Effect of hyperviscosity on the geometry of the vorticity

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Abstract. We consider the two-dimensional modified Navier–Stokes equations by adding a p -Laplacian. We provide upper bounds on the Hausdorff length of the level sets of the vorticity. We express them in terms of the Landau–Lifschitz estimates of the number of degrees of freedom in turbulent flow. The proof, based on very standard methods, relies on a straightforward relation between the size of the nodal set of the vorticity and the growth of hyperdissipation.

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1. Introduction

In this paper, we provide upper bounds on the size of the nodal set $N_\omega^0 = \{x \in \Omega \subset \mathbb{R}^2 / \omega(x, t) = 0\}$ of the vorticity ω for solutions of modified 2D periodic Navier–Stokes equation. We consider a “regularized” Navier–Stokes system was proposed by Lions [18], who added an artificial hyper-viscosity term to the Navier–Stokes equations

$$\begin{aligned} \frac{du}{dt} + \varepsilon (-\Delta)^p u - \nu \Delta u + (u \cdot \nabla) u &= f(x), \quad \text{in } \Omega \times (0, \infty) \\ \operatorname{div} u &= 0, \quad \text{in } \Omega \times (0, \infty), \\ u(x + Le_i, t) &= u(x, t) \quad i = 1, \dots, d \quad t \in (0, \infty) \\ u(x, 0) &= u_0(x), \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

on $\Omega = (0, L)^d$ with periodic boundary conditions and $(e_1, \dots, e_d)$ is the natural basis of $\mathbb{R}^d$. Here $\varepsilon > 0$ is the artificial dissipation parameter and $\nu > 0$ is the kinematic viscosity of the fluid, $p > 1$. The function u is the velocity vector field, and f is a given force field. For $\varepsilon = 0$, the model is reduced to the Navier–Stokes system.

Mathematical model for such fluid motion has been used extensively in turbulence simulations (see e.g. [7]) also see Borue and Orsag [2, 3]. For further discussion of theoretical results concerning (1.1), see [1, 18, 24].

For such a modified problem considered in a bounded domain, Lions was able to prove (cf. [18, Chap.1, Remarque 6.11]) the existence of a unique regular solution provided $(p \geq (d + 2)/4$ for the d -dimensional problem).

In the work [24], the strong convergence of the solution of this problem to the solution of the conventional system as the regularization parameter goes to zero, was established in each dimension $d \leq 4$.

For the Navier–Stokes system weak solutions of problem are known to exist by a basic result by Leray from 1934 [16], only the uniqueness of weak solutions remains as an open problem in 3D. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier–Stokes system.

In particular, in case one accepts the point of view that the dimension of a global attractor for the Navier–Stokes equations is associated with the number of degrees of freedom in turbulent flows Temam [23], then the Hausdorff measure $\mathcal{H}_p^1(N_\omega^0)$ is an important way to the understanding of turbulence theory [24].

Turbulence is a really hard problem. In this paper, we use analytic methods to describe and analysis the effect of hyper viscosity on turbulent flow. We are interested, in particular, in the treatment of the force term and the connections with the turbulence. Specifically, we prove this result, without using the Dirichlet or Rayleigh quotient [13].

Let P_m be the projection onto the first m eigenspaces of the Stokes operator $A = -\Delta$ and let $N_\omega^0 = \{x \in \Omega \subset \mathbb{R}^2 / \omega(x, t) = 0\}$ the nodal sets of the vorticity ω for solutions of the Eq. (1.1). We provide an upper bound on the size of the nodal sets $\mathcal{H}_p^1(N_\omega^0)$ and we show that, the bounds necessarily depend on m and p this dependence is a fractional power of p . We also obtain here scale-invariant estimates on the one-dimensional Hausdorff measure $\mathcal{H}_p^1(N_\omega^0)$ in terms of the Landau–Lifschitz theory of the number of degrees of freedom in turbulent flow. Since expressing the above estimates in terms of the (dimensionless) Grashoff number G_r .

