Mean curvature flow with a constant forcing

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Abstract. In this paper, the motion of surface by its mean curvature and a forcing term θ was studied. We show that to each uniformly convex bounded initial surface M_0 , there exists a unique θ^* such that the surface shrinks or expands depending on whether $\theta < \theta^*$ or $\theta > \theta^*$. Also, we show that the surface with $\theta = \theta^*$ converges to a limiting surface.

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1. Introduction

The study of curvature flow has been of great interest recently. One says that a surface moves by mean curvature, if at each point its normal velocity is proportional to the mean curvature. This evolution mode has been named mean curvature motion (or flow).

A more general law is also of interest. In several physical systems, under stationary conditions the mean curvature of a surface is constrained to coincide with some field. As the latter evolves, a natural evolution mode consists in the surface moving with normal velocity proportional to the difference between the mean curvature and the field. This will be referred to as mean curvature flow with a forcing term. For instance, let us consider phase transition in a solid–liquid system [8, 14, 25]. Let us denote by θ the temperature, by M_t the interface between phases, by v its normal velocity, by H the sum of the principal curvatures. Let us assume H positive for a solid ball, v positive for melting and negative for freezing. The mean curvature flow (with a forcing term and with normalized coefficients) then reads

$$v = H - \theta \quad \text{on} \quad M_t. \quad (1.1)$$

Now we briefly describe some mathematical advances concerning this problem.

Case I $\theta = 0$. When the initial surface M_0 is convex, then as $t \rightarrow T$ the surfaces M_t become spherical and contract to a point. T denote a finite critical time at which the curvature of the surface becomes unbounded. This result was obtained by Gage and Hamilton [12] in one-dimensional case and by Huisken [16] in higher dimensional cases; it was then extended by Grayson [13] to arbitrary embedded curves. On the other hand, in higher dimensional cases, nonconvex surfaces may become singular without shrinking to point in the embedded case. Then one can analyze the behavior near a singularity with rescaling techniques [1–3, 17–19, 26, 28–30]. In order to describe the behavior of surface M_t after singularity happens, Evans and Spruck studied mean curvature flow by viscosity solution theory [10, 22]. Almgren, Taylor, Wang, Brakke and Ilmanen studied mean curvature flow by geometric measure theory [4, 6, 21].

Case II $\theta \neq 0$. In one-dimensional case, when the initial curve M_0 is convex, Chou and Zhu studied existence of global classical solutions to mean curvature flow with a forcing term [9]. Rather complete results are obtained by them. In higher dimensional cases, existence of global weak solutions to this flow was obtained by Chen et al. [7], Luckhaus and Sturzenhecker [23], Sonner [27], respectively.

In this paper, using the techniques of Huisken [16], and the viscosity solution method of Evans and Spruck [10], we obtain existence of global classical solutions to (1.1) for uniformly convex initial interface in higher dimensions. Our work borrows many ingredients from [10, 16, 27]. In order to state our theorem, we need to transform (1.1) into partial differential equation. Assume M_0 is a compact, uniformly convex n -dimensional hypersurface without boundary, which is smoothly imbedded in $\mathbb{R}^{n+1}$, and suppose that M_0 is represented locally by some diffeomorphism

$$F_0 : \mathbb{R}^n \supset U \rightarrow F_0(U) \subset M_0 \subset \mathbb{R}^{n+1}.$$

We transform (1.1) into the following equations

$$\frac{\partial}{\partial t} F(p, t) = (-H(x, t) + \theta)\nu(x, t), \quad x \in U, \quad t \geq 0, \quad (1.2)$$

$$F(\cdot, 0) = F_0, \quad (1.3)$$

where H is the mean curvature and ν the outer unit normal, respectively at the point $F(x, t)$ of the surface $M_t = F(\cdot, t)(U)$. The signs are chosen such that the mean curvature of a convex surface is positive. We have the following main theorem.

Theorem 1.1. *For any given smooth and uniformly convex M_0 , i.e., the eigenvalues of its second fundamental form are strictly positive everywhere, there exists a positive θ^* such that the maximal solution M_t of (1.2) starting at M_0 is uniformly convex for $t \in (0, t_{\max})$, and the following hold:*

- (a) *When $0 < \theta < \theta^*$, $t_{\max}$ is finite and M_t shrinks to a round point as $t \rightarrow t_{\max}$.*
- (b) *When $\theta = \theta^*$, $t_{\max} = +\infty$ and M_t converges to a sphere with radius $\frac{n}{\theta^*}$.*

- (c) When $\theta > \theta^*$, $t_{\max} = +\infty$ and M_t expands to infinity as t approaches infinity. Moreover, $\frac{F(\cdot, t)}{t}$ converges smoothly to a sphere with radius θ .

This paper is organized as follows. In Sect. 2, we derive various evolution equations. In Sect. 3, we give a pinching estimate. In Sects. 4, 5, we derive some derivative estimate. In Sect. 6, we prove the claim(b) of Theorem 1.1. In Sect. 7, we prove the claim(c) of Theorem 1.1.

2. Evolution equations and convexity properties

We will use the same notation as in [16]. In particular we write $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$ for the metric and the second fundamental form and use the following notations

$$H = g^{ij}h_{ij}, \quad |A|^2 = g^{ij}g^{kl}h_{ik}h_{jl},$$

$$C = g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}, \quad Z = HC - |A|^4.$$

Lemma 2.1. *We have the equations*

$$\frac{\partial}{\partial t}g_{ij} = 2(\theta - H)h_{ij}, \quad (2.1)$$

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{im}h^m{}_j + \theta h_{im}h^m{}_j + |A|^2h_{ij}. \quad (2.2)$$

Proof. We can derive (2.1), (2.2) by proceeding exactly as in [16], from the basic equation (1.2). $\square$

From (2.1), we have

$$\frac{\partial}{\partial t}g^{ij} = -2(\theta - H)g^{im}g^{jn}h_{mn}.$$

By contraction we easily obtain from Lemma 2.1 evolution equations for other quantities formed from g and A .

Corollary 2.2. *We have*

$$\frac{\partial}{\partial t}H = \Delta H + (H - \theta)|A|^2, \quad (2.3)$$

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2\theta C, \quad (2.4)$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(|A|^2 - \frac{1}{n}H^2\right) &= \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - 2\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right) \\ &\quad + 2|A|^2\left(|A|^2 - \frac{1}{n}H^2\right) + \frac{2}{n}\theta(H|A|^2 - nC), \end{aligned} \quad (2.5)$$

$$\frac{\partial}{\partial t}\sqrt{\det g_{ij}} = -H(H - \theta)\sqrt{\det g_{ij}}. \quad (2.6)$$

Since the initial hypersurface M_0 is uniformly convex, there is $0 < \varepsilon \leq \frac{1}{n}$ such that at $t = 0$

$$h_{ij} \geq \varepsilon H g_{ij} \quad (2.7)$$

holds everywhere on M_0 . Whereas the evolution equation for H does not yield an immediate lower bound for the mean curvature, we can show that inequality (2.7) is preserved.

Theorem 2.3. *If the initial hypersurface M_0 is uniformly convex, then M_t stays uniformly convex and inequality (2.7) remains true with a uniformly $0 < \varepsilon \leq \frac{1}{n}$ for all $t \geq 0$ where the solution of (1.2), (1.3) exists.*

Proof. It follows immediately from Lemma 2.1 that uniform convexity is preserved since the absolute terms in the evolution equation for A vanish at null-eigenvectors of A . To prove that inequality (2.7) is preserved, we will use Hamilton's maximum principle for tensors [15], and consider

$$M_{ij} = h_{ij} - \varepsilon H g_{ij}.$$

From Lemma 2.1 and Corollary 2.2 (i) we obtain

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + N_{ij},$$

$$\begin{aligned} N_{ij} &= -2H h_{im} h^m{}_{j} + \theta h_{im} h^m{}_{j} + |A|^2 h_{ij} \\ &\quad - \varepsilon |A|^2 (H - \theta) g_{ij} - 2\varepsilon H (\theta - H) h_{ij}. \end{aligned}$$

We have only to check that N_{ij} is non-negative on the null-eigenvectors of M_{ij} at some (x_0, t_0) we may arrange coordinates such that at (x_0, t_0) , $X = e_1, g_{ij} = \delta_{ij}$ and h_{ij} is diagonal. Then

$$\begin{aligned} N_{ij} X^i X^j &= N_{11} \\ &= -2\varepsilon^2 H^3 + \theta \varepsilon^2 H^2 + \varepsilon H |A|^2 - \varepsilon |A|^2 H \\ &\quad + \varepsilon |A|^2 \theta - 2\varepsilon^2 \theta H^2 + 2\varepsilon^2 H^3 \\ &= \varepsilon \theta (|A|^2 - \varepsilon H^2). \end{aligned}$$

This is non-negative since $\theta > 0$, $0 < \varepsilon \leq \frac{1}{n}$ and always $|A|^2 \geq \frac{1}{n} H^2$. $\square$

Lemma 2.4. ([16]) *If inequality (2.7) holds with $H > 0$, $\varepsilon > 0$ at some point of a hypersurface, then we have at that point*

$$(i) \quad Z \geq n\varepsilon^2 H^2 (|A|^2 - \frac{1}{n} H^2), \quad (2.8)$$

$$(ii) \quad |\nabla_i h_{kl} H - \nabla_i H h_{kl}|^2 \geq \frac{1}{2} \varepsilon^2 H^2 |\nabla H|^2, \quad (2.9)$$

$$(iii) \quad nC - H|A|^2 \geq 2n\varepsilon H (|A|^2 - \frac{1}{n} H^2). \quad (2.10)$$

3. A pinching estimate

Theorem 3.1. *There are $\delta > 0$ depending only on M_0 , n , and $C_0 < \infty$ depending only on M_0 , such that*

$$|A|^2 - \frac{1}{n}H^2 \leq C_0 H^{2-\delta}. \quad (3.1)$$

holds for all $0 \leq t < T_0 < +\infty$ where the solution of problem (1.2)–(1.3) exists.

We want to bound the function

$$f_\sigma = |A|^2 - \frac{1}{n}H^2/H^{2-\sigma}$$

for some small $\sigma > 0$ and begin with an evolution equation for f_σ . First, we prove several lemmas.

