$(L^\infty + \text{Bolza})$ control problems as dynamic differential games

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Abstract. We consider a $(L^\infty + \text{Bolza})$ control problem, namely a problem where the payoff is the sum of a L^∞ functional and a classical Bolza functional (the latter being an integral plus an end-point functional). Owing to the $\langle L^1, L^\infty \rangle$ duality, the $(L^\infty + \text{Bolza})$ control problem is rephrased in terms of a static differential game, where a new variable k plays the role of *maximizer* (we regard $1 - k$ as the *available fuel* for the maximizer). The relevant fact is that this static game is equivalent to the corresponding *dynamic differential game*, which allows the (upper) value function to verify a boundary value problem. This boundary value problem involves a Hamilton–Jacobi equation whose Hamiltonian is continuous. The *fueled value function* $\mathcal{W}(t, x, k)$ —whose restriction to $k = 0$ coincides with the value function of the reference $(L^\infty + \text{Bolza})$ problem—is continuous and solves the established boundary value problem. Furthermore, $\mathcal{W}$ is the unique viscosity solution in the class of (not necessarily continuous) bounded solutions.

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1. Introduction

Let us consider the functional

$$\mathcal{F}^{\infty+\mathcal{B}}_{\tau,y}(a) \doteq \mathcal{F}^\infty_{\tau,y}(a) + \mathcal{F}^\mathcal{B}_{\tau,y}(a), \quad (1)$$

where $(\tau, y) \in [0, T] \times \mathbb{R}^n$ is a given initial data,

$$\mathcal{F}^\infty_{\tau,y}(a) \doteq \|l(\cdot, x[a; (\tau, y)](\cdot), a(\cdot))\|_\infty$$

and

$$\mathcal{F}^\mathcal{B}_{\tau,y}(a) \doteq \int_\tau^T L(t, x[a; (\tau, y)](t), a(t))dt + g(x[a; (\tau, y)](T)).$$

Here $a(\cdot)$ is a control function starting from time τ , that is a (Borel-)measurable map from $[\tau, T]$ into a compact subset A of $\mathbb{R}^n$, and $x[a; (\tau, y)](\cdot) : [\tau, T] \rightarrow \mathbb{R}^n$ denotes the (corresponding) solution of the Cauchy control problem

$$\dot{x}(t) = f(t, x(t), a(t)) \quad x(\tau) = y. \quad (2)$$

The set of all control functions starting from time τ is written $\mathcal{A}_\tau$. The precise assumptions on the dynamics f and on the maps h, g, L will be specified later.

For any initial data $(\tau, y) \in [0, T] \times \mathbb{R}^n$ the reference optimization problem is to minimize the functional $\mathcal{F}^{\infty+\mathcal{B}}_{\tau,y}$ over all controls $a(\cdot) \in \mathcal{A}_\tau$ (and the associated trajectories $x[a; (\tau, y)](\cdot)$). We aim to establish a continuous Hamilton–Jacobi equation for the value function

$$v^{\infty+\mathcal{B}}(\tau, y) \doteq \inf_{a(\cdot) \in \mathcal{A}_\tau} \mathcal{F}^{\infty+\mathcal{B}}_{\tau,y}(a). \quad (3)$$

Notice that $\mathcal{F}^{\infty+\mathcal{B}}$ is the sum of the L^∞ (or *minimax*) control functional $\mathcal{F}^\infty$ and the Bolza functional $\mathcal{F}^\mathcal{B}$.

The L^∞ (or *minimax*) control problem (i.e. $L = 0, g = 0$) has been extensively studied from various point of views, including dynamic programming, numerical approximations, and necessary conditions (see e.g. [4–11, 14–16, 19, 20]). In particular, in [7] Barron and Ishii established a Hamilton–Jacobi equation by regarding a L^∞ problem as the limit of standard L^p optimal control problems. Among the merits of this paper, we point out the fact that it provides an idea of how to get approximations of the value function. However, the Hamiltonian established in [7] is highly discontinuous in the unknown solution variable. For instance the authors assert that “*the minimum in the Hamiltonian is calculated over a set which depends on the solution*” [7].

Subsequently, Di Marco and González [12] have extended Barron–Ishii’s equation to the case where a terminal cost is added to the minimax functional. Also in this case the involved Hamiltonians are discontinuous in the solution variable.

The minimum problem investigated here is akin to the one studied in [12]. In fact, we add a Bolza cost to the L^∞ term. The value function $v^{\infty+\mathcal{B}}(\tau, y)$ turns out to be the restriction to the value $k = 0$ of a new value function $\mathcal{W}(\tau, y, k)$, here called *fueled value function*, which involves an additional variable k [see (4)]. The function $\mathcal{W}(\tau, y, k)$ turns out to be the unique solution of a boundary value problem that involves a Hamilton–Jacobi equation **(HJ)** with a *continuous Hamiltonian*.

Our approach is based on a somewhat natural idea¹ from duality theory: the L^∞ norm of a function ℓ is the maximum among the integrals of the product of ℓ with a positive functions with unit L^1 norm. Practically, this means the introduction of a new variable $k \in [0, 1]$ which we will regard as *the consumed fuel*, so that $1 - k$ is the *available fuel*.² More precisely, we shall consider the *fueled value function* $\mathcal{W} : [0, T] \times \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}$ defined as

$$\mathcal{W}(\tau, y, k) \doteq \inf_{a \in \mathcal{A}_\tau} \left((1 - k) \mathcal{F}^\infty_{\tau,y}(a) + \mathcal{F}^\mathcal{B}_{\tau,y}(a) \right). \quad (4)$$

The fueled value function $\mathcal{W}$ is shown to be continuous on $[0, T] \times \mathbb{R}^n \times [0, 1]$. Clearly, it can be continuously extended to $[0, T] \times \mathbb{R}^n \times [0, 1]$ by setting

¹ Actually, a hint in this direction can already be found [9].

² Also in [12] we find an additional variable, called ρ , which however has a different meaning compared to the variable k of our paper.

(cf. Theorem 3.1)

$$\mathcal{W}(T, y, k) = (1 - k)\tilde{l}(y) + g(y), \quad \forall (y, k) \in \mathbb{R}^n \times [0, 1]. \quad (5)$$

where

$$\tilde{l}(y) \doteq \min_{a \in A} l(T, y, a).$$

Of course, one has

$$v^{\infty+\mathcal{B}}(\tau, y) = \mathcal{W}(\tau, y, 0) \quad \forall (\tau, y) \in [0, T] \times \mathbb{R}^n.$$

(I.e. the value function $v^{\infty+\mathcal{B}}$ coincides with the fueled value function $\mathcal{W}$ when we allow the maximum available fuel, namely $k = 0$.)

Let us define the *fueled Hamiltonian* $\mathbf{H} : [0, T] \times \mathbb{R}^n \times [0, 1] \times \mathbb{R}^{1+n+1} \rightarrow \mathbb{R}$ by setting³

$$\begin{aligned} \mathbf{H}(t, x, k, p_t, p_x, p_k) \\ = \min_{a \in A} \max_{w \in [0, 1]} \{ (p_t + p_x \cdot f(t, x, a) + L(t, x, a)) (1 - w) + (p_k + l(t, x, a)) w \}, \end{aligned} \quad (6)$$

and let us consider the sets

$$\Omega \doteq [0, T] \times \mathbb{R}^n \times [0, 1[, \quad \partial_T \Omega = \{T\} \times \mathbb{R}^n \times [0, 1]$$

while we use $\partial \Omega$ to denote the whole boundary of Ω .

The main result of the paper consists in the fact that $\mathcal{W}$ is the unique bounded function such that:

(a) u is a viscosity supersolution of

$$-\mathbf{H}\left(t, x, k, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial k}\right) = 0 \quad (\text{HJ})$$

in Ω ;

- (b) u is a viscosity subsolution of (HJ) in $\bar{\Omega} \setminus \partial_T \Omega$;
- (c) u is continuous on the boundary $\partial \Omega$;
- (d) u verifies the Dirichlet-type condition

$$u(T, x, k) = (1 - k)\tilde{l}(x) + g(x) \quad \forall (x, k) \in \mathbb{R}^n \times [0, 1].$$

Let us point out that *the Hamiltonian $\mathbf{H}$ is continuous* and is obtained as min-maximization of functions over the fixed control set $A \times [0, 1]$. It is meant that that we are using the notion of viscosity solution for non continuous functions (cf. [1, 3] and the references therein). In addition, the uniqueness result is obtained from a comparison result (Theorem 2.1) for bounded (not necessarily continuous) functions. Clearly, the fact that the comparison result is valid for locally bounded (and not necessarily continuous) functions, is essential when one considers numerical approximations.

³ We write $\mathbf{H} = \mathbf{H}(t, x, k, p_t, p_x, p_k)$ even though $\mathbf{H}$ does not depend explicitly on k . Actually, k is a state variable (verifying the equation $\frac{dk}{ds} = w$) and the fact that $\mathbf{H}$ is k -independent is not chart invariant.

To understand the meaning of the fueled value function, it is crucial to regard it as a inf-sup value, which comes from the already mentioned duality approach. More precisely, one has the following result.

