Non-uniform dependence on initial data for the periodic modified Camassa–Holm equation

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Abstract. In this paper, we study the periodic Cauchy problem for the modified Camassa–Holm equation

$$m_t + um_x + 2u_xm = 0, \quad m = (1 - \partial_x^2)^2 u,$$

and show that the solution map is not uniformly continuous in Sobolev spaces $H^s(\mathbb{T})$ for $s > 7/2$. Our proof is based on the method of approximate solutions and well-posedness estimates for the actual solutions.

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1. Introduction

Consider the periodic Cauchy problem for the modified Camassa–Holm (mCH) equation

$$m_t + um_x + 2u_xm = 0, \quad m = (1 - \partial_x^2)^k u, \quad (1.1)$$

$$u(x, 0) = u_0(x) \in H^s(\mathbb{T}), \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad t > 0. \quad (1.2)$$

With $k = 1$, Eq. (1.1) becomes the well-known Camassa–Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \quad (1.3)$$

The CH equation was first derived by Fokas and Fuchssteiner [7] as a bi-Hamiltonian system, and then by Camassa and Holm [4] as a model for shallow water waves. As shown in [4], the CH equation is completely integrable and bi-Hamiltonian, and possesses an infinite number of conservation laws and peaked soliton solutions (peakons). Peakons are true solitons that interact via elastic

collisions under the CH dynamics. By introducing the notion of weak solution, Constantin and Escher [5] provided a mathematical framework for the study of soliton interaction. Moreover, the CH equation has wave breaking phenomena (i.e. the solution remains bounded while its slope becomes unbounded in finite time [6]), while KdV equation does not have [14]. Wave breaking is one of the most intriguing long-standing problems of water wave theory [22]. As mentioned by Whitham [22], it is intriguing to know which mathematical models for shallow water waves exhibit both phenomena of soliton interaction and wave breaking. The CH equation is a such equation.

The well-posedness of the CH equation has been intensively studied. In the periodic case, the local well-posedness was studied by Misiolek [18] in $H^s(\mathbb{T})$ for $s > 3/2$. Non-uniform dependence on initial data in $H^s(\mathbb{T})$ was proved by Himonas et al. [9] for $s > 3/2$. In the non-periodic case, the local well-posedness was studied in $H^s(\mathbb{R})$ for $s > 3/2$ by Li and Olver [16] and Rodríguez-Blanco [19]. Non-uniform dependence in $H^s(\mathbb{R})$ was proved by Himonas and Kenig [8] for $s > 3/2$. The CH equation possesses not only global solutions but also wave-breaking solutions. It is worth noticing that Bressan and Constantin developed a new approach to the analysis of the CH equation, and proved the existence of the global conservative solutions [2] and global dissipative solutions [3] in $H^1(\mathbb{R})$. This new approach resolved all singularities due to possible wave breaking. From a Lagrangian point of view, Holden and Raynaud also obtained the global conservative solutions [11, 12] and global dissipative solutions [10].

Since Arnold [1] introduced the ideas that view the Euler fluid equations as the geodesic equations on diffeomorphism groups, a new stage of development of the Euler equations appeared. Now the geodesic equations on Lie groups are called generalized Euler equations. In [15], Khesin and Misiolek showed that the CH equation is the geodesic equation associated to $H^1(\mathbb{T})$ metric on the diffeomorphism group of the unit circle $\mathbb{T}$, which is the Euler–Poincaré equation by using the Lagrangian associated with the H^1 metric for the fluid velocity.

Recently, McLachlan and Zhang [17] derived the mCH equation as the geodesic equation on diffeomorphism group with respect to the invariant H^k metric, and studied the well-posedness and weak solutions. For $k = 2$, they showed that the Cauchy problem (1.1)–(1.2) is locally well-posed in $H^s(\mathbb{T})$ for $s > 7/2$ and the solution depends continuously on initial data. As shown in [17], the mCH equation has some significant difference from that of the CH equation, e.g., it does not admit finite time blowup solutions.

In this paper, we study the Cauchy problem (1.1)–(1.2) with $k = 2$ and show that the dependence on initial data is not uniformly continuous in $H^s(\mathbb{T})$ for $s > 7/2$. The proof is based on the method of approximate solutions and well-posedness estimates for the actual solutions. Our main result is as follows.

Theorem 1.1. *If $k = 2$ and $s > 7/2$, then the solution map $u_0 \rightarrow u(t)$ of the periodic mCH equation is not uniformly continuous from any bounded subset of $H^s(\mathbb{T})$ into $C([0, T]; H^s(\mathbb{T}))$. More precisely, there exist two sequences of*

solutions $u_n(t)$ and $v_n(t)$ in $C([0, T]; H^s(\mathbb{T}))$ such that

$$\|u_n(t)\|_{H^s} + \|v_n(t)\|_{H^s} \lesssim 1, \quad (1.4)$$

$$\lim_{n \rightarrow \infty} \|u_n(0) - v_n(0)\|_{H^s} = 0, \quad (1.5)$$

and

$$\liminf_{n \rightarrow \infty} \|u_n(t) - v_n(t)\|_{H^s} \gtrsim |\sin t|, \quad 0 \leq t \leq T. \quad (1.6)$$

