta)}$ defined by

$$\mathbf{h}_{(\nu^b, \nu^\#, \omega, \delta)}(p) := \begin{cases} \nu^b p & \text{if } p \leq \beta^b \\ \delta & \text{if } p \in [\beta^b, \beta^\#] \\ \nu^\#(\omega - p) & \text{if } p \geq \beta^\# \end{cases} \quad (30)$$

The associated Lagrangian $\mathbf{l}_{(\nu^b, \nu^\#, \omega, \delta)}$ associated with the trapezoidal Hamiltonian is equal to

$$\mathbf{l}_{(\nu^b, \nu^\#, \omega, \delta)}(u) = \begin{cases} \frac{\delta}{\nu^b}(\nu^b - u) & \text{if } u \in [0, \nu^b] \\ \delta - \frac{\omega \nu^\# - \delta}{\nu^\#} u & \text{if } u \in [-\nu^\#, 0] \\ +\infty & \text{if } u \notin [-\nu^\#, \nu^b] \end{cases} \quad (31)$$

It is piecewise affine (affine on $[-\nu^\#, 0]$ and $[0, +\nu^b]$) and satisfies $\mathbf{l}(+\nu^b) = 0, \mathbf{l}(0) = \delta$ and $\mathbf{l}(-\nu^\#) = \omega \nu^\#$.

The Hamiltonians and Lagrangians $\mathbf{l}$ associated with the parameters $\nu^b \geq 0, \nu^\# \geq 0, \omega > 0, \delta > 0$ share common properties (Fig. 2):

Proposition 7.3. (Lagrangians) Let $\mathbf{h}$ be a concave Hamiltonian associated with the parameters $\nu^b > 0, \nu^\# \geq 0, \omega > 0, \delta \leq \bar{\delta}$. Then the associated convex Lagrangian $\mathbf{l}$ satisfies

$$\begin{cases} 0 \leq \mathbf{l}(u) \leq \frac{\omega \nu^\#}{(\nu^b + \nu^\#)}(\nu^b - u) & \text{if } u \in [0, \nu^b] \\ -\omega u \leq \mathbf{l}(u) \leq \frac{\omega \nu^\#}{(\nu^b + \nu^\#)}(\nu^b - u) & \text{if } u \in [-\nu^\#, 0] \\ +\infty & \text{otherwise} \end{cases}$$

Therefore the function $u \mapsto \mathbf{l}(u)$ is decreasing, satisfies

$$0 = \mathbf{l}(\nu^b) \leq \max(0, -\omega u) \leq \mathbf{l}(u) \leq \frac{\omega \nu^\#}{(\nu^b + \nu^\#)}(\nu^b - u) \leq \mathbf{l}(-\nu^\#) = \omega \nu^\#$$

and the function $u \mapsto \frac{\mathbf{l}(u)}{|u|}$ is decreasing on the interval $[0, \nu^b]$, increasing on the interval $[-\nu^\sharp, 0]$, and, consequently, satisfies

$$\begin{cases} 0 = \frac{\mathbf{l}(\nu^b)}{\nu^b} \leq \frac{\mathbf{l}(u)}{|u|} & \text{if } 0 < u \leq \nu^b \\ \omega = \frac{\mathbf{l}(-\nu^\sharp)}{\nu^\sharp} \leq \frac{\mathbf{l}(u)}{|u|} & \text{if } -\nu^\sharp \leq u < 0 \end{cases}$$

Furthermore, $\mathbf{l}(0) = \sup_p \mathbf{h}(p) = \delta$ is the maximal flux and its subdifferential $\partial \mathbf{l}(0)$ its critical interval.

Proof. Proposition 7.2, p. 202 implies that

$$\mathbf{h}_{(\nu^b, \nu^\sharp, \omega, 0)}(p) \leq \mathbf{h}(p) \leq \mathbf{h}_{(\nu^b, \nu^\sharp, \omega, \bar{\delta})}(p)$$

We infer from the definition that

$$\max(0, -\omega u) = \mathbf{l}_{(\nu^b, \nu^\sharp, \omega, 0)}(u) \leq \mathbf{l}(u) \leq \mathbf{l}_{(\nu^b, \nu^\sharp, \omega, \bar{\delta})}(u) = \frac{\omega \nu^\sharp}{(\nu^b + \nu^\sharp)}(\nu^b - u)$$

by (31), p. 202 with $\delta := 0$ and $\delta := \bar{\delta}$ respectively.