Lemma 1.1. *For every $(\tau, y, k) \in [0, T[\times \mathbb{R}^n \times [0, 1]$ one has*

$$\mathcal{W}(\tau, y, k) = \inf_{a \in \mathcal{A}_\tau} \sup_{c \in C_{\tau, k}} \mathcal{G}_{(\tau, y)}(a, c) \quad (7)$$

where

$$\begin{aligned} \mathcal{G}_{(\tau, y)}(a, c) &= \int_{\tau}^T (l(t, x[a; (\tau, y)](t), a(t)) \cdot c(t) + L(t, x[a; (\tau, y)](t), a(t))) dt \\ &\quad + g(x[a; (\tau, y)](T)) \end{aligned}$$

and, for every $(\tau, k) \in [0, T[\times [0, 1]$, the sets $C_{\tau, k}$ of $(1 - k)$ -fuel controls on $[\tau, T]$ are defined as

$$C_{\tau, k} \doteq \{c \in L^1(\mathbb{R}, \mathbb{R}) : \text{supp } c \subseteq [\tau, T], \ c \geq 0, \ \|c\|_1 = 1 - k\}.$$

This result is a straightforward consequence of the fact that, for every map $\phi \in L^\infty([\tau, T], \mathbb{R})$ such that $\phi(t) \geq 0$ for a.e. $t \in [\tau, T]$, one has $\text{ess sup } \phi = \sup_{c \in C_{\tau, 0}} \int_{\mathbb{R}} \phi(t) c(t) dt$, which, in turn, implies

$$(1 - k) \cdot \text{ess sup } \phi = \sup_{c \in C_{\tau, k}} \int_{\mathbb{R}} \phi(t) c(t) dt.$$

Let us point out that various authors have already observed that the value function can be regarded as an inf-sup (see e.g. [12] and references therein). Yet, this is not enough for $\mathcal{W}$ to be a solution of a Hamilton–Jacobi equation. Indeed a inf-sup problem is a *static game*, whose upper value— $\mathcal{W}$ in our case—in general *fails to verify* a dynamic programming principle. On one hand, let us remark that in general the upper value of a *static game* is strictly larger than the value of the corresponding *dynamic game* (we call *dynamic* the game where maximizing controls are replaced with maximizing *strategies*). On the other hand, let us recall that the upper value of a dynamic game *does verify* a Hamilton–Jacobi(–Isaacs) equation. In fact, the main tool to achieve our results consists in showing that the case examined here represents an exception: we prove that the upper value $\mathcal{W}$ of our static game coincides with the upper value $\mathcal{U}$ of the corresponding dynamic game, namely,

$$\mathcal{U} = \mathcal{W}. \quad (8)$$

As a matter of fact, $\mathcal{U}$ (and hence $\mathcal{W}$) verifies a dynamic programming principle whose infinitesimal version consists in the boundary value problem described in (a)–(d) above. Most of the paper is devoted to the proof of the equality (8), which in particular requires the introduction of an auxiliary (space–time) differential game. We refer to Sect. 3 for a synopsis of the steps that lead to (8).

The paper is organized as follows. In Sect. 2 we state our hypotheses, establish the (reference) boundary value problem, and provide a comparison

result and a uniqueness theorem. In Sect. 3 we state our main result and give the outline of the proof in four steps. Sections 4, 5, 6, 7 will lay these four steps out.

We conclude this section by introducing the definitions of *viscosity subsolution* and *viscosity supersolution* for possibly discontinuous functions. For this purpose, we need first the notion of upper and lower semicontinuous envelopes. We shall use $B(x, \rho)$ to denote the open unit ball centered in x with radius $\rho > 0$ in the Euclidean space.

Definition 1.1. Let q be a positive integer and let $r : E \rightarrow \mathbb{R}$ be a real function defined on a subset $E \subset \mathbb{R}^q$. The upper [resp. lower] semicontinuous envelope r^* [resp. r_*] of r is defined by

$$r^*(x) = \inf_{\rho > 0} \sup_{z \in B(x, \rho) \cap E} r(z) \quad \forall x \in E$$

[respectively,

$$r_*(x) = \sup_{\rho > 0} \inf_{z \in B(x, \rho) \cap E} r(z)].$$

Definition 1.2. (*Viscosity solution*) Let q be a positive integer and let u be a real function defined on a subset $E \subset \mathbb{R}^q$. Let $K : E \times \mathbb{R}^q \rightarrow \mathbb{R}$ be a continuous map. A locally bounded function $u : E \rightarrow \mathbb{R}$ is a viscosity supersolution (resp. subsolution) at $\bar{x} \in E$ of the equation

$$K\left(x, \frac{\partial u}{\partial x}\right) = 0 \tag{9}$$

if for every $\varphi \in C^1(\mathbb{R}^q)$ such that $u_* - \varphi$ (resp. $u^* - \varphi$) has a local minimum (resp. maximum) at $\bar{x}$ on E , one has

$$K\left(\bar{x}, \frac{\partial \varphi}{\partial x}(\bar{x})\right) \geq 0 \quad \left[\text{resp. } K\left(\bar{x}, \frac{\partial \varphi}{\partial x}(\bar{x})\right) \leq 0\right]$$

A locally bounded map u is called a viscosity solution of (9) on E at $\bar{x} \in E$ if it is both a supersolution and a subsolution of (9) at $\bar{x} \in E$.

2. The fueled value function and a boundary value problem

2.1. Assumptions on the data

Throughout this paper, we shall assume the following hypotheses on the data of the problem:

- The control set A is a compact subset of $\mathbb{R}^m$, where m is a positive integer. $T > 0$ will denote the final time. For every $\tau \in [0, T]$, let $\mathcal{A}_\tau$ denote the set of Borel measurable maps from $[\tau, T]$ into A .
- The control vector field $f : [0, T] \times \mathbb{R}^n \times A \rightarrow \mathbb{R}^n$, the output function $l : [0, T] \times \mathbb{R}^n \times A \rightarrow \mathbb{R}$, the (ordinary) Lagrangian $L : [0, T] \times \mathbb{R}^n \times A \rightarrow \mathbb{R}$, and the terminal cost $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous maps such that

- (i) for every compact set $Q \in \mathbb{R}^n$ there exists a non-negative real number C such that

$$|(f, l, L)(t', x', \alpha) - (f, l, L)(t, x, \alpha)| \leq C|(t', x') - (t, x)|$$

for all $(t', x'), (t, x) \in [0, T] \times \mathbb{R}^n$ and $\alpha \in A$, and

- (ii) for a suitable $M \geq 0$ one has⁴

$$|f(t, x, \alpha)| \leq M(1 + |x|), \quad 0 \leq l(t, x, \alpha) \leq M,$$

$$|L(t, x, \alpha)| \leq M, \quad |g(x)| \leq M$$

for all $(t, x, \alpha) \in [0, T] \times \mathbb{R}^n \times A$.

Notice that under the above assumptions, for every $\tau \in [0, T], y \in \mathbb{R}^n$ and every control $a \in \mathcal{A}_\tau$ a solution to the Cauchy problem

$$\dot{x}(t) = f(t, x(t), a(t)) \quad x(\tau) = y \quad (10)$$

exists on $[\tau, T]$ and is unique. This solution is written $x[a; (\tau, y)](\cdot)$.

2.2. A boundary value problem

Let us recall the definition of the Hamiltonian:⁵

$$\begin{aligned} \mathbf{H}(t, x, k, p_t, p_x, p_k) \\ = \min_{a \in A} \max_{w \in [0, 1]} \{ (p_t + p_x \cdot f(t, x, a) + L(t, x, a)) (1 - w) + (p_k + l(t, x, a)) w \}. \end{aligned}$$

The boundary value problem **(BVP)** below refers to the notion of *non continuous viscosity solution* (see Definition 1.2). We shall see that **(BVP)** verifies comparison (hence uniqueness) properties, and eventually we shall show that the fueled value function $\mathcal{W}$ verifies this boundary value problem. Even though the value function turns out to be continuous it is crucial—particularly for numerical purposes—that the comparison results refer to the larger class of (possibly discontinuous) locally bounded functions. In order to define the boundary value problem, we recall the definition of the continuous map $\tilde{l}$:

$$\tilde{l}(x) \doteq \max_{a \in A} l(T, x, a).$$

Definition 2.1 (*Boundary Value Problem (BVP)*). We shall say that a function $u : \bar{\Omega} \rightarrow \mathbb{R}$ is a solution **(BVP)** if the following statements hold true:

- (a) u is a viscosity supersolution of

$$-\mathbf{H}\left(t, x, k, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial k}\right) = 0 \quad (\text{HJ})$$

in Ω ;

- (b) u is a viscosity subsolution of **(HJ)** in $\bar{\Omega} \setminus \partial_T \Omega$;

- (c) u is continuous on the boundary $\partial \Omega$;

⁴ by adding a suitable constant to the payoff functional $\mathcal{F}^{\infty+\mathcal{B}}$ we can replace this condition with the weaker hypothesis $|l(t, x, \alpha)| \leq M$.

⁵ Clearly, one can also write:

$$\mathbf{H}(t, x, k, p_t, p_x, p_k) = \min_{a \in A} [\max \{ p_t + p_x \cdot f(t, x, a) + L(t, x, a), p_k + l(t, x, a) \}].$$

(d) u verifies the Dirichlet-type condition

$$u(T, x, k) = (1 - k)\tilde{l}(x) + g(x) \quad \forall (x, k) \in \mathbb{R}^n \times [0, 1], \quad (11)$$

which must be interpreted in the viscosity sense, that is,

$$u_*(T, x, k) \geq (1 - k)\tilde{l}(x) + g(x) \geq u^*(T, x, k).$$

Theorem 2.1 (Comparison). *Let u_1, u_2 be real, bounded, functions on $\bar{\Omega}$ such that u_1 is a u.s.c. and a viscosity subsolution of (HJ) on Ω and u_2 is l.s.c. and a viscosity supersolution of (HJ) on Ω . Moreover, let us assume that u_2 is continuous on $\partial\Omega$ and, for every $(t, x, k) \in \partial\Omega$, either $u_1(t, x, k) \leq u_2(t, x, k)$ or u_1 is a subsolution of (HJ) at (t, x, k) . Then*

$$u_1(t, x, k) \leq u_2(t, x, k) \quad \forall (t, x, k) \in \bar{\Omega}.$$

Proof. We recall that by assumption there exists $M > 0$ such that $\|L\|_\infty \leq M$. Let G be a lower bound for both u_1 and u_2 , and let us consider the maps

$$v_i \doteq \frac{1}{1+t+k} \log(u_i(T-t, x, 1-k) + Mt + G + 1), \quad i = 1, 2.$$

Let us define the Hamiltonian $\mathbf{K}$ by setting, for every $(t, x, k, p_t, p_x, p_k) \in \bar{\Omega} \times \mathbb{R}^{1+n+1}$ and $r \in \mathbb{R}$

$$\mathbf{K}(t, x, k, r, p_t, p_x, p_k) \doteq \max_{a \in A} \min_{w \in [0, 1]} K(t, x, k, r, p_t, p_x, p_k, a, w),$$

where

$$\begin{aligned} & K(t, x, k, r, p_t, p_x, p_k, a, w) \\ & \doteq p_k w + (1 - w)(p_t - p_x \cdot f(t, x, a)) \\ & \quad - \frac{1}{(1+t+k)e^{(1+t+k)r}} (l(t, x, a) + L(t, x, a) + M). \end{aligned}$$

It is straightforward to check that v_1 and v_2 are a viscosity subsolution and a viscosity supersolution of

$$v + \mathbf{K}\left(t, x, k, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial k}\right) = 0 \quad (12)$$

in $]0, T] \times \mathbb{R}^n \times]0, 1]$, respectively.