We find here that the upper bounds on the Hausdorff length of N_ω^0 converge to the corresponding upper bounds of the usual 2D Navier–Stokes as $p = 1$ and under certain hypothesis we recover the estimates for the usual 2D Navier–Stokes equations with potential force [13].

This result extends the work in [13] to the Navier–Stokes equations with general forcing modified by p -Laplacian. Related results for the 3D Navier–Stokes equations (with general forcing) can be found in [4]. The upper bounds on the Hausdorff measures of the level sets associated with solutions of some other partial differential equations were obtained in [5, 9, 10, 15, 17].

The paper is organized as follows. In Sect. 2, we present the relevant mathematical framework for the paper. In Sect. 3, we provide upper bounds for the one-dimensional Hausdorff measure $\mathcal{H}_p^1$ of the level sets associated with the vorticity of the Navier–Stokes system with hyperdissipation.

The question of the Hausdorff length of the level sets of the vorticity of solutions is thus completely answered for the two dimensional case. Our future investigations will be concerned with the corresponding problem for the three dimensional case. We have already reported partial results in this direction in Sect. 2 and in [24], where it was shown that the weak solution of the 3D regularized system converges strongly to a solution of the conventional system under certain conditions.

The methods validated in this paper certainly give us guidelines to elaborate the general three dimensional case.

2. Notations and preliminaries

In this section we introduce notations and the definitions of standard functional spaces that will be used throughout the paper. We denote by $H_{per}^m(\Omega)$, the Sobolev space of periodic functions. These spaces are endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \quad \text{and the norm} \quad \|u\|_m = \sum_{|\beta| \leq m} \left(\|D^\beta u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

We define the spaces V_m as completions of smooth, divergence-free, periodic, zero-average functions with respect to the H_{per}^m norms. V'_m denote the dual space of V_m and V denote the space V_0 .

Let P be the orthogonal projection in $L_{per}^2(\mathbb{R}^2)^2$ with the range V . Let $A = -P\Delta$ the Stokes operator. It is easy to check that $Au = -\Delta u$ for every $u \in D(A)$. We recall that the operator A is a closed positive self-adjoint unbounded operator.

The eigenvalues of A are $\{\lambda_j\}_{j=1}^{j=\infty}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and the corresponding orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{j=\infty}$ is complete in V_0

$$Aw_j = \lambda_j w_j, \quad w_j \in D(A), \forall j.$$

The spectral theory of A allows us to define the powers A^p of A for $p \geq 1$, A^p is an unbounded self-adjoint operator in V_0 with a domain $D(A^p)$ dense in $V_2 \subset V_0$. We set here

$$A^p u = (-\Delta)^p u \quad \text{for} \quad u \in D(A^p) = V_{2p}.$$

The space $D(A^p)$ is endowed with the scalar product and the norm

$$(u, v)_{D(A^p)} = (A^p u, A^p v), \quad \|u\|_{D(A^p)} = \left\{ (u, v)_{D(A^p)} \right\}^{\frac{1}{2}}. \quad (2.1)$$

In the case for $p > 0$, we have $D(A^p) = \{u \in V, \sum_{j=1}^{\infty} \lambda_j^{2p} (u, w_j)^2 < \infty\}$. For $p \in \mathbb{R}$ the scalar product and the norm in (2.1) can be written alternatively as

$$(u, v)_{D(A^p)} = \sum_{j=1}^{\infty} \lambda_j^{2p} (u, w_j)(v, w_j), \quad \|u\|_{D(A^p)} = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2p} (u, w_j)^2 \right\}^{\frac{1}{2}} \quad (2.2)$$

and for $u \in D(A^p)$ we can write

$$A^p u = \sum_{j=1}^{\infty} \lambda_j^p(u, w_j) w_j.$$

Let us now define the trilinear form $b(\cdot, \cdot, \cdot)$ associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