In this paper, the symbols $\lesssim$ and $\gtrsim$ are used to express the corresponding inequality that includes a universal constant. For example, $f(x) \lesssim g(x)$ means that $f(x) \leq cg(x)$ for some $c > 0$. The remainder of this paper is organized as follows. In Sect. 2 we recall some known results, which are necessary for later proof. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section, we introduce some known results for later proof. For $\varepsilon \in (0, 1]$, let J_ε be the Friedrichs mollifier defined by

$$J_\varepsilon f(x) = j_\varepsilon * f(x), \quad (2.1)$$

where $j(x)$ is a C^∞ function supported in the interval $[-1, 1]$ such that $j(x) \geq 0$, $\int_{\mathbb{R}} j(x) dx = 1$ and $j_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon})$. For $s \in \mathbb{R}$, the operator $D^s = (1 - \partial_x^2)^{s/2}$ is defined by

$$\widehat{D^s f}(\xi) = (1 + \xi^2)^{s/2} \hat{f}(\xi), \quad (2.2)$$

where $\hat{f}$ is the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{T}} e^{-ix\xi} f(x) dx. \quad (2.3)$$

The inverse relation is given by

$$f(x) = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} \hat{f}(\xi) e^{ix\xi}. \quad (2.4)$$

Then, for $f \in H^s(\mathbb{T})$ we have

$$\|f\|_{H^s}^2 = \sum_{\xi \in \mathbb{Z}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 = \|D^s f\|_{L^2}^2. \quad (2.5)$$

Lemma 2.1. ([17]) *Let $k = 2$, $u_0 \in H^s(\mathbb{T})$, $s > 7/2$. Then there exist a $T > 0$ depending on $\|u_0\|_{H^s}$ and a unique solution $u(t)$ to the Cauchy problem (1.1)–(1.2) such that*

$$u(t) \in C([0, T]; H^s(\mathbb{T})) \cap C^1([0, T]; H^{s-1}(\mathbb{T})).$$

Moreover, the solution map

$$u_0 \rightarrow u(t) : H^s(\mathbb{T}) \rightarrow C([0, T]; H^s(\mathbb{T})) \cap C^1([0, T]; H^{s-1}(\mathbb{T}))$$

is continuous.

Lemma 2.2. (Kato-Ponce commutator estimate, [13])

If $s > 0$, then there is a $C_s > 0$ such that for any $f, g \in H^s(\mathbb{T})$

$$\|D^s(fg) - fD^s g\|_{L^2} \leq C_s(\|D^s f\|_{L^2}\|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty}\|D^{s-1}g\|_{L^2}). \quad (2.6)$$

Lemma 2.3. ([8]) Let $u(x)$ be a function such that $\|\partial_x u\|_{L^\infty} < \infty$. Then, there is a $c > 0$ such that for any $f \in L^2(\mathbb{T})$

$$\|[J_\varepsilon, u]\partial_x f\|_{L^2} \leq c\|\partial_x u\|_{L^\infty}\|f\|_{L^2}. \quad (2.7)$$

Lemma 2.4. (Moser's estimate, [21]) If $s > 0$, then there is a $C_s > 0$ such that for any $f, g \in H^s(\mathbb{T})$

$$\|fg\|_{H^s} \leq C_s(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}). \quad (2.8)$$

Lemma 2.5. ([9]) Let $\sigma, \alpha \in \mathbb{R}$. If $n \in \mathbb{Z}^+$ and $n \gg 1$, then

$$\|\cos(nx - \alpha)\|_{H^\sigma} = \sqrt{2}\pi(1 + n^2)^{\sigma/2} \approx n^\sigma. \quad (2.9)$$

Relation (2.9) is also true if $\cos(nx - \alpha)$ is replaced by $\sin(nx - \alpha)$.

Lemma 2.6. ([20]) If $s > 3/2$ and $0 \leq \sigma + 1 \leq s$, then there is a $c > 0$ such that

$$\|[D^\sigma \partial_x, f]v\|_{L^2} \leq c\|f\|_{H^s}\|v\|_{H^\sigma}. \quad (2.10)$$

Lemma 2.7. ([9]) For $1/2 < \sigma < 1$, we have

$$\|fg\|_{H^{\sigma-1}} \leq C\|f\|_{H^\sigma}\|g\|_{H^{\sigma-1}}. \quad (2.11)$$

3. Non-uniform dependence for $s > 7/2$

In this section we prove Theorem 1.1. Following the approach in [9], we choose approximate solutions to the mCH equation such that the size of the difference between approximate and actual solutions with identical initial data is negligible. Hence, to understand the degree of dependence, it suffices to focus on the behavior of approximate solutions, which are simple in form, rather than actual solutions.

3.1. Well-posedness estimates for actual solutions

In order for the method of approximate solutions to go through, we need well-posedness estimates for the size of the actual solutions.

