Existence of conformal metrics with prescribed scalar curvature on the four dimensional half sphere

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Abstract. In this paper, we consider the problem of prescribing the scalar curvature under minimal boundary conditions on the four dimensional half sphere. Using dynamical and topological methods involving the study of the critical points at infinity of the associated variational structure, we prove some existence results like Bahri-Coron theorem. Furthermore, we consider the approximate subcritical problem and we construct some solutions which blow up at two different points, one of them lay on the boundary and the other one is an interior point.

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1. Introduction

In this paper we study some nonlinear problem arising from conformal geometry. Precisely, let (M^n, g) be a n -dimensional Riemannian manifold with boundary, $n \geq 3$ and let $\tilde{g} = u^{4/(n-2)}g$, be a conformal metric to g , where u is a smooth positive function, then the following equations relate the scalar curvatures $R_g, R_{\tilde{g}}$ and the mean curvatures of the boundary $h_g, h_{\tilde{g}}$, with respect to g and $\tilde{g}$ respectively.

$$\begin{cases} -c_n \Delta_g u + R_g u = R_{\tilde{g}} u^{(n+2)/(n-2)} & \text{in } M, \\ \frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u = h_{\tilde{g}} u^{n/(n-2)} & \text{on } \partial M, \end{cases} \quad (1)$$

where $c_n = 4(n-1)/(n-2)$ and ν denotes the outward normal vector with respect to g .

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In view of the above equations, the following problem naturally arises: given two functions $K : M \rightarrow \mathbb{R}$ and $H : \partial M \rightarrow \mathbb{R}$, does there exist a metric $\tilde{g}$ conformally equivalent to g such that $R_{\tilde{g}} = K$ and $h_{\tilde{g}} = H$? According to equation (1), the problem is equivalent to finding a smooth positive solution u of the following equation

$$\begin{cases} -c_n \Delta_g u + R_g u = K u^{(n+2)/(n-2)} & \text{in } M, \\ \frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u = H u^{n/(n-2)} & \text{on } \partial M. \end{cases} \quad (2)$$

Such a problem was studied in [1, 9, 10, 14–18, 21] and the references therein. In this paper, we consider the case of the standard four dimensional half sphere under minimal boundary conditions. More precisely, let K be a C^3 positive Morse function on $\overline{S^4_+}$, we look for conditions on K to ensure the existence of a positive solution of the problem

$$\begin{cases} L_g u := -\Delta_g u + 2u = K u^3, & u > 0 & \text{in } S^4_+, \\ \partial u / \partial \nu = 0 & & \text{on } \partial S^4_+, \end{cases} \quad (P)$$

where g is the standard metric of $S^n_+ = \{x \in \mathbb{R}^{n+1} / |x| = 1, x_{n+1} > 0\}$.

Li [21], and Djadli–Malchiodi–Ould Ahmedou [15] studied this problem when the manifold is the three dimensional standard half sphere. Their approach involves a fine blow up analysis of some subcritical approximations and the use of the topological degree tools. Ben Ayed–El Mehdi–Ould Ahmedou [9], [10] gave some topological conditions on K to prescribe the scalar curvature under minimal boundary conditions on half spheres of dimension $n \geq 4$.

Note that problem (P) is in some sense related to the well known scalar curvature problem

$$-\Delta_g u + (n(n-2)/4)u = K u^{(n+2)/(n-2)}, \quad u > 0 \quad \text{in } S^n \quad (3)$$

to which much work has been devoted (see [3, 4, 7, 13, 14, 19–21] and the references therein).

As for (3), there are topological obstructions of Kazdan–Warner type to solve (P) (see [11]) and so a natural question arises: under which conditions on K , (P) has a positive solution?

Regarding problem (3), Bahri [3] observed that a new phenomenon appears in dimension $n \geq 5$ due to the fact that the self interaction of the functions failing the Palais–Smale condition dominates the interaction of two of those functions. In the three dimension case, the reverse happens (see [4]). In the four dimensional case, we have a balance phenomenon, that is, the self interaction and the interaction are of the same size (see [7]).

In the case of (P), Ben Ayed et al. proved that the balance phenomenon appears if we assume that $(\partial K / \partial \nu)(y) < 0$ where y is any critical point of $K_1 = K|_{\partial S^4_+}$. Moreover, it is proved that this phenomenon appears also when the manifold is the three dimensional half sphere (see [15]).

Note that the embedding of $H^1(S_+^4)$ into $L^4(S_+^4)$ is noncompact. Hence, for the study of problem (P), it is interesting to approach it by a family of subcritical problems (P_ε)

$$\begin{cases} -\Delta_g u + 2u = Ku^{3-\varepsilon}, & u > 0 & \text{in } S_+^4 \\ \partial u / \partial \nu = 0 & & \text{on } \partial S_+^4, \end{cases} \quad (P_\varepsilon)$$

and we need to study the asymptotic behavior of the solutions (u_ε) as $\varepsilon \rightarrow 0$. Observe that, since $\varepsilon > 0$, problem (P_ε) has always a positive solution (u_ε) . It is well known that, for the minimizing solutions, (u_ε) has to converge to a solution of (P) or to blow up at one point which is the maximum of K or of K_1 . For the other solutions (u_ε) (not minimizing solutions) and in the case of the three dimensional half sphere, Djadli et al. [15] proved that (u_ε) can blow up at $(x_1, \dots, x_p)$ where the points x_i 's are different critical points of K_1 with $(\partial K / \partial \nu)(x_i) > 0$. Furthermore they proved that the x_i 's are isolated simple blow ups (see [19, 20] for the definition) which implies that, writing $u_\varepsilon = \sum_{i \leq p} \alpha_i^\varepsilon \delta_{(a_i^\varepsilon, \lambda_i^\varepsilon)} + v_\varepsilon$, we have that $|a_i - a_j| \geq c > 0$ for $i \neq j$ (the function $\delta_{(a, \lambda)}$ is defined in (1.1)). Hence, the tower bubble solutions do not exist. Moreover Ben Ayed et al. [10] proved that there are critical points at infinity (following the terminology of A. Bahri) for the functional J_0 (defined by (1.2)) associated to the problem (P).

Recall that the critical point at infinity is the limit of the orbits of the gradient flow of J_0 which remain in $V(p, \eta(s))$, where $\eta(s)$ is a given function which tends to 0 as s tends to infinity and $V(p, \eta)$ is defined by (3.1) below (for more details, see Definition 4.1 below). In other words, this implies the existence of solutions (u_ε) which blow up at $(y_1, \dots, y_p)$ where the points y_i 's are different critical points of K in S_+^4 . Note that, in [10], the authors assumed that $(\partial K / \partial \nu) < 0$ at each critical point of K_1 . This assumption allows them to cancel the influence of the boundary. However, up to the knowledge of the authors, there is no result about the solutions which blow up at a mixed type of the critical points of K and K_1 i.e. critical point z of K_1 with $(\partial K / \partial \nu)(z) > 0$ and critical point y of K .

In this paper, we generalize the results of [10]. We cancel the assumption of [10] which is $(\partial K / \partial \nu) < 0$ at each critical point of K_1 that is the function $K_1 = K|_{\partial S_+^4}$ can have some critical points with $(\partial K / \partial \nu) > 0$. Our problem becomes more difficult, we need to understand the influence of the boundary in the characterization of the critical points at infinity.

The goal of our first result is the construction of some solutions (u_ε) of (P_ε) which blow up at two different points, one of them lies on the boundary and the other is an interior point. This result implies that there are more critical points at infinity than the authors found in [10].

Before stating the result, we need to introduce some notations. For $a \in \overline{S_+^4}$ and $\lambda > 0$, let

$$\delta_{(a, \lambda)}(x) = c_0 \frac{\lambda}{(\lambda^2 + 1 + (1 - \lambda^2) \cos d(a, x))} \quad (1.1)$$

where d is the geodesic distance on $(\overline{S_+^4}, g)$ and c_0 is chosen so that $\delta_{(a,\lambda)}$ is a solution of

$$-\Delta u + 2u = u^3, \quad u > 0, \quad \text{in } S^4.$$

We denote by $P\delta_{(a,\lambda)}$ the projection of the function $\delta_{(a,\lambda)}$ defined by

$$\begin{aligned} -\Delta P\delta_{(a,\lambda)} + 2P\delta_{(a,\lambda)} &= -\Delta\delta_{(a,\lambda)} + 2\delta_{(a,\lambda)}, \quad \text{in } S_+^4, \\ \partial P\delta_{(a,\lambda)}/\partial\nu &= 0 \text{ on } \partial S_+^4. \end{aligned}$$

Let G be the Green's function of $L_g := -\Delta + 2Id$ on S_+^4 and H be its regular part defined by

$$\begin{cases} G(x, y) = (1 - \cos(d(x, y)))^{-1} + H(x, y), \\ L_g H = 0 \text{ in } S_+^4; \quad \partial G/\partial\nu = 0 \text{ on } \partial S_+^4. \end{cases}$$

It is known that H is a positive function and $H(x, x) \rightarrow +\infty$ as x goes to the boundary.

The space $H^1(S_+^4)$ is equipped with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\begin{aligned} \|u\|^2 &= \int_{S_+^4} |\nabla u|^2 + 2 \int_{S_+^4} u^2, \quad \text{and} \\ \langle u, v \rangle &= \int_{S_+^4} \nabla u \nabla v + 2 \int_{S_+^4} uv, \quad u, v \in H^1(S_+^4). \end{aligned}$$

Theorem 1.1. *Let z be a nondegenerate critical point of $K_1 = K|_{\partial S_+^4}$ with $(\partial K/\partial\nu)(z) > 0$ and y be a nondegenerate critical point of K with $(-\Delta K(y)/3K(y) - 4H(y, y)) > 0$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) has a solution (u_ε) of the form*

$$u_\varepsilon = \alpha_1^\varepsilon \delta_{(x_1^\varepsilon, \lambda_1^\varepsilon)} + \alpha_2^\varepsilon P\delta_{(x_2^\varepsilon, \lambda_2^\varepsilon)} + v_\varepsilon, \quad \text{with}$$

$$\begin{aligned} \alpha_1^\varepsilon &\rightarrow K(z)^{-1/2}; \quad \alpha_2^\varepsilon \rightarrow K(y)^{-1/2}; \quad \|v_\varepsilon\| \rightarrow 0; \quad x_1^\varepsilon \rightarrow z; \quad x_2^\varepsilon \rightarrow y, \\ x_1^\varepsilon &\in \partial S_+^4; \quad \lambda_1^\varepsilon \rightarrow +\infty; \quad \lambda_2^\varepsilon = c(\lambda_2^\varepsilon)^2(1 + o(1)), \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Here, c is a positive fixed constant.

In other words, Theorem 1.1 implies that, we have more critical points at infinity (see definition below) than the authors found in [10]. More precisely, for each two critical points $z \in \partial S_+^4$ with $\partial K/\partial\nu(z) > 0$ and $y \in S_+^4$ with $-\Delta K(y)/3K(y) - 4H(y, y) > 0$, corresponds a critical point at infinity for the associated variational functional J_0 (see Proposition 1.3), where

$$J_0(u) = \left(\int_{S_+^4} K u^4 \right)^{-1/2}, \quad u \in \Sigma^+ := \{v \in H^1(S_+^4) : \|v\| = 1, v > 0\}. \quad (1.2)$$

In the following, we need to characterize all the new critical points at infinity and using the ideas introduced by Bahri [2], we derive some existence results for (P) .