Indeed, suppose that $(\bar{t}, \bar{x}, \bar{k})$ is a local minimum for $v_1 - \psi$ where ψ is a $\mathcal{C}^1$ function. It is not restrictive to assume that $(v_1 - \psi)(\bar{t}, \bar{x}, \bar{k}) = 0$ and, so, locally we have $v_1 \geq \psi$. From the monotonicity of the exponential function, we obtain that $(T - \bar{t}, \bar{x}, 1 - \bar{k})$ is a local minimum for $(t, x, k) \rightarrow (u_1 - \phi)(T - t, x, 1 - k)$, where we have set $\phi(T - t, x, 1 - k) = e^{(1+t+k)\psi(t, x, k)} - G - 1 - Mt$. As a consequence, since u_1 is a viscosity supersolution of (HJ), we obtain that

$$\begin{aligned} 0 \leq & -\min_{a \in A} \max_{w \in [0, 1]} \left\{ \left(\frac{\partial \phi}{\partial t}(T - \bar{t}, \bar{x}, 1 - \bar{k}) + \frac{\partial \phi}{\partial x}(T - \bar{t}, \bar{x}, 1 - \bar{k}) \cdot f(T - \bar{t}, \bar{x}, a) \right. \right. \\ & \left. \left. + L(T - \bar{t}, \bar{x}, a) \right) (1 - w) + \left(\frac{\partial \phi}{\partial k}(T - \bar{t}, \bar{x}, 1 - \bar{k}) + l(T - \bar{t}, \bar{x}, a) \right) w \right\}. \end{aligned}$$

On the other hand we have:

$$\begin{aligned}\frac{\partial \phi}{\partial t}(T - \bar{t}, \bar{x}, 1 - \bar{k}) &= -e^{(1+\bar{t}+\bar{k})\psi(\bar{t}, \bar{x}, \bar{k})} \left[(1 + \bar{t} + \bar{k}) \frac{\partial \psi}{\partial t}(\bar{t}, \bar{x}, \bar{k}) + \psi(\bar{t}, \bar{x}, \bar{k}) \right] + M \\ \frac{\partial \phi}{\partial x}(T - \bar{t}, \bar{x}, 1 - \bar{k}) &= e^{(1+\bar{t}+\bar{k})\psi(\bar{t}, \bar{x}, \bar{k})} (1 + \bar{t} + \bar{k}) \frac{\partial \psi}{\partial x}(\bar{t}, \bar{x}, \bar{k}) \\ \frac{\partial \phi}{\partial k}(T - \bar{t}, \bar{x}, 1 - \bar{k}) &= -e^{(1+\bar{t}+\bar{k})\psi(\bar{t}, \bar{x}, \bar{k})} \left[(1 + \bar{t} + \bar{k}) \frac{\partial \psi}{\partial k}(\bar{t}, \bar{x}, \bar{k}) + \psi(\bar{t}, \bar{x}, \bar{k}) \right].\end{aligned}$$

Then we obtain that

$$\psi(\bar{t}, \bar{x}, \bar{k}) + \mathbf{K} \left(\bar{t}, \bar{x}, \bar{k}, \psi(\bar{t}, \bar{x}, \bar{k}), \frac{\partial \psi}{\partial t}(\bar{t}, \bar{x}, \bar{k}), \frac{\partial \psi}{\partial x}(\bar{t}, \bar{x}, \bar{k}), \frac{\partial \psi}{\partial k}(\bar{t}, \bar{x}, \bar{k}) \right) \geq 0$$

which yields that v_1 is a viscosity supersolution of (12).

The proof that v_2 is a subsolution of (12) is similar.

Moreover, on $\bar{\Omega} v_2$ is continuous, and, for every $(t, x, k) \in \partial\Omega$, either $v_1(t, x, k) \leq v_2(t, x, k)$ or v_1 is a subsolution of (12) at (t, x, k) . Finally, since $l(t, x, a) + L(t, x, a) + M \geq 0$ for all $(t, x, a) \in [0, T] \times \mathbb{R}^n \times A$, for every $(t, x, k, p_t, p_x, p_k) \in \bar{\Omega} \times \mathbb{R}^{1+n+1}$ the function

$$r \rightarrow \mathbf{K}(t, x, k, r, p_t, p_x, p_k)$$

is increasing. Therefore, the hypotheses of [2, Theorem 1.1] are verified, and this allows us to conclude that $v_1 \leq v_2$ on $\bar{\Omega}$, which in turn implies $u_1 \leq u_2$. $\square$

The comparison result stated in Theorem 2.1 will be used in next sections to prove various inequalities. Here we derive the uniqueness of solutions:

Theorem 2.2 (Uniqueness). *There exists at most one bounded solution of (BVP).*

Proof. Let V_1, V_2 be bounded solutions of the boundary value problem (BVP). Hence:

- (i) V_{2*} is a supersolution of (HJ) on Ω and is continuous on the boundary $\partial\Omega$;
- (ii) V_1^* is a subsolution of (HJ) on $\bar{\Omega} \setminus \partial_T \Omega$. Furthermore $V_1^*(T, x, k) \leq V_{2*}(T, x, k)$, for all $(x, k) \in \mathbb{R}^n \times [0, 1]$.

Therefore $u_1 \doteq V_1^*, u_2 \doteq V_{2*}$ verify the hypotheses of Theorem 2.1. It follows that

$$V_1^* \leq V_{2*}$$

on $\bar{\Omega}$. Since the roles of V_1 and V_2 can be interchanged, one also have

$$V_{2*} \leq V_{1*}$$

on $\bar{\Omega}$. Thus one can conclude that $V_1 = V_2$, and that this function is continuous. $\square$

3. Main results

Theorem 3.1 (Continuity of $\mathcal{W}$). *The fueled value function $\mathcal{W}$ is continuous on $\bar{\Omega}$.*

Theorem 3.2 [$\mathcal{W}$ solves (BVP)]. *The fueled value function $\mathcal{W} : \bar{\Omega} \rightarrow \mathbb{R}$ is a solution of (BVP).*

Remark 3.1. In view of Theorem 2.2 $\mathcal{W}$ is in fact the unique solution of the boundary value problem (BVP).

Proofs of Theorems 3.1 and 3.2. Our proofs are built upon a series of steps displayed in the next sections. To begin with, observe that the fact that $\mathcal{W}$ is the value function of a inf-sup problem—see Lemma 1.1—is clearly not enough for our purposes. Indeed a inf-sup problem generally does not verify a dynamic programming principle. To overcome this drawback, we will show that

$$\mathcal{W} = \mathcal{U}, \quad (13)$$

where $\mathcal{U}$ is the upper value of a certain related differential game, where the controls c are suitably replaced by the corresponding *non-anticipating strategies*.

We point out that equality (13) is somehow unexpected and non-trivial. Indeed, it is unusual that a inf-sup problem, namely a *static* game has the same value as the upper value of the corresponding *dynamic game*: as a rule the value of the static game is larger than the upper value of the dynamic game.

In the next sections we shall prove Steps I–IV below, which in turn imply equality (13):

Step (I) (Sect. 4) As we have mentioned above, the inequality

$$\mathcal{U} \leq \mathcal{W} \quad (14)$$

is somehow standard. However we shall prove it in detail both for the sake of completeness and because the differential game at issue involves *unbounded controls*.

Step (II) (Sect. 5) The proof of the inequality $\mathcal{U} \geq \mathcal{W}$ turns out to be not straightforward. However it becomes simpler and (possibly) more expressive when the output function l is independent of the control a . In this case we use just a control (direct) approach to prove the inequality $\mathcal{U} \geq \mathcal{W}$. While, if l depends also on a , we shall use a different (indirect) approach which combines analytical tools from both control and viscosity solutions theory [see Step (III)]. For this reason we shall treat this particular case separately.

Step (III) (Sect. 6) In order to prove the inequality $\mathcal{U} \geq \mathcal{W}$ in its full generality—i.e. when l depends on the control a as well—we shall introduce a new (dynamic) differential game whose upper value will be denoted by $\hat{\mathcal{U}}$. In this game the class of admissible controls a is slightly restricted, while the class of (maximizing) controls c is enlarged to allow graph-like objects.

By means of control theoretical arguments it will be shown (see Proposition 6.1) that

$$\mathcal{W} \leq \hat{\mathcal{U}}. \quad (15)$$

Step (IV) (Sect. 7) We shall prove that $\hat{\mathcal{U}}$ is a subsolution of (HJ) on $\bar{\Omega} \setminus \partial_T \Omega$. By means of the comparison result stated in Theorem 2.1 this implies

$$(\hat{\mathcal{U}} \leq) \hat{\mathcal{U}}^* \leq \mathcal{U}. \quad (16)$$

Assuming (Step I)–(Step IV) are proved, one can put (14)–(16) together to obtain

$$\hat{\mathcal{U}} \leq \mathcal{U} \leq \mathcal{W} \leq \hat{\mathcal{U}} \quad \text{on } \bar{\Omega} \setminus \partial_T \Omega,$$

hence

$$\mathcal{W} = \mathcal{U} = \hat{\mathcal{U}} \quad \text{on } \bar{\Omega}. \quad (17)$$

From Theorem 4.1 below one has that *the function $\mathcal{U}$ is continuous*. Therefore, the same property is valid for the fueled value $\mathcal{W}$. As a consequence the assertion of Theorem 3.1 is verified.