Proposition 3.1. For $k = 2$ and $s > 7/2$, the following results hold:

1. If $u_0 \in H^s(\mathbb{T})$, then there exists a unique solution $u(t)$ to the Cauchy problem (1.1)–(1.2) in $C([0, T_0]; H^s(\mathbb{T}))$, where the life span T_0 depends on $\|u_0\|_{H^s}$. Moreover, we have

$$T_0 > T = \frac{1}{2c_s\|u_0\|_{H^s}}. \quad (3.1)$$

2. The solution map $u_0 \rightarrow u(t)$ is continuous from any bounded subset of $H^s(\mathbb{T})$ into $C([0, T]; H^s(\mathbb{T}))$, and $u(t)$ satisfies the estimate

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad 0 \leq t \leq T. \quad (3.2)$$

Proof. According to Lemma 2.1, we need only to prove the estimates (3.1) and (3.2). For this purpose, it is more convenient to rewrite the Cauchy problem (1.1)–(1.2) in the following form

$$\partial_t u = -u\partial_x u - D^{-4}\partial_x \left(u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} \right), \quad (3.3)$$

$$u(x, 0) = u_0(x) \in H^s(\mathbb{T}), \quad x \in \mathbb{T}, \quad t > 0. \quad (3.4)$$

Starting with (3.3), we want to show that $u(t)$ satisfies the following differential inequality

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^s}^2 \leq c_s \|u(t)\|_{H^s}^3, \quad 0 \leq t < T_0. \quad (3.5)$$

We can get $\frac{d}{dt} \|u(t)\|_{H^s}^2$ by applying the operator D^s to both sides of (3.3), then multiplying the resulting equation by $D^s u$ and integrating it with respect to x . However the term $u\partial_x u$ is only in $H^{s-1}(\mathbb{T})$. To overcome this obstacle, we consider the following mollification of (3.3)

$$\begin{aligned} \partial_t J_\varepsilon u &= -J_\varepsilon(u\partial_x u) - D^{-4}\partial_x J_\varepsilon u^2 - D^{-4}\partial_x J_\varepsilon(u_x^2) + \frac{7}{2}D^{-4}\partial_x J_\varepsilon(u_{xx}^2) \\ &\quad + 3D^{-4}\partial_x J_\varepsilon(u_x u_{xxx}). \end{aligned} \quad (3.6)$$

Applying the operator D^s to both sides of (3.6), then multiplying the resulting equation by $D^s J_\varepsilon u$ and integrating it with respect to x , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|J_\varepsilon u\|_{H^s}^2 &= - \int_{\mathbb{T}} D^s J_\varepsilon(u\partial_x u) \cdot D^s J_\varepsilon u \, dx - \int_{\mathbb{T}} D^{s-4}\partial_x J_\varepsilon u^2 \cdot D^s J_\varepsilon u \, dx \\ &\quad - \int_{\mathbb{T}} D^{s-4}\partial_x J_\varepsilon(u_x^2) \cdot D^s J_\varepsilon u \, dx + \frac{7}{2} \int_{\mathbb{T}} D^{s-4}\partial_x J_\varepsilon(u_{xx}^2) \\ &\quad \times D^s J_\varepsilon u \, dx + 3 \int_{\mathbb{T}} D^{s-4}\partial_x J_\varepsilon(u_x u_{xxx}) \cdot D^s J_\varepsilon u \, dx. \end{aligned} \quad (3.7)$$

Noticing

$$D^s J_\varepsilon = J_\varepsilon D^s, \quad (3.8)$$

$$(J_\varepsilon f, g)_{L^2} = (f, J_\varepsilon g)_{L^2}, \quad (3.9)$$

$$\|J_\varepsilon u\|_{H^s} \leq \|u\|_{H^s}, \quad (3.10)$$

we can estimate the five integrals in the right-hand side of (3.7). For the first integral, we have

$$\begin{aligned} \int_{\mathbb{T}} D^s J_\varepsilon(u\partial_x u) \cdot D^s J_\varepsilon u \, dx &= \int_{\mathbb{T}} D^s(u\partial_x u) \cdot J_\varepsilon D^s J_\varepsilon u \, dx \\ &= \int_{\mathbb{T}} [D^s(u\partial_x u) - uD^s\partial_x u] \cdot J_\varepsilon D^s J_\varepsilon u \, dx \\ &\quad + \int_{\mathbb{T}} uD^s\partial_x u \cdot J_\varepsilon D^s J_\varepsilon u \, dx. \end{aligned} \quad (3.11)$$

According to Lemma 2.2, we have

$$\begin{aligned} & \left| \int_{\mathbb{T}} [D^s(u\partial_x u) - uD^s\partial_x u] \cdot J_\varepsilon D^s J_\varepsilon u dx \right| \\ & \leq \|D^s(u\partial_x u) - uD^s\partial_x u\|_{L^2} \|J_\varepsilon D^s J_\varepsilon u\|_{L^2} \\ & \leq 2C_s \|\partial_x u\|_{L^\infty} \|u\|_{H^s}^2. \end{aligned} \quad (3.12)$$