The main difficulty is to understand the location of the concentration points which lie on the boundary. Indeed, the first question which arises is: Do there exist tower bubble solutions for (P_ε) ? In another sense, is it possible to have two concentration points or more close to each other? This is the aim of Theorem 1.2 below.

In order to state our result, we need to introduce some notations and assumptions.

(H₁) We assume that $K_1 = K|_{\partial S^4_+}$ has only nondegenerate critical points $z_1, \dots, z_r$ with

$$\frac{\partial K}{\partial \nu}(z_i) > 0, \quad \text{for } 1 \leq i \leq r_1, \quad \frac{\partial K}{\partial \nu}(z_i) < 0, \quad \text{for } i \geq r_1 + 1.$$

We remark that r_1 can be zero and in this case we are in the same situation than [10].

(H₂) For each $i \leq r_1$, if z_i is not a local maximum point, we assume that

$$\frac{4c_3^2}{K_1(z_i)} \left(\frac{\partial K}{\partial \nu}(z_i) \right)^2 < \gamma c_5 c_2^2, \quad (1.3)$$

where γ is the least of the absolute values of the eigenvalues of $D^2 K_1(z_i)$ and the c_i 's are positive constants defined in the Appendix.

Let z_k be a critical point of K_1 with $\partial K / \partial \nu(z_k) > 0$, and for $q \geq 2$, we introduce the set

$$\begin{aligned} V(q, \varepsilon, k) = \{ & \alpha_1 \delta_{(a_1, \lambda_1)} + \dots + \alpha_q \delta_{(a_q, \lambda_q)} : a_i \in \partial S^4_+, \lambda_i > \varepsilon^{-1}, \\ & \alpha_i > 0 \text{ for } i = 1, \dots, q, |\alpha_i^2 K(a_i)(\alpha_j^2 K(a_j))^{-1} - 1| < \varepsilon, \\ & \varepsilon_{ij} < \varepsilon \text{ and } \forall i \ a_i \in B(z_k, 2\mu) \}, \end{aligned}$$

where $\varepsilon_{ij}^{-1} = \lambda_i / \lambda_j + \lambda_j / \lambda_i + \lambda_i \lambda_j (1 - \cos d(a_i, a_j)) / 2$ and ε, μ are small positive parameters.

Theorem 1.2. *Assume that K satisfies (H_1) and (H_2) . In $V(q, \varepsilon, k)$, there exists a pseudo gradient Y_k so that the following holds: There exists a constant $c > 0$ independent of $u = \sum_{i=1}^q \alpha_i \delta_{(a_i, \lambda_i)} + v \in V(q, \varepsilon, k)$, so that*

$$\langle -\nabla J_0(u), Y_k \rangle \geq c \sum_{i \leq q} \left(\sum_{j \neq i} \varepsilon_{ij}^{3/2} + \frac{1}{\lambda_i^{3/2}} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)|^{3/2} \right).$$

Furthermore, $|Y_k|$ is bounded and the maximum of the λ_i 's is not an increasing function along the flow lines of Y_k (hence the Palais-Smale condition is satisfied along each flow line in this set).

Note that, Theorem 1.2 implies that, in $V(q, \varepsilon, k)$, there is no critical point and no critical point at infinity. In other words, this result implies that the blow up point z_k is an isolated simple blow up point (see [19, 20] for the definition) for the problem (P_ε) .

We notice that this result is immediate in the case of the three dimensional half sphere S^3_+ . In fact in this case it is clear that $\varepsilon_{ij} = (\lambda_i / \lambda_j + \lambda_j / \lambda_i + \lambda_i \lambda_j (1 - \cos(d(a_i, a_j))) / 2)^{-1/2}$ dominates $1 / \lambda_i$ where $\lambda_i = \max(\lambda_i, \lambda_j)$. Hence, (ordering the λ_i 's : $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_q$) we decrease all the λ_i 's for $i \geq 2$ and we

increase λ_1 if it is very small with respect to λ_2 . Thus the result follows in this case. In our paper the situation is more delicate since it is possible to get that ε_{ij} and λ_i^{-1} are of the same order.

To characterize the critical points at infinity, we need to add some assumptions for the interior critical points of K .

(H₃) All the critical points y_i 's of K in S_+^4 satisfy : $(-\Delta K(y_i)/3K(y_i)) - 4H(y_i, y_i) \neq 0$.

Without loss of generality, we can assume that $-\Delta K(y_i)/3K(y_i) - 4H(y_i, y_i) > 0$, for $1 \leq i \leq r_2$.

For $s \in \{1, \dots, r_2\}$ and for any s -tuple $\tau_s = (i_1, \dots, i_s) \in \{1, \dots, r_2\}^s$ such that $i_p \neq i_q$ if $p \neq q$, we define a matrix $M(\tau_s) = (M_{pq})_{1 \leq p, q \leq s}$, by

$$M_{pp} = \frac{-\Delta K(y_{i_p})}{3K(y_{i_p})^2} - 4 \frac{H(y_{i_p}, y_{i_p})}{K(y_{i_p})}, \quad M_{pq} = -\frac{4G(y_{i_p}, y_{i_q})}{(K(y_{i_p})K(y_{i_q}))^{1/2}} \quad \text{for } p \neq q,$$

and we denote by $\rho(\tau_s)$ the least eigenvalue of $M(\tau_s)$.

(H₄) For any $s \in \{1, \dots, r_2\}$, $M(\tau_s)$ is nondegenerate.

Now we are able to give the key point in the theory of the critical points at infinity. Using the notation of A. Bahri (see also Definition 4.1 below), we have the following result.

Proposition 1.3. *Assume that J_0 has no critical points in Σ^+ . Then, under the assumptions $(H_1) - (H_4)$, the only critical points at infinity of J_0 correspond to*

$$\sum_{i \leq q} K_1(z_{j_i})^{-\frac{1}{2}} \delta_{(z_{j_i}, \infty)} + \sum_{i \leq s} K(y_{j_i})^{-\frac{1}{2}} P \delta_{(y_{j_i}, \infty)}, \quad \text{with} \\ \frac{\partial K}{\partial \nu}(z_{j_i}) > 0 \quad \text{and} \quad \rho(y_{j_1}, \dots, y_{j_s}) > 0, \quad (1.4)$$

where $p \in \mathbb{N}^*$, $(q, s) \in \mathbb{N}^2$ and $p = q + s$.

Furthermore, such a critical point at infinity has a Morse index equal to

$$5s + 4q - 1 - \sum_{i \leq q} \text{index}(K_1, z_{j_i}) - \sum_{i \leq s} \text{index}(K, y_{j_i}). \quad (1.5)$$

Using some topological arguments and some deformation lemmas, Proposition 1.3 implies:

Theorem 1.4. *Under the assumptions $(H_1) - (H_4)$, and assume further that (H_5) For each $i \leq r_1$, the Morse index of K_1 at z_i is equal to 0 or 2.*

$$\text{If} \quad -1 \neq \sum_{s=1}^{r_2} \sum_{\tau_s=(i_1, \dots, i_s)/\rho(\tau_s)>0} (-1)^{s-\sum_{j=1}^s k_{i_j}}, \quad (1.6)$$

where $k_{i_j} := \text{index}(K, y_{i_j})$, then problem (P) has a solution.

Note that our result is a generalization of Theorem 1.1 of [10]. In fact, here, the function K_1 can have some critical points with $(\partial K / \partial \nu) > 0$.

The proof of Theorem 1.4 is based on the characterization of the critical points at infinity of the associated variational structure J_0 . Assuming that (P) has no solution, using [6], we know that Σ^+ (defined in (1.2)) retracts by deformation onto set X which is the union of the unstable manifolds of

the critical points at infinity. Since Σ^+ is a contractible set, we derive that the Euler-Poincaré characteristic of Σ^+ is equal to 1. Furthermore, under the assumptions (H_1) – (H_5) , we prove that the contribution of the critical points at infinity in the Euler-Poincaré characteristic depends on the right hand side of (1.6). Hence, from their contribution formula, we obtain a contradiction with (1.6).

Remark 1.5. The assumption on the Morse index of K_1 in (H_5) is very important. In fact, if there is one point $z \in \partial S_+^4$ with $(\partial K/\partial \nu)(z) > 0$ and $\text{index}(K_1, z) \in \{1, 3\}$, then we derive that the contribution of the critical points at infinity in the Euler-Poincaré characteristic is always equal to 1. Hence, we find the same phenomenon as in the scalar curvature problem in the sphere S^n , $n \geq 5$ (see Bahri [2]).

Note that, if

$$-\Delta K(y) - 12K(y)H(y, y) < 0 \quad \text{for each } y \in S_+^4 \text{ such that } \nabla K(y) = 0, \quad (1.7)$$

then we have only *boundary* critical points at infinity. Thus under (H_1) , (H_2) and (H_5) , as a consequence of Theorem 1.4, we derive that (P) has a solution. However, we can improve this result and, under some assumptions, we prove that the functional J_0 has to satisfy the *Palais-Smale* condition under a given level. In fact we have

Theorem 1.6. *Let z_0, y_0 be such that*

$$\begin{aligned} K_1(z_0) &= \max\{K_1(z) : z \in \partial S_+^4\} \quad \text{and} \\ K(y_0) &= \max\{K(y) : y \in S_+^4, \nabla K(y) = 0\}. \end{aligned}$$

If one of the following conditions

$$(i) \quad (\partial K/\partial \nu)(z_0) < 0, \quad \text{and} \quad 2K(z_0) \geq K(y_0)$$

$$(ii) \quad -\Delta K(y) - 12K(y)H(y, y) < 0, \quad \text{and} \quad 2K(z_0) \leq K(y_0)$$

holds, then (P) has at least one solution w . Furthermore, we have $J_0(w) \leq \bar{\mu}_0$ where $\bar{\mu}_0 := (S^4)^{1/2} \min\{(2K(z_0))^{-1/2}, K(y_0)^{-1/2}\}$.

We remark that if we have $(\partial K/\partial \nu)(z_0) < 0$ and (1.7) is satisfied for y_0 then one of the above conditions in Theorem 1.6 is satisfied and therefore (P) has a solution. We note that, part of this result is proved in [8] (when the condition (i) holds).

The remainder of the present paper is organized as follows. In Sect. 2, we prove Theorem 1.1. In Sect. 3, we set up the variational structure of (P) and we recall some preliminaries. In Sect. 4, we perform the construction of a pseudo gradient of J_0 which satisfies some required properties. While Sect. 5 is devoted to the proofs of the other results. The proofs require some careful expansions which are given in Appendices.

2. Proof of Theorem 1.1

First, let us introduce the general setting. For $\varepsilon > 0$, we define the functional

$$I_\varepsilon(u) = \frac{1}{2} \int_{S_+^4} |\nabla u|^2 + \int_{S_+^4} u^2 - \frac{1}{4-\varepsilon} \int_{S_+^4} K|u|^{4-\varepsilon}, \quad u \in H^1(S_+^4).$$

Note that if u is a positive critical point of I_ε , then u is a solution of (P_ε) , and inversely.