The continuity property of the trilinear form enables us to define (using Riesz representation Theorem) a bilinear continuous operator $B(u, v); V_2 \times V_2 \rightarrow V_2'$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall w \in V_2. \quad (2.3)$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have

$$b(u, u, u) = 0 \quad \text{and} \quad b(u, v, w) = -b(u, w, v). \quad (2.4)$$

We recall some well known inequalities that we will be using in what follows. Young's inequality

$$ab \leq \frac{\sigma}{p} a^p + \frac{1}{q \sigma^{\frac{q}{p}}} b^q, \quad a, b, \sigma > 0, \quad p > 1, q = \frac{p}{p-1}. \quad (2.5)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|A^{\frac{1}{2}} u\|^2 \quad \text{for all } u \in V_0. \quad (2.6)$$

Denoting

$$\|u\|_{G(t)}^2 = \|e^{tA^{\frac{1}{2}}} u\|^2 \quad \text{and} \quad (u, v)_{G(t)} = (e^{tA^{\frac{1}{2}}} u, e^{tA^{\frac{1}{2}}} v).$$

The set $D(e^{\alpha A})$ is called the Gevrey class of operator of order $\alpha \geq 0$, see [8]. Our use of Gevrey classes shall be based on the following consideration.

Recall that for a function $h: \Omega \rightarrow \mathbb{R}$, $N_h^c = \{x \in \Omega : h(x) = c\}$, and let $\mathcal{H}^1$ be the one-dimensional Hausdorff measure operating on subsets of $\mathbb{R}^2$.

3. Level sets of the vorticity function

Using the operators defined above, we can write (1.1) in the evolution form

$$\begin{aligned} \frac{du}{dt} + \varepsilon A^p u + \nu A u + B(u, u) &= f(x), \quad \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u &= 0 \text{ and } u_0(x) = u_0, \quad \text{in } \Omega. \end{aligned} \quad (3.1)$$

The existence and uniqueness results for initial value problem (1.1) can be found in [18]. The following theorem collects the main result in this work.

Theorem 3.1. *For $p \geq 1$, for $\varepsilon > 0$ fixed, $f \in L^2(0, T; V_0')$ and $u_0 \in V_0$ be given, there exists a unique weak solution of (1.1) which satisfies $u \in L^2(0, T; V_p) \cap L^\infty(0, T; V_0)$, $\forall T > 0$.*

The modern understanding of turbulence is that it is a collection of weakly correlated vortical motions, which, despite their intermittent and chaotic distribution over a wide range of space and time scales, actually consist of local characteristic ‘eddy’ patterns that persist as they move around under the influences of their own and other eddies’ vorticity fields [12].

In fluid mechanics, the Reynolds number is important in analyzing any type of flow when there is substantial velocity gradient (i.e. shear.). It indicates the relative significance of the viscous effect compared to the inertia effect. The Reynolds number is proportional to inertial force divided by viscous force (see [7])

$$Re = \frac{UL}{\nu}, \quad U^2 = L^{-2} \langle \|u\|^2 \rangle \quad (3.2)$$

where L the characteristic scale of the forcing and $\langle \cdot \rangle$ is the long-time-average

$$\langle g(\cdot) \rangle = \lim_{T \rightarrow \infty} \sup \left(\frac{1}{T} \int_0^T g(t) dt \right). \quad (3.3)$$

With Reynolds number calculator we can analyze what makes fluid flow regime laminar and what is needed to force the fluid to flow in turbulent regime. Experimental observations show that for ‘fully developed’ flow, laminar flow occurs when $Re < R_{le}$ and turbulent flow occurs when $Re > R_{te}$. In the interval between R_{le} and R_{te} , laminar and turbulent flows are possible (‘transition’ flows) [7] and references therein. The nature of the vortex formed in the fluid flow depends strongly on the Reynolds number ([22]; and references therein). These transition Reynolds numbers are also called critical Reynolds numbers, and were studied by Osborne Reynolds around 1895 [21]. The transition to turbulence and the construction of vortex are delayed by increasing the critical Reynolds number. If we assume that the critical Reynolds number R_{ce} for the onset of vortex shedding is reached for

$$\|u\| = \nu R_{ce} L, \quad (3.4)$$

then for each $Re \geq R_{ce}$ the associate velocity u satisfies the inequality

$$\|u\| \geq \nu R_{ce} L = \mu, \quad (3.5)$$

μ is a positive constant.