Since $\mathbf{h}(0) = 0$ and $\nu^b = \mathbf{h}'(0)$ (or $\nu^b \in \partial_+ \mathbf{h}(0)$), we deduce from the Lemma 3.1, p. 190 that for all u , $\langle 0, \nu^b \rangle = \mathbf{h}(0) - \mathbf{l}(\nu^b) \geq \mathbf{h}(0) - \mathbf{l}(u)$, which boils down to $0 = \mathbf{l}(\nu^b) = \inf_u \mathbf{l}(u)$. In the case when $\omega > 0$ and $\nu^\sharp > 0$ are strictly positive and finite, conditions $\mathbf{h}(\omega) = 0$ and $-\nu^\sharp = \mathbf{h}'(\omega)$ (or $-\nu^\sharp \in \partial_+ \mathbf{h}(\omega)$) imply that

$$\langle -\nu^\sharp, \omega \rangle = \mathbf{h}(\omega) - \mathbf{l}(-\nu^\sharp) \text{ and } \forall u \in [-\nu^\sharp, \nu^b], \langle \omega, u \rangle \geq \mathbf{h}(\omega) - \mathbf{l}(u) = -\mathbf{l}(u)$$

which can be written

$$\nu^\sharp \omega = \mathbf{l}(-\nu^\sharp) \text{ and } \forall u \in [-\nu^\sharp, \nu^b], -\omega u \leq \mathbf{l}(u)$$

Since $0 = \mathbf{l}(\nu^b) \leq \mathbf{l}(u)$ and since $\mathbf{l}$ is convex, we deduce that $\mathbf{l}$ is decreasing: take any $u \in [-\nu^\sharp, \nu^b]$ and $w := \alpha u + (1 - \alpha)\nu^b \in [u, \nu^b]$ for $\alpha \in [0, 1]$. Therefore $\mathbf{l}(w) \leq \alpha \mathbf{l}(u) + (1 - \alpha)\mathbf{l}(\nu^b) = \alpha \mathbf{l}(u) \leq \mathbf{l}(u)$. Hence, if $w \geq u$, $\mathbf{l}(w) \leq \mathbf{l}(u)$.

Since $\mathbf{l}$ is positive, then, for any $u \in [0, \nu^b]$, we observe that $0 = \frac{\mathbf{l}(\nu^b)}{\nu^b} \leq \frac{\mathbf{l}(u)}{|u|}$. On the other hand, for any $u \in [-\nu^\sharp, 0]$, we deduce from inequalities $-\omega u \leq \mathbf{l}(u)$ that, by dividing by $|u| > 0$, $\omega = \frac{\mathbf{l}(-\nu^\sharp)}{\nu^\sharp} \leq \frac{\mathbf{l}(u)}{|u|}$ since $u \leq 0$.

It is actually easy to check that the function $u \mapsto \frac{\mathbf{l}(u)}{|u|}$ is decreasing on the interval $[0, \nu^b]$ and increasing on the interval $[-\nu^\sharp, 0]$. $\square$

We derive from properties of the Hamiltonians associated with parameters $\nu^b \geq 0$, $\nu^\sharp \geq 0$, $\omega > 0$, $\delta > 0$ exposed in Proposition 7.3, p. 202 for providing

- lower and upper estimates of the viability solution to the Hamilton–Jacobi problem (4), p. 190,
- the domains of the viability solutions $x \mapsto V(t, x)$, which are proven to be equal to $\mathbf{C}(0) + t[-\nu^\sharp, \nu^b]$ whenever we assume that for every $t \geq 0$, $\mathbf{C}(t) \subset \mathbf{C}(0) + t[-\nu^\sharp, \nu^b]$

couched only in terms of the trajectory condition $\mathbf{c}$, its associated set-valued map $\mathbf{C}$ and the four characteristic parameters $\nu^b \geq 0$, $\nu^\sharp \geq 0$, $\omega > 0$ and $\delta > 0$. They are consequently valid for any Hamiltonian $\mathbf{h}$ associated with those parameters.