Lemma 3.2. *We have with $\alpha = 2 - \sigma$ that*

$$\begin{aligned} \frac{\partial}{\partial t} f_\sigma &= \Delta f_\sigma + \frac{2(\alpha-1)}{H} \langle \nabla_l H, \nabla_l f_\sigma \rangle - \frac{2}{H^{\alpha+2}} |\nabla_l h_{ij} H - \nabla_l H h_{ij}|^2 \\ &\quad - \frac{(2-\alpha)(\alpha-1)}{H^{\alpha+2}} (|A|^2 - \frac{1}{n}H^2) |\nabla H|^2 + \frac{\theta}{H^{3-\sigma}} (|A|^4 - HC) \\ &\quad + \sigma |A|^2 \frac{(H-\theta)}{H} f_\sigma, \end{aligned} \quad (3.2)$$

$$\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2(\alpha-1)}{H} \langle \nabla_i H, \nabla_i f_\sigma \rangle - \frac{\varepsilon^2}{H^\alpha} |\nabla H|^2 + \sigma |A|^2 f_\sigma - 2\varepsilon^2 \theta H f_\sigma. \quad (3.3)$$

Proof. By computing as in [16], we have

$$\begin{aligned} \frac{\partial}{\partial t} f_\sigma &= \frac{1}{H^\alpha} (\Delta |A|^2 - 2|A|^4 - \theta C) + |A|^2 \left(-\frac{\alpha H^{\alpha-1}}{H^{2\alpha}} \right) \cdot (\Delta H + (H-\theta)|A|^2) \\ &\quad - \frac{1}{n} (2-\alpha) H^{1-\alpha} \cdot (\Delta H + (H-\theta)|A|^2) \\ &= (H\Delta |A|^2 - \alpha |A|^2 \Delta H) \frac{1}{H^{\alpha+1}} - \frac{2-\alpha}{n} H^{1-\alpha} \Delta H - \frac{2}{H^\alpha} |\nabla A|^2 \\ &\quad + (2-\alpha) |A|^2 f_\sigma + \theta \frac{|A|^2}{H^{1+\alpha}} \left(\alpha |A|^2 - \frac{2-\alpha}{n} H^2 \right) \frac{2\theta}{1+\alpha} C, \end{aligned}$$

$$\nabla_i f_\sigma = (H \nabla_i |A|^2 - \alpha |A|^2 \nabla_i H) / H^{\alpha+1} - \frac{2-\alpha}{n} H^{1-\alpha} \nabla_i H,$$

$$\begin{aligned} \Delta f_\sigma &= \frac{(H\Delta |A|^2 - \alpha |A|^2 \Delta H)}{H^{\alpha+1}} - \frac{2-\alpha}{n} H^{1-\alpha} \Delta H - \frac{2\alpha}{H^{\alpha+1}} \langle \nabla_i |A|^2, \nabla_i H \rangle \\ &\quad + \alpha(\alpha+1) \frac{|A|^2}{H^{\alpha+2}} |\nabla H|^2 - \frac{1}{n} \frac{(2-\alpha)(1-\alpha)}{H^\alpha} |\nabla H|^2, \end{aligned}$$

$$|\nabla_i h_{kl} \cdot H - \nabla_i H \cdot h_{kl}|^2 = H^2 |\nabla A|^2 + |A|^2 |\nabla H|^2 - \langle \nabla_i |A|^2, \nabla_i H \rangle H.$$

The conclusion (3.2) follows. From Lemma 2.4, $\theta > 0$, we derive the inequality (3.3). $\square$

Lemma 3.3. *There exists a constant C_1 such that*

$$\left(\int_{M_t} f_\sigma^p d\mu \right)^{\frac{1}{p}} \leq C_1 \quad (3.4)$$

for $p \geq 200\varepsilon^{-2}$, $\sigma \leq n \cdot 2^{-4}\varepsilon^3 p^{-\frac{1}{2}}$.

Proof. Multiplying (3.3) by pf_σ^{p-1} as in [16], we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \int f_\sigma^p d\mu + p(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu + \varepsilon^2 p \int \frac{1}{H^\alpha} f_\sigma^{p-1} |\nabla H|^2 d\mu \\ & \leq 2(\alpha-1) \int \frac{1}{H} f_\sigma^{p-1} |\nabla H| |\nabla f_\sigma| d\mu + \sigma p \int H^2 f_\sigma^p d\mu \\ & \quad - 2\varepsilon^2 p \theta \int H f_\sigma^p d\mu + \int H(\theta - H) f_\sigma^p d\mu. \end{aligned}$$

Note that $\alpha \leq 2$, $|A|^2 \leq H^2$, $ab \leq \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2$, $f_\sigma \leq H^{2-\alpha}$. We have

$$\begin{aligned} & 2(\alpha-1)p \int \frac{1}{H} f_\sigma^{p-1} \cdot |\nabla H| \cdot |\nabla f_\sigma| \\ & \leq \frac{1}{2}p(p-1) \int f_\sigma^{p-1} |\nabla_\sigma|^2 d\mu + \frac{2p}{p-1} \int f_\sigma^{p-1} \cdot \frac{1}{H^\alpha} \cdot |\nabla H|^2 d\mu. \end{aligned}$$

Take p satisfying $p-1 \geq 200\varepsilon^{-2} - 1 \geq 100\varepsilon^{-2}$, we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \int f_\sigma^p d\mu + \frac{1}{2}p(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu + \frac{49}{50}\varepsilon^2 p \int \frac{1}{H^\alpha} f_\sigma^{p-1} |\nabla H|^2 d\mu \\ & \leq \sigma p \int H^2 f_\sigma^p d\mu - 2\varepsilon^2 p \theta \int H f_\sigma^p d\mu + \int H(\theta - H) f_\sigma^p d\mu. \end{aligned}$$

Now by using Lemma 5.4 in [16] with $\eta = \varepsilon p^{-\frac{1}{2}}/4$, we derive for all $p \geq 200\varepsilon^{-2}$, $\sigma \leq n \cdot 2^{-4} \cdot \varepsilon^3 p^{-\frac{1}{2}}$,

$$\frac{\partial}{\partial t} \int f_\sigma^p d\mu \leq 0.$$

Hence

$$\left(\int f_\sigma^p d\mu \right)^{\frac{1}{p}} \leq C_1.$$

□

The proof of Theorem 3.1. By the method used by Huisken [16], we let $f_{\sigma,k} = \max(f_\sigma - k, 0)$ and $A(k) = \{x \in M : f_\sigma > k\}$. Multiplying (3.3) with $pf_{\sigma,k}^{p-1}$, we derive for $p\varepsilon^2$ large

$$\begin{aligned} & \frac{\partial}{\partial t} \int f_{\sigma,k}^p d\mu + \frac{1}{2}p(p-1) \int_{A(k)} |\nabla f_\sigma|^2 f_{\sigma,k}^{p-2} d\mu \\ & \leq \sigma p \int_{A(k)} H^2 f_{\sigma,k}^{p-1} f_\sigma d\mu - \varepsilon^2 p \int_{A(k)} \theta H f_{\sigma,k}^{p-1} f_\sigma d\mu - \int_{A(k)} H^2 f_{\sigma,k}^{p-1} d\mu. \end{aligned} \quad (3.5)$$

Proceeding exactly as in [16], we see that there are a fixed finite p_0 and a $\sigma_0 > 0$ such that for all $0 < \sigma \leq \sigma_0$, $0 < T_0 < \infty$ and $k \geq k_1 > 0$

$$f_\sigma \leq k + d, \quad d^{p_0} = C_2 2^{p_0 \gamma / \gamma + 1} \|A(k)\|_{T_0}^{\gamma-1}$$

where k_1 and C_2 are constants depending only on n, ε, C_1, M_0 and

$$\|A(k)\|_{T_0} = \int_0^{T_0} \int_{A(k)} d\mu dt.$$

To show that this term is bounded independently of T we show that $\|A(k)\| = \int_{A(k)} d\mu$ decays exponentially for some $k \geq k_1$.

Choose a fixed p such that (3.5) holds and then $\sigma \leq \frac{1}{2}p^{-1}$, such that

$$\begin{aligned} \frac{\partial}{\partial t} \int_{A(k_1)} f_{\sigma, k_1}^p d\mu &\leq -\frac{1}{2} \int_{A(k_1)} H^2 f_{\sigma, k_1}^{p-1} f_\sigma d\mu \\ &\leq -\frac{1}{2} \int_{A(k_1)} H^2 f_{\sigma, k_1}^p d\mu \end{aligned}$$

since $f_\sigma \geq f_{\sigma, k_1}$. Moreover, on $A(k_1)$ we have

$$k_1 \leq \frac{|A|^2 - \frac{1}{n} H^2}{H^{2-\sigma}} \leq H^\sigma$$

and therefore

$$\frac{\partial}{\partial t} \int_{A(k_1)} f_{\sigma, k_1}^p d\mu \leq -\frac{1}{2} k_1^{2/\sigma} \int_{A(k_1)} f_{\sigma, k_1}^p d\mu,$$

so we get

$$\int_{A(k_1)} f_{\sigma, k_1}^p d\mu \leq e^{-\delta t} \int_{A(k_1)} f_{\sigma, k_1}^p d\mu|_{t=0},$$

where $\delta = \frac{1}{2} k_1^{2/\sigma}$. Then let $k = k_1 + 1$ and observe that $1 \leq (f_\sigma - k_1)^p$ on $A(k_1 + 1)$. Thus we have

$$\begin{aligned} |A(k_1 + 1)| &= \int_{A(k_1+1)} d\mu \leq \int_{A(k_1+1)} (f_\sigma - k_1)^p d\mu \\ &\leq \int_{A(k_1)} (f_\sigma - k_1)^p d\mu = \int_{A(k_1)} f_{\sigma, k_1}^p d\mu \leq C e^{-\delta t}. \end{aligned}$$

Then the proof of Theorem 3.1 is completed.

4. Gradient estimate for the mean curvature

Theorem 4.1. *For all $\eta > 0$ there is a constant $C_3 < \infty$ depending only on η, M_0 and θ such that*

$$|\nabla H|^2 \leq \eta H^4 + C_3 \quad (4.1)$$

holds on M_t for all $0 \leq t \leq T$.

In order to prove this theorem, we will use the method used by Huisken [16] and need the following lemmas.