According to Lemma 2.3, we have

$$\begin{aligned} & \left| \int_{\mathbb{T}} u D^s \partial_x u \cdot J_\varepsilon D^s J_\varepsilon u dx \right| \\ & = \left| \int_{\mathbb{T}} J_\varepsilon (u D^s \partial_x u) \cdot D^s J_\varepsilon u dx \right| \\ & = \left| \int_{\mathbb{T}} ([J_\varepsilon, u] D^s \partial_x u + u J_\varepsilon D^s \partial_x u) \cdot D^s J_\varepsilon u dx \right| \\ & \leq \left| \int_{\mathbb{T}} [J_\varepsilon, u] D^s \partial_x u \cdot D^s J_\varepsilon u dx \right| + \left| \int_{\mathbb{T}} u J_\varepsilon D^s \partial_x u \cdot D^s J_\varepsilon u dx \right| \\ & \leq \| [J_\varepsilon, u] D^s \partial_x u \|_{L^2} \|D^s J_\varepsilon u\|_{L^2} + \frac{1}{2} \left| \int_{\mathbb{T}} \partial_x u \cdot (D^s J_\varepsilon u)^2 dx \right| \\ & \leq c \|\partial_x u\|_{L^\infty} \|u\|_{H^s}^2 + \frac{1}{2} \|\partial_x u\|_{L^\infty} \|u\|_{H^s}^2 \\ & \leq \left(c + \frac{1}{2} \right) \|\partial_x u\|_{L^\infty} \|u\|_{H^s}^2. \end{aligned} \quad (3.13)$$

Combining (3.11), (3.12) and (3.13), we have

$$\left| \int_{\mathbb{T}} D^s J_\varepsilon (u \partial_x u) \cdot D^s J_\varepsilon u dx \right| \leq c_s \|\partial_x u\|_{L^\infty} \|u\|_{H^s}^2 \leq c_s \|u\|_{C^3} \|u\|_{H^s}^2. \quad (3.14)$$

The estimates of the remaining four integrals are similar, and thus, for simplicity we only estimate the fifth integral. According to Lemma 2.4, we have

$$\begin{aligned} & \left| 3 \int_{\mathbb{T}} D^{s-4} \partial_x J_\varepsilon (u_x u_{xxx}) \cdot D^s J_\varepsilon u dx \right| \\ & \leq 3 \|D^{s-4} \partial_x J_\varepsilon (u_x u_{xxx})\|_{L^2} \|D^s J_\varepsilon u\|_{L^2} \\ & \leq 3 \|u_x u_{xxx}\|_{H^{s-3}} \|u\|_{H^s} \\ & \leq 3C_s [\|u_x\|_{H^{s-3}} \|u_{xxx}\|_{L^\infty} + \|u_x\|_{L^\infty} \|u_{xxx}\|_{H^{s-3}}] \|u\|_{H^s} \\ & \leq 3C_s [\|u_{xxx}\|_{L^\infty} + \|u_x\|_{L^\infty}] \|u\|_{H^s}^2 \\ & \leq 6C_s \|u\|_{C^3} \|u\|_{H^s}^2. \end{aligned} \quad (3.15)$$

Hence

$$\frac{1}{2} \frac{d}{dt} \|J_\varepsilon u(t)\|_{H^s}^2 \leq c_s \|u(t)\|_{C^3} \|u(t)\|_{H^s}^2, \quad 0 \leq t < T_0. \quad (3.16)$$

Integrating (3.16) from 0 to t , we have

$$\frac{1}{2} \|J_\varepsilon u(t)\|_{H^s}^2 - \frac{1}{2} \|J_\varepsilon u_0\|_{H^s}^2 \leq c_s \int_0^t \|u(\tau)\|_{C^3} \|u(\tau)\|_{H^s}^2 d\tau. \quad (3.17)$$

Let ε tend to 0 in (3.17), then it follows that

$$\frac{1}{2}\|u(t)\|_{H^s}^2 - \frac{1}{2}\|u_0\|_{H^s}^2 \leq c_s \int_0^t \|u(\tau)\|_{C^3} \|u(\tau)\|_{H^s}^2 d\tau. \quad (3.18)$$

Using Gronwall's inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^s}^2 \leq c_s \|u(t)\|_{C^3} \|u(t)\|_{H^s}^2, \quad 0 \leq t < T_0. \quad (3.19)$$

Since $s > 7/2$, using Sobolev's inequality

$$\|u(t)\|_{C^3} \leq c_s \|u(t)\|_{H^s}, \quad (3.20)$$

we get the desired inequality (3.5).

Now let $y(t) = \|u(t)\|_{H^s}^2$, then (3.5) takes the form

$$\frac{1}{2} y^{-\frac{3}{2}} \frac{dy}{dt} \leq c_s. \quad (3.21)$$

Integrating (3.21) from 0 to t gives

$$\frac{1}{\sqrt{y_0}} - \frac{1}{\sqrt{y(t)}} \leq c_s t. \quad (3.22)$$

It follows that

$$\|u(t)\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{1 - c_s \|u_0\|_{H^s} t}. \quad (3.23)$$

Then $\|u(t)\|_{H^s}$ is finite if

$$t < \frac{1}{c_s \|u_0\|_{H^s}}. \quad (3.24)$$

This implies that the solution $u(t)$ certainly exists for $0 \leq t \leq T$, where

$$T = \frac{1}{2c_s \|u_0\|_{H^s}}. \quad (3.25)$$

Therefore the lifespan T_0 satisfies the estimate (3.1), and the estimate (3.2) is directly derived by (3.23). This completes the proof of Proposition 3.1. $\square$