Proposition 7.4. (Estimates of the viability solution)

Proposition 7.5. (Estimates of the Moskowitz function) *Let us associate with the function $\mathbf{c}$ and parameters $\nu^b \geq 0$, $\nu^\sharp \geq 0$, $\omega > 0$ and $\delta > 0$ the following subset*

$$\mathbf{E}(t, x) := \left\{ (s, u) \in [0, t] \times [-\nu^\sharp, +\nu^b] \text{ such that } x \in \mathbf{C}(t - s) + su \right\} \quad (32)$$

Hence the viability traffic function to Moskowitz traffic problem (4), p. 190 is given by

$$V(t, x) := \inf_{(s, u) \in \mathbf{E}(t, x)} [\mathbf{c}(t - s, x - su) + s\mathbf{l}(u)] \quad (33)$$

Let us set

$$\underline{\mathbf{c}}(t, x) := \inf_{(s, u) \in \mathbf{E}(t, x)} (\mathbf{c}(t - s, x - su) + \max(0, -s\omega u))$$

and

$$\overline{\mathbf{c}}(t, x) := \inf_{(s, u) \in \mathbf{E}(t, x)} \left(\mathbf{c}(t - s, x - su) + s \frac{\omega \nu^\sharp}{(\nu^b + \nu^\sharp)} (\nu^b - u) \right)$$

Then, the viability traffic function V satisfies the estimates

$$\underline{\mathbf{c}}(t, x) \leq V(t, x) \leq \overline{\mathbf{c}}(t, x) \quad (34)$$

Proof. Since the celerity flux function $\mathbf{l}$ satisfies

$$\max(0, -\omega u) \leq \mathbf{l}(u) \leq \frac{\omega \nu^\sharp}{(\nu^b + \nu^\sharp)} (u - \nu^b)$$

thanks to Proposition 7.3, p. 202, then inequality $\underline{\mathbf{c}}(t, x) \leq V(t, x) \leq \overline{\mathbf{c}}(t, x)$ is straightforward from Lax–Hopf formula. $\square$

Let us associate with the function $\mathbf{c}$ and parameters $\nu^b \geq 0$, $\nu^\sharp \geq 0$, $\omega > 0$ and $\delta > 0$ the following subset

$$\mathbf{E}(t, x) := \left\{ (s, u) \in [0, t] \times [-\nu^\sharp, +\nu^b] \text{ such that } x \in \mathbf{C}(t - s) + su \right\} \quad (35)$$

Hence the viability solution to Hamilton–Jacobi trajectory-valued problem (4), p. 190 is given by

$$V(t, x) := \inf_{(s, u) \in \mathbf{E}(t, x)} [\mathbf{c}(t - s, x - su) + s\mathbf{l}(u)] \quad (36)$$

Let us set

$$\underline{\mathbf{c}}(t, x) := \inf_{(s, u) \in \mathbf{E}(t, x)} (\mathbf{c}(t - s, x - su) + \max(0, -s\omega u))$$

and

$$\overline{\mathbf{c}}(t, x) := \inf_{(s, u) \in \mathbf{E}(t, x)} \left(\mathbf{c}(t - s, x - su) + s \frac{\omega \nu^\sharp}{(\nu^b + \nu^\sharp)} (\nu^b - u) \right)$$

Then, the viability solution V satisfies the estimates

$$\underline{\mathbf{c}}(t, x) \leq V(t, x) \leq \bar{\mathbf{c}}(t, x) \quad (37)$$

Proof. Since the Lagrangian $\mathbf{l}$ satisfies

$$\max(0, -\omega u) \leq \mathbf{l}(u) \leq \frac{\omega \nu^\sharp}{(\nu^\flat + \nu^\sharp)}(u - \nu^\flat)$$

thanks to Proposition 7.3, p. 202, then inequality $\underline{\mathbf{c}}(t, x) \leq V(t, x) \leq \bar{\mathbf{c}}(t, x)$ is straightforward from Lax–Hopf formula. $\square$