Lemma 4.2. *We have*

- (i) $\frac{\partial}{\partial t}|\nabla H|^2 = \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2|A|^2|\nabla H|^2 + 2\langle \nabla_i H \cdot h_{mj}, \nabla_j H \cdot h_{im} \rangle$
 $+ 2(H - \theta)\langle \nabla_i |A|^2, \nabla_j H \rangle - 2\theta\langle h_{ij}, \nabla_i H \cdot \nabla_j H \rangle,$
- (ii) $\frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta|\nabla H|^2 + 8nH(H + \theta)|\nabla A|^2.$

Proof. (i) By computing as in [16]

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla H|^2 &= \frac{\partial}{\partial t}(g^{ij}\nabla_i H \cdot \nabla_j H) \\ &= 2\langle \nabla_i H, \nabla_i(\Delta H + (H - \theta)|A|^2) \rangle + 2(H - \theta)\langle h_{ij}, \nabla_i H \nabla_j H \rangle, \\ \Delta|\nabla H|^2 &= 2g^{kl}\Delta(\nabla_k H)\nabla_l H + 2|\nabla^2 H|^2, \\ \Delta(\nabla_k H) &= \nabla_k(\Delta H) + g^{ij}\nabla_i H(Hh_{kj} - h_{km}g^{mn}h_{mj}), \\ 2g^{ij}\nabla_i(\Delta H)\nabla_j H &= 2g^{ij}\Delta(\nabla_i H)\nabla_j H - 2g^{ij}(g^{sp}\nabla_s H(Hh_{ip} - h_{im}g^{mn}h_{np}))\nabla_j H \\ &= \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 - 2g^{ij}g^{sp}\nabla_s H \cdot Hh_{ip} \cdot \nabla_j H \\ &\quad + 2g^{ij}g^{sp}\nabla_s H \cdot h_{im}g^{mn}h_{np} \cdot \nabla_j H \\ &= \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 - 2H\langle h_{ij}, \nabla_i H \cdot \nabla_j H \rangle \\ &\quad + 2\langle \nabla_i H \cdot h_{mj}, \nabla_j H \cdot h_{im} \rangle, \end{aligned}$$

hence conclusion (i) follows.

(ii) Note that

$$\begin{aligned} |2(H - \theta)\langle \nabla_i H, \nabla_i |A|^2 \rangle| &= 2|(H - \theta)\langle \nabla_i H, \nabla_i(h_{kl}h^{kl}) \rangle| \\ &\leq 4nH^2|\nabla A|^2 + 4n\theta H|\nabla A|^2, \end{aligned}$$

$$h_{ij} \geq 0, \quad |\nabla H|^2 \leq n|\nabla A|^2.$$

Using Hölder inequality, we obtain

$$\frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta|\nabla H|^2 + 8nH(H + \theta)|\nabla A|^2.$$

Then conclusion (ii) follows. $\square$

Lemma 4.3. *For all $0 \leq t \leq T$, we have the estimate*

- (i) $\frac{\partial}{\partial t}\left(|A|^2 - \frac{1}{n}H^2\right) \leq \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - \frac{2(n-1)}{3n}|\nabla A|^2 + 2|A|^2\left(|A|^2 - \frac{1}{n}H^2\right),$
- (ii) $\frac{\partial}{\partial t}\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leq \Delta\left(H^2\left(|A|^2 - \frac{1}{n}H^2\right)\right) - \frac{2(n-1)}{3n}H^2|\nabla A|^2$
 $+ 4H^2|A|^2\left(|A|^2 - \frac{1}{n}H^2\right) - 4H\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right) \rangle,$
- (iii) $\frac{\partial}{\partial t}\left(H\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leq \Delta\left(H\left(|A|^2 - \frac{1}{n}H^2\right)\right) - \frac{2(n-1)}{3n}H|\nabla A|^2$
 $+ 3H|A|^2\left(|A|^2 - \frac{1}{n}H^2\right) - 2\left\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right) \right\rangle.$

Proof. Claim (i), (ii) follow from Lemma 3.4 (i) in [16]. So we only need to prove claim (iii). By computing as in [16], (2.3) and (2.5), we have

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left(\left(|A|^2 - \frac{1}{n} H^2 \right) H \right) \\
 &= \left(|A|^2 - \frac{1}{n} H^2 \right) (\Delta H + (H - \theta) |A|^2) + H \left[\Delta \left(|A|^2 - \frac{1}{n} H^2 \right) \right. \\
 &\quad \left. - 2 \left(|\nabla A|^2 - \frac{1}{n} |\nabla H|^2 \right) + 2 |A|^2 \left(|A|^2 - \frac{1}{n} H^2 \right) - 2\theta \left(C - \frac{1}{n} H |A|^2 \right) \right] \\
 &= \Delta \left(\left(|A|^2 - \frac{1}{n} H^2 \right) H \right) - 2 \left\langle \nabla_i H, \nabla_i \left(|A|^2 - \frac{1}{n} H^2 \right) \right\rangle \\
 &\quad + (3H - \theta) |A|^2 \left(|A|^2 - \frac{1}{n} H^2 \right) - 2H \left(|\nabla A|^2 - \frac{1}{n} |\nabla H|^2 \right) \\
 &\quad - 2\theta H \left(C - \frac{1}{n} H |A|^2 \right). \tag{4.2}
 \end{aligned}$$

From Lemma 2.2 (ii) in [16], we know the inequality

$$|\nabla A|^2 - \frac{1}{n} |\nabla H|^2 \geq \frac{2(n-1)}{3n} |\nabla A|^2. \tag{4.3}$$

Combining (4.2) with (4.3), we have

$$\begin{aligned}
 \frac{\partial}{\partial t} \left(\left(|A|^2 - \frac{1}{n} H^2 \right) H \right) &\leq \Delta \left(\left(|A|^2 - \frac{1}{n} H^2 \right) H \right) - \frac{2(n-1)}{3n} H |\nabla A|^2 \\
 &\quad + 3H |A|^2 \left(|A|^2 - \frac{1}{n} H^2 \right) - 2 \left\langle \nabla_i H, \nabla_i \left(|A|^2 - \frac{1}{n} H^2 \right) \right\rangle.
 \end{aligned}$$

□

Now we are in a position to prove Theorem 4.1.

Proof. Let $h_{ij}^0 = h_{ij} - \frac{1}{n} g_{ij}$, using Theorem 3.1, we have

$$\begin{aligned}
 H \left| \left\langle \nabla_i H, \nabla_i \left(|A|^2 - \frac{1}{n} H^2 \right) \right\rangle \right| &= H |\langle \nabla_i H, \nabla_i (h_{kl}^0 h^{0kl}) \rangle| \\
 &\leq 2H |\nabla H| |h_{ij}^0| |\nabla_k h_{ij}^0| \leq 2nC_0^{\frac{1}{2}} |\nabla A|^2 H^{2-\frac{\delta}{2}} \\
 &\leq \frac{(n-1)}{3n} \frac{1}{4} H^2 |\nabla A|^2 + C_4(C_0, n, \delta) |\nabla A|^2,
 \end{aligned}$$

and similarly

$$\theta \left| \left\langle \nabla_i H, \nabla_i \left(|A|^2 - \frac{1}{n} H^2 \right) \right\rangle \right| \leq \frac{n-1}{3n} \frac{1}{4} \theta H |\nabla A|^2 + C_5(C_0, n, \delta) \theta |\nabla A|^2.$$

Let

$$g_1 = (H + \theta) H \left(|A|^2 - \frac{1}{n} H^2 \right), \quad g_2 = (1 + \theta) \left(|A|^2 - \frac{1}{n} H^2 \right),$$

then

$$\begin{aligned}\frac{\partial}{\partial t}g_1 &\leq \Delta g_1 - \frac{(n-1)}{3n}(H+\theta)H|\nabla A|^2 + (C_4 + \theta C_5)|\nabla A|^2 \\ &\quad + 4H^2|A|^2 \left(|A|^2 - \frac{1}{n}H^2\right) + 3\theta H|A|^2 \left(|A|^2 - \frac{1}{n}H^2\right), \\ \frac{\partial}{\partial t}g_2 &\leq \Delta g_2 - \frac{2(n-1)}{3n}(1+\theta)|\nabla A|^2 + 2(1+\theta)|A|^2 \left(|A|^2 - \frac{1}{n}H^2\right).\end{aligned}$$

Let N_1 be so large that $N_1 \cdot \frac{2(n-1)}{3n} \geq 2\max(C_4, C_5)$, then

$$\begin{aligned}\frac{\partial}{\partial t}(g_1 + N_1g_2) &\leq \Delta(g_1 + N_1g_2) - \frac{(n-1)}{3n}(H+\theta)H|\nabla A|^2 \\ &\quad - N_1 \cdot \frac{(n-1)}{3n}(1+\theta)|\nabla A|^2 \\ &\quad + [4(H+\theta)H|A|^2 + 2N_1(1+\theta)|A|^2] \cdot \left(|A|^2 - \frac{1}{n}H^2\right).\end{aligned}$$

Now let $N_2 \cdot \frac{(n-1)}{3n} \geq 16n$, then we see from Lemma 4.2 (ii) that the function

$$g_3 = |\nabla H|^2 + N_2g_1 + N_2N_1g_2$$

satisfies the inequality

$$\begin{aligned}\frac{\partial}{\partial t}g_3 &\leq \Delta g_3 - N_2 \frac{n-1}{6n}(H+\theta)H|\nabla A|^2 - N_1N_2 \frac{(n-1)}{3n}(1+\theta)|\nabla A|^2 \\ &\quad + N_2[4(H+\theta)H|A|^2 + 2N_1(1+\theta)|A|^2] \left(|A|^2 - \frac{1}{n}H^2\right).\end{aligned}$$

From Theorem 3.1, we have

$$\begin{aligned}&N_2[4(H+\theta)H|A|^2 + 2N_1(1+\theta)|A|^2] \left(|A|^2 - \frac{1}{n}H^2\right) \\ &\leq N_2[4H^2 + 4\theta H + 2N_1(1+\theta)]H^2C_0 \cdot H^{2-\delta} \\ &\leq \eta H^2H^4 + C_6H^2 + C_7H^3\end{aligned}$$

with C_6, C_7 depending only on $\delta, N_1, N_2, C_0, \theta$.

Thus we derive

$$\begin{aligned}\frac{\partial}{\partial t}g_3 &\leq \Delta g_3 - N_2 \frac{n-1}{6n^2}(H+\theta)H|\nabla A|^2 - N_1N_2 \frac{n-1}{3n^2}(1+\theta)|\nabla A|^2 \\ &\quad + \eta H^2H^4 + C_8H^3.\end{aligned}$$

We want to show now that for any $\tilde{\eta} > 0$, there is $C_9(\tilde{\eta})$ such that

$$g_3 < \tilde{\eta}H^4 + C_9 \quad \text{for } 0 \leq t \leq T.$$

Choose C_9 so large that this inequality holds at $t = 0$ and then suppose there is a first time $t = t_0 \leq T$ where $g_3 = \tilde{\eta}H^4 + C_9$ at x_0 . At this point we have $\Delta g_3 \leq 0$, $\frac{\partial}{\partial t}g_3 \geq 0$ and therefore

$$\begin{aligned}0 &\leq -[N_2(H+\theta)H + 2N_2N_1(1+\theta)] \frac{n-1}{6n^2} \cdot \left[\tilde{\eta}H^4 + C_9\right. \\ &\quad \left.- N_2(H^2 + \theta H + N_1(1+\theta)) \left(|A|^2 - \frac{1}{n}H^2\right)\right] + \eta H^2H^4 + C_9H^3.\end{aligned}$$

Using Theorem 3.1 as before we have

$$\tilde{\eta}H^4 + C_9 - N_2 \left\{ (H^2 + \theta H + N_1(1 + \theta)) \left(|A|^2 - \frac{1}{n}H^2 \right) \geq \frac{1}{2}\tilde{\eta}H^4 + \frac{1}{2}C_9 \right.$$

provided $C_9 = C_9(\tilde{\eta}, \delta, N_1, N_2, C_0, M_0)$ is large enough. So we get

$$0 \leq -\tilde{\eta}\frac{n-1}{12n^2}N_2H^2H^4 - N_2N_1\frac{n-1}{6n^2}C_9 - N_2N_1\frac{n-1}{6n^2}\tilde{\eta}H^4 + \eta H^2H^4 + C_9(\eta)H^3$$

and we yield a contradiction if we first choose $\eta \leq \tilde{\eta}\frac{n-1}{12n^2}N_2$ and then $C_9 = C_9(C_\delta, \tilde{\eta})$ large. This completes the proof of Theorem 4.1. $\square$

5. Higher derivatives estimate

Let $S*T$ denote any linear combination of tensors formed by contraction with g from S and T , $\nabla^m T$ denote m -th covariant derivative of tensor T .