3.2. Approximate solutions

The approximate solutions are of the form

$$u^{\omega,n}(x,t) = \omega n^{-1} + n^{-s} \cos(nx - \omega t), \quad (3.26)$$

where ω is in a bounded subset of $\mathbb{R}$ and $n \in \mathbb{Z}^+$. Now we compute the error of the approximate solutions. Note that

$$\begin{aligned} \partial_t u^{\omega,n} + u^{\omega,n} \partial_x u^{\omega,n} &= \omega n^{-s} \sin(nx - \omega t) + [\omega n^{-1} + n^{-s} \cos(nx - \omega t)] \\ &\quad \times [-n^{-s+1} \sin(nx - \omega t)] \\ &= -\frac{1}{2} n^{-2s+1} \sin(2nx - 2\omega t) \\ &= F_1(t), \end{aligned} \quad (3.27)$$

$$\begin{aligned}
D^{-4}\partial_x [(u^{\omega,n})^2] &= D^{-4}\partial_x [\omega^2 n^{-2} + n^{-2s}\cos^2(nx - \omega t) + 2\omega n^{-s-1}\cos(nx - \omega t)] \\
&= D^{-4}\partial_x \left[\frac{1}{2}n^{-2s}\cos(2nx - 2\omega t) + 2\omega n^{-s-1}\cos(nx - \omega t) \right] \\
&= -n^{-2s+1}D^{-4}[\sin(2nx - 2\omega t)] - 2\omega n^{-s}D^{-4}[\sin(nx - \omega t)] \\
&= F_2(t) + F_3(t),
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
D^{-4}\partial_x [(u^{\omega,n})^2_x] &= D^{-4}\partial_x [(-n^{-s+1}\sin(nx - \omega t))^2] \\
&= n^{-2s+2}D^{-4}\partial_x [\sin^2(nx - \omega t)] \\
&= n^{-2s+3}D^{-4}[\sin(2nx - 2\omega t)] \\
&= F_4(t),
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
-D^{-4}\partial_x \left[\frac{7}{2}(u^{\omega,n})^2_{xx} \right] &= -\frac{7}{2}D^{-4}\partial_x [(-n^{-s+1}\sin(nx - \omega t))^2_x] \\
&= -\frac{7}{2}D^{-4}\partial_x [(-n^{-s+2}\cos(nx - \omega t))^2] \\
&= -\frac{7}{2}n^{-2s+4}D^{-4}\partial_x \left[\frac{1}{2} + \frac{1}{2}\cos(2nx - 2\omega t) \right] \\
&= \frac{7}{2}n^{-2s+5}D^{-4}[\sin(2nx - 2\omega t)] \\
&= F_5(t),
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
-D^{-4}\partial_x [3(u^{\omega,n})_x(u^{\omega,n})_{3x}] &= -3D^{-4}\partial_x [(-n^{-s+1}\sin(nx - \omega t)) \\
&\quad \times (-n^{-s+1}\sin(nx - \omega t))_{xx}] \\
&= 3D^{-4}\partial_x [(n^{-s+1}\sin(nx - \omega t))(-n^{-s+2}\cos(nx - \omega t))_x] \\
&= 3n^{-2s+4}D^{-4}\partial_x [\sin^2(nx - \omega t)] \\
&= 3n^{-2s+5}D^{-4}[\sin(2nx - 2\omega t)] \\
&= F_6(t).
\end{aligned} \tag{3.31}$$

Thus the error F of the approximate solutions is

$$F(t) = F_1(t) + F_2(t) + \cdots + F_6(t). \tag{3.32}$$

According to Lemma 2.5, for $n \gg 1$ we have

$$\begin{aligned}
\|F_1(t)\|_{H^\sigma} &= \left\| -\frac{1}{2}n^{-2s+1}\sin(2nx - 2\omega t) \right\|_{H^\sigma} \\
&= \frac{1}{2}n^{-2s+1}\|\sin(2nx - 2\omega t)\|_{H^\sigma} \\
&\lesssim n^{-2s+1+\sigma},
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
\|F_2(t)\|_{H^\sigma} &= \|n^{-2s+1}D^{-4}[\sin(2nx - 2\omega t)]\|_{H^\sigma} \\
&= n^{-2s+1}\|\sin(2nx - 2\omega t)\|_{H^{\sigma-4}} \\
&\lesssim n^{-2s-3+\sigma},
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
\|F_3(t)\|_{H^\sigma} &= \|2\omega n^{-s} D^{-4}[\sin(nx - \omega t)]\|_{H^\sigma} \\
&= 2|\omega| n^{-s} \|\sin(nx - \omega t)\|_{H^{\sigma-4}} \\
&\lesssim n^{-s-4+\sigma},
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
\|F_4(t)\|_{H^\sigma} &= \|n^{-2s+3} D^{-4}[\sin(2nx - 2\omega t)]\|_{H^\sigma} \\
&= n^{-2s+3} \|\sin(2nx - 2\omega t)\|_{H^{\sigma-4}} \\
&\lesssim n^{-2s-1+\sigma},
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
\|F_5(t)\|_{H^\sigma} &= \left\| \frac{7}{2} n^{-2s+5} D^{-4}[\sin(2nx - 2\omega t)] \right\|_{H^\sigma} \\
&= \frac{7}{2} n^{-2s+5} \|\sin(2nx - 2\omega t)\|_{H^{\sigma-4}} \\
&\lesssim n^{-2s+1+\sigma},
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
\|F_6(t)\|_{H^\sigma} &= \|3n^{-2s+5} D^{-4}[\sin(2nx - 2\omega t)]\|_{H^\sigma} \\
&= 3n^{-2s+5} \|\sin(2nx - 2\omega t)\|_{H^{\sigma-4}} \\
&\lesssim n^{-2s+1+\sigma}.
\end{aligned} \tag{3.38}$$

Putting the above estimates together gives the following estimate for the error $F(t)$.