Remark. Observe that

$$\inf_{(s,u) \in \mathbf{E}(t,x)} \mathbf{c}(t-s, x-su) \leq \underline{\mathbf{c}}(t, x)$$

1. the lower estimate $\underline{\mathbf{c}}$ is the solution to the Hamilton–Jacobi problem (4), p. 190 when the Hamiltonian is defined by:

$$\forall p \in [0, \omega], \quad \mathbf{h}_{(\nu^\flat, \nu^\sharp, \omega, 0)}(p) = 0$$

the conjugate function of which is defined by

$$\forall u \in [-\nu^\sharp, +\nu^\flat], \quad \mathbf{l}_{(\nu^\flat, \nu^\sharp, \omega, 0)}(u) = \max(0, -\omega u)$$

2. the upper estimate $\bar{\mathbf{c}}$ is the solution to the Hamilton–Jacobi problem (4), p. 190 when the Hamiltonian $\mathbf{h}$ is the Hamiltonian

$$\mathbf{h}_{(\nu^\flat, \nu^\sharp, \omega, \bar{\beta})}(p) = \begin{cases} \nu^\flat p & \text{if } p \leq \bar{\beta} \\ \nu^\sharp(\omega - p) & \text{if } p \geq \bar{\beta} \end{cases}$$

the conjugate function of which is the Lagrangian defined by

$$\forall u \in [-\nu^\sharp, +\nu^\flat], \quad \mathbf{l}_{(\nu^\flat, \nu^\sharp, \omega, \bar{\beta})}(u) = \frac{\omega \nu^\sharp}{(\nu^\flat + \nu^\sharp)}(\nu^\flat - u)$$

3. by taking the values $0, -\nu^\sharp$ and $\nu^\flat$, we obtain the further estimate

$$\bar{\mathbf{c}}(t, x) \leq \min \left[\begin{array}{l} \inf_{(s, \nu^\flat) \in \mathbf{E}(t, x)} \mathbf{c}(t-s, x-s\nu^\flat), \\ \inf_{(s, 0) \in \mathbf{E}(t, x)} \mathbf{c}(t-s, x) + s \frac{\omega \nu^\sharp \nu^\flat}{(\nu^\flat + \nu^\sharp)} \\ \inf_{(s, -\nu^\sharp) \in \mathbf{E}(t, x)} (\mathbf{c}(t-s, x+s\nu^\sharp) + s\omega \nu^\sharp) \end{array} \right] \quad \square$$

We provide some examples of trajectory conditions:

1. **Cauchy initial conditions** The *Cauchy initial condition* requires that at initial time $t = 0$, the initial viability solution $\gamma_0(\cdot) : x \mapsto \gamma_0(x) \in \mathbb{R} \cup \{+\infty\}$ is given. We extend the Cauchy trajectory condition by setting $\mathbf{c}(0, x) := \gamma_0(x)$ and $\mathbf{c}(t, x) = +\infty$ whenever $t > 0$. In this case, $\mathbf{C}(0) = \text{Dom}(\gamma_0)$ and $\mathbf{C}(t) = \emptyset$.

Since $\mathcal{E}p(\mathbf{c}) \subset \mathcal{E}p(V)$, we infer that

$$\forall x \in \mathbb{R}, \quad V(0, x) \leq \gamma_0(x)$$

Proposition 7.6. (Cauchy initial conditions) *The viability solution to the Hamilton–Jacobi problem (4), p. 190 with Cauchy condition $\gamma_0(\cdot)$ satisfies actually condition*

$$\forall x \in \mathbb{R}, \quad V(0, x) = \gamma_0(x)$$

The domains of its viability solutions are equal to

$$\forall t \geq 0, \quad \text{Dom}(V(t, \cdot)) = \text{Dom}(\gamma_0) + t[-\nu^\sharp, \nu^\flat]$$