Another nondimensional quantity that we use often is the so-called Grashof number, which is proportional to the forcing term f . Hence, we define the Grashof numbers in the 2-dimensional case, as in Foias et al. [7] by

$$G_r(f) = \frac{1}{\nu^2 \lambda_1} \|f\|. \quad (3.6)$$

The effects of variation in Grashof number on vortex have been shown in the work of Olson and Titi [19], they keep the spatial structure of the forcing function fixed and vary the Grashof number by varying the amplitude of the forcing function. Namely, they vary the Grashof number by rescaling the forcing function by a multiplicative factor. This is equivalent to changing the viscosity or the size of the domain. As G_r increases, or equivalently as the

viscosity decreases, the turbulent flow becomes more energetic and one would expect the number of numerically determining modes to increase as well. There are many reasons to suppose that the existence and intensity of vortex in our work should increase as the Grashof number increases [11, 19, 20]. In [11], zero forcing implies that the attractor has been reduced to zero. Since all solutions decay eventually to zero in the unforced case.

This intuition is supported by existing theoretical critical number $G_r^c(f)$ for the existence of level curves of representative vorticity fields.

Note the flow for $G_r(f) \geq G_r^c(f)$ has noticeably more large scale structure compared to the flow for $G_r(f) \leq G_r^c(f)$. This is consistent with the energy spectra, where most of the energy is in the lowest modes, that is, in the large spatial scales and eddies when the Grashof number is large [19].

The effect of a body force on the vorticity production and the turbulence generation in a fluid flow is described by the Grashof number.

In addition, we assume without loss of generality that $\|f\|$ is bounded. Than, there exist a maximum Grashof number $G_r^{\max}(f)$ and a positive constant ρ such that the body force f satisfies the following inequality

$$\|f\| \leq \nu^2 \lambda_1 G_r^{\max}(f) = \rho. \quad (3.7)$$

Since $\|f\|$ is strictly positive we get

$$\frac{\|u\|}{\|f\|} \geq \frac{\mu}{\rho} = \frac{LR_c e}{\nu \lambda_1 G_r^{\max}(f)} = \beta, \quad (3.8)$$

this gives a relation between $\|u\|$ and $\|f\|$

$$\|u\| \geq \beta \|f\|. \quad (3.9)$$

According to the definition of the Gevrey norm and the relation (3.9) we get

$$\|u\|_G \geq \beta \|f\|_G. \quad (3.10)$$

The above inequality is valid for the 3D case for some modifications of the Grashof number. The vorticity, $\omega = \nabla \times u$ satisfies the equation

$$\left(\frac{d}{dt} + u \cdot \nabla + \nu \Delta + \varepsilon (-\Delta)^p \right) \omega = \omega \cdot \nabla u + F \quad (3.11)$$

where $F = \nabla \times f$.

Theorem 3.2. [14]. *Suppose that a nonzero function $h \in V_1$ satisfies $\|e^{\alpha A} h\|_1 \leq M \|h\|_1$, then*

$$\mathcal{H}^1(N_h^0) \leq C_1 L^2 (1 + \log M) e^{C_2 L/\alpha}.$$

Hereafter, C_i for $i \in \mathbb{N}$, stand for universal constants. The above statement will be used in conjunction with the following statement:

Lemma 3.3. [13]. *Let $u \in V_0$, and let ω be its vorticity. If*

$$\|A^{\frac{1}{2}} e^{\alpha A^{\frac{1}{2}}} u\| \leq M \|A^{\frac{1}{2}} u\| \quad (3.12)$$

for some $M > 0$, then, for every $c \in \mathbb{R}$

$$\|e^{\alpha A^{\frac{1}{2}}}(\omega - c)\|_1 \leq M\|\omega - c\|_1. \quad (3.13)$$

For the rest of the paper, let $u(t)$ be an arbitrary solution of the system (1.1).