Furthermore, we mention that it will be convenient to perform some stereographic projection in order to reduce our problem to $\mathbb{R}_+^4$. Let $D^{1,2}(\mathbb{R}_+^4)$ denote the completion of $C_c^\infty(\overline{\mathbb{R}_+^4})$ with respect to Dirichlet norm. The stereographic projection π_a through a point $a \in \partial S_+^4$ induces an isometry $\iota : H^1(S_+^4) \rightarrow D^{1,2}(\mathbb{R}_+^4)$ according to the following formula

$$(\iota v)(x) = 2(1 + |x|^2)^{-1} v(\pi_a^{-1}(x)), \quad v \in H^1(S_+^4), x \in \mathbb{R}_+^4. \quad (2.1)$$

In particular, one can check that the following holds true, for every $v \in H^1(S_+^4)$

$$\int_{S_+^4} (|\nabla v|^2 + 2v^2) = \int_{\mathbb{R}_+^4} |\nabla(\iota v)|^2 \quad \text{and} \quad \int_{S_+^4} |v|^4 = \int_{\mathbb{R}_+^4} |\iota v|^4.$$

In the sequel, we will identify the function K and its composition with the stereographic projection π_a . We will also identify a point b of S_+^4 and its image by π_a . Moreover, it is easy to see that, using (2.1) with π_{-a} , the function $\iota\delta_{(a,\lambda)}$ is equal to

$$\iota\delta_{(a,\lambda)} = c_0 \frac{\lambda}{1 + \lambda^2|x - a|^2}.$$

For sake of simplicity, we will write $\delta_{(a,\lambda)}$ instead of $\iota\delta_{(a,\lambda)}$. These facts will be assumed as understood in the sequel. Let $\partial_\nu K$ denote the normal derivative of K and let

$$\begin{aligned} M_\varepsilon = \{m = (\alpha, \lambda, x_1, x_2, v) \in \mathbb{R}^2 \times (\mathbb{R}_+^*)^2 \times \partial S_+^4 \times (S_+^4) \times H^1(S_+^4) : v \in E_{(x,\lambda)}, \\ \|v\| < \nu_0; |\alpha_i^2 K(x_i) \alpha_j^{-2} K(x_j)^{-1} - 1| < \nu_0, \lambda_i > \nu_0^{-1}, \varepsilon \log \lambda_i < \nu_0, \forall i; \\ c_0 \lambda_2^2 < \lambda_1 < c_0^{-1} \lambda_2^2; |x_1 - x_2| > d_0; \\ |16c_3 \partial_\nu K(x_1) \lambda_1^{-1} - \varepsilon K(x_1) S_4| < \varepsilon^{1+\sigma/2}\}, \end{aligned}$$

where σ, c_0, d_0 are some suitable positive constants, ν_0 is a small positive constant and

$$\begin{aligned} E_{(x,\lambda)} = \{w \in H^1(S_+^4) / \langle w, \varphi \rangle = 0 \\ \forall \varphi \in \{P\delta_i, \partial P\delta_i / \partial \lambda_i, \partial P\delta_i / \partial x_i^j, i = 1, 2; j \leq 4\}\}. \end{aligned}$$

Here, x_i^j denotes the j -th component of x_i and it is easy to get that $P\delta_{(x_1, \lambda_1)} = \delta_{(x_1, \lambda_1)}$ since $x_1 \in \partial S_+^4$. Let us define the function

$$\Psi_\varepsilon : M_\varepsilon \rightarrow \mathbb{R}; \quad m = (\alpha, \lambda, x, v) \mapsto I_\varepsilon(\alpha_1 \delta_{(x_1, \lambda_1)} + \alpha_2 P\delta_{(x_2, \lambda_2)} + v).$$

For sake of simplicity, we will write $P\delta_i$ instead of $P\delta_{(x_i, \lambda_i)}$.

As said before, if a belongs to ∂S_+^4 , then we have $P\delta_{(a,\lambda)} = \delta_{(a,\lambda)}$ and therefore in our case, for $m = (\alpha, \lambda, x, v) \in M_\varepsilon$ we can write $u = \alpha_1\delta_1 + \alpha_2P\delta_2 + v$ as $u = \alpha_1P\delta_1 + \alpha_2P\delta_2 + v$.

As in [5], using the Euler-Lagrange's coefficients, it is easy to get the following proposition.

Proposition 2.1. *Let $m = (\alpha, \lambda, x, v) \in M_\varepsilon$. m is a critical point of Ψ_ε if and only if $u = \alpha_1\delta_1 + \alpha_2P\delta_2 + v$ is a critical point of I_ε , i.e. if and only if there exists $(A, B, C) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (\mathbb{R}^4)^2$ such that the following holds:*

$$\frac{\partial \Psi_\varepsilon}{\partial \alpha_i} = 0, \quad \forall i = 1, 2 \quad (E_{\alpha_i})$$

$$\frac{\partial \Psi_\varepsilon}{\partial \lambda_i} = B_i \left\langle \frac{\partial^2 P\delta_i}{\partial \lambda_i^2}, v \right\rangle + \sum_{j=1}^4 C_{ij} \left\langle \frac{\partial^2 P\delta_i}{\partial x_i^j \partial \lambda_i}, v \right\rangle, \quad \forall i = 1, 2 \quad (E_{\lambda_i})$$

$$\frac{\partial \Psi_\varepsilon}{\partial x_i} = B_i \left\langle \frac{\partial^2 P\delta_i}{\partial \lambda_i \partial x_i}, v \right\rangle + \sum_{j=1}^4 C_{ij} \left\langle \frac{\partial^2 P\delta_i}{\partial x_i^j \partial x_i}, v \right\rangle, \quad \forall i = 1, 2 \quad (E_{x_i})$$

$$\frac{\partial \Psi_\varepsilon}{\partial v} = \sum_{i=1,2} \left(A_i P\delta_i + B_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum_{j=1}^4 C_{ij} \frac{\partial P\delta_i}{\partial x_i^j} \right). \quad (E_v)$$

The results of Theorem 1.1 will be obtained through a careful analysis of the previous equations on M_ε . As usual in this type of problems, we first deal with the v -part of u , in order to show that it is negligible with respect to the concentration phenomenon. The study of (E_v) yields:

Proposition 2.2. *There exists a smooth map which to any $(\varepsilon, \alpha, \lambda, x)$ such that $(\alpha, \lambda, x, 0)$ in M_ε associates $\bar{v} \in E_{(x,\lambda)}$ such that $\|\bar{v}\| < \nu_0$ and (E_v) is satisfied for some $(A, B, C) \in \mathbb{R}^2 \times \mathbb{R}^2 \times (\mathbb{R}^4)^2$. Such a $\bar{v}$ is unique, minimizes $\Psi_\varepsilon(\alpha, \lambda, x, v)$ with respect to v in $\{v \in E_{(x,\lambda)} / \|v\| < \nu_0\}$, and we have the following estimate*

$$\|\bar{v}\| = O \left(\varepsilon + \frac{1}{\lambda_1} + \frac{|\nabla K(x_2)|}{\lambda_2} + \frac{1}{\lambda_2^2} + \varepsilon_{12}(\log \varepsilon_{12}^{-1})^{1/2} \right). \quad (2.2)$$

The proof becomes standard, hence we will omit it.

Proof of Theorem 1.1. The proof follows the ideas introduced in [5]. Once $\bar{v}$ is defined by Proposition 2.2, we estimate the numbers A, B, C by taking the scalar product of (E_v) with $\delta_1, P\delta_2, \partial\delta_1/\partial\lambda_1, \partial P\delta_2/\partial\lambda_2, \partial\delta_1/\partial x_1$ and $\partial P\delta_2/\partial x_2$ respectively. Thus, as in [5], we get a quasi-diagonal system in the variables A, B and C_i 's.

The other hand side is given by

$$\frac{\partial \Psi_\varepsilon}{\partial \alpha_i} = \left\langle \frac{\partial \Psi_\varepsilon}{\partial v}, P\delta_i \right\rangle; \quad \frac{1}{\alpha_i} \frac{\partial \Psi_\varepsilon}{\partial \lambda_i} = \left\langle \frac{\partial \Psi_\varepsilon}{\partial v}, \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle; \quad \frac{1}{\alpha_i} \frac{\partial \Psi_\varepsilon}{\partial x_i} = \left\langle \frac{\partial \Psi_\varepsilon}{\partial v}, \frac{\partial P\delta_i}{\partial x_i} \right\rangle.$$

Using Propositions 6.2 and 6.3, some computations yield

$$\frac{\partial \Psi_\varepsilon}{\partial \alpha_1} = -S_4 \beta_1 + V_{\alpha_1}(\varepsilon, \alpha, \lambda, x), \quad \frac{\partial \Psi_\varepsilon}{\partial \alpha_2} = -2S_4 \beta_2 + V_{\alpha_2}(\varepsilon, \alpha, \lambda, x), \quad (2.3)$$

with $\beta_1 = \alpha_1 - 1/K(z)^{\frac{1}{2}}$, $\beta_2 = \alpha_2 - 1/K(y)^{\frac{1}{2}}$ and V_{α_i} is a smooth function which satisfies

$$V_{\alpha_i} = O\left(\beta_i^2 + \varepsilon \log \lambda_i + (\text{if } i = 1)(\lambda_1^{-1} + |x_1 - z|^2) + (\text{if } i = 2)(\lambda_2^{-2} + |x_2 - y|^2)\right).$$

In the same way we get

$$\frac{\partial \Psi_\varepsilon}{\partial \lambda_1} = \frac{1}{K(z)} \left(\frac{2c_3}{\lambda_1^2} \frac{\partial K}{\partial \nu}(x_1) + \frac{\varepsilon K(x_1) S_4}{8\lambda_1} \right) + V_{\lambda_1}(\varepsilon, \alpha, \lambda, x), \quad (2.4)$$

$$\begin{aligned} \frac{\partial \Psi_\varepsilon}{\partial \lambda_2} &= \frac{1}{K(y)} \left(\frac{\varepsilon K(x_2) S_4}{4\lambda_2} + \frac{c_2}{4} \left(\frac{\Delta K(x_2)}{3K(x_2)} + 4H(x_2, x_2) \right) \frac{1}{\lambda_2^3} \right) \\ &\quad + V_{\lambda_2}(\varepsilon, \alpha, \lambda, x), \end{aligned} \quad (2.5)$$

$$\begin{aligned} \frac{\partial \Psi_\varepsilon}{\partial x_1} &= -2c_5 \nabla_T K(x_1) + V_{x_1}(\varepsilon, \alpha, \lambda, x) \quad \text{and} \\ \frac{\partial \Psi_\varepsilon}{\partial x_2} &= -c_8 \nabla K(x_2) + V_{x_2}(\varepsilon, \alpha, \lambda, x), \end{aligned} \quad (2.6)$$

where the constants c_i are defined in the appendices, V_{λ_i} and V_{x_i} are smooth functions satisfying

$$\begin{aligned} V_{\lambda_i} &= O\left(\lambda_i^{-1} (\lambda_i^{-i-1} + |x_2 - y|^2 \lambda_2^{-2} + \varepsilon^2 \log \lambda_i + \varepsilon \lambda_i^{-i} \log \lambda_i) \right. \\ &\quad \left. + (|\beta| + \varepsilon + |x_i - \bar{a}_i|^2) (\varepsilon \lambda_i^{-1} + \lambda_i^{-i-1})\right), \end{aligned} \quad (2.7)$$

$$\begin{aligned} V_{x_i} &= O(\lambda_i^{-1} + (|\beta| + \varepsilon \log \lambda_i + |x_i - \bar{a}_i|^2) |x_i - \bar{a}_i|), \\ &\quad \text{where } \bar{a}_1 = z \text{ and } \bar{a}_2 = y. \end{aligned} \quad (2.8)$$