The viability solution is equal to

$$V(t, x) = \inf_{u \in [-\nu^\sharp, \nu^\flat]} (\gamma_0(x - tu) + t\mathbf{l}(u))$$

and satisfies the estimates

$$\begin{cases} \inf_{u \in [-\nu^\sharp, \nu^\flat]} (\gamma_0(x - tu) + t \max(0, -\omega u)) \leq V(t, x) \\ \leq \inf_{u \in [-\nu^\sharp, \nu^\flat]} \left(\gamma_0(x - tu) + t \frac{\omega \nu^\sharp}{(\nu^\flat + \nu^\sharp)} (\nu^\flat - u) \right) \end{cases} \quad (38)$$

Proof. Since 0 belongs to $[-\nu^\sharp, \nu^\flat]$, condition

$$\forall t > 0, \quad \mathbf{C}(t) = \emptyset \subset \mathbf{C}(0) + t[-\nu^\sharp, \nu^\flat]$$

holds true. Then the domains of the associated traffic profiles are equal to

$$\forall t \geq 0, \quad \text{Dom}(V)(t, \cdot) = \mathbf{C}(0) + t[-\nu^\sharp, \nu^\flat] = \text{Dom}(\gamma_0) + t[-\nu^\sharp, \nu^\flat]$$

thanks to Theorem 5.3, p. 197. The estimates follow from Proposition 7.5, p. 204. $\square$

2. **Lagrangian conditions** Lagrangian trajectory conditions are associated with
 - (a) the trajectory of an increasing “nominal” evolution $t \mapsto \gamma(t)$ of a given mobile,
 - (b) a Lagrangian condition $t \mapsto \mathbf{c}(t, \gamma(t))$ defined on this trajectory.

Definition 7.7. (*Lagrangian Conditions on Nominal Evolution*) We say that an evolution $t \mapsto \gamma(t)$ is *consistent* with the controls $\nu^\sharp$ and $\nu^\flat$ if it satisfies the “speed limit”

$$\forall t \geq 0, \quad \gamma'(t) \in [-\nu^\sharp, +\nu^\flat] \quad (39)$$

We observe that a nominal evolution is consistent with the controls $\nu^\flat$ and $\nu^\sharp$ if and only if

$$\forall t \geq 0, \quad \forall s \in [0, t], \quad -s\nu^\sharp \leq \gamma(t) - \gamma(t-s) \leq s\nu^\flat$$

In this case, the associated tube $\mathbf{C}(t) := \{\gamma(t)\}$ satisfies property (23), so that the domain of the viability solution is equal to

$$\forall t \geq 0, \quad \text{Dom}(V(t, \cdot)) = [\gamma(0) - t\nu^\sharp, \gamma(0) + t\nu^\flat]$$

The subset $\mathbf{E}(t, x)$ defined by (35), p. 204 is equal to

$$\mathbf{E}(t, x) := \left\{ (d, u) \in [0, t] \times [-\nu^\sharp, +\nu^\flat] \text{ such that } u = \frac{x - \gamma(t-d)}{d} \right\}$$

Proposition 7.8. (Lagrangian conditions) *Assume that the nominal evolution is consistent with the controls ν^b and $\nu^\sharp$. Then the viability solution to partial differential equation (4), p. 190 associated with Lagrangian trajectory condition $t \mapsto \mathbf{c}(t, \gamma(t))$ is defined by*

$$V(t, x) = \inf_{d \in [0, t]} \left[\mathbf{c}(t - d, \gamma(t - d)) + d \mathbf{l} \left(\frac{x - \gamma(t - d)}{d} \right) \right] \quad (40)$$

and satisfies the estimates

$$\begin{cases} \inf_{d \in [0, t]} (\mathbf{c}(t - d, \gamma(t - d)) + \max(0, \omega(\gamma(t - d) - x))) \leq V(t, x) \\ \leq \inf_{d \in [0, t]} \left(\mathbf{c}(t - d, \gamma(t - d)) + \frac{\omega \nu^\sharp}{(\nu^b + \nu^\sharp)} (s \nu^b + \gamma(t - d) - x) \right) \end{cases} \quad (41)$$