In addition, by Theorems 4.1 and 7.1 below we have that the function $\mathcal{U} = \hat{\mathcal{U}}$ solves the boundary value problem (BVP). Therefore the same holds true for the fueled value $\mathcal{W}$. Hence, Theorem 3.2 is proved as well. $\square$

4. Step (I): the value $\mathcal{U}$ and the inequality $\mathcal{U} \leq \mathcal{W}$

4.1. A differential game and its upper value $\mathcal{U}$

We introduce the notion of *strategy*.

Definition 4.1. For any $\tau \in [0, T[$ we say that a map

$$\gamma : \mathcal{A}_\tau \rightarrow L^1(\mathbb{R}, \mathbb{R})$$

is a strategy—in the Varaya–Roxin–Elliott–Kalton’s sense, see e.g. [13]—if it is non-anticipative. This means that for every $\bar{t} \in [\tau, T]$ and every pair $a, \tilde{a} \in \mathcal{A}_\tau$ such that

$$a(t) = \tilde{a}(t) \quad \forall t \in [\tau, \bar{t}] \quad (18)$$

one has

$$\gamma[a](t) = \gamma[\tilde{a}](t) \quad \forall t \in [\tau, \bar{t}] \quad (19)$$

Remark 4.1. Definition 4.1 gives a “everywhere” version of strategy. Actually, the classical notion of non-anticipative strategy in the Varaya–Roxin–Elliott–Kalton’s sense is defined considering the usual equivalence relation on measurable maps (cf. for instance [13] or [1]). So $\gamma : \mathcal{A}_\tau \rightarrow L^1(\mathbb{R}, \mathbb{R})$ is non-anticipative in the ‘standard sense’ if (18) and (19) are valid almost everywhere: if $a(\cdot) = \tilde{a}(\cdot)$ a.e. on $[\tau, \bar{t}]$, then $\gamma[a](\cdot) = \gamma[\tilde{a}](\cdot)$ a.e. on $[\tau, \bar{t}]$. The reason why we need a “everywhere” version of *non-anticipativity* notion relies essentially on the fact that in this paper ‘impulsive type’ controls play a crucial role.

Let us specialize the notion of strategy to that of “fuel strategy”.

Definition 4.2 (*Fuel strategy*). Let the pair (τ, k) belong to $[0, T] \times [0, 1]$. We say that a functional

$$\gamma : \mathcal{A}_\tau \rightarrow L^1(\mathbb{R}, \mathbb{R})$$

is a $(1 - k)$ -fuel strategy on $[\tau, T]$ if:

- γ is a strategy;
- for every $a \in \mathcal{A}_\tau$ $\gamma[a]$ belongs to $C_{\tau, k}$, i.e., $\text{supp}(\gamma[a]) \subseteq [t, T]$, $\gamma[a] \geq 0$, $\|\gamma[a]\|_1 = 1 - k$.

The set of $(1 - k)$ -fuel strategies on $[\tau, T]$ will be denoted by $\Gamma_{\tau, k}$.

Let us define the *upper value* $\mathcal{U}$ of the dynamic differential game obtained by replacing fuel controls with fuel strategies in (7). Precisely, for every $(\tau, y, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$, let us set

$$\mathcal{U}(\tau, y, k) \doteq \sup_{\gamma \in \Gamma_{\tau, k}} \inf_{a \in \mathcal{A}_\tau} \mathcal{G}_{(\tau, y)}(a, \gamma[a])$$

and

$$\mathcal{U}(T, y, k) \doteq (1 - k)\tilde{l}(x) + g(x)$$

Eventually we shall show that $\mathcal{W} = \mathcal{U}$. For this purpose let us begin with recalling the following result from [17, 18]:⁶

Theorem 4.1. *The function $\mathcal{U}$ is bounded and continuous. Moreover $\mathcal{U}$ is a supersolution of (HJ) on Ω .*

Later, the supersolution property will be exploited to show, via comparison theorem, that $\hat{\mathcal{U}} \leq \mathcal{U}$ (see Corollary 7.1), where $\hat{\mathcal{U}}$ is the upper value function of an auxiliary, *impulsive*, differential game.

4.2. The inequality $\mathcal{U} \leq \mathcal{W}$

We prove the inequality $\mathcal{U} \leq \mathcal{W}$ just for the sake of completeness. Indeed, it is well-known that the upper value of a static game (i.e. a inf-sup value, minimizing and maximizing over controls) is greater than or equal to the upper value of the corresponding dynamic game (i.e. a sup-inf value, maximizing over strategies and minimizing over controls).

Proposition 4.1. *For any $(\tau, y, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$ one has*

$$\mathcal{U}(\tau, y, k) \leq \mathcal{W}(\tau, y, k) \quad (20)$$

Proof. In view of (7), inequality (20) coincides with the inequality

$$\inf_{a \in \mathcal{A}_\tau} \sup_{c \in C_{\tau, k}} \mathcal{G}_{(\tau, y)}(a, c) \geq \sup_{\gamma \in \Gamma_{\tau, k}} \inf_{a \in \mathcal{A}_\tau} \mathcal{G}_{(\tau, y)}(a, \gamma[a]). \quad (21)$$

This relation is trivial. Indeed, for any $a \in \mathcal{A}_\tau$ and any $\gamma \in \Gamma_{\tau, k}$, one has

$$\sup_{c \in C_{\tau, k}} \mathcal{G}_{(\tau, y)}(a, c) \geq \mathcal{G}_{(\tau, y)}(a, \gamma[a]).$$

⁶ Actually, in [17, 18] a more general result, involving vector valued, unbounded controls, is proved.

This implies

$$\inf_{a \in \mathcal{A}_\tau} \sup_{c \in C_{\tau,k}} \mathcal{G}_{(\tau,y)}(a, c) \geq \inf_{a \in \mathcal{A}_\tau} \mathcal{G}_{(\tau,y)}(a, \gamma[a]),$$

so we get (21) just by taking the supremum value on the right-hand side over all strategies γ . $\square$

5. Step (II): the inequality $\mathcal{W} \leq \mathcal{U}$ when $l = l(t, x)$

In the special case when the L^∞ cost l does not depend on the control a , the inequality $\mathcal{W} \leq \mathcal{U}$ can be proved in a quite direct way: we use a mere control approach, which somehow highlight the impulsive nature of the problem. For the general case see Sect. 6.

Proposition 5.1. *Let us assume that l is independent of the control a . Then, for any $(\tau, y, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$, one has*

$$\mathcal{W}(\tau, y, k) \leq \mathcal{U}(\tau, y, k).$$

Proof. Fix $(\tau, y, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$. Recall that by the definition of $\mathcal{W}$, we have

$$\begin{aligned} \mathcal{W}(\tau, y, k) = \inf_{a \in \mathcal{A}_\tau} \left\{ (1-k) \operatorname{ess\,sup}_{\tau \leq t \leq T} [l(s, x[a; (\tau, y)](s))] \right. \\ \left. + \int_\tau^T L(t, x[a; (\tau, y)](t), a(t)) \, dt + g(x[a; (\tau, y)](T)) \right\}. \quad (22) \end{aligned}$$

If $k = 1$, the thesis is trivial. Therefore, let us assume $k \in [0, 1)$.

For any $a \in \mathcal{A}_\tau$ and any $t \in [\tau, T]$ let $\sigma_{a,t} \in [\tau, t]$ be the first time such that

$$l(\sigma_{a,t}, x[a; (\tau, y)](\sigma_{a,t})) = \operatorname{ess\,sup}_{\tau \leq s \leq t} [l(s, x[a; (\tau, y)](s))]. \quad (23)$$

Now, for every $\epsilon > 0, t \in [\tau, T]$ and $a \in \mathcal{A}_\tau$ define the interval

$$I_{\epsilon,a,t} \doteq [\sigma_{a,t} - \epsilon, \sigma_{a,t} + \epsilon] \cap [\tau, T].$$

Notice that, since f has linear growth in x and l is Lipschitz continuous with respect to t and x (cf. Sect. 2.1), there exists a positive constant C_1 such that for all $t \in [\tau, T]$ and $a \in \mathcal{A}_\tau$ we have

$$\begin{aligned} |l(\sigma_{a,t}, x[a; (\tau, y)](\sigma_{a,t})) - l(s, x[a; (\tau, y)](s))| &\leq C_1 |\sigma_{a,t} - s| \leq C_1 \epsilon \\ \text{for all } s &\in I_{\epsilon,a,t}. \end{aligned}$$

Since $t \leq t'$ implies $\sigma_{a,t} \leq \sigma_{a,t'}$, we can consider the strategy $\gamma_\epsilon \in \Gamma_{\tau,k}$ by setting⁷

$$\gamma_\epsilon[a](t) \doteq \frac{1-k}{\operatorname{Meas}(I_{\epsilon,a,t})} \chi_{I_{\epsilon,a,t}}(t).$$

⁷ For any subset $E \subset \mathbb{R}$ χ_E will denote the indicator map of E . This means that χ_E is defined on $\mathbb{R}$ and $\chi_E(x) = 1$ if $x \in E$, while $\chi_E(x) = 0$ if $x \in \mathbb{R} \setminus E$.