Using the previous estimates, the solution of the system in A, B and C shows that

$$\begin{cases} A_i = O\left(|\beta| + \varepsilon \log \lambda_i + (\text{if } i = 1)(\frac{1}{\lambda_1} + |x_1 - z|^2) + (\text{if } i = 2)(\frac{1}{\lambda_2^2} + |x_2 - y|^2)\right), \\ B_1 = O\left(\varepsilon^{1+\sigma/2} \lambda_1\right); \quad B_2 = O\left(\frac{1}{\lambda_2} + \varepsilon \lambda_2\right), \\ C_1 = O\left(\frac{|x_1 - z|}{\lambda_1^2} + \frac{1}{\lambda_1^3}\right); \quad C_2 = O\left(\frac{|x_2 - y|}{\lambda_2^2} + \frac{1}{\lambda_2^3}\right). \end{cases}$$

This allows us to evaluate the right hand side in the equations (E_{λ_i}) and (E_{x_i}) . Now, we consider a point $(z, y) \in \partial S_+^4 \times S_+^4$ such that z and y are nondegenerate critical points of K_1 and K respectively. We set

$$\begin{aligned} \frac{1}{\lambda_1} &= \frac{\varepsilon S_4}{16c_3} K(z) \left(\frac{\partial K}{\partial \nu}(z) \right)^{-1} (1 + \zeta_1); \quad x_1 = z + \xi_1; \\ \frac{1}{\lambda_2} &= \varepsilon^{\frac{1}{2}} \Lambda (1 + \zeta_2); \quad x_2 = y + \xi_2, \end{aligned} \quad (2.9)$$

where Λ verifies : $c_2 \Lambda^2 (\frac{\Delta K(y)}{3K(y)} + 4H(y, y)) + S_4 K(y) = 0$, $\zeta_i \in \mathbb{R}$ and $(\xi_1, \xi_2) \in \mathbb{R}^3 \times \mathbb{R}^4$ are assumed to be small. With these changes of variables and using

(2.3), (E_{α_i}) is equivalent to

$$\beta_i = V_{\alpha_i}(\varepsilon, \beta, \zeta, \xi) = O(\beta^2 + \varepsilon|\log \varepsilon| + |\xi|^2).$$

Now, using (2.4), an easy computation shows that

$$\begin{aligned} & \frac{2c_3}{\lambda_1^2} \frac{\partial K}{\partial \nu}(z + \xi_1) + \frac{\varepsilon K(z + \xi_1)S_4}{8\lambda_1} \\ &= \frac{\varepsilon^2 S_4^2 K(z)^2}{128c_3} \left(- \left(\frac{\partial K}{\partial \nu}(z) \right)^{-1} \zeta_1 + \left(\frac{\partial K}{\partial \nu}(z) \right)^{-2} D^2 K_1(z)(e_4, \xi_1) \right) \\ &+ O(\varepsilon^2(\zeta_1^2 + |\xi_1|^2)). \end{aligned}$$

This implies that (E_{λ_1}) is equivalent, while using (2.7) and (2.9), to

$$\begin{aligned} & -\zeta_1 + \left(\frac{\partial K}{\partial \nu}(z) \right)^{-1} D^2 K_1(z)(e_4, \xi_1) = V_{\lambda_1}(\varepsilon, \beta, \zeta, \xi) \\ &= O(\varepsilon|\log \varepsilon| + |\beta|^2 + \zeta_1^2 + |\xi|^2). \end{aligned} \quad (2.10)$$

Using (2.9), an easy computation shows that

$$\begin{aligned} & \frac{\varepsilon K(y + \xi_2)S_4}{4\lambda_2} + \frac{c_2}{4} \left(\frac{\Delta K(y + \xi_2)}{3K(y + \xi_2)} + 4H(y + \xi_2, y + \xi_2) \right) \frac{1}{\lambda_2^3} \\ &= \frac{\varepsilon^{3/2} \Lambda^3 c_2}{2} \left(\frac{\Delta K(y)}{3K(y)} + 4H(y, y) \right) \zeta_2 \\ &+ \frac{\varepsilon^{3/2} \Lambda^3 c_2}{4} \left(\frac{1}{3K(y)} \nabla(\Delta K)(y) + 8 \frac{\partial H}{\partial x_2}(y, y) \right) \xi_2 \\ &+ O(\varepsilon^{3/2}(\zeta_2^2 + |\xi_2|^2)). \end{aligned}$$

This implies that (E_{λ_2}) is equivalent, while using (2.7) and (2.9), to

$$\begin{aligned} & \left(\frac{\Delta K(y)}{3K(y)} + 4H(y, y) \right) \zeta_2 + \frac{1}{2} \left(\frac{1}{3K(y)} \nabla(\Delta K)(y) + 8 \frac{\partial H}{\partial x_2}(y, y) \right) \xi_2 \\ &= V_{\lambda_2}(\varepsilon, \beta, \zeta, \xi) = O(|\beta|^2 + \zeta_2^2 + |\xi|^2 + \varepsilon^{1/2}). \end{aligned} \quad (2.11)$$

Using (2.6), (2.8) and (2.9), (E_{x_1}) is equivalent to

$$D^2 K_1(z)\xi_1 = V_{x_1}(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2). \quad (2.12)$$

Lastly, using (2.6), (2.8) and (2.9), (E_{x_2}) is equivalent to

$$D^2 K(y)\xi_2 = V_{x_2}(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2). \quad (2.13)$$

This system may also be written as

$$\beta = V(\varepsilon, \beta, \zeta, \xi), \quad \text{and} \quad L(\zeta, \xi) = W(\varepsilon, \beta, \zeta, \xi), \quad (2.14)$$

where L is a fixed linear operator on $\mathbb{R}^9$ defined by the left hand sides of (2.10)–(2.13), and V, W are smooth functions satisfying

$$\begin{cases} V(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\xi|^2), \\ W(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2). \end{cases}$$

Moreover, a simple computation shows that the determinant of L is not equal to zero. Hence L is invertible, and Brouwer's fixed point theorem shows that (2.14) has a solution $(\beta^\varepsilon, \zeta^\varepsilon, \xi^\varepsilon)$ for ε small enough, such that

$$|\beta^\varepsilon| = O(\varepsilon^{1/2}); \quad |\zeta^\varepsilon| = O(\varepsilon^{1/2}); \quad |\xi^\varepsilon| = O(\varepsilon^{1/2}).$$

Hence, we have constructed $m^\varepsilon = (\alpha_1^\varepsilon, \alpha_2^\varepsilon, \lambda_1^\varepsilon, \lambda_2^\varepsilon, x_1^\varepsilon, x_2^\varepsilon)$ such that $u_\varepsilon := \sum \alpha_i^\varepsilon P\delta_{(x_i^\varepsilon, \lambda_i^\varepsilon)} + \bar{v}_\varepsilon$, is a critical point of I_ε (by Proposition 2.1), i.e u_ε satisfies

$$-\Delta u_\varepsilon + 2u_\varepsilon = K|u_\varepsilon|^{2-\varepsilon}u_\varepsilon \quad \text{in } S_+^4, \quad \partial u_\varepsilon / \partial \nu = 0 \quad \text{on } \partial S_+^4. \quad (2.15)$$

Since $\|u_\varepsilon^-\|$ is very small (where $u_\varepsilon^- = \max(0, -u_\varepsilon)$), multiplying the Eq. (2.15) by u_ε^- and integrating on S_+^4 we derive that $u_\varepsilon^- = 0$ and by the maximum principle we obtain: $u_\varepsilon > 0$. $\square$

3. Preliminary results

We define the functional

$$I_0(u) = \frac{1}{2} \int_{S_+^4} |\nabla u|^2 + \int_{S_+^4} u^2 - \frac{1}{4} \int_{S_+^4} K|u|^4, \quad u \in H^1(S_+^4).$$

Note that if u is a positive critical point of I_0 , then u is a solution of (P) and inversely.

For $u \in \Sigma^+$, in a first step, we can minimize, on the radial direction and it is easy to prove that, for each $u \in \Sigma^+$, there exists only one critical point which will be denoted by $\pi(u)$. Hence the study will be reduced in the sphere Σ^+ of the functional $I_0(\pi(u)u)$ which has the same critical points as the functional J_0 associated to (P) defined by (1.2).

The Palais-Smale condition fails to be satisfied for J_0 on Σ^+ . Its failure has been studied by various authors (see Brezis-Coron [12], Lions [22], Struwe [24]) and it becomes well known that: assuming that J_0 has no critical points in Σ^+ , then it satisfies the Palais-Smale condition outside $\cup_p V(p, \varepsilon)$, where for $\varepsilon > 0$ small enough and $p \in \mathbb{N}^*$

$$V(p, \varepsilon) := \left\{ u \in \Sigma^+ / \exists a_i \in \bar{S}_+^4, \exists \lambda_i > \varepsilon^{-1}, \exists \alpha_i > 0, \text{ for } i=1, \dots, p \text{ such that} \right. \\ \left. \left\| u - \sum_{i=1}^p \alpha_i \delta_i \right\| < \varepsilon, \left| \frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1 \right| < \varepsilon, \varepsilon_{ij} < \varepsilon \text{ and } \lambda_i d_i < \varepsilon \text{ or} \right. \\ \left. \lambda_i d_i > \varepsilon^{-1} \right\}, \quad (3.1)$$

where $\delta_i = \delta_{(a_i, \lambda_i)}$, $d_i = d(a_i, \partial S_+^4)$ and $\varepsilon_{ij}^{-1} = \lambda_i / \lambda_j + \lambda_j / \lambda_i + \lambda_i \lambda_j (1 - \cos d(a_i, a_j)) / 2$.

Remark 3.1. Note that the corresponding ε_{ij} when we use the variables a_i in $\mathbb{R}_+^4$ is defined by

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{-1}.$$

Furthermore, the set $V(p, \varepsilon)$ is parametrized as: For each $u \in V(p, \varepsilon)$, there exist a unique $(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p, \lambda_1, \dots, \lambda_p)$ (up to permutation) and $v \in H^1(S_+^4)$ satisfying

$$\|v\| \leq \varepsilon, \quad \langle v, \psi \rangle = 0 \text{ for } \psi \in \left\{ \delta_i, \frac{\partial \delta_i}{\partial \lambda_i}, \frac{\partial \delta_i}{\partial a_i}, P\delta_j, \frac{\partial P\delta_j}{\partial \lambda_j}, \frac{\partial P\delta_j}{\partial a_j} : i \leq q, j > q \right\}, \quad (V_0)$$

such that

$$u = \sum_{i=1}^q \alpha_i \delta_{(a_i, \lambda_i)} + \sum_{i=q+1}^p \alpha_i P\delta_{(a_i, \lambda_i)} + v.$$

We also have the following proposition whose proof is similar, up to modifications, to the corresponding statements in [2].