Proposition 3.2. *For $n \gg 1$, we have*

$$\|F(t)\|_{H^\sigma} \lesssim n^{-2s+1+\sigma} + n^{-2s-1+\sigma} + n^{-2s-3+\sigma} + n^{-s-4+\sigma}. \tag{3.39}$$

In particular, if $s > (1 + \sigma)/2$, then

$$\|F(t)\|_{H^\sigma} \lesssim n^{-r_s}, \tag{3.40}$$

where $r_s > 0$ and

$$r_s = \begin{cases} 2s - 1 - \sigma, & \text{if } \frac{1+\sigma}{2} < s \leq 5, \\ s + 4 - \sigma, & \text{if } s > 5. \end{cases} \tag{3.41}$$

3.3. Estimating the difference between approximate and actual solutions

Consider the following Cauchy problem

$$\partial_t u = -u\partial_x u - D^{-4}\partial_x \left(u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} \right), \tag{3.42}$$

$$u(x, 0) = u^{\omega, n}(x, 0) = \omega n^{-1} + n^{-s} \cos(nx), \quad x \in \mathbb{T}, \quad t > 0. \tag{3.43}$$

From Lemma 2.5, we have

$$\begin{aligned}
\|u^{\omega, n}(x, t)\|_{H^s} &= \|\omega n^{-1} + n^{-s} \cos(nx - \omega t)\|_{H^s} \\
&\leq 2\pi|\omega| n^{-1} + \sqrt{2}\pi n^{-s} (1 + n^2)^{s/2} \\
&\leq 2\pi|\omega| + 2^{\frac{1+s}{2}} \pi \\
&\lesssim 1.
\end{aligned} \tag{3.44}$$

According to Proposition 3.1, let $u_{\omega,n}(x, t)$ be the unique solution to (3.42)–(3.43) in $C([0, T]; H^s(\mathbb{T}))$ and

$$v(x, t) = u^{\omega,n}(x, t) - u_{\omega,n}(x, t). \quad (3.45)$$

Then it follows that v satisfies the following Cauchy problem

$$\begin{aligned} \partial_t v &= F - \frac{1}{2} \partial_x [(u^{\omega,n} + u_{\omega,n})v] - D^{-4} \partial_x [(u^{\omega,n} + u_{\omega,n})v] \\ &\quad - D^{-4} \partial_x [\partial_x (u^{\omega,n} + u_{\omega,n}) \partial_x v] + \frac{7}{2} D^{-4} \partial_x [\partial_x^2 (u^{\omega,n} + u_{\omega,n}) \partial_x^2 v] \\ &\quad + 3 D^{-4} \partial_x (\partial_x^3 u_{\omega,n} \partial_x v) + 3 D^{-4} \partial_x (\partial_x u^{\omega,n} \partial_x^3 v), \end{aligned} \quad (3.46)$$

$$v(x, 0) = 0. \quad (3.47)$$

Proposition 3.3. *If $n \gg 1$, $s > 7/2$ and $5/2 < \sigma < \min\{3, s - 1\}$, then*

$$\|v(t)\|_{H^\sigma} \lesssim n^{-r_s}, \quad 0 \leq t \leq T. \quad (3.48)$$

Proof. Applying D^σ to both sides of (3.46), multiplying the resulting equation by $D^\sigma v$ and integrating it with respect to x , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma}^2 &= \int_{\mathbb{T}} D^\sigma F \cdot D^\sigma v dx - \frac{1}{2} \int_{\mathbb{T}} D^\sigma \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot D^\sigma v dx \\ &\quad - \int_{\mathbb{T}} D^{\sigma-4} \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot D^\sigma v dx \\ &\quad - \int_{\mathbb{T}} D^{\sigma-4} \partial_x [\partial_x (u^{\omega,n} + u_{\omega,n}) \partial_x v] \cdot D^\sigma v dx \\ &\quad + \frac{7}{2} \int_{\mathbb{T}} D^{\sigma-4} \partial_x [\partial_x^2 (u^{\omega,n} + u_{\omega,n}) \partial_x^2 v] \cdot D^\sigma v dx \\ &\quad + 3 \int_{\mathbb{T}} D^{\sigma-4} \partial_x (\partial_x^3 u_{\omega,n} \partial_x v) \cdot D^\sigma v dx \\ &\quad + 3 \int_{\mathbb{T}} D^{\sigma-4} \partial_x (\partial_x u^{\omega,n} \partial_x^3 v) \cdot D^\sigma v dx. \end{aligned} \quad (3.49)$$

Now we estimate the seven integrals in the right-hand side of (3.49).