Theorem 5.1. *We have*

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^m A|^2 &= \Delta |\nabla^m A|^2 - 2 |\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \\ &\quad + \theta \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A. \end{aligned} \quad (5.1)$$

In order to prove Theorem 5.1, we will use the method used by Huisken [16] and need the following lemmas.

Lemma 5.2. *Assume $\bar{A}$ and B are two tensors, and $\frac{\partial}{\partial t} \bar{A} = \Delta \bar{A} + B$. Then*

$$\frac{\partial}{\partial t} \nabla \bar{A} = \Delta \nabla \bar{A} + \nabla R_m * \bar{A} + R_m * \nabla \bar{A} + \bar{A} * A * \nabla A + \theta \bar{A} * \nabla A + \nabla B$$

where $R_m = \{R_{ijkl}\}$.

Proof. By computing as in [16]

$$\begin{aligned} \frac{\partial}{\partial t} \Gamma_{jk}^i &= \frac{1}{2} g^{il} \left\{ \nabla_j \left(\frac{\partial}{\partial t} g_{kl} \right) + \nabla_k \left(\frac{\partial}{\partial t} g_{jl} \right) - \nabla_l \left(\frac{\partial}{\partial t} g_{jk} \right) \right\} \\ &= -g^{il} \{ \nabla_j ((H - \theta) h_{kl}) + \nabla_k ((H - \theta) h_{jl}) - \nabla_l ((H - \theta) h_{jk}) \} \\ &= A * \nabla A + \theta (g^{il} \nabla_j h_{kl} + g^{jl} \nabla_k h_{jl} - g^{il} \nabla_l h_{jk}). \end{aligned}$$

One has

$$\frac{\partial}{\partial t} \nabla \bar{A} = \nabla \frac{\partial \bar{A}}{\partial t} + \bar{A} * A * \nabla A + \theta \bar{A} * \nabla A.$$

Again, one has

$$\Delta \nabla \bar{A} = \Delta \nabla \bar{A} + \nabla R_m * \bar{A} + R_m * \nabla \bar{A}.$$

Hence, the conclusion follows. $\square$

Lemma 5.3. Suppose $A = \{h_{ij}\}$, then

$$\frac{\partial}{\partial t} \nabla^m A = \Delta(\nabla^m A) + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A + \theta \sum_{i+j=m} \nabla^i A * \nabla^j A. \quad (5.2)$$

Proof. For $m = 0$, the conclusion follows from Lemma 2.1. Assume that (5.2) holds for m . By Lemma 5.2, we have

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^{m+1} A &= \Delta \nabla^{m+1} A + \nabla R_m * \nabla^m A + R_m * \nabla^{m+1} A \\ &\quad + \nabla \left(\sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A + \sum_{i+j=m} \nabla^i A * \nabla^j A \right) \\ &= \Delta \nabla^{m+1} A + \sum_{i+j+k=m+1} \nabla^i A * \nabla^j A * \nabla^k A \\ &\quad + \theta \sum_{i+j=m+1} \nabla^i A * \nabla^j A, \end{aligned}$$

i.e. (5.2) is true for $m + 1$. Hence the proof of Lemma 5.3 is completed. $\square$

The proof of Theorem 5.1. By computing as in [16]

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^m A|^2 &= 2 \left\langle \nabla^m A, \frac{\partial}{\partial t} \nabla^m A \right\rangle + A * A * \nabla^m A * \nabla^m A + \theta A * \nabla^m A * \nabla^m A \\ &= 2 \langle \nabla^m A, \Delta \nabla^m A \rangle + 2 \left\langle \nabla^m A, \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A \right\rangle \\ &\quad + \left\langle \nabla^m A, \theta \sum_{i+j=m} \nabla^i A * \nabla^j A \right\rangle + A * A * \nabla^m A * \nabla^m A \\ &\quad + \theta A * \nabla^m A * \nabla^m A \\ &= 2 \langle \nabla^m A, \Delta \nabla^m A \rangle + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \\ &\quad + \theta \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A, \\ \Delta |\nabla^m A|^2 &= 2 \langle \nabla^m A, \Delta \nabla^m A \rangle + 2 |\nabla^{m+1} A|^2. \end{aligned}$$

We obtain

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^m A|^2 &= \Delta |\nabla^m A|^2 - 2 |\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \\ &\quad + \theta \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A. \end{aligned}$$

The proof of Theorem 5.1 is completed.

Theorem 5.4. *We have*

$$\begin{aligned} & \frac{d}{dt} \int |\nabla^m A|^2 d\mu + 2 \int |\nabla^{m+1} A|^2 d\mu \\ & \leq C(n, m) (\max_{M_t} |A|^2 + \theta \max_{M_t} |A|) \int |\nabla^m A|^2 d\mu. \end{aligned}$$

Proof. By computing as in [16]

$$\begin{aligned} & \int \frac{\partial}{\partial t} |\nabla^m A|^2 + 2 \int |\nabla^{m+1} A|^2 \\ & = \int \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A + \theta \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A, \end{aligned}$$

we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \int |\nabla^m A|^2 + 2 \int |\nabla^{m+1} A|^2 \\ & \leq N \left\{ \int |\nabla^i A|^{2m/j} \right\}^{j/2m} \left\{ \int |\nabla^j A|^{2m/k} \right\}^{k/2m} \left\{ \int |\nabla^k A|^{2m/k} \right\}^{k/2m} \\ & \quad \times \left\{ \int |\nabla^m A|^2 \right\}^{1/2} + N\theta \left\{ \int |\nabla^i A|^{2m/i} \right\}^{i/2m} \\ & \quad \times \left\{ \int |\nabla^j A|^{2m/j} \right\}^{j/2m} \left\{ \int |\nabla^m A|^2 \right\}^{1/2}. \end{aligned}$$

Using the following inequality in [15]

$$\int_M |\nabla^i A|^{2m/i} d\mu \leq C(m, n) \max |A|^{2(m/i-1)} \int_M |\nabla^m A|^2 d\mu,$$

we obtain the conclusion. $\square$

From Theorem 5.4 and the proof of Lemma 8.3 in [16], we have

Corollary 5.5. *If $\max_{M_t} |A|^2 \leq N$, $\forall 0 \leq t < T < +\infty$, then $|\nabla^m A| \leq N_m$ for all m . The constant C_m depends on n , M_0 and N .*

Theorem 5.6. *The maximal solution of problem (1.2)–(1.3) with uniformly convex initial hypersurface either exists for all time or shrinks to a point in finite time. Moreover, if $t_{\max} < \infty$, then we have*

$$\max_{M_t} |A|^2 \rightarrow +\infty \quad \text{as } t \rightarrow t_{\max}. \quad (5.3)$$

Proof. By the method used by Evans and Spruck [11], we have that the equation (1.2), (1.3) has a unique solution M_t for a short time with any smooth compact initial surface M_0 at $t = 0$.

Let $0 \leq t < t_{\max}$ be the maximal time interval where the solution exists. If $t_{\max} < \infty$, by the proof of Theorem 8.1 in [16], we have

$$\max_{M_t} |A|^2 \rightarrow +\infty \quad \text{as } t \rightarrow t_{\max}. \quad (5.4)$$

From Theorem 3.1, Theorem 5.1, Corollary 5.5, (5.4), and the proof of Theorem 8.1 in [16, section 8], one yields the conclusion. $\square$

6. Convergence to a stationary solution

In this section, we show that to each uniformly convex without boundary initial surface M_0 , there exists a unique θ^* such that the solution of problem (1.2)–(1.3) shrinks or expands depending on whether $\theta < \theta^*$ or $\theta > \theta^*$. Also, we show that the solution of problem (1.2)–(1.3) with $\theta = \theta^*$ converges to a limiting surface.

We begin by stating a comparison principle that is a direct consequence of the comparison principle for viscosity solutions of our mean curvature evolution PDE problem (1.2)–(1.3) [7, 10].

Proposition 6.1. *Let M_t^1 and M_t^2 be a solution of problem (1.2)–(1.3) with uniformly convex initial surface M_0^1 , M_0^2 , respectively. Let U_t^1 , U_t^2 be the domain bounded by M_t^1 , M_t^2 , respectively. Suppose $U_0^1 \subset U_0^2$, then $U_t^1 \subset U_t^2$ for $t > 0$.*

Assume M_0 is a compact, uniformly convex n -dimensional hypersurface without boundary, which is smoothly imbedded in $\mathbb{R}^{n+1}$, and suppose that M_0 is represented locally by some diffeomorphism

$$F_0 : \mathbb{R}^n \supset U \rightarrow F_0(U) \subset M_0 \subset \mathbb{R}^{n+1}.$$

Consider the following equations

$$\begin{cases} F_t = (-H + \theta)\vec{\nu}, & x \in U, t > 0, \\ F(\cdot, 0) = F_0, & x \in U, \end{cases} \quad (6.1)_\theta$$

$$\begin{cases} F_t^1 = (-H^1 + \theta^1)\vec{\nu}^1, & x \in U, t > 0, \\ F^1(\cdot, 0) = F_0, & x \in U \end{cases} \quad (6.3)_{\theta^1}$$

where H, H^1 is mean curvature of the surface M_t, M_t^1 , respectively.