Proposition 3.2. *There exists a C^1 map which, to each $(\alpha_1, \dots, \alpha_p, a_1, \dots, a_p, \lambda_1, \dots, \lambda_p)$ such that $w := \sum_{i=1}^q \alpha_i \delta_{(a_i, \lambda_i)} + \sum_{i=q+1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$ associates $\bar{v} = \bar{v}_{(\alpha_i, a_i, \lambda_i)}$ satisfying*

$$J_0(w + \bar{v}) = \min\{J_0(w + v), v \text{ satisfies } (V_0)\}.$$

Moreover, there exists a constant $c > 0$ such that the following holds

$$\|\bar{v}\| \leq c \left(\sum_{i \leq q} \frac{1}{\lambda_i} + \sum_{i > q} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i > q} \frac{1}{(\lambda_i d_i)^2} + \sum_{k \neq r} \varepsilon_{kr} \log(\varepsilon_{kr}^{-1})^{1/2} \right).$$

4. Construction of a pseudo gradient flow in $V(p, \varepsilon)$

In this section we are going to construct a global pseudo gradient flow for the functional J_0 under assumptions (H_1-H_4) on K . Along its flow lines there can be only finitely many isolated blow up points. Such a flow is defined by combining two basic facts. On the one hand, the first one comes from the Morse Lemma at infinity which moves points and concentrations as follows: points move according to ∇K if they are interior points and along ∇K_1 if they are boundary points, concentrations move so as to decrease the functional J_0 . On the other hand, there is another pseudo gradient when the points are very close and the total interaction $\sum \varepsilon_{ij}$ is large with respect to $\sum \frac{1}{\lambda_i}$. We need to convex-combine both flows. The obtained pseudo gradient has the following property: when the flow lines, starting from $V(p, \varepsilon/2)$, leave $V(p, \varepsilon)$ the functional J_0 will decrease by a fixed constant which depends only on ε . Hence the kind of travel cannot exist infinite times.

Following A. Bahri we introduce the following definitions and notations

Definition 4.1. A critical point at infinity of J_0 in $V(p, \varepsilon)$ is a limit of a flow line $u(s)$ of

$$\frac{\partial u}{\partial s} = -\nabla J_0(u) \quad \text{and} \quad u(0) = u_0 \in \Sigma^+ \quad (4.1)$$

such that $u(s)$ remains in $V(p, \varepsilon(s))$ for $s \geq s_0$, where $\varepsilon(s)$ is some function tending to zero when $s \rightarrow \infty$. Using Remark 3.1, $u(s)$ can be written as:

$$u(s) = \sum_{i \leq p} \alpha_i(s) P\delta_{(a_i(s), \lambda_i(s))} + v(s).$$

Denoting $a_i := \lim_{s \rightarrow \infty} a_i(s)$ and $\alpha_i := \lim_{s \rightarrow \infty} \alpha_i(s)$, following the notation of A. Bahri, we denote such a critical point at infinity by

$$(a_1, \dots, a_p)_\infty \text{ or } \sum_{i \leq p} \alpha_i P\delta_{(a_i, \infty)}.$$

For such a *critical point at infinity* once a Morse type reduction is performed, are associated stable and unstable manifolds see [2, 3].

As a by product of the construction of our pseudo gradient, we are able to perform a Morse Lemma at infinity and therefore, we identify the critical points at infinity of our problem.

Proposition 4.2. *For $p \geq 1$, there exists a pseudo gradient W so that the following holds : There is a constant $c > 0$ independent of $u = \sum_{i=1}^q \alpha_i \delta_i + \sum_{j=q+1}^p \alpha_j P\delta_j \in V(p, \varepsilon)$ so that*

- (i) $\langle -\nabla J_0(u), W \rangle$

$$\geq c \sum_{i \leq q} \left(\sum_{j \neq i} \varepsilon_{ij}^{3/2} + \frac{1}{\lambda_i^{3/2}} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)|^{\frac{3}{2}} \right)$$

$$+ c \sum_{i > q} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} + \frac{|\nabla K(a_i)|}{\lambda_i} \right)$$
- (ii) $\langle -\nabla J_0(u + \bar{v}), W + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W) \rangle$

$$\geq c \sum_{i \leq q} \left(\sum_{j \neq i} \varepsilon_{ij}^{3/2} + \frac{1}{\lambda_i^{3/2}} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)|^{\frac{3}{2}} \right)$$

$$+ c \sum_{i > q} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} + \frac{|\nabla K(a_i)|}{\lambda_i} \right)$$
- (iii) $|W|$ is bounded. Furthermore, the only case where the maximum of the λ_i 's is not bounded is when each point a_i of the boundary is close to a critical point z_{j_i} of K_1 with $(\partial K / \partial \nu)(z_{j_i}) > 0$ and $z_{j_i} \neq z_{j_k}$ for $i \neq k$, and each point a_i in S_+^4 is close to a critical point y_{j_i} of K with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{j_1}, \dots, y_{j_s}) > 0$, where $s = p - q$ and $\rho(y_{j_1}, \dots, y_{j_s})$ denotes the least eigenvalue of $M(y_{j_1}, \dots, y_{j_s})$.

Before proving Proposition 4.2, we prove Theorem 1.2 which gives a construction of a pseudo gradient when the concentration points are close to the same critical point of K_1 with $\partial K / \partial \nu > 0$.

Proof of Theorem 1.2. The required pseudo gradient will be defined by using the variables α_i , a_i and λ_i . Concerning the α_i 's variables, we define the vector field

$$Z_i^\alpha = \psi(\lambda_i(1 - J_0(u)^2 \alpha_i^2 K(a_i))) \delta_i, \quad (4.2)$$

where ψ is a C^∞ function which satisfies

$$\begin{aligned} t\psi(t) \leq 0, \quad \psi(t) = -1 \quad \text{if } t > 2, \quad \psi(t) = 0 \quad \text{if } |t| \leq 1 \quad \text{and } \psi(t) = 1 \\ \text{if } t < -2. \end{aligned} \quad (4.3)$$

Using the Appendix 2 [Proposition 7.2], we derive

$$\begin{aligned} \langle -\nabla J_0(u), Z_i^\alpha \rangle &\geq c|\psi(\lambda_i(1 - J_0(u)^2 \alpha_i^2 K(a_i)))| \\ &\quad \left[|1 - J_0(u)^2 \alpha_i^2 K(a_i)| + O\left(\frac{1}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij}\right) \right]. \end{aligned} \quad (4.4)$$

Without loss of generality, we can assume that $\lambda_1 \leq \dots \leq \lambda_q$. Let

$$I := \{1\} \cup \{i \leq q : \lambda_k \leq M\lambda_{k-1}, \forall k \leq i\},$$

where M is a large positive constant.

The set I corresponds to the indices i so that λ_i/λ_1 is bounded. Four cases may occur

Case 1. $I = \{1\}$. We define the following vector fields

$$Y_{11} = \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1}, \quad Y_{12} = -\sum_{i>1} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} 2^i \quad \text{and} \quad Y_{13} = \sum_{i \leq q} Z_i^\alpha.$$

We remark that for each $i \neq j$ with $\lambda_i \leq c\lambda_j$ we have

$$-\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} = \varepsilon_{ij} \left(1 - 2 \frac{\lambda_i}{\lambda_j} \varepsilon_{ij} \right) \geq \varepsilon_{ij} (1 + o(1)). \quad (4.5)$$

Moreover, the above formula is also true if we have $\lambda_l |a_i - a_j| \rightarrow +\infty$ for $l = i, j$ (the above assumption implies that $\varepsilon_{ij} \lambda_i / \lambda_j$ is small). However, in the general case, we cannot control the sign of $-\lambda_j \partial \varepsilon_{ij} / \partial \lambda_j$. But, an easy computation implies that $|\lambda_i \partial \varepsilon_{ij} / \partial \lambda_i| \leq c\varepsilon_{ij}$ and

$$-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - 2\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geq \varepsilon_{ij} (1 + o(1)) \quad \text{for } \lambda_i \leq \lambda_j. \quad (4.6)$$

Thus, using the Appendix 2 [Proposition 7.2], we get

$$\langle -\nabla J_0(u), Y_{11} \rangle \geq \frac{c}{\lambda_1} + O\left(\sum_{j \neq 1} \varepsilon_{1j} + \sum_{k \neq r} \varepsilon_{kr}^2\right), \quad (4.7)$$

$$\langle -\nabla J_0(u), Y_{12} \rangle \geq c \sum_{i>1} \left(\sum_{j \neq i} \varepsilon_{ij} + O\left(\frac{1}{\lambda_i}\right) \right) + O\left(\sum_{k \neq r} \varepsilon_{kr}^2\right). \quad (4.8)$$

Observe that, if for some indices i we have $|\psi(\lambda_i(1 - J_0(u)^2 \alpha_i^2 K(a_i)))| < 1$, then we get $\lambda_i |1 - J_0(u)^2 \alpha_i^2 K(a_i)| \leq 2$ and therefore we can make appear $|1 - J_0(u)^2 \alpha_i^2 K(a_i)|$ from $1/\lambda_1$ which exists in the lower bound of (4.7).

In the other case, we have $|\psi(\lambda_i(1 - J_0(u)^2\alpha_i^2K(a_i)))| \geq 1$ and therefore $|1 - J_0(u)^2\alpha_i^2K(a_i)|$ appears in (4.4).

Our global vector field in this case is $Y_k^1 = \sqrt{M}Y_{11} + MY_{12} + Y_{13}$. Thus, we derive

$$\langle -\nabla J_0(u), Y_k^1 \rangle \geq c \sum_{i \leq q} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{\lambda_i} + |1 - J_0(u)^2\alpha_i^2K(a_i)| \right). \quad (4.9)$$

We remark that λ_q which is the maximum of the λ_i 's is a decreasing function along the flow lines generated by Y_k^1 .

Case 2. I contains at least two indices and there exists $j_0 \in I$ satisfying

$$c_2 \sum_{i \in I, i \neq j_0} \varepsilon_{ij_0} \geq \frac{1}{(1 - \varrho/2)} \frac{4c_3}{K(z_k)} \frac{1}{\lambda_{j_0}} \frac{\partial K}{\partial \nu}(z_k), \quad (4.10)$$

where ϱ is a positive constant which will be defined below (see (4.31)) and the constants c_i are defined in Proposition 6.2.

Let $F := \{i \in I : i \text{ satisfies (4.10)}\}$. We define the following vector fields

$$\begin{aligned} Y_{21} &= - \sum_{i \in F} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}, \quad Y_{22} = -m \sum_{i \in I \setminus F} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} - \sqrt{M} \sum_{i \notin I} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} 2^i \quad \text{and} \\ Y_{23} &= \sum_{i \leq q} Z_i^\alpha, \end{aligned}$$

where m is a small positive constant such that $m = o(\varrho/M^q)$ and ϱM is large. Using the Appendix 2 [Proposition 7.2], we get

$$\begin{aligned} \langle -\nabla J_0(u), Y_{21} \rangle &= J_0(u) \sum_{i \in F} \left(-c_2 \sum_{j \leq q, j \neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - 4c_3 J_0(u)^2 \frac{\alpha_i^3}{\lambda_i} \frac{\partial K}{\partial \nu}(a_i) \right) \\ &\quad + R, \end{aligned} \quad (4.11)$$

where $R = O(\sum_{k \neq r} \varepsilon_{kr}^2 + \sum_{i \leq q} \frac{1}{\lambda_i^2})$.