1. Estimate of the first integral.

$$\left| \int_{\mathbb{T}} D^\sigma F \cdot D^\sigma v dx \right| \leq \|D^\sigma F\|_{L^2} \|D^\sigma v\|_{L^2} \leq \|F\|_{H^\sigma} \|v\|_{H^\sigma}. \quad (3.50)$$

2. Estimate of the second integral.

$$\begin{aligned} &\int_{\mathbb{T}} D^\sigma \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot D^\sigma v dx \\ &= \int_{\mathbb{T}} [D^\sigma \partial_x (u^{\omega,n} + u_{\omega,n})] v \cdot D^\sigma v dx + \int_{\mathbb{T}} (u^{\omega,n} + u_{\omega,n}) D^\sigma \partial_x v \cdot D^\sigma v dx. \end{aligned} \quad (3.51)$$

According to Lemma 2.6, we have

$$\begin{aligned} \left| \int_{\mathbb{T}} [D^\sigma \partial_x (u^{\omega,n} + u_{\omega,n})] v \cdot D^\sigma v dx \right| &\leq \| [D^\sigma \partial_x (u^{\omega,n} + u_{\omega,n})] v \|_{L^2} \| D^\sigma v \|_{L^2} \\ &\leq c \| u^{\omega,n} + u_{\omega,n} \|_{H^s} \| v \|_{H^\sigma}^2. \end{aligned} \quad (3.52)$$

Since

$$\begin{aligned} \left| \int_{\mathbb{T}} (u^{\omega,n} + u_{\omega,n}) D^\sigma \partial_x v \cdot D^\sigma v dx \right| &= \frac{1}{2} \left| \int_{\mathbb{T}} \partial_x (u^{\omega,n} + u_{\omega,n}) \cdot (D^\sigma v)^2 dx \right| \\ &\leq \frac{1}{2} \| \partial_x (u^{\omega,n} + u_{\omega,n}) \|_{L^\infty} \| v \|_{H^\sigma}^2 \\ &\lesssim \| u^{\omega,n} + u_{\omega,n} \|_{C^1} \| v \|_{H^\sigma}^2 \\ &\lesssim \| u^{\omega,n} + u_{\omega,n} \|_{H^s} \| v \|_{H^\sigma}^2, \end{aligned} \quad (3.53)$$

we obtain

$$\left| \frac{1}{2} \int_{\mathbb{T}} D^\sigma \partial_x [(u^{\omega,n} + u_{\omega,n}) v] \cdot D^\sigma v dx \right| \lesssim \| u^{\omega,n} + u_{\omega,n} \|_{H^s} \| v \|_{H^\sigma}^2. \quad (3.54)$$

3. Estimate of the seventh integral.

Since $1/2 < \sigma - 2 < 1$, according to Lemma 2.7, we have

$$\begin{aligned} \left| 3 \int_{\mathbb{T}} D^{\sigma-4} \partial_x [\partial_x u^{\omega,n} \partial_x^3 v] \cdot D^\sigma v dx \right| &\leq 3 \| D^{\sigma-4} \partial_x [\partial_x u^{\omega,n} \partial_x^3 v] \|_{L^2} \| D^\sigma v \|_{L^2} \\ &\leq 3 \| \partial_x u^{\omega,n} \partial_x^3 v \|_{H^{\sigma-3}} \| v \|_{H^\sigma} \\ &\leq 3C \| \partial_x u^{\omega,n} \|_{H^{\sigma-2}} \| \partial_x^3 v \|_{H^{\sigma-3}} \| v \|_{H^\sigma} \\ &\lesssim \| u^{\omega,n} \|_{H^s} \| v \|_{H^\sigma}^2. \end{aligned} \quad (3.55)$$

4. Estimates of the remaining four integrals. The estimates of the four integrals are similar, and thus, for simplicity we only estimate the fifth integral. Since $5/2 < \sigma < 3$, we have

$$\begin{aligned} &\left| \frac{7}{2} \int_{\mathbb{T}} D^{\sigma-4} \partial_x [\partial_x^2 (u^{\omega,n} + u_{\omega,n}) \partial_x^2 v] \cdot D^\sigma v dx \right| \\ &= \frac{7}{2} \left| \int_{\mathbb{T}} [\partial_x^2 (u^{\omega,n} + u_{\omega,n}) \partial_x^2 v] \cdot D^{2\sigma-4} \partial_x v dx \right| \\ &\leq \frac{7}{2} \| \partial_x^2 (u^{\omega,n} + u_{\omega,n}) \partial_x^2 v \|_{L^2} \| D^{2\sigma-4} \partial_x v \|_{L^2} \\ &\lesssim \| \partial_x^2 (u^{\omega,n} + u_{\omega,n}) \|_{L^\infty} \| \partial_x^2 v \|_{L^2} \| v \|_{H^{2\sigma-3}} \\ &\lesssim \| u^{\omega,n} + u_{\omega,n} \|_{H^s} \| v \|_{H^\sigma}^2. \end{aligned} \quad (3.56)$$

Therefore

$$\frac{1}{2} \frac{d}{dt} \| v \|_{H^\sigma}^2 \lesssim [\| u^{\omega,n} \|_{H^s} + \| u_{\omega,n} \|_{H^s}] \| v \|_{H^\sigma}^2 + \| F \|_{H^\sigma} \| v \|_{H^\sigma}. \quad (3.57)$$