Theorem 3.4. *Let $\|u\| \geq \beta\|f\|$ for any $\alpha \leq \frac{\nu\lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{4\sqrt{2}}{\nu}$, then there exists a universal constant C_3 such that if $\|A^{\frac{1}{2}}u\| \leq C_3$, then*

$$\|A^{\frac{1}{2}}e^{\alpha t A^{\frac{1}{2}}}u\| \leq 2\|A^{\frac{1}{2}}u_0\|, \quad t \geq 0. \quad (3.14)$$

Proof. For any $\alpha, t \geq 0$, we take the inner product of (3.1) with u , to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}}u\|_{G(t)}^2 &= \alpha \|A^{\frac{3}{4}}u\|_{G(t)}^2 + (Au, u)_{G(t)} \\ &= \alpha \|A^{\frac{3}{4}}u\|_{G(t)}^2 - \varepsilon \|A^{\frac{p+1}{2}}u\|_{G(t)}^2 \\ &\quad - \nu \|Au\|_{G(t)}^2 - b(u, u, Au)_{G(t)} + (f, Au)_{G(t)}, \end{aligned} \quad (3.15)$$

then using the Young's inequality (2.5) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}}u\|_{G(t)}^2 &\leq -\varepsilon \|A^{\frac{p+1}{2}}u\|_{G(t)}^2 + \frac{\nu}{4} \|Au\|_{G(t)}^2 + \frac{\alpha^2}{\nu} \|A^{\frac{1}{2}}u\|_{G(t)}^2 - \nu \|Au\|_{G(t)}^2 \\ &\quad + \left(\frac{\nu}{2} \|Au\|_{G(t)}^2 + \frac{1}{2\nu} \|f\|_{G(t)}^2 \right) + b(u, u, Au)_{G(t)}. \end{aligned}$$

Applying the Poincaré's inequality we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}}u\|_{G(t)}^2 &\leq -\varepsilon \|A^{\frac{p+1}{2}}u\|_{G(t)}^2 + \frac{\nu}{4} \|Au\|_{G(t)}^2 + \frac{\alpha^2}{\nu} \|A^{\frac{1}{2}}u\|_{G(t)}^2 - \nu \|Au\|_{G(t)}^2 \\ &\quad + \left(\frac{\nu}{2} \|Au\|_{G(t)}^2 + \frac{\beta^2}{2\lambda_1\nu} \|A^{\frac{1}{2}}u\|_{G(t)}^2 \right) + b(u, u, Au)_{G(t)} \\ &\leq -\varepsilon \|A^{\frac{p+1}{2}}u\|_{G(t)}^2 + \frac{-\lambda_1\nu}{4} \|A^{\frac{1}{2}}u\|_{G(t)}^2 + \frac{\alpha^2}{\nu} \|A^{\frac{1}{2}}u\|_{G(t)}^2 \\ &\quad + \frac{\beta^2}{2\lambda_1\nu} \|A^{\frac{1}{2}}u\|_{G(t)}^2 + b(u, u, Au)_{G(t)}. \end{aligned}$$

We get for $\beta^2 = \frac{1}{2\lambda_1\alpha^2}$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}}u\|_{G(t)}^2 &\leq -\varepsilon \|A^{\frac{p+1}{2}}u\|_{G(t)}^2 - \frac{\lambda_1\nu}{4} \|A^{\frac{1}{2}}u\|_{G(t)}^2 \\ &\quad + \frac{2\alpha^2}{\nu} \|A^{\frac{1}{2}}u\|_{G(t)}^2 + b(u, u, Au)_{G(t)}. \end{aligned}$$

We use the following inequality from [8] and [13, Section 4]

$$b(u, u, Au)_{G(t)} \leq C_4 \|A^{\frac{1}{2}}u\|_{G(t)}^2 \|Au\|_{G(t)} \left(1 + \log \frac{\|Au\|_{G(t)}^2}{\lambda_1 \|A^{\frac{1}{2}}u\|_{G(t)}^2} \right)^{\frac{1}{2}} \quad (3.16)$$

to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2 &\leq -\varepsilon \left\| A^{\frac{p+1}{2}} u \right\|_{G(t)}^2 - \frac{\lambda_1 \nu}{4} \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2 + \frac{2\alpha^2}{\nu} \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2 \\ &\quad + C_4 \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2 \|Au\|_{G(t)} \left(1 + \log \frac{\|Au\|_{G(t)}^2}{\lambda_1 \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2} \right)^{\frac{1}{2}}. \end{aligned}$$