Observe that for each $i \leq q$, we have $K(a_i) = K(z_k) + o(1)$, $\alpha_i = \alpha_j + o(1)$, $J_0(u)^2\alpha_i^2K(a_i) = 1 + o(1)$ and $(\partial K/\partial \nu)(a_i) = (\partial K/\partial \nu)(z_k) + o(1) > c > 0$ if we choose μ small enough. Using (4.5) for $i, j \in I$, thus, (4.11) becomes

$$\begin{aligned} \langle -\nabla J_0(u), Y_{21} \rangle &= \sum_{i \in F} \left(\frac{c_2(1 + o(1))}{K(z_k)^{1/2}} \sum_{j \in I} \varepsilon_{ij} - \frac{4c_3(1 + o(1))}{\lambda_i K(z_k)^{3/2}} \frac{\partial K}{\partial \nu}(z_k) \right. \\ &\quad \left. + O\left(\sum_{j \notin I} \varepsilon_{ij}\right) \right) + R. \end{aligned}$$

From (4.10), we derive

$$\langle -\nabla J_0(u), Y_{21} \rangle \geq \sum_{i \in F} \left(c \sum_{j \in I} \varepsilon_{ij} + \frac{c}{\lambda_i} + O(\sum_{j \notin I} \varepsilon_{ij}) \right) + O\left(\sum_{k \neq r} \varepsilon_{kr}^2\right). \quad (4.12)$$

For the vector Y_{22} , using (4.6) and the Appendix 2 [Proposition 7.2], we get

$$\begin{aligned} \langle -\nabla J_0(u), Y_{22} \rangle &\geq m \sum_{i \in I \setminus F} \left(\sum_{j \neq i} \varepsilon_{ij} + O\left(\frac{1}{\lambda_i}\right) \right) + \sqrt{M} \sum_{i \notin I} \left(\sum_{j \neq i} \varepsilon_{ij} + O\left(\frac{1}{\lambda_i}\right) \right) \\ &\quad + O\left(\sum_{k \neq r} \varepsilon_{kr}^2 \right). \end{aligned} \quad (4.13)$$

Let $Y_k^2 = M'(Y_{21} + Y_{22}) + Y_{23}$, where M' is a large positive constant. Note that all the λ_i 's variables are decreasing functions along the flow lines. Using (4.12), (4.13) and (4.4) as in the proof of (4.9), it is easy to show that

$$\langle -\nabla J_0(u), Y_k^2 \rangle \geq c \sum_{i \leq q} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{\lambda_i} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)| \right). \quad (4.14)$$

Case 3. I contains at least two indices, z_k is a local maximum point of K_1 and we have

$$c_2 \sum_{i \in I, i \neq j} \varepsilon_{ij} < \frac{1}{(1-\varrho)} \frac{4c_3}{K(z_k)} \frac{1}{\lambda_j} \frac{\partial K}{\partial \nu}(z_k) \quad \text{for all } j \in I. \quad (4.15)$$

In this case, we need to introduce the sets

$$\Gamma_1 = \{j \notin I \text{ s.t. } c_2 \sum_{i \neq j} \varepsilon_{ij} \geq \frac{1}{(1-\varrho/2)} \frac{4c_3}{K(z_k)} \frac{1}{\lambda_j} \frac{\partial K}{\partial \nu}(z_k)\} \quad \text{and}$$

$$\Gamma_2 = \{j \leq q : j \notin I \cup \Gamma_1\}.$$

We define the vector field

$$Y_{31} = \sum_{i \in I} \frac{1}{\lambda_h} \frac{\partial \delta_i}{\partial a_i} \frac{\nabla_T K(a_i)}{\sum_{l \in I} |\nabla_T K(a_l)|},$$

where $h \in I$ such that $|\nabla_T K(a_h)| = \max\{|\nabla_T K(a_i)|, i \in I\}$.

From the Appendix 2 [Proposition 7.2], we get

$$\begin{aligned} \langle -\nabla J_0(u), Y_{31} \rangle &\geq \frac{c}{\lambda_h \sum_{l \in I} |\nabla_T K(a_l)|} \\ &\quad \times \sum_{i \in I} \left(8c_5 \frac{|\nabla_T K(a_i)|^2}{K_1(z_k)} + c_2 \sum_{j \in I \cup \Gamma_2} \frac{\partial \varepsilon_{ij}}{\partial a_i} \nabla_T K(a_i) \right) \\ &\quad + O\left(\sum_{i \in I, j \in \Gamma_1} \varepsilon_{ij} \right) + R. \end{aligned} \quad (4.16)$$

Note that, for $i \neq j$, then $\partial \varepsilon_{ij} / \partial a_i = -\partial \varepsilon_{ij} / \partial a_j$, and therefore using the fact that, z_k is a local maximum point, we have, for $i, j \in I$

$$\begin{aligned} \frac{\partial \varepsilon_{ij}}{\partial a_i} \nabla_T K(a_i) + \frac{\partial \varepsilon_{ji}}{\partial a_j} \nabla_T K(a_j) &= \frac{\partial \varepsilon_{ij}}{\partial a_i} (\nabla_T K(a_i) - \nabla_T K(a_j)) \\ &= 2\lambda_i \lambda_j (a_j - a_i) \varepsilon_{ij}^2 (D^2 K(z_k)(a_i - a_j) + o(|a_i - z_k| + |a_j - z_k|)) \\ &\geq 2\lambda_i \lambda_j |a_j - a_i| \varepsilon_{ij}^2 o(|a_i - z_k| + |a_j - z_k|). \end{aligned} \quad (4.17)$$

But, since λ_i, λ_j and λ_h are of the same order, we get

$$\begin{aligned} \frac{\lambda_i \lambda_j |a_j - a_i| \varepsilon_{ij}^2}{\lambda_h \sum_{l \in I} |\nabla_T K(a_l)|} o(|a_i - z_k| + |a_j - z_k|) &= c \varepsilon_{ij}^{3/2} o\left(\frac{|a_i - z_k| + |a_j - z_k|}{\sum_{l \in I} |\nabla_T K(a_l)|}\right) \\ &= o(\varepsilon_{ij}^{3/2}). \end{aligned}$$

Now, using the definition of Γ_2 , for $i \in I, j \in \Gamma_2$, we have

$$\varepsilon_{ij} = \frac{1}{\lambda_i \lambda_j |a_i - a_j|^2} + o(\varepsilon_{ij}). \quad (4.18)$$

Moreover, since $j \in \Gamma_2$ (which implies that $\lambda_j \geq \lambda_h$ and $\varepsilon_{ij} \leq c/\lambda_j$), an easy computation shows

$$\begin{aligned} \left| \frac{1}{\lambda_h} \frac{\partial \varepsilon_{ij}}{\partial a_i} \frac{\nabla_T K(a_i)}{\sum_{l \in I} |\nabla_T K(a_l)|} \right| &\leq c \frac{\sqrt{\lambda_i \lambda_j}}{\lambda_h} \varepsilon_{ij}^{3/2} \leq c \frac{\sqrt{\lambda_i}}{\lambda_h} \frac{1}{\lambda_j} \leq \frac{c}{\sqrt{M}} \frac{1}{\lambda_h \sqrt{\lambda_j}} \\ &= o\left(\frac{1}{\lambda_h^{3/2}}\right), \end{aligned}$$

where we will choose M large enough. Thus, (4.16) becomes

$$\begin{aligned} \langle -\nabla J_0(u), Y_{31} \rangle &\geq c \sum_{i \in I} \frac{8c_5 |\nabla_T K(a_i)|^2}{K_1(z_k) \lambda_h \sum_{l \in I} |\nabla_T K(a_l)|} + o\left(\frac{1}{\lambda_h^{3/2}}\right) \\ &\quad + O\left(\sum_{i \in I, j \in \Gamma_1} \varepsilon_{ij}\right) + R \\ &\geq \frac{c}{\sharp I} \frac{8c_5 |\nabla_T K(a_h)|}{K_1(z_k) \lambda_h} + o\left(\frac{1}{\lambda_h^{3/2}}\right) + O\left(\sum_{i \in I, j \in \Gamma_1} \varepsilon_{ij}\right) + R, \end{aligned} \quad (4.19)$$

where $\sharp I$ denotes the cardinal of I . Note that, there exists $i_0 \in I$, such that

$$|\nabla_T K(a_h)| \geq \gamma |a_h - a_{i_0}|/2, \quad \text{where } \gamma \text{ is defined in assumption } (H_2), \quad (4.20)$$

and from (4.15), we have $\varepsilon_{hi_0} \leq c/\lambda_{i_0}$, which implies that $(\lambda_h \lambda_{i_0} |a_h - a_{i_0}|^2)^{-1} \leq c/\lambda_{i_0}$. Hence, $|a_h - a_{i_0}| \geq c/\sqrt{\lambda_h}$. Therefore, using again (4.19), (4.20) and the definition of Γ_2 , we get

$$\begin{aligned} \langle -\nabla J_0(u), Y_{31} \rangle &\geq c \sum_{i \leq q} \frac{1}{\lambda_i^{3/2}} + c \sum_{i \in \Gamma_2, j \neq i} \varepsilon_{ij}^{3/2} + c \sum_{i, j \in I, i \neq j} \varepsilon_{ij}^{3/2} \\ &\quad + O\left(\sum_{i \in I, j \in \Gamma_1} \varepsilon_{ij}\right) + R. \end{aligned} \quad (4.21)$$

It remains to make appear the ε_{ij} for $i \in \Gamma_1$. For this aim, we define the following vector fields

$$Y_{32} = - \sum_{i \in \Gamma_1} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} 2^i \quad \text{and} \quad Y_{33} = \sum_{i \leq q} \psi \left(\frac{\lambda_i}{C} (1 - J_0(u)^2 \alpha_i^2 K(a_i)) \right) \delta_i,$$

where C is a large positive constant and ψ is defined in (4.3). As in the proof of (4.12), we get

$$\langle -\nabla J_0(u), Y_{32} \rangle \geq c \sum_{i \in \Gamma_1} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{\lambda_i} \right) + O \left(\sum_{k \neq r} \varepsilon_{kr}^2 + \sum_{i \leq q} \frac{1}{\lambda_i^2} \right). \quad (4.22)$$

Using also the Appendix 2 [Proposition 7.2] and the definition of ψ , we derive

$$\begin{aligned} \langle \nabla J_0(u), Y_{33} \rangle &\geq \sum_{i \leq q} \left| \psi \left(\frac{\lambda_i}{C} (1 - J_0(u)^2 \alpha_i^2 K(a_i)) \right) \right| \\ &\quad \left[|1 - J_0(u)^2 \alpha_i^2 K(a_i)| + O \left(\frac{1}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \right) \right]. \end{aligned}$$

Observe that, choosing C large enough, if for some indices i we have

$$\psi \left(\frac{\lambda_i}{C} (1 - J_0(u)^2 \alpha_i^2 K(a_i)) \right) \neq 0, \quad \text{then} \quad \frac{1}{\lambda_i} = o(|1 - J_0(u)^2 \alpha_i^2 K(a_i)|). \quad (4.23)$$

In the other case, we have: $|1 - J_0(u)^2 \alpha_i^2 K(a_i)| = O(\lambda_i^{-1})$. Hence, we can make appear this term from $\lambda_i^{-3/2}$.