Notice that, for $\epsilon \rightarrow 0^+$, $\gamma_\epsilon[a](\cdot)$ provides a representation of the Dirac delta function concentrated at $\sigma_{a,t}$ (multiplied by $(1-k)$). Since

$$\begin{aligned} \operatorname{ess\,sup}_{\tau \leq t \leq T} [l(t, x[a; (\tau, y)](t))] &= \frac{1}{\operatorname{Meas}(I_{\epsilon, a, t})} \int_{I_{\epsilon, a, t}} l(\sigma_{a, t}, x[a; (\tau, y)](\sigma_{a, T})) \, dt \\ &\leq \frac{1}{\operatorname{Meas}(I_{\epsilon, a, t})} \int_{I_{\epsilon, a, t}} (l(t, x[a; (\tau, y)](t)) + C_1 \epsilon) \, dt \\ &\leq \frac{1}{\operatorname{Meas}(I_{\epsilon, a, t})} \int_{I_{\epsilon, a, t}} l(t, x[a; (\tau, y)](t)) \, dt + C_1 \epsilon \\ &= \frac{1}{1-k} \left(\int_{[\tau, T]} l(t, x[a; (\tau, y)](t)) \gamma_\epsilon[a](t) \, dt + C_1(1-k)\epsilon \right), \end{aligned}$$

we have

$$\begin{aligned} (1-k) \operatorname{ess\,sup}_{\tau \leq t \leq T} [l(t, x[a; (\tau, y)](t))] \\ \leq \int_{[\tau, T]} l(t, x[a; (\tau, y)](t)) \gamma_\epsilon[a](t) \, dt + C_1(1-k)\epsilon. \end{aligned}$$

Therefore, in view of (22) we obtain

$$\mathcal{W}(\tau, y, k) \leq \inf_{a \in \mathcal{A}_\tau} \left\{ \int_{[\tau, T]} l(t, x[a; (\tau, y)](t)) \gamma_\epsilon[a](t) \, dt \right. \quad (24)$$

$$\begin{aligned} &\left. + \int_{\tau}^T L(t, x[a; (\tau, y)](t), a(t)) \, dt + g(x[a; (\tau, y)](T)) \right\} + \epsilon \\ &\leq \mathcal{U}(\tau, y, k) + C_1(1-k)\epsilon. \end{aligned} \quad (25)$$

Letting ϵ go to zero we get the result. $\square$

6. Step (III): the value function $\hat{\mathcal{U}}$ and the inequality $\mathcal{W} \leq \hat{\mathcal{U}}$

In the proof of Proposition 5.1 we have used the continuity of the function $t \rightarrow l(t, x[a; (\tau, y)](t))$. Such argument cannot be exploited when l depends also on a , since in that case $t \rightarrow l(t, x[a; (\tau, y)](t), a(t))$ is just measurable. For the purpose of proving the non trivial relation $\mathcal{W} \leq \mathcal{U}$ in the general case, we are going to define an auxiliary differential game, which is obtained by suitably modifying the sets of controls and strategies in the game introduced in the Sect. 4. More specifically, the new *minimizing* controls a will be restricted to those such that the maps $t \mapsto l(t, x[a; (\tau, y)](t), a(t))$ are upper semi-continuous, while the class of *maximizing* controls will be enlarged in order to include graph-like objects.

Definition 6.1. If $\tau < T$ and $y \in \mathbb{R}^n$, we shall use $\hat{\mathcal{A}}_{\tau, y}$ to denote the subset of the controls $a \in \mathcal{A}_\tau$ (defined everywhere on $[\tau, T]$ and) such that the map

$$t \mapsto l(t, x[a; (\tau, y)](t), a(t))$$

is upper semicontinuous.

Remark 6.1. It is trivial to verify that, for any $(\tau, y) \in [0, T] \times \mathbb{R}^n$, the set $\hat{\mathcal{A}}_{\tau, y}$ is not empty. For instance, constant controls belong to $\hat{\mathcal{A}}_{\tau, y}$.

Let us introduce the set of fuel *space-time controls*, which, in view of Remark 6.2 below, can be considered as an extension of the class of fuel controls.

Definition 6.2. For any $(\tau, k) \in [0, T] \times [0, 1]$ let us set

$$S_{\tau, k} \doteq T + 1 - \tau - k.$$

We say that

$$w : [0, S_{\tau, k}] \rightarrow [0, 1]$$

is a *fuel space-time control from* (τ, k) if $w \in L^\infty([0, S_{\tau, k}], [0, 1])$ and

$$\int_0^{S_{\tau, k}} w(s) \, ds = 1 - k.$$

We shall use $\hat{C}_{\tau, k}$ to denote the set of all fuel space-time controls from (τ, k) .

Notice that if $w \in \hat{C}_{\tau, k}$, then we have

$$\int_0^{S_{\tau, k}} (1 - w(s)) \, ds = T - \tau.$$

Remark 6.2. For any $(\tau, k) \in [0, T] \times [0, 1]$, the set $C_{\tau, k}$ can be naturally embedded in $\hat{C}_{\tau, k}$. Indeed, for every $c \in C_{\tau, k}$ let us define a reparameterization $s_c(\cdot) : [\tau, T] \rightarrow [0, S_{\tau, k}]$ by setting, for every $t \in [\tau, T]$,

$$s_c(t) \doteq \int_\tau^t (1 + c(\xi)) \, d\xi.$$

The map $s_c(\cdot)$ is absolutely continuous and strictly increasing. Let us define the map $t_c : [0, S_{\tau, k}] \rightarrow [\tau, T]$ as the inverse of s_c , so that, in particular,

$$\frac{dt_c}{ds}(s) = \frac{1}{1 + c \circ t_c(s)}.$$

Finally, let us set $w_c(\cdot) \doteq 1 - \frac{dt_c}{ds}(\cdot)$ and

$$J(c)(s) \doteq w_c(s) = \frac{c \circ t_c(s)}{1 + c \circ t_c(s)} \quad \forall s \in [0, S_{\tau, k}].$$

Clearly, J can be thought as an embedding of $C_{\tau, k}$ in $\hat{C}_{\tau, k}$. Notice also that $J(C_{\tau, k})$ coincides with the set of space-time controls $w \in \hat{C}_{\tau, k}$ such that $w(s) < 1$ for almost all $s \in [0, S_{\tau, k}]$.

Let $(\tau, y, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$, $w \in \hat{C}_{\tau, k}$, and $a \in \hat{\mathcal{A}}_{\tau, y}$. By the hypotheses on the data (see Sect. 2.1), a solution to the Cauchy problem

$$\begin{cases} \dot{t}(s) = 1 - w(s) \\ \dot{z}(s) = f(t(s), z(s), a \circ t(s))(1 - w(s)) \\ \dot{k}(s) = w(s) \\ (t, z, k)(0) = (\tau, y, k) \end{cases} \quad (26)$$

exists on the whole interval $[0, S_{\tau,k}]$ and is unique.⁸

Together with the dynamics (26) we shall consider a new functional $\hat{\mathcal{G}}_{\tau,y,k}$. For any $(\tau, y, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$,

$$\hat{\mathcal{G}}_{\tau,y,k} : \hat{\mathcal{A}}_{\tau,y} \times \hat{C}_{\tau,k} \rightarrow \mathbb{R}$$

is defined by⁹

$$\begin{aligned} \hat{\mathcal{G}}_{\tau,y,k}(a, w) &\doteq \int_0^{S_{\tau,k}} l(t(s), z(s), a \circ t(s)) \cdot w(s) \, ds \\ &\quad + \int_0^{S_{\tau,k}} L(t(s), z(s), a \circ t(s)) \cdot (1 - w(s)) \, ds + g(z(S_{\tau,k})). \end{aligned} \quad (27)$$

When $\tau = T$ (so that $S_{\tau,k} = 1 - k$), we set

$$\begin{aligned} \hat{\mathcal{G}}_{T,y,k} : A &\rightarrow \mathbb{R} \\ \hat{\mathcal{G}}_{T,y,k}(b) &\doteq \int_0^{S_{\tau,k}} l(T, y, b)w(s) \, ds + g(y) \equiv (1 - k)l(T, y, b) + g(y). \end{aligned} \quad (28)$$

Remark 6.3. From the definition above, for any $(\tau, y, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$, $c \in C_{\tau,k}$, and $a \in \hat{\mathcal{A}}_{\tau,y}$ we have

$$\mathcal{G}_{(\tau,y)}(a, c) = \hat{\mathcal{G}}_{\tau,y,k}(a, J(c)).$$

To define the notion of *space-time strategy* we need to define a right inverse of the time reparameterization $t(\cdot)$. First we observe that, once we have fixed a fuel space-time control from a point (τ, k) , say $w(\cdot) \in \hat{C}_{\tau,k}$, from the first equation of (26) we obtain

$$t[w; \tau, k](s) = \tau + \int_0^s (1 - w(\sigma)) \, d\sigma.$$

Here, to be more precise (and highlight the dependence with respect to τ and k) we write $t[w; \tau, k](\cdot)$ instead of just $t(\cdot)$.

Definition 6.3. Fix any $(\tau, k) \in [0, T] \times [0, 1]$ and $w(\cdot) \in \hat{C}_{\tau,k}$. Let $s[w; \tau, k](\cdot)$ be determined by

$$s[w; \tau, k](t) = \sup\{s \in [0, S_{\tau,k}] : t[w; \tau, k](s) = t\}, \quad \forall t \in [\tau, T].$$

Observe that $s[w; \tau, k](\tau, T) = [s[w; \tau, k](\tau), S_{\tau,k}]$. Moreover $s[w; \tau, k](\cdot)$ is strictly increasing and right-continuous, and satisfies

$$t[w; \tau, k] \circ s[w; \tau, k](t) = t, \quad \forall t \in [\tau, T].$$

⁸ $a \circ t(\cdot)$ is Lebesgue measurable, for a is assumed Borel measurable.