To establish (3.14) we use the estimate [13]

$$a\mu \left(1 + \log \frac{\mu^2}{b^2} \right)^{\frac{1}{2}} \leq d\mu^2 + \frac{a^2}{d^2} \log \frac{2a}{bd} \quad a, d > 0, \mu \geq b > 0. \quad (3.17)$$

By applying the Poincaré's inequality (2.6), we have that for $\mu = \|Au\|_{G(t)}$ and $d = \frac{\nu}{8}$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2 + \varepsilon \left\| A^{\frac{p+1}{2}} u \right\|_{G(t)}^2 &\leq -\frac{\lambda_1 \nu}{8} \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2 + \frac{2\alpha^2}{\nu} \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2 \\ &\quad + C_5 \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^4 \log \frac{C_6 \left\| A^{\frac{1}{2}} u \right\|_{G(t)}}{\lambda_1^{\frac{1}{2}}}. \end{aligned}$$

Letting $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ we have for $\beta \leq \frac{4\sqrt{2}}{\nu}$ that

$$\frac{1}{2} \frac{d}{dt} \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^2 \leq C_5 \left\| A^{\frac{1}{2}} u \right\|_{G(t)}^4 \log \frac{C_6 \left\| A^{\frac{1}{2}} u \right\|_{G(t)}}{\lambda_1^{\frac{1}{2}}}. \quad (3.18)$$

If $\|A^{\frac{1}{2}} u\| < \frac{\lambda_1^{\frac{1}{2}}}{C_6} = C_3$, the term with a logarithm in (3.18) is negative, and thus (3.18) implies that $\|A^{\frac{1}{2}} u\|_{G(t)}$ is a decreasing function of t . $\square$

Let Ω be a periodic box, for simplicity assume $\Omega = (0, L)^2$, A has eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ with corresponding eigenspaces $E_1, E_2, \dots$. Let P_m be the projection on the eigenspaces $E_1 \oplus E_2 \oplus \dots \oplus E_m$ and let $Q_m = I - P_m$ we have $\|u\|^2 = \|P_m u\|^2 + \|Q_m u\|^2$ and we also have from (2.2) that

$$\|A^p u\| \leq \lambda_m^p \|u\| \quad \text{for every } p \geq 0 \text{ and } u \in D(A^p). \quad (3.19)$$

For any $t \geq 0$, let $\omega(t)$ be the vorticity of $u(t)$. We shall, for any fixed $t > 0$, estimate the quantity

$$\ell(\omega(t)) = \sup_{c \in \mathbb{R}} \mathcal{H}_p^1(N_\omega^c). \quad (3.20)$$

Lemma 3.5. *Let $\|u\| \geq \beta \|f\|$ for any $\beta \geq 0$, then*

$$\|u(t)\| \geq \|u(0)\| \exp(\eta t) \text{ for every } t \geq 0, \quad (3.21)$$

with $\eta = -(\varepsilon \lambda_m^{\frac{p}{2}} + \frac{1+\beta^2}{2\beta^2})$.

Proof. Taking the scalar product of both sides of (1.1) by $u(t)$ and using (2.4), we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|A^{\frac{1}{2}} u\|^2 + \varepsilon \|A^{\frac{p}{2}} u\|^2 = (f, u) \quad \text{for } t \geq 0. \quad (3.22)$$

Using (3.19) and the following inequality

$$(f, u) \geq -\frac{1}{2} \|f\|^2 - \frac{1}{2} \|u\|^2. \quad (3.23)$$

Because the increasing sequence $0 \leq \lambda_1 \leq \lambda \leq \lambda_m$ we have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq -\nu \|A^{\frac{1}{2}} u\|^2 - \left(\varepsilon \lambda_m^{\frac{p}{2}} + \frac{1 + \beta^2}{2\beta^2} \right) \|u\|^2 \quad (3.24)$$

note that since $\|A^{\frac{1}{2}} u\| \leq C_3$ we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq -\nu C_3 - \left(\varepsilon \lambda_m^{\frac{p}{2}} + \frac{1 + \beta^2}{2\beta^2} \right) \|u\| \quad (3.25)$$

if we set $\eta = -(\varepsilon \lambda_m^{\frac{p}{2}} + (1 + \beta^2)/2\beta^2)$, then we have from (3.25) that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq -\nu C_3 + \eta \|u\|.$$