Our vector field in this case is $Y_k^3 = \sqrt{M''} Y_{31} + M'' Y_{32} + Y_{33}$, where M'' is a large positive constant. By (4.21)–(4.23), we obtain

$$\langle -\nabla J_0(u), Y_k^3 \rangle \geq c \sum_{i \leq q} \left(\sum_{j \neq i} \varepsilon_{ij}^{3/2} + \frac{1}{\lambda_i^{3/2}} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)|^{3/2} \right). \quad (4.24)$$

Case 4. I contains at least two indices, z_k is not a local maximum point of K_1 and for all $j \in I$, (4.15) holds.

We introduce the subset $I' = \{i \in I : \lambda_q |\nabla_T K(a_i)| > \bar{d}\}$ where $\bar{d}$ is a small positive constant.

Observe that $I \setminus I'$ contains at most one point. In this case, two subcases may occur.

Case 4.1. $I = I'$. We define the vector field

$$Y_{41}^1 = \sum_{i \in I} \frac{1}{\lambda_1} \frac{\partial \delta_i}{\partial a_i} \frac{\nabla_T K(a_i)}{|\nabla_T K(a_i)|}.$$

Thus, using the Appendix 2 [Proposition 7.2], we get

$$\begin{aligned} \langle -\nabla J_0(u), Y_{41}^1 \rangle &\geq \frac{c}{\lambda_1} \sum_{i \in I} \left(8c_5 \frac{|\nabla_T K(a_i)|}{K_1(z_k)} (1 + o(1)) + c_2 \sum_{l \in I} \frac{\partial \varepsilon_{li}}{\partial a_i} \frac{\nabla_T K(a_i)}{|\nabla_T K(a_i)|} \right. \\ &\quad \left. + O \left(\frac{1}{\lambda_1} \sum_{l \in \Gamma_2} \left| \frac{\partial \varepsilon_{li}}{\partial a_i} \right| \right) \right) + O \left(\sum_{i \in I, j \in \Gamma_1} \varepsilon_{ij} \right) + R. \end{aligned} \quad (4.25)$$

First observe that, for $i \in I$ and $l \in \Gamma_2$, using (4.18) and (4.15), we get

$$\begin{aligned} \left| \frac{\partial \varepsilon_{il}}{\partial a_i} \right| &\leq \frac{c \varepsilon_{il}}{|a_i - a_l|} \leq \frac{c}{\lambda_l |a_i - a_l|} \leq \frac{1}{\sqrt{M}} \frac{c}{\sqrt{\lambda_i \lambda_l} |a_i - a_l|} \leq \frac{c \sqrt{\varepsilon_{il}}}{\sqrt{M}} \leq \frac{c}{\sqrt{M \lambda_l}} \\ &\leq \frac{c}{M} \frac{1}{\sqrt{\lambda_i}}. \end{aligned} \quad (4.26)$$

Moreover, from (4.15), we get that for $i, j \in I$,

$$\begin{aligned} \max \left(\frac{1}{\lambda_i}, \frac{1}{\lambda_j} \right) \frac{1 + o(1)}{|a_i - a_j|^2} &< \frac{1}{c_2} \frac{1}{1 - \varrho} \frac{4c_3}{K_1(z_k)} \frac{\partial K}{\partial \nu}(z_k) := \gamma_k, \\ |a_i - a_j| &\geq \gamma_k^{-1/2} \max(\lambda_i^{-1}, \lambda_j^{-1})^{1/2} \geq \frac{1}{2} \gamma_k^{-1/2} (\lambda_i^{-1/2} + \lambda_j^{-1/2}) (1 + o(1)), \end{aligned} \quad (4.27)$$

$$|\nabla_T K(a_i)| + |\nabla_T K(a_j)| \geq \gamma |a_i - a_j| \geq \frac{1}{2} \gamma \gamma_k^{-1/2} (\lambda_i^{-1/2} + \lambda_j^{-1/2}) (1 + o(1)). \quad (4.28)$$

Thus we get

$$\sum_{i \in I} |\nabla_T K(a_i)| \geq \frac{1}{2} \gamma \gamma_k^{-1/2} (1 + o(1)) \sum_{i \in I} \frac{1}{\lambda_i^{1/2}}. \quad (4.29)$$

Now, observe that, using (4.15) and (4.27), we have

$$\begin{aligned} \sum_{l \in I, l \neq i} \left| \frac{\partial \varepsilon_{il}}{\partial a_i} \right| &= \sum_{l \in I, l \neq i} \frac{(1 + o(1))}{|a_i - a_l|} \varepsilon_{il} \leq (1 + o(1)) \gamma_k^{1/2} \sqrt{\lambda_i} \sum_{l \in I, l \neq i} \varepsilon_{il} \\ &\leq \frac{\gamma_k^{3/2}}{\sqrt{\lambda_i}} \left(\frac{1}{c_2} + o(1) \right). \end{aligned} \quad (4.30)$$

Combining (4.29), (4.30), (4.25) and (4.26), we get

$$\begin{aligned} \langle -\nabla J_0(u), Y_{41}^1 \rangle &\geq \frac{c}{\lambda_1} \gamma_k^{-1/2} \sum_{i \in I} \frac{1}{\sqrt{\lambda_i}} \left(\gamma \frac{4c_5}{K_1(z_k)} - \gamma_k^2 + O \left(\frac{1}{M} \right) \right) \\ &\quad + O \left(\sum_{i \in I, j \in \Gamma_1} \varepsilon_{ij} \right) + R. \end{aligned}$$

From the assumption (H_2) , we can choose ϱ so that

$$\frac{\gamma_k^2}{1 - \varrho} < \gamma \frac{4c_5}{K_1(z_k)}. \quad (4.31)$$

Therefore

$$\begin{aligned} \langle -\nabla J_0(u), Y_{41}^1 \rangle &\geq c \sum_{i \leq q} \frac{1}{\lambda_i^{3/2}} + c \sum_{i \in \Gamma_2, j \neq i} \varepsilon_{ij}^{3/2} + c \sum_{i, j \in I, i \neq j} \varepsilon_{ij}^{3/2} \\ &\quad + O \left(\sum_{i \in I, j \in \Gamma_1} \varepsilon_{ij} \right) + R. \end{aligned} \quad (4.32)$$

Case 4.2. $I \neq I'$. We define the vector field

$$Y_{41}^2 = \sum_{i \in I'} \frac{1}{\lambda_1} \frac{\partial \delta_i}{\partial a_i} \frac{\nabla_T K(a_i)}{|\nabla_T K(a_i)|}.$$

Let $i_0 \in I \setminus I'$, using the first part of (4.28), we get : $|\nabla_T K(a_i)| \geq \gamma |a_i - a_{i_0}|/2$ for each $i \in I'$.

Using the Appendix 2 [Proposition 7.2], we get

$$\begin{aligned} \langle -\nabla J_0(u), Y_{41}^2 \rangle &\geq \frac{c}{\lambda_1} \sum_{i \in I'} \left(8c_5 \frac{|\nabla_T K(a_i)|}{K_1(z_k)} + c_2 \sum_{l \in I} \frac{\partial \varepsilon_{li}}{\partial a_i} \frac{\nabla_T K(a_i)}{|\nabla_T K(a_i)|} \right. \\ &\quad \left. + O \left(\sum_{l \in \Gamma_2} \left| \frac{\partial \varepsilon_{li}}{\partial a_i} \right| + \lambda_1 \sum_{j \in \Gamma_1} \varepsilon_{ij} \right) \right) + R \\ &\geq \frac{c}{\lambda_1} \sum_{i \in I'} \left(\frac{4c_5 \gamma}{K_1(z_k)} |a_i - a_{i_0}| + c_2 \sum_{l \in I} \frac{\partial \varepsilon_{li}}{\partial a_i} \frac{\nabla_T K(a_i)}{|\nabla_T K(a_i)|} \right. \\ &\quad \left. + O \left(\sum_{l \in \Gamma_2} \left| \frac{\partial \varepsilon_{li}}{\partial a_i} \right| + \lambda_1 \sum_{j \in \Gamma_1} \varepsilon_{ij} \right) \right) + R. \end{aligned}$$

From (4.30) and the same arguments than the case 4.1, we derive

$$\begin{aligned} \langle -\nabla J_0(u), Y_{41}^2 \rangle &\geq c \sum_{i \leq q} \frac{1}{\lambda_i^{3/2}} + c \sum_{i \in \Gamma_2, j \neq i} \varepsilon_{ij}^{3/2} + c \sum_{i, j \in I, i \neq j} \varepsilon_{ij}^{3/2} \\ &\quad + O \left(\sum_{i \in I', j \in \Gamma_1} \varepsilon_{ij} \right) + R. \end{aligned} \quad (4.33)$$

Now, let us define the vector field Y_{41} by a convex combination of Y_{41}^1 and Y_{41}^2 . To make appear the ε_{ij} for $i \in \Gamma_1$, we define the same vector fields in the last case

$$Y_{42} = - \sum_{i \in \Gamma_1} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} 2^i \quad \text{and} \quad Y_{43} = \sum_{i \leq q} \psi \left(\frac{\lambda_i}{C} (1 - J_0(u)^2 \alpha_i^2 K(a_i)) \right) \delta_i,$$

where C is a large positive constant. Our vector field in this case is $Y_k^4 = \sqrt{M_4} Y_{41} + M_4 Y_{42} + Y_{43}$, where M_4' is a large positive constant. By (4.32), (4.33), (4.22) and (4.23), we have

$$\langle -\nabla J_0(u), Y_k^4 \rangle \geq c \sum_{i \leq q} \left(\sum_{j \neq i} \varepsilon_{ij}^{3/2} + \frac{1}{\lambda_i^{3/2}} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)|^{3/2} \right). \quad (4.34)$$

Note that, we separated the cases 3 and 4, because (4.17) does not hold if z_k is not a local maximum point of K_1 . Moreover, (1.3) is not assumed if z_k is a local maximum point.

Let Y_k be a convex combination of $Y_k^1 - Y_k^4$. Using (4.9), (4.14), (4.24) and (4.34), we get

$$\langle -\nabla J_0(u), Y_k \rangle \geq c \sum_{i \leq q} \left(\sum_{j \neq i} \varepsilon_{ij}^{3/2} + \frac{1}{\lambda_i^{3/2}} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)|^{3/2} \right).$$

It is easy to see that $\dot{\lambda}_{\max} \leq 0$ and $\|Y_k\|$ is bounded. Hence the proof is completed. $\square$

Proof of Proposition 4.2. Let

$$u = \sum_{i \leq q} \alpha_i \delta_{(a_i, \lambda_i)} + \sum_{i > q} \alpha_i P \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon).$$

Without loss of generality, we can assume that $\lambda_1 \leq \dots \leq \lambda_q$ and $\lambda_{q+1} \leq \dots \leq \lambda_p$.