⁹ Here and in what follows we adopt the following notational rule: *as soon as initial data (τ, y, x) and controls $a(\cdot) \in \hat{\mathcal{A}}_{\tau,y}$, $w(\cdot) \in \hat{C}_{\tau,k}$ appear in a formula together with a trajectory $(t(\cdot), z(\cdot), k(\cdot))$, we mean that the latter is the unique solution of system (26) corresponding to those controls and initial data.* This requires some extra attention from the reader but has the advantage to avoid cumbersome expressions.

Definition 6.4. Let us fix $(\tau, y, k) \in [0, T[\times \mathbb{R}^n \times [0, 1]$. We say that a map

$$\psi : \hat{\mathcal{A}}_{\tau, y} \rightarrow \hat{C}_{\tau, k}$$

is a *space-time strategy* if, for every $\bar{t} \in [\tau, T]$ and every pair $a, \tilde{a} \in \hat{\mathcal{A}}_{\tau, y}$ such that

$$a(t) = \tilde{a}(t) \quad \forall t \in [\tau, \bar{t}],$$

one has

$$\psi[a] \circ s[\psi[a], \tau, k](t) = \psi[\tilde{a}] \circ s[\psi[\tilde{a}], \tau, k](t) \quad \forall t \in [\tau, \bar{t}].$$

The set of space-time strategies from $\hat{\mathcal{A}}_{\tau, y}$ to $\hat{C}_{\tau, k}$ will be denoted by $\hat{\Gamma}_{\tau, k, y}$.

Definition 6.5. Let us define the *space-time upper value*

$$\hat{\mathcal{U}} : [0, T] \times \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}$$

by setting

$$\hat{\mathcal{U}}(\tau, y, k) \doteq \sup_{\psi \in \hat{\Gamma}_{\tau, y, k}} \inf_{a \in \hat{\mathcal{A}}_{\tau, y}} \hat{\mathcal{G}}_{\tau, y, k}(a, \psi[a]) \quad \forall \tau < T,$$

and

$$\hat{\mathcal{U}}(T, y, k) = (1 - k) \inf_{b \in A} l(T, y, b) + g(y) = (1 - k) \tilde{l}(y) + g(y).$$

Proposition 6.1. For any $(\tau, y, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$ one has

$$\mathcal{W}(\tau, y, k) \leq \hat{\mathcal{U}}(\tau, y, k) \tag{29}$$

Proof. When $k = 1$ and $t < T$ we have:

$$\begin{aligned} \hat{\mathcal{U}}(\tau, y, 1) &= \inf_{a \in \hat{\mathcal{A}}_{\tau, y}} \int_0^{T-\tau} L(t(s), z(s), a \circ t(s)) \, ds + g(z[a; (\tau, y)](T - \tau)) \\ &= \inf_{a \in \hat{\mathcal{A}}_{\tau}} \int_{\tau}^T L(t, x[a; (\tau, y)](t), a(t)) \, dt + g(x[a; (\tau, y)](T)) \\ &= \mathcal{W}(\tau, y, 1). \end{aligned}$$

If $t = T$, there is nothing to prove, since $\mathcal{W}(T, y, k) = \hat{\mathcal{U}}(T, y, k)$, by definition.

Let us consider the case when $(t, k) \in [0, T[\times [0, 1]$. For every $t \in [\tau, T]$, $a \in \hat{\mathcal{A}}_{\tau, y}$ and any $k \in [0, 1]$, we set

$$\begin{aligned} \mathcal{G}_{\tau, y, k}(a)(t) &\doteq (1 - k)l(t, x[a; (\tau, y)](t), a(t)) \\ &\quad + \int_{\tau}^T L(s, x[a; (\tau, y)](s), a(s)) \, ds + g(x[a; (\tau, y)](T)). \end{aligned}$$

Moreover, for every $a \in \hat{\mathcal{A}}_{\tau, y}$, let $t_a \in [\tau, T]$ be defined by

$$t_a \doteq \min\{t \in [\tau, T] \mid \mathcal{G}_{\tau, y, k}(a)(t) \geq \mathcal{W}(\tau, y, k)\}.$$

Notice that t_a is well defined, for the set $B \doteq \{t \in [\tau, T] \mid \mathcal{G}_{\tau, y, k}(a)(t) \geq \mathcal{W}(\tau, y, k)\}$ is non empty and compact. Indeed, from the upper semicontinuity of the map $t \rightarrow l(t, x[a; (\tau, y)](t), a(t))$, not only we know that

$$\begin{aligned}
\max_{t \in [\tau, T]} \mathcal{G}_{\tau, y, k}(a)(t) &\geq (1-k) \|l(\cdot, x[a; (\tau, y)](\cdot), a(\cdot))\|_\infty \\
&\quad + \int_\tau^T L(s, x[a; (\tau, y)](s), a(s)) \, ds + g(x[a; (\tau, y)](T)) \\
&\geq \inf_{a \in \hat{\mathcal{A}}_\tau} \mathcal{F}^{\infty+B}_{\tau, y}(a) = \mathcal{W}(\tau, y, k),
\end{aligned}$$

(that is, B is non-empty) but we also deduce that B is closed and, consequently, compact. Let us define the map

$$\psi^* : \hat{\mathcal{A}}_{\tau, y} \rightarrow \hat{C}_{\tau, k}$$

by setting, for every $a \in \hat{\mathcal{A}}_{\tau, y}$,

$$\psi^*[a](s) \doteq \chi_{[t_a - T, S_{\tau, k} - T + t_a]}(s), \quad \text{for all } s \in [0, S_{\tau, k}].$$

Notice that

$$1 - \psi^*[a](s) = \chi_{[0, t_a - \tau] \cup (S_{\tau, k} - T + t_a, S_{\tau, k}]}(s) \quad \text{for all } s \in [0, S_{\tau, k}].$$

It is straightforward to verify that the map ψ^* is in fact a space-time strategy. Moreover, for every $a \in \hat{\mathcal{A}}_{\tau, y}$, we have

$$\begin{aligned}
&\hat{\mathcal{G}}_{\tau, y, k}(a, \psi^*[a]) \\
&= \int_0^{S_{\tau, k}} l(t(s), z(s), a \circ t(s)) \cdot \chi_{[t_a - \tau, S_{\tau, k} - T + t_a]}(s) \, ds \\
&\quad + \int_0^{S_{\tau, k}} L(t(s), z(s), a \circ t(s)) \cdot (\chi_{[0, t_a - \tau] \cup (S_{\tau, k} - T + t_a, S_{\tau, k}]}(s)) \, ds + g(z(S_{\tau, k})) \\
&= \int_\tau^T l(t_a, x[a; (\tau, y)](t_a), a(t_a)) \cdot \chi_{[t_a, t_a + 1 - k]}(t) \, dt \\
&\quad + \int_\tau^T L(t, x[a; (\tau, y)](t), a(t)) \, dt + g(x[a; (\tau, y)](T)) \\
&= (1-k) l(t_a, x[a; (\tau, y)](t_a), a(t_a)) \\
&\quad + \int_\tau^T L(t, x[a; (\tau, y)](t), a(t)) \, dt + g(x[a; (\tau, y)](T)) \\
&= \mathcal{G}_{\tau, y, k}(a)(t_a) \geq \mathcal{W}(\tau, y, k).
\end{aligned}$$

Hence

$$\hat{\mathcal{U}}(\tau, y, k) \geq \inf_{a \in \hat{\mathcal{A}}_{\tau, y}} \hat{\mathcal{G}}_{\tau, y, k}(a, \psi^*[a]) \geq \mathcal{W}(\tau, y, k),$$

so the proposition is proved. $\square$

7. Step (IV): the inequality $\hat{\mathcal{U}} \leq \mathcal{U}$

In the next subsections we will prove Theorem 7.1 below.

Theorem 7.1. *The function $\hat{\mathcal{U}}$ is a viscosity subsolution of (HJ) in $\overline{\Omega} \setminus \partial_T \Omega (= [0, T] \times \mathbb{R}^n \times [0, 1])$.*

We get the desired inequality, namely $\hat{\mathcal{U}} \leq \mathcal{U}$, by a comparison argument:

Corollary 7.1. *For every $(t, x, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$ we have*

$$\hat{\mathcal{U}}(t, x, k) \leq \mathcal{U}(t, x, k).$$

Proof. Let us set $u_1 \doteq \hat{\mathcal{U}}, u_2 = \mathcal{U}$. In view of Theorem 4.1, u_2 is continuous and is a supersolution of equation (HJ) on $[0, T] \times \mathbb{R}^n \times [0, 1]$. On the other hand, by Theorem 7.1 u_1 is a subsolution of equation (HJ) on $[0, T] \times \mathbb{R}^n \times [0, 1]$. Finally, let us observe that

$$u_2(T, x, k) = \mathcal{U}(T, x, k) = (1 - k) \inf_{b \in A} l(T, y, b) + g(y) = \hat{\mathcal{U}}(T, x, k) = u_1(T, x, k) \quad (30)$$

for all $(x, k) \in \mathbb{R}^n \times [0, 1]$. Hence, by Theorem 2.1, one has

$$\hat{\mathcal{U}} = u_1 \leq u_2 = \mathcal{U}$$

on $[0, T] \times \mathbb{R}^n \times [0, 1]$. $\square$

7.1. Dynamic programming for $\hat{\mathcal{U}}$

Theorem 7.2. *For every $(\tau, x, k) \in [0, T] \times \mathbb{R}^n \times [0, 1]$ and $0 < \sigma < S_{\tau, k}$ we have*

$$\begin{aligned} \hat{\mathcal{U}}(\tau, x, k) = & \sup_{\psi \in \hat{\Gamma}_{\tau, x, k}} \inf_{a \in \hat{\mathcal{A}}_{\tau, x}} \left\{ \int_0^\sigma l(t(s), z(s), a \circ t(s)) \cdot \psi[a](s) ds \right. \\ & \left. + \int_0^\sigma L(t(s), z(s), a \circ t(s)) \cdot (1 - \psi[a](s)) ds \right\} + \hat{\mathcal{U}}(t(\sigma), z(\sigma), k(\sigma)). \end{aligned} \quad (31)$$