Integrating the above inequality from 0 to t , we get

$$\|u\|^2 \geq \frac{-\nu C_3}{\eta} (1 - \exp(\eta t)) + \|u(0)\|^2 \exp(\eta t) \quad (3.26)$$

or, since $\frac{-\nu C_3}{\eta} (1 - \exp(\eta t)) \geq 0$. Thus, we have the inequality (3.21). $\square$

Proposition 3.6. Let $\|u\| \geq \beta \|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3 \nu \lambda_1^{\frac{1}{2}}$, then we have that

$$\ell(\omega(t)) \leq C_1 L \left(1 + \frac{1}{2} \text{Log} \frac{\lambda_m}{\lambda_1} + \left(\varepsilon \lambda_m^{\frac{p}{2}} + \frac{1 + \beta^2}{2\beta^2} \right) t \right) e^{\frac{C_2 L}{\alpha t}} \quad \text{for } t \geq 0, \quad (3.27)$$

for any $\alpha \leq \frac{\nu \lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{4\sqrt{2}}{\nu}$.

Proof. By Theorem 3.4 and Lemma 3.5, we get for $t \geq 0$ the following

$$\begin{aligned} \|A^{\frac{1}{2}} e^{\alpha t A^{\frac{1}{2}}} u\| &\leq 2 \|A^{\frac{1}{2}} u_0\| \\ &\leq 2 \lambda^{\frac{1}{2}} \|u_0\| \end{aligned} \quad (3.28)$$

and use the inequality (3.21) to get

$$\begin{aligned} \|A^{\frac{1}{2}} e^{\alpha t A^{\frac{1}{2}}} u\| &\leq 2 \lambda^{\frac{1}{2}} \|u(t)\| \exp\left(\varepsilon \lambda_m^{\frac{p}{2}} + \frac{1 + \beta^2}{2\beta^2} t\right) \\ &\leq 2 \left(\frac{\lambda}{\lambda_1} \right)^{\frac{1}{2}} \|A^{\frac{1}{2}} u(t)\| \exp\left(\varepsilon \lambda_m^{\frac{p}{2}} + \frac{1 + \beta^2}{2\beta^2} t\right). \end{aligned} \quad (3.29)$$

The rest follows by combining (3.29) with Lemma 3.3 and Theorem 3.2. $\square$

The foundational result for our one-dimensional Hausdorff measure estimates of the level sets of the vorticity of solutions is

Theorem 3.7. *Let $\|u\| \geq \beta\|f\|$, $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3\nu\lambda_1^{\frac{1}{2}}$, then*

$$\ell(\omega(t)) \leq C_7\lambda_m^{\frac{p}{2}} \text{ for } t \geq t_0, \quad (3.30)$$

with $t_0 = \frac{2C_2L}{\nu\lambda_1^{\frac{1}{2}}}$ for any $\alpha \leq \frac{\nu\lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{4\sqrt{2}}{\nu}$.

Proof. For $t \geq 2C_2L/\nu\lambda_1^{\frac{1}{2}}$ the inequality (3.27) implies

$$\ell(\omega(t)) \leq C_1 e \left(2 + \frac{1}{2} \text{Log} \frac{\lambda_m}{\lambda_1} + \left(\varepsilon \lambda_m^{\frac{p}{2}} + \frac{1 + \beta^2}{2\beta^2} \right) \frac{C_2L}{\alpha} \right) \text{ for } t \geq 0. \quad (3.31)$$

Since $\lambda_m \geq \lambda_1$, (3.30) follows directly from the above inequality. $\square$

The estimate of $\mathcal{H}_p^1$ grows in m at a rate lower than $\frac{p}{2}$.