By (3.2) and (3.44), we have

$$\| u^{\omega,n} \|_{H^s} + \| u_{\omega,n} \|_{H^s} \leq \| u^{\omega,n}(x, t) \|_{H^s} + 2 \| u^{\omega,n}(x, 0) \|_{H^s} \lesssim 1, \quad 0 \leq t \leq T. \quad (3.58)$$

According to Proposition 3.2, we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^\sigma}^2 \lesssim \|v\|_{H^\sigma}^2 + n^{-r_s} \|v\|_{H^\sigma}. \quad (3.59)$$

It follows that

$$\frac{d}{dt} \|v\|_{H^\sigma} \lesssim \|v\|_{H^\sigma} + n^{-r_s}. \quad (3.60)$$

Finally, writing (3.60) in the form

$$\frac{d}{dt} (e^{-t} \|v\|_{H^\sigma}) \lesssim e^{-t} n^{-r_s}, \quad (3.61)$$

integrating (3.61) from 0 to t and noticing that $v(0) = 0$, we obtain (3.48). This completes the proof of Proposition 3.3. $\square$

Proof of Theorem 1.1 It suffices to show that $u_{1,n}(x, t)$ and $u_{-1,n}(x, t)$ are two sequences of solutions satisfying the three conditions (1.4), (1.4) and (1.6). We verify it item by item in the following.

1. By (3.2) and (3.44), we have

$$\|u_{1,n}(t)\|_{H^s} + \|u_{-1,n}(t)\|_{H^s} \leq 2(\|u^{1,n}(0)\|_{H^s} + \|u^{-1,n}(0)\|_{H^s}) \lesssim 1. \quad (3.62)$$

2. Moreover,

$$\lim_{n \rightarrow \infty} \|u_{1,n}(0) - u_{-1,n}(0)\|_{H^s} = \lim_{n \rightarrow \infty} \|2n^{-1}\|_{H^s} = 0. \quad (3.63)$$

3. By (2.9) and (3.2), we get

$$\begin{aligned} & \|u^{\pm 1,n}(t)\|_{H^{2s-\sigma}} + \|u_{\pm 1,n}(t)\|_{H^{2s-\sigma}} \\ & \leq \|\pm n^{-1} + n^{-s} \cos(nx \mp t)\|_{H^{2s-\sigma}} + 2\|\pm n^{-1} + n^{-s} \cos(nx)\|_{H^{2s-\sigma}} \\ & \leq 6\pi n^{-1} + 3n^{-s} \sqrt{2\pi} (1 + n^2)^{s-\frac{\sigma}{2}} \\ & \leq 6\pi n^{-1} + 3\sqrt{2\pi} \left(1 + \frac{1}{n^2}\right)^{s-\frac{\sigma}{2}} \cdot n^{s-\sigma} \\ & \leq 6\pi + 3\sqrt{2\pi} 2^{s-\frac{\sigma}{2}} \cdot n^{s-\sigma} \\ & \lesssim n^{s-\sigma}. \end{aligned} \quad (3.64)$$

According to the interpolation inequality

$$\|\psi\|_{H^s} \leq (\|\psi\|_{H^\sigma} \|\psi\|_{H^{2s-\sigma}})^{\frac{1}{2}} \quad (3.65)$$

and Proposition 3.3, we obtain

$$\begin{aligned} & \|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^s} \\ & \leq (\|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^\sigma} \|u^{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^{2s-\sigma}})^{\frac{1}{2}} \\ & \lesssim (n^{-r_s} n^{s-\sigma})^{\frac{1}{2}} \\ & \lesssim n^{-\frac{5}{4}}, \end{aligned} \quad (3.66)$$

where the last step follows from (3.41) and $s > 7/2$. Using the identity

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right), \quad (3.67)$$

we have

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s} &= \liminf_{n \rightarrow \infty} \|2n^{-1} + 2n^{-s} \sin(nx) \sin t\|_{H^s} \\
 &\geq \liminf_{n \rightarrow \infty} (\|2n^{-s} \sin(nx) \sin t\|_{H^s} - \|2n^{-1}\|_{H^s}) \\
 &\geq \liminf_{n \rightarrow \infty} (2n^{-s} \|\sin(nx)\|_{H^s} |\sin t|) \\
 &\quad - \limsup_{n \rightarrow \infty} \|2n^{-1}\|_{H^s} \\
 &\gtrsim |\sin t|,
 \end{aligned} \tag{3.68}$$

where the last step follows from Lemma 2.5. Therefore from (3.66) and (3.68), we have

$$\begin{aligned}
 \liminf_{n \rightarrow \infty} \|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s} &\geq \liminf_{n \rightarrow \infty} (\|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s} - \|u^{1,n}(t) \\
 &\quad - u_{1,n}(t)\|_{H^s} - \|u^{-1,n}(t) - u_{-1,n}(t)\|_{H^s}) \\
 &\geq \liminf_{n \rightarrow \infty} \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s} \\
 &\quad - \limsup_{n \rightarrow \infty} (\|u^{1,n}(t) - u_{1,n}(t)\|_{H^s} \\
 &\quad + \|u^{-1,n}(t) - u_{-1,n}(t)\|_{H^s}) \\
 &\gtrsim |\sin t|.
 \end{aligned} \tag{3.69}$$

This completes the proof of Theorem 1.1.

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