Proof. Let $R(\tau, x, k)$ denote the right-hand side of (31), and let us prove that

$$\hat{\mathcal{U}}(\tau, x, k) \geq R(\tau, x, k) \quad (32)$$

For any $\delta > 0$ there exists a strategy $\psi^1 \in \hat{\Gamma}_{\tau, x, k}$ such that

$$\begin{aligned} R(\tau, x, k) \leq & \inf_{a \in \hat{\mathcal{A}}_{\tau, x}} \left\{ \int_0^\sigma l(t(s), z(s), a \circ t(s)) \cdot \psi^1[a](s) ds \right. \\ & \left. + \int_0^\sigma L(t(s), z(s), a \circ t(s)) \cdot (1 - \psi^1[a](s)) ds \right\} + \hat{\mathcal{U}}(t(\sigma), z(\sigma), k(\sigma)) + \delta. \end{aligned}$$

Now, for every $a \in \hat{\mathcal{A}}_{\tau, x}$ there exists a strategy $\psi^a \in \hat{\Gamma}_{t(\sigma), z(\sigma), k(\sigma)}$ such that

$$\hat{\mathcal{U}}(t(\sigma), z(\sigma), k(\sigma)) \leq \inf_{b \in \hat{\mathcal{A}}_{t(\sigma), z(\sigma)}} \hat{\mathcal{G}}_{t(\sigma), z(\sigma), k(\sigma)}(a, \psi^a[b]) + \delta.$$

Let us define a strategy $\psi^2 \in \hat{\Gamma}_{\tau, x, k}^\sigma$ by setting, for every $a \in \hat{\mathcal{A}}_{\tau, x}$

$$\psi^2[a](s) \doteq \begin{cases} \psi^1[a](s) & \forall s \in [0, \sigma] \\ \psi^a[a|_{[t(\sigma), T]}](s - \sigma) & \forall s \in [\sigma, \sigma + S_{\tau, k}]. \end{cases}$$

We claim that that ψ^2 is in fact a space-time strategy belonging to $\hat{\Gamma}_{\tau, x, k}$. Indeed, clearly $\psi^2[a] \in \hat{\mathcal{C}}_{\tau, k}$ for each $a \in \hat{\mathcal{A}}_{\tau, x}$. Moreover, ψ^2 is non-anticipative (in the sense of Definition 6.4). To see this take $a_1, a_2 \in \hat{\mathcal{A}}_{\tau, x}$ such that $a_1(t) = a_2(t)$ for all $s \in [\tau, \tau_1]$ for some $\tau_1 \in [\tau, T]$. If $\sigma_1 \doteq s[\psi^2[a_1], \tau, k](\tau_1) =$

$s[\psi^2[a_2], \tau, k](\tau_1)$ —see Definition 6.3—is such that $\sigma_1 \leq \sigma$, then, since $\psi^1 \in \hat{\Gamma}_{\tau, x, k}$, we obtain, for all $t \in [\tau, \tau_1]$,

$$\begin{aligned}\psi^2[a_1] \circ s[\psi^2[a_1], \tau, k](t) &= \psi^1[a_1] \circ s[\psi^1[a_1], \tau, k](t) = \psi^1[a_2] \circ s[\psi^1[a_2], \tau, k](t) \\ &= \psi^2[a_2] \circ s[\psi^2[a_2], \tau](t).\end{aligned}$$

So, the non anticipativity property is proved for $\sigma_1 \leq \sigma$. As for the case $\sigma_1 \geq \sigma$, nonanticipativity of ψ^2 follows from the fact that $\psi^a \in \hat{\Gamma}_{t(\sigma), z(\sigma), k(\sigma)}$.

Now observe that

$$\begin{aligned}R(\tau, x, k) &\leq \inf_{a \in \hat{\mathcal{A}}_{\tau, x}} \left\{ \int_0^\sigma l(t(s), z(s), a \circ t(s)) \cdot \psi^1[a](s) ds \right. \\ &\quad \left. + \int_0^\sigma L(t(s), z(s), a \circ t(s)) \cdot (1 - \psi^1[a](s)) ds \right\} \\ &\quad + \inf_{b \in \hat{\mathcal{A}}_{t(\sigma), z(\sigma), k(\sigma)}} \left\{ \hat{\mathcal{G}}_{t(\sigma), z(\sigma), k(\sigma)}(b, \psi^a[b]) \right\} + 2\delta \\ &\leq \inf_{a \in \hat{\mathcal{A}}_{\tau, x}} \left\{ \hat{\mathcal{G}}_{\tau, x, k}(a, \psi^2[a]) \right\} + 2\delta \\ &\leq \hat{\mathcal{U}}(\tau, x, k) + 2\delta.\end{aligned}$$

By the arbitrariness of δ , (32) is proved.

Let us prove the inequality

$$\hat{\mathcal{U}}(\tau, x, k) \leq R(\tau, x, k) \quad (33)$$

Let $\delta > 0$ and let us select a strategy $\psi^1 \in \hat{\Gamma}_{\tau, x, k}$ such that

$$-\delta + \hat{\mathcal{U}}(\tau, x, k) \leq \inf_{a \in \hat{\mathcal{A}}_{\tau, x}} \left\{ \hat{\mathcal{G}}_{\tau, x, k}(a, \psi^1[a]) \right\}. \quad (34)$$

Then,

$$\begin{aligned}R(\tau, x, k) &\geq \inf_{a \in \hat{\mathcal{A}}_{\tau, x}} \left\{ \int_0^\sigma l(t(s), z(s), a \circ t(s)) \cdot \psi^1[a](s) ds \right. \\ &\quad \left. + \int_0^\sigma L(t(s), z(s), a \circ t(s)) \cdot (1 - \psi^1[a](s)) ds \right\} + \hat{\mathcal{U}}(t(\sigma), z(\sigma), k(\sigma)).\end{aligned}$$

We choose $a_1 \in \hat{\mathcal{A}}_{\tau, x}$ such that

$$\begin{aligned}R(\tau, x, k) &\geq \left\{ \int_0^\sigma l(t(s), z(s), a_1 \circ t(s)) \cdot \psi^1[a_1](s) ds \right. \\ &\quad \left. + \int_0^\sigma L(t(s), z(s), a_1 \circ t(s)) \cdot (1 - \psi^1[a_1](s)) ds \right\} + \hat{\mathcal{U}}(t(\sigma), z(\sigma), k(\sigma)) - \delta.\end{aligned} \quad (35)$$

Let us construct a strategy $(\psi_0^2, \psi^2) \in \hat{\Gamma}_{t(\sigma), z(\sigma), k(\sigma)}$ as follows. For every $a \in \hat{\mathcal{A}}_{t(\sigma), z(\sigma)}$ let us define $\hat{a} \in \hat{\mathcal{A}}_{\tau, x}$ by setting

$$\hat{a}(t) = \begin{cases} a_1(t) & \forall t \in [\tau, t(\sigma)] \\ a(t) & \forall t \in]t(\sigma), T] \end{cases}$$

while we set $\hat{a}(t(\sigma))$ to be any value $b \in \{a_1(t(\sigma)), a(t(\sigma))\}$ such that

$$l(t(\sigma), z(t(\sigma)), b) = \max \{l(t(\sigma), z(t(\sigma)), a_1(t(\sigma))), l(t(\sigma), z(t(\sigma)), a(t(\sigma)))\}.$$

(Notice that the ambiguity of the definition of $\hat{a}$ at σ does not affect the value of the functional $\hat{\mathcal{G}}$).

Let us define $\psi^2 \in \hat{\Gamma}_{t(\sigma), z(\sigma), k(\sigma)}$ by setting, for every $a \in \hat{\mathcal{A}}_{t(\sigma), y(\sigma)}$,

$$\psi^2[a](s) = (\psi_0^1, \psi^1)[\hat{a}](s + \sigma).$$

As before we can easily verify that ψ^2 is in fact a strategy belonging to $\hat{\Gamma}_{t(\sigma), z(\sigma), k(\sigma)}$. In particular one has

$$\hat{\mathcal{U}}(t(\sigma), z(\sigma), k(\sigma)) \geq \inf_{a \in \hat{\mathcal{A}}_{t(\sigma), z(\sigma)}} \left\{ \hat{\mathcal{G}}_{t(\sigma), z(\sigma), k(\sigma)}(a, \psi^2[a]) \right\}$$

so there exists $a_2 \in \hat{\mathcal{A}}_{t(\sigma), z(\sigma)}$ such that

$$\hat{\mathcal{U}}(t(\sigma), z(\sigma), k(\sigma)) \geq \left\{ \hat{\mathcal{G}}_{t(\sigma), z(\sigma), k(\sigma)}(a_2, \psi^2[a_2]) \right\} - \delta.$$

Therefore, in view of (34) and (35) one obtains

$$\begin{aligned} R(\tau, x, k) &\geq \int_0^\sigma l(t(s), z(s), a_1 \circ t(s)) \cdot \psi^1[a_1](s) ds \\ &+ \int_0^\sigma L(t(s), z(s), a_1 \circ t(s)) \cdot (1 - \psi^1[a_1](s)) ds + \hat{\mathcal{G}}_{t(\sigma), z(\sigma), k(\sigma)} \\ & (a_2, \psi^2[a_2]) - 2\delta = \hat{\mathcal{G}}_{\tau, x, k}(\hat{a}_2, \psi^1[\hat{a}_2]) - 2\delta \geq \hat{\mathcal{U}}(\tau, x, k) - 3\delta, \end{aligned}$$

which, by the arbitrariness of δ , concludes the proof of (33). $\square$

7.2. Proof of Theorem 7.1

We show that the function $\hat{\mathcal{U}}$ is a viscosity subsolution of (HJ) on $[0, T] \times \mathbb{R}^n \times [0, 1]$ (in the sense of Definition 1.2).