Proposition 6.2. *Assume that M_0 is a compact, uniformly convex n -dimensional hypersurface without boundary. Suppose that $\{M_t\}_{t>0}$ is the solution of (6.1) $_{\theta}$ –(6.2), $\{M_t^1\}_{t>0}$ is the solution of (6.3) $_{\theta^1}$ –(6.4). Let U_t and U_t^1 is domain bounded by M_t, M_t^1 , respectively. Suppose that $\theta < \theta^1$. Then $U_t \subset U_t^1$ for $t > 0$.*

In order to prove Proposition 6.2, we will use the viscosity solution theory as in [7, 10]. Consider the following problem

$$\begin{cases} u_t = \frac{1}{n}(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2})u_{x_i x_j} - \theta|\nabla u|, & (x, t) \in \mathbb{R}^{n+1} \times (0, T) \end{cases} \quad (6.5)$$

$$\begin{cases} u|_{t=0} = g, & x \in \mathbb{R}^{n+1} \end{cases} \quad (6.6)$$

where $\nabla u = (u_{x_1}, u_{x_2}, \dots, u_{x_{n+1}})$.

Definition 6.1. A function $u \in C(\mathbb{R}^{n+1} \times [0, T)) \cap L^\infty(\mathbb{R}^{n+1} \times (0, T))$ is a weak subsolution of equation (6.5) provided that if $u - \phi$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^{n+1} \times (0, T)$ for each $\phi \in C^2(\mathbb{R}^{n+1}) \times (0, T)$, then

$$\begin{cases} \phi_t \leq \frac{1}{n}(\delta_{ij} - \phi_{x_i}\phi_{x_j}/|\nabla\phi|^2)\phi_{x_i x_j} - \theta|\nabla\phi| & \text{at } (x_0, t_0) \\ \text{if } \nabla\phi(x_0, t_0) \neq 0 \end{cases} \quad (6.7)$$

and

$$\begin{cases} \phi_t \leq \frac{1}{n}(\delta_{ij} - \eta_i \eta_j) \phi_{x_i x_j} & \text{at } (x_0, t_0) \\ \text{for some } \eta \in \mathbb{R}^{n+1} \text{ with } |\eta| \leq 1, & \text{if } \nabla \phi(x_0, t_0) = 0. \end{cases} \quad (6.8)$$

Definition 6.2. A function $u \in C(\mathbb{R}^{n+1} \times [0, T))$ is a weak supersolution of equation (6.5) provided that if $u - \phi$ has a local minimum at a point $(x_0, t_0) \in \mathbb{R}^{n+1} \times (0, T)$ for each $\phi \in C^2(\mathbb{R}^{n+1} \times (0, T))$, then

$$\begin{cases} \phi_t \geq \frac{1}{n}(\delta_{ij} - \frac{\phi_{x_i} \phi_{x_j}}{|\nabla \phi|^2}) \phi_{x_i x_j} - \theta |\nabla \phi| & \text{at } (x_0, t_0) \\ \text{if } \nabla \phi(x_0, t_0) \neq 0, \end{cases} \quad (6.9)$$

and

$$\begin{cases} \phi_t \geq \frac{1}{n}(\delta_{ij} - \eta_i \eta_j) \phi_{x_i x_j} & \text{at } (x_0, t_0) \\ \text{for some } \eta \in \mathbb{R}^{n+1} \text{ with } |\eta| \neq 1, & \text{if } \nabla \phi(x_0, t_0) = 0. \end{cases} \quad (6.10)$$

Definition 6.3. A function $u \in C(\mathbb{R}^{n+1} \times [0, T))$ is a weak solution of (6.5) provided u is both a weak subsolution and a weak supersolution.

Lemma 6.1. Assume that u is a weak subsolution in $\mathbb{R}^{n+1} \times [0, T]$ of the following equation

$$u_t = \frac{1}{n} \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2} \right) u_{x_i x_j} - \theta_1 |\nabla u|$$

and v is a weak supersolution in $\mathbb{R}^{n+1} \times [0, T]$ of the following equation

$$v_t = \frac{1}{n} \left(\delta_{ij} - \frac{v_{x_i} v_{x_j}}{|\nabla v|^2} \right) v_{x_i x_j} - \theta_2 |\nabla v|.$$

Suppose further

$$\begin{aligned} \theta_1 &\geq \theta_2, \\ u &\leq v \text{ on } \mathbb{R}^{n+1} \times \{t = 0\}. \end{aligned}$$

Finally assume

$$\begin{cases} u \text{ and } v \text{ are constant, with } u \leq v, \\ \text{on } \mathbb{R}^{n+1} \times [0, T] \cap \{|x| \geq R(T)\} \end{cases} \quad (6.11)$$

for some constant $R(T) \geq 0$. Then

$$u \leq v \text{ on } \mathbb{R}^{n+1} \times [0, T]. \quad (6.12)$$

Proof. The key technical devices in the proof are taken from [10]. For the sake of completeness, we sketch it here.

1. Should (6.12) fail, then

$$\max_{(x,t) \in \mathbb{R}^{n+1} \times [0,T]} (u - v) \equiv a > 0;$$

and so for $\alpha > 0$ small enough,

$$\max_{(x,t) \in \mathbb{R}^{n+1} \times [0,T]} (u - v - \alpha t) \geq \frac{a}{2} > 0.$$

Define

$$\begin{aligned} u^\varepsilon(x, t) &= \sup_{y \in \mathbb{R}^{n+1}, s \in [0, T]} \{u(y, s) - \varepsilon^{-1}(|x - y|^2 + (t - s)^2)\}, \\ v_\varepsilon(x, t) &= \inf_{y \in \mathbb{R}^{n+1}, s \in [0, T]} \{v(y, s) + \varepsilon^{-1}(|x - y|^2 + (t - s)^2)\}, \end{aligned}$$

According to (6.11) we have

$$u^\varepsilon = u, \quad v_\varepsilon = v, \quad \text{on } \{|x| \geq 2R(T)\} \quad (6.13)$$

for all small $\varepsilon > 0$. Note further $u^\varepsilon \rightarrow u$ and $v_\varepsilon \rightarrow v$ uniformly by Lemma 3.1 in [10]. Consequently if we fix $\varepsilon > 0$ small enough,

$$\max_{(x, t) \in \mathbb{R}^{n+1} \times [0, T]} (u^\varepsilon - v_\varepsilon - \alpha t) \geq \frac{\alpha}{4} > 0. \quad (6.14)$$

2. Given $\delta > 0$, define

$$\Phi(x, y, t, s) = u^\varepsilon(x + y, t + s) - v_\varepsilon(x, t) - \alpha t - \delta^{-1}(|y|^4 + s^4) - \eta \frac{1}{T-t}. \quad (6.15)$$

Owing to (6.14) we see, if $\eta > 0$ small enough,

$$\max_{(x, t), (x+y, t+s) \in \mathbb{R}^{n+1} \times [0, T]} \Phi \geq \frac{\alpha}{8} > 0. \quad (6.16)$$

Choose now $(x_1, t_1), (x_1 + y_1, t_1 + s_1) \in \mathbb{R}^{n+1} \times [0, T]$ so that

$$\Phi(x_1, y_1, t_1, s_1) = \max_{(x, t), (x+y, t+s) \in \mathbb{R}^{n+1} \times [0, T]} \Phi. \quad (6.17)$$

Note in view of (6.13), (6.15) and Lemma 3.1 (ii) in [10] that such points exists. Since $\Phi(x_1, y_1, t_1, s_1) > 0$, (6.15) implies

$$|y_1|, |s_1| \leq C\delta^{1/4}.$$

3. If $\varepsilon, \delta > 0$ are fixed small enough, we have

$$\sigma(\varepsilon) < t_1, \quad t_1 + s < T,$$

with $\sigma(\varepsilon)$ defined in (3.3) of [10]. In fact, $\Phi \rightarrow -\infty$ as $t \rightarrow T^-$ uniformly on $\{|x| \geq 2R(T)\} \times \{|x| \geq 2R(T)\}$. Hence $t_1, t_1 + s < T$. The proof of $t_1, t_1 + s < \sigma(\varepsilon)$ is identical that of (3.17) in [10]. Hereafter in the proof, $\alpha, \varepsilon, \delta, \eta > 0$ are fixed.

According to Lemma 3.1 (vii) in [10], u^ε is a weak subsolution near

$$(x_1 + y_1, t_1 + s_1), \quad (6.18)$$

v_ε is a weak supersolution near

$$(x_1, t_1). \quad (6.19)$$

4. Claim: $y_1 \neq 0$.

Assume that $y_1 = 0$. Then (6.15), (6.17) imply

$$\begin{aligned} u^\varepsilon(x_1, t_1 + s_1) - v_\varepsilon(x_1, t_1) - \alpha t_1 - \delta^{-1}s_1^4 - \frac{\eta}{T-t_1} \\ \geq u^\varepsilon(x + y, t + s) - v_\varepsilon(x, t) - \alpha t - \delta^{-1}(|y|^4 + s^4) - \frac{\eta}{T-t} \end{aligned}$$

for all $(x, t), (x + y, t + s) \in \mathbb{R}^{n+1} \times [0, T]$.

First, put $x = x_1$, $t = t_1$ as above, and simplify to obtain

$$u^\varepsilon(x_1 + y, t_1 + s) \leq u^\varepsilon(x_1, t_1 + s_1) + \delta^{-1}|y^4| + \delta^{-1}(s^4 - s_1^4)$$

for $(x_1 + y, t_1 + s) \in \mathbb{R}^{n+1} \times [0, T]$. By the same computing of (3.20) in [10], we have

$$\frac{4s_1^3}{\delta} \leq 0.$$

Second, put $y = x_1 - x$, $s = t_1 + s_1 - t$, then

$$\begin{aligned} v_\varepsilon(x, t) &\geq v_\varepsilon(x_1, t_1) + \left(\frac{4s_1^3}{\delta} - \alpha - \frac{\eta}{(T - t_1)(T - t)} \right) (t - t_1) \\ &\quad - 6s_1^2 \frac{(t - t_1)^2}{\delta} + O((x - x_1)^4 + |t - t_1|^4) \text{ as } (x, t) \rightarrow (x_1, t_1). \end{aligned}$$

Note that v_ε is a weak supersolution near (x_1, t_1) . By the same computing of (3.23) in [10], we have

$$\frac{4s_1^3}{\delta} - \alpha - \frac{\eta}{(T - t_1)(T - t)} \geq 0.$$

This yields a contradiction with $\frac{4s_1^3}{\delta} \leq 0$.