Proof. To say that $(d, u) \in \mathbf{E}(t, x)$ amounts to saying that $\mathbf{E}(t, x)$ is the graph of the map $s \mapsto \frac{x - \gamma(t - d)}{d}$. Since the Lagrangian trajectory condition is defined only of the graph of the nominal evolution, we deduce that $\mathbf{c}(t - d, x - du)$ is finite if and only if $(d, u) \in \mathbf{E}(t, x)$, in which case $u := \frac{x - \gamma(t - d)}{d}$, so that general formula (36), p. 204 boils down to

$$V(t, x) = \inf_{d \in [0, t]} \left[\mathbf{c}(t - d, \gamma(t - d)) + d \mathbf{l} \left(\frac{x - \gamma(t - d)}{d} \right) \right] \quad \square$$

3. **Combined trajectory conditions** We have listed particular trajectory conditions, the Cauchy conditions, as well as the Dirichlet, Eulerian and Lagrangian conditions.

We now turn our attention to the combination of Cauchy and several Lagrangian conditions: the trajectory conditions involve

- a Cauchy condition $\mathbf{c}_0$ such that $\mathbf{c}_0(0, x) = \gamma_0(x)$ and $\mathbf{c}_0(t, x) = +\infty$ whenever $t > 0$
- and/or Lagrangian conditions $\mathbf{c}_i(t, \gamma_i(t))$ satisfying $\mathbf{c}_i(t, x) = +\infty$ whenever $x \neq \gamma_i(t), i \in I$.

We assume that these conditions satisfy

$$\forall i \in I, \begin{cases} \gamma_i(0) \in \text{Dom}(\gamma_0(\cdot)) \\ \gamma_i'(t) \in [-\nu^\sharp, +\nu^b] \end{cases} \quad (42)$$

We introduce the combined trajectory condition

$$\mathbf{c}(t, x) := \min [\gamma_0(x), \mathbf{c}_i(t, x)]$$

The domain $\mathbf{C}(t) := \text{Dom}(\mathbf{c}(t, \cdot))$ of its profile is equal to

$$\mathbf{C}(t) = \begin{cases} \text{Dom}(\gamma_0) & \text{if } t = 0 \\ \bigcup_{i \in I} \{\gamma_i(t)\} & \text{if } t > 0 \end{cases}$$

The Min Inf-Convolution Morphism Theorem 6.3, p. 200 implies

Proposition 7.9. (The Min-Morphism property) *Let us denote by*

- V_0 the viability solution associated with the Cauchy condition $\mathbf{c}_0$,
- V_i the viability solution associated with the Lagrangian conditions $\mathbf{c}_i$
- $V_{\mathbf{c}}$ the viability solution associated with the combined condition $\mathbf{c}$.

FIGURE 3. Schematic view of the influence zones of two Lagrangian conditions $\lambda(\cdot)$ and μ . For a given (T, x) , the figure displays the departure set (of initial states from which consistent evolutions arrive at x) and the set of their graphs ranging (triangle zone, convex hull of the deset and of (T, x)). It contains a conflict zone of evolutions starting from $\lambda(0)$ or from $\mu(0)$ and arriving at x

Then

$$V_c(t, x) = \min[V_0(t, x), V_i(t, x)]$$

and is defined on $\text{Dom}(V_c(t, \cdot)) = \text{Dom}(\gamma_0) + t[-\nu^\sharp, \nu^\flat]$. It satisfies trajectory condition

$$V_c(t, x) = \min[V_0(t, x), V_i(t, x)] \leq c(t, x) := \min[\gamma_0(x), c_i(t, x)]$$

We deduce that if the traffic condition is increasing on time and decreasing in position, so is its associated traffic evolution (Fig. 3).

Let us denote by $P := \mathbb{R}_- \times \mathbb{R}_+$ the order relation under which a traffic solution is decreasing: $(t, x) \succeq (s, y)$ if and only if $t \leq s$ and $y \geq x$. Therefore, the traffic function V is decreasing along this preorder if and only if $\mathcal{E}p(V) = \mathcal{E}p(V) + P \times \mathbb{R}_+$, i.e., if $x_1 \leq x_2$ and $t_1 \geq t_2$, then $V(t_2, x_2) \leq V(t_1, x_1)$. Its *decreasing envelope* is defined by

$$V_\searrow(t, x) = \inf_{s \geq t, y \leq x} V(s, y)$$

and its epigraph is equal to $\mathcal{E}p(V_\searrow) := \mathcal{E}p(V) + P \times \mathbb{R}_+$.