Let $(\bar{t}, \bar{x}, \bar{k}) \in \bar{\Omega}$ with $\bar{t} < T$, and let φ be a C^1 map on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ such that the function $\hat{\mathcal{U}}^* - \varphi$ has a local strict maximum at $(\bar{t}, \bar{x}, \bar{k})$ in $\bar{B}((\bar{t}, \bar{x}, \bar{k}), \bar{r}) \cap \bar{\Omega}$, for some radius $\bar{r} > 0$. It is not restrictive to assume that

$$\hat{\mathcal{U}}^*(\bar{t}, \bar{x}, \bar{k}) = \varphi(\bar{t}, \bar{x}, \bar{k}),$$

so that $\hat{\mathcal{U}}^*(t, x, k) \leq \varphi(t, x, k)$, for every $(t, x, k) \in \bar{B}((\bar{t}, \bar{x}, \bar{k}), \bar{r}) \cap \bar{\Omega}$.

Let us define the (un-max-minimized) Hamiltonian map $\mathcal{H}(t, x, k, p_t, p_x, p_k, a, w)$ by setting

$$\begin{aligned} \mathcal{H}(t, x, k, p_t, p_x, p_k, a, w) &\doteq (p_t(t, x, k) + p_x(t, x, k) \cdot f(t, x, a) + L(t, x, a)) \\ &\times (1 - w) + (l(t, x, a) + p_k(t, x, k)) w. \end{aligned}$$

We have to prove that

$$\mathbf{H} \left(\bar{t}, \bar{x}, \bar{k}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k} \right) = \min_{a \in A} \max_{w \in [0, 1]} \mathcal{H} \left(\bar{t}, \bar{x}, \bar{k}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, a, w \right) \geq 0.$$

Let us assume, by contradiction, that there exists $\Theta \in (0, 1)$ such that

$$\min_{a \in A} \max_{w \in [0, 1]} \mathcal{H} \left(\bar{t}, \bar{x}, \bar{k}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, a, w \right) \leq -\Theta.$$

By continuity of $\mathcal{H}$, there exists a radius $\bar{\rho} > 0$ such that, for any $(t, x, k) \in \overline{B}((\bar{t}, \bar{x}, \bar{k}), \bar{\rho}) \cap \Omega$,

$$\min_{a \in A} \max_{w \in [0, 1]} \mathcal{H} \left(\bar{t}, \bar{x}, \bar{k}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, a, w \right) \leq -\Theta/2.$$

Let (t_n, x_n, k_n) be a sequence in $\overline{B}((\bar{t}, \bar{x}, \bar{k}), \bar{\rho}) \cap \overline{\Omega}$ such that $(t_n, x_n, k_n) \rightarrow (\bar{t}, \bar{x}, \bar{k})$ and $\hat{\mathcal{U}}(t_n, x_n, k_n) \rightarrow \hat{\mathcal{U}}^*(\bar{t}, \bar{x}, \bar{k})$.

Since $(t_n, x_n, k_n) \rightarrow (\bar{t}, \bar{x}, \bar{k})$, by Lemma 7.1 below we have that there is $\hat{\sigma}$ such that for each n there exists a control $\hat{a}_n \in \hat{\mathcal{A}}_{t_n, x_n}$ which, for every strategy $\psi^n \in \hat{\Gamma}_{t_n, x_n, k_n}$ and any $\sigma \in [0, \hat{\sigma}]$, verifies

$$\int_0^\sigma \mathcal{H} \left(t_n(s), z_n(s), k_n(s), \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, \hat{a}_n \circ t_n(s), \psi^n[\hat{a}_n](s) \right) ds \leq -\Theta\sigma/4. \quad (36)$$

Let us set

$$\epsilon_n \doteq |\hat{\mathcal{U}}(t_n, x_n, k_n) - \varphi(t_n, x_n, k_n)|.$$

Clearly, $\epsilon_n \rightarrow 0$. By Theorem 7.2, (36), and the trivial inequality $\hat{\mathcal{U}} \leq \hat{\mathcal{U}}^*$, one obtains

$$\begin{aligned} 0 &= \sup_{\psi \in \hat{\Gamma}_{t_n, x_n, k_n}} \inf_{a \in \hat{\mathcal{A}}_{t_n, x_n}} \left\{ \int_0^\sigma l(t_n(s), z_n(s), a \circ t_n(s)) \cdot \psi[a](s) ds \right. \\ &\quad \left. + \int_0^\sigma L(t_n(s), z_n(s), a \circ t_n(s)) \cdot (1 - \psi[a](s)) ds \right\} \\ &\quad + \hat{\mathcal{U}}(t_n(\sigma), z_n(\sigma), k_n(\sigma)) - \hat{\mathcal{U}}(t_n, x_n, k_n) \\ &\leq \sup_{\psi \in \hat{\Gamma}_{t_n, x_n, k_n}} \inf_{a \in \hat{\mathcal{A}}_{t_n, x_n}} \left\{ \int_0^\sigma l(t_n(s), z_n(s), a \circ t_n(s)) \cdot \psi[a](s) ds \right. \\ &\quad \left. + \int_0^\sigma L(t_n(s), z_n(s), a \circ t_n(s)) \cdot (1 - \psi[a](s)) ds \right\} \\ &\quad + \hat{\mathcal{U}}^*(t_n(\sigma), z_n(\sigma), k_n(\sigma)) - \hat{\mathcal{U}}(t_n, x_n, k_n) \\ &\leq \sup_{\psi \in \hat{\Gamma}_{t_n, x_n, k_n}} \inf_{a \in \hat{\mathcal{A}}_{t_n, x_n}} \left\{ \int_0^\sigma l(t_n(s), z_n(s), a \circ t_n(s)) \cdot \psi[a](s) ds \right. \\ &\quad \left. + \int_0^\sigma L(t_n(s), z_n(s), a \circ t_n(s)) \cdot (1 - \psi[a](s)) ds \right\} \\ &\quad + \varphi(t_n(\sigma), z_n(\sigma), k_n(\sigma)) - \varphi(t_n, x_n, k_n) + \epsilon_n \\ &\leq \sup_{\psi \in \hat{\Gamma}_{t_n, x_n, k_n}} \int_0^\sigma \mathcal{H} \left(t_n(s), z_n(s), k_n(s), \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, \hat{a}_n \circ t_n(s), \psi^n[\hat{a}_n](s) \right) ds \\ &\quad + \epsilon_n \leq -\frac{\Theta\sigma}{4} + \epsilon_n. \end{aligned}$$

Passing to the limit for $n \rightarrow \infty$, we get $0 \leq -\frac{\Theta\sigma}{4}$, a contradiction. $\square$

Lemma 7.1. *Suppose that all the hypotheses of Sect. 2.1 are verified and fix a C^1 function $\varphi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$. Let us assume that for a point $(t_0, x_0, k_0) \in [0, T] \times \mathbb{R}^n \times [0, 1]$ there exists $\eta > 0$ such that*

$$\min_{a \in A} \max_{w \in [0, 1]} \mathcal{H} \left(t_0, x_0, k_0, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, a, w \right) \leq -\eta.$$

Then, there exist numbers $r_0 > 0$ and $\hat{\sigma} \in (0, T - t)$ such that for any $(t, x, k) \in \overline{B}((t_0, x_0, k_0), r_0) \cap \overline{\Omega}$ we can find a control $\hat{a} \in \hat{\mathcal{A}}_{t,x}$ with the following property: for every strategy $\psi \in \hat{\Gamma}_{t,x,k}$ and for every $\sigma \in [0, \hat{\sigma}]$ we have

$$\int_0^\sigma \mathcal{H} \left(t(s), z(s), k(s), \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, \hat{a} \circ t(s), \psi[\hat{a}](s) \right) ds \leq -\frac{\eta \sigma}{2}.$$

Proof. Assume that

$$\min_{a \in A} \max_{w \in [0, 1]} \mathcal{H} \left(t_0, x_0, k_0, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, a, w \right) \leq -\eta.$$

Then, since A is compact and $\mathcal{H}$ is continuous, there exists $\tilde{a} \in A$ such that

$$\max_{w \in [0, 1]} \mathcal{H} \left(t_0, x_0, k_0, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, \tilde{a}, w \right) \leq -\eta.$$

Now notice that from the assumptions on the data (see Sect. 2.1), once we have fixed a radius $\rho_0 = |(t_0, x_0, k_0)| + 1$, the number $R_0 = e^{(M+2)T}(1 + \rho_0)$ is an upper bound for the magnitude of all trajectories $\sigma \rightarrow (t(\sigma), z(\sigma), k(\sigma))$ starting at $(t, x, k) \in \overline{B}((t_0, x_0, k_0), 1)$, which are solution to system (26). Thus $M_0 = (M_0 + 2)(1 + R_0)$ is an upper bound for the speed of $(t(\cdot), z(\cdot), k(\cdot))$. As a consequence for any $a \in \hat{\mathcal{A}}_{t,x}$ and for every strategy $\psi \in \hat{\Gamma}_{t,x,k}$, for the associated trajectory $(t(\cdot), z(\cdot), k(\cdot))$ we have the estimate

$$|(t(\sigma), z(\sigma), k(\sigma)) - (t, x, k)| \leq M_0 \sigma.$$

Then, there exists $r_0 \in (0, 1)$ and $\hat{\sigma} \in (0, T - t)$ such that if we take the control $\hat{a}(\cdot) \equiv \tilde{a} \in \hat{\mathcal{A}}_{t,x}$ we have

$$\mathcal{H} \left(t(s), z(s), k(s), \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial k}, \hat{a} \circ t(s), \psi[\hat{a}](s) \right) \leq -\frac{\eta}{2}$$

for every $s \in [0, \hat{\sigma}]$ and for all strategy $\psi \in \hat{\Gamma}_{t,x,k}$. Then the assertion of the lemma immediately follows. $\square$

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