5. By Lemma 3.1 (vi) in [10], we have

$$(x, y, t, s) \rightarrow u^\varepsilon(x + y, t + s) + \varepsilon^{-1}(|x - y|^2 + (t + s)^2)$$

is convex. As

$$(x, t) \rightarrow -v_\varepsilon(x, t) + \varepsilon^{-1}(|x|^2 + t^2)$$

is convex as well, we have

$$(x, y, t, s) \rightarrow \Phi(x, y, t, s) + C(|x|^2 + |y|^2 + t^2 + s^2)$$

is convex near (x_1, y_1, t_1, s_1) , for some sufficiently large constant $C = C(\varepsilon, \delta)$. Since Φ attains its maximum at (x_1, y_1, t_1, s_1) , we may use Jensen theorem: there exist points $\{(x^k, y^k, t^k, s^k)\}_{k=1}^\infty$ such that

$$(x^k, y^k, t^k, s^k) \rightarrow (x_1, y_1, t_1, s_1) \text{ as } k \rightarrow \infty, \quad (6.20)$$

$$\begin{aligned} \Phi, u^\varepsilon \text{ and } v_\varepsilon \text{ are each twice differentiable at } (x^k, y^k, t^k, s^k) \\ (k = 1, 2, \dots), \end{aligned} \quad (6.21)$$

$$\nabla_{x,y,t,s} \Phi(x^k, y^k, t^k, s^k) \rightarrow 0 \text{ as } k \rightarrow \infty, \quad (6.22)$$

$$\nabla_{x,y,t,s}^2 \Phi(x^k, y^k, t^k, s^k) \leq o(1)I_{2(n+1)+2} \text{ as } k \rightarrow \infty. \quad (6.23)$$

6. By (6.15), (6.21), we have

$$\nabla_x \Phi(x^k, y^k, t^k, s^k) = \nabla u^\varepsilon(x^k + y^k, t^k + s^k - \nabla v_\varepsilon(x^k, t^k)) \equiv p^k - \bar{p}^k, \quad (6.24)$$

$$\begin{aligned} \nabla_y \Phi(x^k, y^k, t^k, s^k) &= \nabla u^\varepsilon(x^k + y^k, t^k + s^k) - 4|y^k|^2 y^k / \delta \\ &= p^k - 4|y^k|^2 y^k / \delta. \end{aligned} \quad (6.25)$$

Since $y^k \rightarrow y_1$, we deduce from (6.22) that

$$p^k, \bar{p}^k \rightarrow 4|y_1|^2 y_1 / \delta \equiv p \text{ in } \mathbb{R}^{n+1} \quad (6.26)$$

by step 4, we have $p \neq 0$. Hence $p^k, \bar{p}^k \neq 0$ for large enough k .

From (6.15), (6.22) we note

$$\begin{aligned} \Phi_t(x^k, y^k, t^k, s^k) &= u_t^\varepsilon(x^k, y^k, t^k, s^k) - v_{\varepsilon t}(x^k, t^k) - \alpha - \frac{\eta}{(T - t^k)^2} \\ &\equiv q^k - \bar{q}^k - \alpha - \frac{\eta}{(T - t^k)^2}. \end{aligned} \quad (6.27)$$

As u^ε and v^ε are Lipschitz, we may assume (passing to a subsequence if necessary) that

$$q^k \rightarrow q, \quad \bar{q}^k \rightarrow \bar{q} \text{ in } \mathbb{R}. \quad (6.28)$$

Then (6.22), (6.27) ensure

$$q - \bar{q} \geq \alpha > 0. \quad (6.29)$$

7. (6.15), (6.21) imply

$$\begin{aligned} \nabla_x^2 \Phi(x^k, y^k, t^k, s^k) &= \nabla^2 u^\varepsilon(x^k + y^k, t^k + s^k) - \nabla^2 v_\varepsilon(x^k, t^k) \\ &\equiv R^k - \bar{R}^k. \end{aligned}$$

Now (6.23) forces

$$R^k - \bar{R}^k \leq \varepsilon_k I_{(n+1)}, \quad (6.30)$$

where $\varepsilon_k \rightarrow 0$. Furthermore, by Lemma 3.1 (vi) in [10], one has $R^k \geq -CI_{(n+1)}$ and $\bar{R}^k \leq CI_{(n+1)}$, for $C = C(\varepsilon)$. Thus

$$-CI_{(n+1)} \leq R^k \leq \bar{R}^k + \varepsilon_k I_{(n+1)} \leq CI_{(n+1)}.$$

We may consequently suppose that (passing as necessary to subsequence)

$$R^k \rightarrow R, \quad \bar{R}^k \rightarrow \bar{R} \text{ in } S^{n \times n}. \quad (6.31)$$

with

$$R \leq \bar{R}. \quad (6.32)$$

8. Note that $p, \bar{p} \neq 0$, hence $p^k = \nabla u^\varepsilon(x^k + y^k, t^k + s^k)$, $\bar{p}^k = \nabla v_\varepsilon(x^k, t^k)$ are nonzero for large k . Since u^ε is a weak subsolution near $(x_1 + y_1, t_1 + s_1)$ and v_ε is a weak supersolution near (x_1, t_1) , we thus have

$$q^k \leq \frac{1}{n}(\delta_{ij} - p_i^k p_j^k / |p^k|^2) r_{ij}^k - \theta_1 |p^k|,$$

$$\bar{q}^k \geq \frac{1}{n}(\delta_{ij} - \bar{p}_i^k \bar{p}_j^k / |\bar{p}^k|^2) \bar{r}_{ij}^k - \theta_2 |\bar{p}^k|$$

for all large k . We send k to infinity, and obtain

$$q \leq \frac{1}{n}(\delta_{ij} - p_i p_j / |p|^2) r_{ij} - \theta_1 |p|, \quad \bar{q} \geq \frac{1}{n}(\delta_{ij} - p_i p_j / |p|^2) \bar{r}_{ij} - \theta_2 |p|.$$

Hence

$$q - \bar{q} \leq \frac{1}{n}(\delta_{ij} - p_i p_j / |p|^2)(r_{ij} - \bar{r}_{ij}) - (\theta_1 - \theta_2)|p|.$$

Now the matrix $((\delta_{ij} - p_i p_j / |p|^2))$ is a nonnegative and $R - \bar{R}$ is nonpositive, $\theta_1 \geq \theta_2$. Consequence $q - \bar{q} \leq 0$, a contradiction to (6.29). The proof of Lemma 6.1 is complete. $\square$

Lemma 6.2. *Assume that u_θ is a weak solution of*

$$\begin{cases} u_t = \frac{1}{n}(\delta_{ij} - u_{x_i} u_{x_j} / |\nabla u|^2) u_{x_i x_j} - \theta |\nabla u| & \text{in } \mathbb{R}^{n+1} \times [0, T] \\ u(x, 0) = g(x) & \text{on } \mathbb{R}^{n+1}. \end{cases} \quad (6.33)$$

Suppose that g is a constant on $\{|x| \geq R(T)\}$ and u is a constant on $\mathbb{R}^{n+1} \times [0, T] \cap \{|x| \geq R(T)\}$ for some constant $R > 0$. Then

$$\lim_{\theta \rightarrow \theta_0} \|u_\theta - u_{\theta_0}\|_{L^\infty(\mathbb{R}^{n+1} \times [0, T])} = 0.$$

Proof. 1. Let $\theta_n \rightarrow \theta_0$ as $n \rightarrow \infty$, then there exist two numbers $\bar{\theta}, \underline{\theta}$ such that

$$\underline{\theta} \leq \theta_n \leq \bar{\theta}, \quad n = 1, 2, \dots$$

Let $u_n, \underline{u}, \bar{u}$ be the solution of (6.33) with $\theta = \theta_n, \underline{\theta}, \bar{\theta}$, respectively. By Lemma 6.1, we have

$$\underline{u}(x, t) \leq u_n(x, t) \leq \bar{u} \quad \text{in } \mathbb{R}^{n+1} \times [0, T].$$

Note that $u_n, \underline{u}, \bar{u}$ are constants on $\mathbb{R}^{n+1} \times [0, T] \cap \{|x| \geq R(T)\}$. There is a subsequence of $\{u_n\}, \{u_{n_k}\}$ such that

$$u_{n_k} \rightarrow v \quad \text{uniformly on } \mathbb{R}^{n+1} \times [0, T], \quad (6.34)$$

where v is a constant on $\{|x| \geq 2R(T)\}$.

2. Now we prove that v is a weak solution of (6.33) with $\theta = \theta_0$.

a. Choose $\phi \in C^2(\mathbb{R}^{n+1} \times (0, T))$ and suppose first $v - \phi$ has a strict local maximum at some point $(x_0, t_0) \in \mathbb{R}^{n+1} \times (0, T)$. As $u_{n_k} \rightarrow v$ uniformly near (x_0, t_0) , $u_{n_k} - \phi$ has a local maximum at a point (x_k, t_k) ($k = 1, 2, \dots$) with

$$(x_k, t_k) \rightarrow (x_0, t_0) \quad \text{as } k \rightarrow \infty.$$

Since u_k is a weak solution, we have either

$$\phi_t \leq \frac{1}{n}(\delta_{ij} - \phi_{x_i} \phi_{x_j} / |\nabla \phi|^2) \phi_{x_i x_j} - \theta |\nabla \phi| \quad \text{at } (x_k, t_k) \quad (6.35)$$

if $\nabla \phi(x_k, t_k) \neq 0$, or

$$\phi_t \leq \frac{1}{n}(\delta_{ij} - \eta_i^k \eta_j^k) \phi_{x_i x_j} \quad \text{at } (x_k, t_k) \quad (6.36)$$

for some $\eta^k \in \mathbb{R}^{n+1}$ with $|\eta^k| \leq 1$, if $\nabla \phi(x_k, t_k) = 0$.

b. Assume first $\nabla \phi(x_0, t_0) \neq 0$. Then $\nabla \phi(x_k, t_k) \neq 0$ for all large enough k . Hence we may pass to limits in the inequality (6.35) to discover

$$\phi_t \leq (\delta_{ij} - \phi_{x_i} \phi_{x_j} / |\nabla \phi|^2) \phi_{x_i x_j} - \theta |\nabla \phi| \quad \text{at } (x_0, t_0).$$

c. Suppose $\nabla \phi(x_0, t_0) = 0$. We set

$$\xi^k = \begin{cases} (\nabla \phi / |\nabla \phi|)(x_k, t_k) & \text{if } \nabla \phi(x_k, t_k) \neq 0, \\ \eta^k & \text{if } \nabla \phi(x_k, t_k) = 0. \end{cases}$$

Passing if necessary to a subsequence we may assume $\xi^k \rightarrow \eta$. Then $|\eta| \leq 1$. Utilizing (6.36), we deduce as well

$$\phi_t \leq \frac{1}{n}(\delta_{ij} - \eta_i \eta_j) \phi_{x_i x_j} \quad \text{at } (x_0, t_0).$$

d. If $v - \phi$ has only a local maximum at (x_0, t_0) , we apply the above argument to

$$\psi(x, t) \equiv \phi(x, t) + |x - x_0|^4 + (t - t_0)^4,$$

so that $v - \psi$ has a strict local maximum at (x_0, t_0) . Hence u is a weak subsolution of (6.33) with $\theta = \theta_0$. Similar reasoning verifies that v is a weak supersolution of (6.33) with $\theta = \theta_0$. Hence $v = u_{\theta_0}$.