Proposition 3.8. *Let $\|u\| \geq \beta\|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3\nu\lambda_1^{\frac{1}{2}}$, then*

$$\sup_{t \rightarrow \infty} \ell(\omega(t)) \leq C_8 m^{\frac{p}{2}} \text{ for } t \geq t_0, \quad (3.32)$$

for any $\alpha \leq \frac{\nu\lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{4\sqrt{2}}{\nu}$.

Proof. Note that in the 2D case we have $\lambda_j \geq C_9\lambda_1 j$ for some positive universal constant (see, for example [23, Lemma VI 2.1]). Therefore, since $\lambda_m \sim \lambda_1 m$ the growth in m of the Hausdorff measure (3.32) is less than $m^{\frac{p}{2}}$. $\square$

If we impose the condition $\lambda_m \leq (\frac{1}{l_\epsilon})^2$ or $\frac{\lambda_m}{\lambda_1} \leq (\frac{l_0}{l_\epsilon})^2$ where $l_0 = \lambda_1^{-\frac{1}{2}}$ represents characteristic macroscopic length, and l_ϵ is the Kolmogorov length scale, i.e. $l_\epsilon = \nu^{\frac{3}{4}}\epsilon^{-\frac{1}{4}}$ where ϵ is Kolmogorov's mean rate of dissipation of energy in turbulent flow (see e.g. [1, 7, 23], and the references contained therein) is defined as

$$\epsilon = \nu\lambda_1 \lim_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t \|A^{\frac{1}{2}}u\|^2 ds.$$

Substituting this in (3.30) gives

$$\sup_{t \rightarrow \infty} \ell(\omega(t)) \leq C_{10} \left(\frac{l_0}{l_\epsilon} \right)^p. \quad (3.33)$$

Since the (dimensionless) Grashoff number G_r (see e.g. [1, 7, 23]) is an upper bound for $(\frac{l_0}{l_\epsilon})^2$. Hence, we obtain for the Hausdorff measure of the equation (1.1) the following estimate in terms of the Grashoff number G_r .

Proposition 3.9. *Let $\|u\| \geq \beta\|f\|$ and $u_0 \neq 0$, and suppose that $\|u_0\| \leq C_3\nu\lambda_1^{\frac{1}{2}}$, then for any $\alpha \leq \frac{\nu\lambda_1^{\frac{1}{2}}}{4}$ and $\beta \leq \frac{\nu\lambda_1}{2\sqrt{2}}$ we have*

$$\sup_{t \rightarrow \infty} \ell(\omega(t)) \leq C_{11} G_r^{\frac{p}{2}} \text{ for } t \geq t_0. \quad (3.34)$$

This result holds independently of m . The estimate grows in p at a rate lower than $\frac{p}{2}$. If we impose $p = 1$, the estimates become $\sup_{t \rightarrow \infty} \ell(\omega(t)) \leq C_{11} G_r^{\frac{1}{2}}$. This result recover the usual 2D Navier–Stokes equations estimates. Here again our results indicate that under certain conditions the upper bounds for $\mathcal{H}_p^1(N_\omega^0)$ converge to the associated upper bounds of the Hausdorff measure $\mathcal{H}_1^1(N_\omega^0)$ estimates for the usual 2D Navier–Stokes equations with potential force.

Proving global regularity for the Navier–Stokes equations is one of the most challenging outstanding problems in nonlinear analysis. The main difficulty in establishing this result lies in controlling certain norms of vorticity. More specifically, the vorticity stretching term in the 3D vorticity equation forms the main obstacle to achieving this control, Foias [6] and estimates on the number of degrees of freedom for the Navier–Stokes equations and its closure models are a measure of the complexity of the system, Avrin [1]. This paper proposed another way to estimate this complexity through bounding the size of the nodal set for the vorticity and expressing this estimate in terms of G_r . This result answering a interesting question raised by Avrin in 2D [1]. This paper gives an overview of the foundational issues in the 3D case.

Another interesting way to study decaying turbulence in the hyperviscous flow is to provide a numerical investigation of our theoretical results on the dependence of turbulence structure and vortex dynamics, as was done in [19] for the number of numerically determining modes in the 2D Navier–Stokes equations.

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