Theorem 7.10. (Monotonicity property of the traffic function and its optimal viability traffic functions) *Assume that the Hamiltonian $\mathbf{h}$ is a flux function defined in Definition 7.1, p. 201. Let $P := -\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, $\mathbf{c} : t \in \mathbb{R}_+ \mapsto$*

FIGURE 4. *Left* Trajectory of a Lagrangian condition and domain of the viability solution. *Right* Trajectory of the increasing envelopes of a Lagrangian condition and domain of the increasing envelope of the viability solution, equal to the viability solution of the increasing envelope of the Lagrangian condition

$\mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous traffic condition, $\mathbf{c}_\searrow$ its decreasing envelope, and $V_{\mathbf{c}}$ and $V_{\mathbf{c}_\searrow}$ the viability traffic functions associated with the traffic conditions $\mathbf{c}$ and $\mathbf{c}_\searrow$. Therefore, the viability traffic function $V_{\mathbf{c}_\searrow}$ associated with the P -decreasing envelope $\mathbf{c}_\searrow$ of the traffic condition $\mathbf{c}$ is the P -decreasing envelope $(V_{\mathbf{c}})_\searrow$ of the viability traffic function associated with the traffic condition $\mathbf{c}$:

$$(V_{\mathbf{c}})_\searrow = V_{\mathbf{c}_\searrow} \quad (43)$$

Consequently, the viability traffic function $V_{\mathbf{c}_\searrow}$ associated with the P -decreasing envelope $\mathbf{c}_\searrow$ is decreasing in position and increasing in time.

Proof. This is a consequence of the second statement of Theorem 6.3, p. 200 by taking $\mathbf{c}_1 \equiv \mathbf{c}$ and $\mathbf{c}_2 := \psi_{\mathbb{R}_+ \times \mathbb{R}_+}$, the indicator of $\mathbb{R}_+ \times \mathbb{R}_+$. Then $\mathbf{c}_1 \equiv \mathbf{c}_2 = \mathbf{c}_\searrow$ and $V_{\mathbf{c}_\searrow} = V_{\mathbf{c}} \star \mathbf{c}_2 = (V_{\mathbf{c}})_\searrow$. $\square$

The question which arises deals with the construction of decreasing traffic condition in position. Equation (26), p. 200 implies that the domain $\text{Dom}(\mathbf{c}_\searrow) =: \text{Graph}(\mathbf{C}_\searrow)$ satisfies

$$\text{Dom}(\mathbf{c}_\searrow) = \text{Dom}(\mathbf{c}) + \mathbb{R}_- \times \mathbb{R}_+$$

When the domain of $\mathbf{c}$ is made of a single traffic condition $\gamma(\cdot) : t \in [\tau_\gamma^b, \tau_\gamma^a] \mapsto \gamma(t)$, we infer that the domain of $\mathbf{c}$ can be regarded as the epigraph of the function $\gamma_{\nearrow}(\cdot)$

$$\mathcal{E}p(\gamma_{\nearrow}) := \mathcal{E}p(\gamma) + \mathbb{R}_- \times \mathbb{R}_+$$

Hence,

$$\gamma_{\nearrow}(t) := \inf_{s \geq t} \gamma(s)$$

Its domain is the interval $[0, \tau_\gamma^\#]$, it is equal to $\gamma(t) := \inf_{s \geq t} \gamma(s)$ on the interval $[0, \tau_\gamma^b]$, to $\gamma(t) := \inf_{s \geq t} \gamma(s)$ on the interval $[\tau_\gamma^b, \tau_\gamma^\#]$ and, if $\tau_\gamma^\# < +\infty$, to $+\infty$ whenever $t > \tau_\gamma^\#$ (Fig. 4).

In the case when γ is increasing on its domain $[\tau_\gamma^b, \tau_\gamma^\#]$, extended to $+\infty$ elsewhere, we obtain the simple formula

$$\forall t \in [0, \tau_\gamma^\#], \quad \gamma_{\nearrow}(t) := \max(\gamma(\tau_\gamma^b), \gamma(t))$$

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