3. $\lim_{\theta \rightarrow \theta_0} \|u_\theta - u_{\theta_0}\|_{L^\infty(\mathbb{R}^{n+1} \times [0, T])} = 0$.

If this conclusion is fail, then there exist a positive number $\varepsilon_0 > 0$ and a sequence $\{\theta_n\}$ such that

$$\theta_n \rightarrow \theta_0 \quad \text{as } n \rightarrow \infty,$$

but

$$\|u_{\theta_n} - u_{\theta_0}\|_{L^\infty(\mathbb{R}^{n+1} \times [0, T])} \geq \varepsilon_0 > 0. \quad (6.37)$$

By step 1,2, there exists a subsequence $\{u_{\theta_{n_k}}\}$ of $\{u_{\theta_n}\}$ such that

$$u_{\theta_{n_k}} \rightarrow u_{\theta_0} \quad \text{uniformly on } \mathbb{R}^{n+1} \times [0, T],$$

as $k \rightarrow \infty$. This yields a contradiction with (6.37). The proof of Lemma 6.4 is completed. $\square$

The proof of Proposition 6.2. Let U_0 be a smooth domain bounded by M_0 . Choose a continuous function $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ satisfying

$$M_0 = \{x \in \mathbb{R}^{n+1} : g(x) = 0\},$$

g is a constant on $\mathbb{R}^{n+1} \cap \{|x| \geq s\}$ for some $s > 0$, and $U_0 = \{x : g(x) < 0\}$.

Let u, u_1 be weak solution of (6.33) with $\theta = \theta, \theta_1$ respectively.

Then, note that a classical solution is a weak solution (see [10]), we have

$$\begin{aligned} M_t &= \{x : u(x, t) = 0\}, \quad 0 \leq t \leq t_{\max}, \\ M_t^1 &= \{x : u_1(x, t) = 0\}, \quad 0 \leq t \leq t_{1\max}, \\ U_t &= \{x : u(x, t) < 0\}, \\ U_t^1 &= \{x : u_1(x, t) < 0\}. \end{aligned}$$

By Lemma 6.3, we have

$$u(x, t) \leq v(x, t) \quad \text{on } \mathbb{R}^{n+1} \times [0, T].$$

Hence

$$U_t \subset U_t^1.$$

The proof of Proposition 6.2 is completed.

Lemma 6.3. Assume M_0 is a compact, uniformly convex n -dimensional hypersurface without boundary. Let F be a solution of problem (1.2)–(1.3) with positive θ . Denotes the mean curvature of M_0 by H_0 . Then, we have

(a) If $\min_{M_0} H_0 > \theta$, then $t_{\max}$ is finite and M_t shrinks to point.

(b) If $\max_{M_0} H_0 < \theta$, then $t_{\max} = \infty$, and the domain U_t bounded by M_t occupies the entire space as t approaches infinity.

Proof. (a) Define $v = H - \theta$, then

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v + |A|^2 v, \\ v(x, 0) > 0. \end{cases}$$

So, by the maximum principle, we have

$$v(x, t) \geq 0, \quad \forall t > 0.$$

Let $v_{\min}(t) = \min_{x \in M_t} (v - \theta)$, then

$$\frac{dv_{\min}}{dt} \geq 0.$$

Hence

$$v(x, t) \geq v_{\min}(t) \geq v_{\min}(0) > 0.$$

So $H(x, t)$ has a uniform positive lower bound. Combining this with Lemma 2.2 (i), one yields

$$\frac{dv_{\min}}{dt} \geq C_{10} v_{\min}, \quad \text{for some } C_{10} > 0.$$

If $t_{\max} = \infty$, $v_{\min}$ grows at least exponentially. From Lemma 2.2 (i), we have

$$\frac{dv_{\min}}{dt} \geq C_{11} v_{\min}^3, \quad \text{for some } C_{11} > 0,$$

for large t . Hence $v_{\min}$ blows up in finite time, and the contradiction holds. So $t_{\max}$ must be finite.

(b) Let $v_{\max}(t) = \max_{M_t} (H - \theta)$, one can see that $v_{\max}(t)$ is nonincreasing.

In particular,

$$v_{\max}(t) \leq v_{\max}(0) < 0.$$

Hence

$$0 \leq H(x, t) \leq \theta, \quad \forall x, \forall t \geq 0.$$

By Theorem 5.6, $t_{\max} = \infty$. Define

$$d(x, t) = \text{dist}(x, M_t), \quad x \in U_t.$$

Note that M_t is convex, there exists a unique point $y \in M_t$ verifying $d(x, t) = |x - y|$. Then

$$d_t(x, t) = (\text{div} \nu)(y, t) + \theta,$$

where $\nu = \nabla d$ is unit normal vector of M_t at (y, t) .

Let $k_1, \dots, k_n$ be the principle curvatures of M_t at the point y , calculated with respect to the unit normal field ν . Thus $(\text{div} \nu)(y, t) = -(k_1 + \dots + k_n)$, and we have

$$d_t = -v \geq -v_{\max}(0) > 0,$$

that is

$$d(x, t) \geq -v_{\max}(0)/t + d(x, 0).$$

Thus M_t encloses any bounded set for sufficiently large t . $\square$

By Proposition 6.1, 6.2 and Lemma 6.3, we have the following

Lemma 6.4. *Assume M_0 is a compact, uniformly convex n -dimensional hypersurface without boundary. Let M_t be the solution of (1.2), (1.3) and $\theta > 0$.*

- (a) *If M_0 is contained inside a ball of radius less than $\frac{1}{\theta}$, then M_t shrinks to a point and $t_{\max}$ is finite.*
- (b) *If M_0 contains a ball of radius greater than $\frac{1}{\theta}$, then M_t expands to infinity.*

Define $I_1 = \{\theta : M_t \text{ is a solution of (6.1)}_\theta, (6.2) \text{ that shrinks to a point}\}$, and $I_2 = \{\theta : M_t \text{ is a solution of (6.1)}_\theta, (6.2) \text{ that expands to infinity as } t \rightarrow \infty\}$.

By Lemma 6.4 (a), we know that $t_{\max}$ is finite if $\theta \in I_1$. By the continuous dependence of F on θ , both I_1 and I_2 are open. By Lemma 6.4, they are also nonempty.

Lemma 6.5. *I_1 and I_2 are open.*

Proof. 1. I_1 is open.

Let $\theta_0 \in I_1$. Choose a continuous function $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned} M_0 &= \{x \in \mathbb{R}^{n+1}, g(x) = 0\}, \\ g &\text{ constant on } \mathbb{R}^{n+1} \setminus \{|x| \geq s\} \end{aligned}$$

for some $s > 0$, and $U_0 = \{x | g(x) < 0\}$, where U_0 is a domain bounded by M_0 . Let u be a unique weak solution of

$$\begin{cases} u_t = \frac{1}{n}(\delta_{ij} - u_{x_i}u_{x_j}/|\nabla u|^2)u_{x_i x_j} - \theta_0 |\nabla u| & \text{in } \mathbb{R}^{n+1} \times (0, T), \\ u(x, 0) = g & \text{on } \mathbb{R}^{n+1}, \end{cases}$$

and v be a unique weak solution of

$$\begin{cases} v_t = \frac{1}{n}(\delta_{ij} - v_{x_i}v_{x_j}/|\nabla v|^2)v_{x_i x_j} - \theta |\nabla v| & \text{in } \mathbb{R}^{n+1} \times (0, T), \\ v(x, 0) = g & \text{on } \mathbb{R}^{n+1}, \end{cases}$$

then $M_t = \{x : u(x, t) = 0\} (0 \leq t \leq t_{\max})$ is the solution of the following equation

$$\begin{cases} \frac{\partial F}{\partial t} = -H + \theta_0, \\ F(\cdot, 0) = F_0, \end{cases} \quad (6.38)$$

$M'_t = \{x : v(x, t) = 0\} (0 \leq t \leq t'_{\max})$ is the solution of the equation

$$\begin{cases} \frac{\partial F'}{\partial t} = -H' + \theta, \\ F'(\cdot, 0) = F_0. \end{cases} \quad (6.39)$$

Denote

$$U_t = \{x \in \mathbb{R}^{n+1} : u(x, t) < 0\},$$

$$U'_t = \{x \in \mathbb{R}^{n+1} : v(x, t) < 0\}.$$

Since M_t tends to a point p as $t \rightarrow t_{\max}$, there exists a $T_0 > 0$ such that

$$U_t \subset\subset B_{1/2\theta}(p), \quad \forall t \geq T_0,$$

where $B_{1/2\theta}(p)$ is a ball of radius $1/2\theta$. Hence, $\forall t \geq T_0$,

$$U(x, t) \geq \alpha_0 > 0, \quad \text{in } \mathbb{R}^{n+1} \setminus B_{1/\theta}(p).$$

By Lemma 6.2, if $|\theta - \theta_0| < \varepsilon$, then, $\forall t \geq T_0$,

$$v(x, t) \geq \frac{1}{2}\alpha_0 \quad \text{in } \mathbb{R}^{n+1} \setminus B_{1/\theta}(p).$$

Hence

$$U'_t \subset B_{1/\theta}(p), \quad \forall t \geq T_0.$$

By Lemma 6.4 (a), M'_t shrinks to a point as $t \rightarrow t'_{\max}$. Thus $(\theta_0 - \frac{1}{2}\alpha_0, \theta_0 + \frac{1}{2}\alpha) \subset I_1$. So I_1 is a open set.

2. Similarly, I_2 is a open set. $\square$

Using comparison principle Proposition 6.1, Proposition 6.2, we conclude $I_1 = (0, \underline{\theta})$ and $I_2 = (\bar{\theta}, +\infty)$, where $0 \leq \underline{\theta} \leq \bar{\theta}$. From Theorem 5.6, we know that for each $\theta \in [\underline{\theta}, \bar{\theta}]$, the flow (6.1) $_{\theta}$, and (6.2) exists for all time.

Lemma 6.6. *Assume M_0 is a compact, uniformly convex n dimensional hypersurface without boundary. Let F satisfy (6.1) $_{\theta}$ and (6.2) for $\theta \in [\underline{\theta}, \bar{\theta}]$. Then $H(\cdot, t)$ is uniformly bounded in $[0, \infty)$.*

Proof. Suppose there is a sequence of times $T_i \rightarrow +\infty$ such that

$$\max_{M_{T_i}} H = H_{T_i} \quad \text{and} \quad H_{T_i} \rightarrow \infty \quad \text{as } i \rightarrow \infty.$$

Then for any $\eta > 0$ there is i such that for all $j \geq i$,

$$|\nabla H| \leq \frac{1}{2}\eta H_{\max}^2 \quad \text{on } M_{T_j}.$$

We can then use Theorem 4.1 and Meyer's theorem as in [16] to conclude that

$$H_{\min} \geq (1 - \eta)H_{\max} \quad \text{on } M_{T_j}.$$

Theorem 3.1 then implies that all principal curvatures on M_{T_j} tend to infinity which is clearly a contradiction since the enclosed volume is large than some positive number. $\square$

Now we begin to study the asymptotic behavior of M_t as $t \rightarrow \infty$ for $\theta \in [\underline{\theta}, \bar{\theta}]$. Evolution equation (2.3) gives

$$\left(\frac{\partial}{\partial t} - \Delta\right)\{e^{\delta t}(H - \theta)^2 - K\} \leq -e^{\delta t}(H - \theta)^2|A|^2 + \delta e^{\delta t}(H - \theta)^2$$

for any constants $\delta > 0$ and $K \in \mathbb{R}$. Choose $\delta = \inf_{M_t} |A|^2$ (which is strictly greater than zero by Theorem 2.3) and $K = \sup_{M_0} \{(H - \theta)^2\}$ (which is bounded by assumption). Then $\{e^{\delta t}(H - \theta)^2 - K\}_{t=0} \leq 0$ and

$$\left(\frac{\partial}{\partial t} - \Delta\right)\{e^{\delta t}(H - \theta)^2 - K\} \leq 0.$$

The maximum principle yields

$$(H - \theta)^2 \leq Ke^{-\delta t}$$

for all $t > 0$ which imply that $\sup_{M_t} |H - \theta|$ tends to zero as $t \rightarrow +\infty$. Thus H is bounded from below by $\frac{\theta}{2}$ for $t \geq T_1$. Hence, from Lemma 3.2 we see that

$$f_0 = \left(|A|^2 - \frac{1}{n} H^2 \right) / H^2$$

satisfies the inequality

$$\frac{\partial}{\partial t} f_0 \leq \Delta f_0 + \frac{2}{H} \langle \nabla_l H, \nabla_l f_0 \rangle - \varepsilon^2 \theta^2 f_0.$$

This shows

$$|A|^2 - \frac{1}{n} H^2 \leq C e^{-\delta_1 t},$$

and a slight modification in the Proof of Theorem 5.1 shows that all higher derivatives of the second functional form decay exponentially as well. In particular

$$\left| \frac{d}{dt} F \right| = |H - \theta| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Thus M_t converges to a limiting hypersurface M_∞^θ , and in view of (6.1) the limiting hypersurface M_∞^θ is a sphere with radius $1/\theta$. We have

Lemma 6.7. *Assume M_0 is a compact, uniformly convex n -dimensional hypersurface without boundary. Let M_t be the solution of (6.1) $_\theta$ and (6.2), $\theta \in [\underline{\theta}, \bar{\theta}]$. Then $M_t \rightarrow M_\infty^\theta$ as $t \rightarrow \infty$.*

Proposition 6.3. *For every uniformly convex initial surface M_0 , there exists $\theta^* > 0$ such that the following hold.*

- (a) *The solution M_t of (6.1) $_\theta$ and (6.2) shrinks to a point in finite time when $0 < \theta < \theta^*$.*
- (b) *The solution of (6.1) $_{\theta^*}$ and (6.2), converges to a sphere with radius $1/\theta^*$.*
- (c) *The solution of (6.1) $_\theta$ and (6.2) expands to infinity as $t \rightarrow \infty$ when $\theta > \theta^*$.*

Proof. We only need to show that $\underline{\theta} = \bar{\theta}$. Let $\underline{M}_t$ and $\overline{M}_t$ be the solution of (6.1) $_\theta$ and (6.2) for $\theta = \underline{\theta}$ and $\bar{\theta}$, respectively.

On one hand, by Proposition 6.2, we have

$$\underline{M}_t \subset \overline{U}_t, \quad t \geq 0.$$

However, on the other hand, let

$$d(x, t) = \begin{cases} -\text{dist}(x, \underline{M}_t) & \text{if } x \in \underline{U}_t \\ \text{dist}(x, \underline{M}_t) & \text{if } x \in \mathbb{R}^{n+1} \setminus \underline{U}_t, \end{cases}$$

then $d(x, t)$ is smooth in

$$Q = \{(x, t) : -\delta_0 \leq d(x, t) \leq \delta_0, \quad 0 \leq t \leq t_0\},$$

where δ_0 and t_0 are small enough, U_t is the bounded region enveloped by $\underline{M}_t$. As [11], we have

$$d_t = F(\nabla^2 d, d) + \underline{\theta} \quad \text{in } Q. \tag{6.40}$$

Similarly the signed distance

$$\bar{d}_t(x, t) = \begin{cases} -\text{dist}(x, \bar{M}_t) & \text{if } x \in \bar{U}_t \\ \text{dist}(x, \bar{M}_t) & \text{if } x \in \mathbb{R}^{n+1} \setminus \bar{U}_t \end{cases}$$

is smooth in Q , provided $\delta_0, t_0 > 0$ are sufficiently small. Also

$$\bar{d}_t = F(\nabla^2 \bar{d}, \bar{d}) + \bar{\theta} \quad \text{in } Q. \quad (6.41)$$

Set

$$w = \bar{d} - \underline{d}.$$

From (6.40), (6.41) it follows that

$$\begin{cases} w_t = a_{ij} w_{x_i x_j} + cw + (\bar{\theta} - \underline{\theta}), \\ w(x, 0) = 0. \end{cases} \quad (6.42)$$

$$(6.43)$$

for

$$\begin{aligned} a_{ij} &= \int_0^1 \frac{\partial F}{\partial r_{ij}}(t \nabla^2 \bar{d} + (1-t) \nabla^2 \underline{d}, t \bar{d} + (1-t) \underline{d}) dt, \\ c &= \int_0^1 \frac{\partial F}{\partial z}(t \nabla^2 \bar{d} + (1-t) \nabla^2 \underline{d}, t \bar{d} + (1-t) \underline{d}) dt, \end{aligned}$$

where F is defined by (2.3) in [11].

According to Lemma 2.1 in [11] the coefficients $((a_{ij}))$ are uniformly positive definite if δ_0, t_0 are small enough. Also, one has that $c \geq 0$ is a bounded function in $\bar{Q}$. If $\bar{\theta} > \underline{\theta}$, set $v(t) = \min_{Q \cap \{t=t\}} w(x, t)$, then $\min_{Q \cap \{t=t\}} (\bar{d} - \underline{d})$ is increasing and becomes positive for sufficiently small $t > 0$. Hence

$$\bar{d}(x, t) > \underline{d}(x, t) \quad \text{for } 0 < t \leq t_0$$

if t_0 is small enough, contradicting $\underline{M}_t \subset \bar{U}_t$ for $t \geq 0$. So we must have $\bar{\theta} = \underline{\theta}$. The proof of proposition 6.3 is completed. $\square$

7. Convergence to a sphere

According to Proposition 6.3, the solution M_t of (6.1) $_{\theta}$, $\theta > \theta^*$, and (6.2) possesses the following property: For any bounded subset K of $\mathbb{R}^{n+1}$, there exists t_K such that K is contained inside M_t for all $t \geq t_K$. In this section, we study the asymptotic behavior of this solution.

Intuitively in this case, the mean curvature of M_t is eventually negligible. In view of (1.2), the volume of U_t grows linearly. Therefore, we consider the normalization given by

$$\tilde{F}(\cdot, t) = F(\cdot, t)/t.$$

Lemma 7.1. *There exists a K_0 that depends on θ and M_0 such that $\tilde{H}$ is uniformly bounded if*

$$\max_{x \in M_0} H(x, 0) \leq K_0,$$

where $\tilde{H}$ is the mean curvature of $\tilde{M}_t = M_t/t$.

Proof. From Lemma 2.2 (i), we have

$$\frac{\partial}{\partial t}H = \Delta H + (H - \theta)|A|^2.$$

Thus

$$H \leq K_0, \quad x \in M_t, \quad t \geq 0.$$

Hence, if $H - \theta \leq K_0 - \theta < 0$, we have

$$\begin{aligned} \frac{\partial}{\partial t}H &= \Delta H + (H - \theta)|A|^2 \\ &\leq \Delta H + (K_0 - \theta)\frac{1}{n}|H|^2 \\ &:= \Delta H - v_0 H^2. \end{aligned}$$

So

$$\frac{\partial}{\partial t}H_{\max} \leq -v_0 H_{\max}^2(t).$$

This implies

$$H_{\max}(t) \leq \frac{H_{\max}(t)}{1 + v_0 + H_{\max}(t)t}, \quad t \in [0, \infty),$$

$$\tilde{H}_{\max}(t) = tH_{\max}(t) \leq \frac{1}{v_0}, \quad t \in [0, \infty).$$

□

Lemma 7.2. *Let $\tilde{m}_c$ be the support function of $\tilde{M}_t$ for $(6.1)_\theta$, $\theta > 0$, of which the initial surface is a large sphere which center is origin with radius R_0 . Then*

$$\tilde{m}_c(\cdot, t) - \theta \rightarrow 0$$

uniformly as $t \rightarrow \infty$.

Proof. Let $R(t) = (\tilde{F}, \tilde{\nu})$, where $\tilde{\nu}$ is the unit normal vector of $\tilde{M}_t$. Then

$$\frac{dR}{dt} = -\frac{1}{t}R - \frac{1}{2t^2R} + \frac{\theta}{t}.$$

Set $\tau = \log t$, we have

$$\frac{dR}{d\tau} = -R - e^{-\tau}\frac{1}{2R} + \theta.$$

Hence

$$\frac{d}{dt}[e^\tau \cdot (R - \theta)] = -\frac{1}{2R}.$$

So

$$e^\tau(R(\tau) - \theta) = R(0) - \theta - \int_0^\tau \frac{1}{2R(t)} dt,$$

$$\begin{aligned}
|R(\tau) - \theta| &= |(R(0) - \theta)e^{-\tau} - e^{-\tau} \int_0^\tau \frac{dt}{2R(t)}| \\
&\leq (R(0) - \theta)e^{-\tau} + e^{-\tau} \int_0^\tau \frac{dt}{2R(t)} \\
&\leq M\tau e^{-\tau} \quad \text{if } R(0) \geq 2\theta,
\end{aligned}$$

where $M = R(0) - \theta + v_0$. This proves the Lemma 7.2. $\square$

Theorem 7.3. $\tilde{m}(\cdot, \tau) - \theta$ tends to zero uniformly as $\tau \rightarrow \infty$.

Proof. Since M_t expands, we may assume, without loss of generality, that M_t is pinched between two large circle. Then the desired result follows from Lemma 7.2. $\square$

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