Since $|\partial K(z_i)/\partial \nu| > 0$ for any critical point z_i of K_1 , there exist $\mu > 0$ and $c > 0$ such that $|\partial K(a)/\partial \nu| > c$ for any $a \in \cup B(z_i, 2\mu)$. In the sequel, we will choose μ such that $B(z_i, 8\mu) \cap B(z_j, 8\mu) = \emptyset$ for each $i \neq j$.

As said before, the construction is based on moving the points a_i if they are away from the critical points of K or K_1 . In the other cases the vector field is basically defined on the variables λ_i . For this reason, we divide $V(p, \varepsilon)$ into some regions $V_i(p, \varepsilon)$, and we will build a vector field W_i on each $V_i(p, \varepsilon)$. Our global vector field W will be built using a convex combination of the vector fields W_i . For $k \leq r_1$, we introduce the set

$$L_k := \{i \leq q / \exists i_1 = i, \dots, i_j, \text{ s.t. } a_{i_j} \in B(z_k, \mu) \text{ and } d(a_{i_l}, a_{i_{l-1}}) < \mu/p, \\ \forall l \leq j\}.$$

It is clear that if $i \in L_k$ and $j \notin L_k$ we have $|a_i - a_j| \geq c > 0$. Let

$$V_1(p, \varepsilon) := \{u \in V(p, \varepsilon) : \forall i, a_i \in \cup_{k \leq r_1} B(z_k, 2\mu) \cup \cup_k B(y_k, 2\mu) \\ \text{and } \#L_k \leq 1, \forall k \leq r_1\}.$$

$$V_2(p, \varepsilon) := \{u \in V(p, \varepsilon) : \exists i_0, a_{i_0} \notin \cup_{k \leq r_1} B(z_k, 2\mu) \cup \cup_k B(y_k, 2\mu) \\ \text{and } \#L_k \leq 1, \forall k \leq r_1\}.$$

$$V_3(p, \varepsilon) := \{u \in V(p, \varepsilon) : \exists k \leq r_1, \text{ such that } \#L_k \geq 2\}.$$

Recall that r_1 satisfies : $\partial K(z_k)/\partial \nu > 0$ for any critical point z_k of K_1 such that $k \leq r_1$.

4.1. Construction of W_1 on $V_1(p, \varepsilon)$

Let $\bar{u} := \sum_{i>q} \alpha_i P \delta_i$. It is clear that $\bar{u} \in V(p - q, \varepsilon)$ and all the concentration points are in S_+^4 . Thus, we can apply the vector field (denoted here Z_1) defined in Proposition 3.1 of [10] (see also [7]). Then, we get

$$\langle -\nabla J_0(u), Z_1 \rangle \geq c \sum_{i>q} \left(\sum_{j>q, j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + O \left(\sum_{j \leq q} \varepsilon_{ij} \right). \quad (4.35)$$

It is interesting to mention that, along the flow lines of Z_1 , the only case where the maximum of the λ_i 's is not bounded is when each point a_i is close to a critical point y_{j_i} of K with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{j_1}, \dots, y_{j_s}) > 0$, where $s = p - q$ and $\rho(y_{j_1}, \dots, y_{j_s})$ denotes the least eigenvalue of $M(y_{j_1}, \dots, y_{j_s})$. For the indices $i \leq q$, we introduce the sets

$$P_1 = \{i \leq q : \lambda_i \geq m' \lambda_{q+1}^2\} \quad \text{and} \quad P_2 = \{i \leq q : i \notin P_1\},$$

where m' is a small positive constant. Note that, if $p = q$ then $P_1 = \emptyset$ and $P_2 = \{1, \dots, p\}$.

We remark that for $i \in P_1$, we can make appear the variables λ_i^{-1} and ε_{ij} from λ_{q+1}^{-2} . In fact, for each $i \neq j$, we have $|a_i - a_j| \geq c > 0$ and therefore,

$$\lambda_i^{-1} = O(\lambda_{q+1}^{-2}), \quad \text{and} \quad \varepsilon_{ij} = \frac{1}{\lambda_i \lambda_j |a_i - a_j|^2} + o(\varepsilon_{ij}) = o(\lambda_{q+1}^{-2}). \quad (4.36)$$

It remains to make appear the indices $i \in P_2$. For this goal, we define

$$Z_2 = \sum_{i \in P_2} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}.$$

Note that, for $i \in P_2$, it is easy to obtain that $\varepsilon_{ij} = o(1/\lambda_i)$ for each $j \neq i$. Hence, using the Appendix 2 [Proposition 7.2], we get

$$\langle -\nabla J_0(u), Z_2 \rangle \geq c \sum_{i \in P_2} \left(\frac{1}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \right) + O \left(\sum_{k \neq r} \varepsilon_{kr}^2 \right). \quad (4.37)$$

Now, we define the vector field $Z_3 = Z_1 + Z_2$. Combining (4.35), (4.36) and (4.37), we get

$$\langle -\nabla J_0(u), Z_3 \rangle \geq c \left(\sum_{j \neq i} \varepsilon_{ij} + \sum_{i>q} \left(\frac{1}{(\lambda_i d_i)^2} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \leq q} \frac{1}{\lambda_i} \right). \quad (4.38)$$

To add the other variables which appear in Proposition 4.2, we introduce the vector field

$$Z_4 = \sum_{i \leq q} Z_i^\alpha, \quad \text{where } Z_i^\alpha \text{ is defined by (4.2).}$$

Our vector field W_1 will be the following

$$W_1 = M_1 Z_3 + Z_4, \quad \text{where } M_1 \text{ is a large positive constant.}$$

By (4.38) and (4.4), we obtain the estimate of Claim (i) for W_1 instead of W . Note that the only case where the maximum of the λ_i 's for $i \leq q$ is not bounded is when $\rho(y_{j_1}, \dots, y_{j_s}) > 0$, where $s = p - q$ or when all the points a_i are in the boundary ($q = p$).

4.2. Construction of W_2 on $V_2(p, \varepsilon)$

Let $P = \cup_{k \leq r_1} L_k$. We can write $u := u_1 + u_2$, where $u_1 = \sum_{i \in P} \alpha_i \delta_i$ and $u_2 = u - u_1$. Observe, if $i \notin P$ then $a_i \notin \cup_{k \leq r_1} B(z_k, 2\mu)$ or $a_i \in S_+^4$. Note that $u_2 \in V(p - \sharp P, \varepsilon)$. Hence, we can use the vector field (denoted here Z_5) defined in [10] (we apply Z_5 only to u_2). Then, we get

$$\begin{aligned} \langle -\nabla J_0(u), Z_5 \rangle &\geq c \sum_{i \leq q, i \notin P} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{\lambda_i} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)| \right) \\ &\quad + c \sum_{i > q} \left(\sum_{j \notin P, j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + O \left(\sum_{j \in P} \varepsilon_{ij} \right) \right), \end{aligned} \quad (4.39)$$

where we use $\varepsilon_{ij} = o(\lambda_i^{-1})$ since we have $|a_i - a_j| \geq c$ for $i \leq q, i \notin P$ and $j \in P$.

Note that, there exists i_0 such that $a_{i_0} \notin \cup_{k \leq r_1} B(z_k, \mu) \cup \cup_k B(y_k, \mu)$, this implies that $c\lambda_{i_0}^{-1}$ is in the lower bound of (4.39). Furthermore, the maximum of the λ_i 's, $i \notin P$, is bounded along the flow lines generated by Z_5 (see Proposition 3.1 of [10] for more details).

To make appear the other variables with indices in P , we need to introduce the following set

$$P' = \{i \in P : M_2 \lambda_i < \lambda_{i_0}\}, \quad \text{where } M_2 \text{ is a large positive constant.}$$

We remark that, for $i \in P \setminus P'$, we can make appear λ_i^{-1} 's and ε_{ij} 's from $\lambda_{i_0}^{-1}$. In fact we have

$$\frac{1}{\lambda_i} = O \left(\frac{1}{\lambda_{i_0}} \right), \quad \text{and} \quad \varepsilon_{ij} = \frac{1 + o(1)}{\lambda_i \lambda_j |a_i - a_j|^2} = o \left(\frac{1}{\lambda_{i_0}} \right), \quad \text{where } i \in P \setminus P'. \quad (4.40)$$

It remains to make appear the indices $i \in P'$. For this aim, we will increase the λ_i 's, $i \in P'$ (note that for $i \in P'$, we have $\lambda_i \leq M_2^{-1} \lambda_{i_0}$).

$$Z_6 = \sum_{i \in P'} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}, \quad Z_7 = \sum_{i \leq q} Z_i^\alpha, \quad \text{where } Z_i^\alpha \text{ is defined by (4.2).}$$

Note that, for $i \in P'$, it is easy to obtain that $\varepsilon_{ij} = o(1/\lambda_i)$ for each $j \neq i$ (since $|a_i - a_j| \geq c$). Hence, using the Appendix 2 [Proposition 7.2], we get

$$\langle -\nabla J_0(u), Z_6 \rangle \geq c \sum_{i \in P'} \left(\frac{1}{\lambda_i} + \sum_{j \neq i} \varepsilon_{ij} \right) + O \left(\sum_{k \neq r} \varepsilon_{kr}^2 \right). \quad (4.41)$$

Our vector field W_2 will be the following

$$W_2 = M'_2 Z_5 + \sqrt{M'_2} Z_6 + Z_7, \quad \text{where } M'_2 \text{ is a large positive constant.}$$

Combining (4.39)–(4.41) and (4.4), we obtain the estimate of Claim (i) for W_2 instead of W .

4.3. Construction of W_3 on $V_3(p, \varepsilon)$

We introduce the following sets

$$P_k := \{i \in L_k : \quad \text{s.t. } \#L_k \geq 2\}, \quad P := \{i \leq q : i \in \cup_{k \leq r_1} L_k \text{ and } i \notin \cup_k P_k\}.$$

In this case, we can write u as $u = \sum_k u_1^k + u_2 + u_3$, where $u_1^k = \sum_{i \in P_k} \alpha_i \delta_i$ and $u_2 = \sum_{i \in P} \alpha_i \delta_i$. Hence, $u_1^k \in V(\#P_k, \varepsilon, k)$ and we can apply Theorem 1.2 for each u_1^k . Thus, there exists $Z_8 = \sum_k Y_k$ which moves only the variables indexed by $i \in \cup_k P_k$ and we obtain

$$\begin{aligned} \langle -\nabla J_0(u), Z_8 \rangle &\geq c \sum_{i \in \cup P_k} \left(\sum_{j \in \cup P_k} \varepsilon_{ij}^{3/2} + \frac{1}{\lambda_i^{3/2}} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)|^{3/2} \right. \\ &\quad \left. + O \left(\sum_{j \notin \cup P_k} \varepsilon_{ij} \right) \right), \end{aligned} \quad (4.42)$$

where we have used $\varepsilon_{ij} = O(\lambda_i^{-2} + \lambda_j^{-2})$ since we have $|a_i - a_j| \geq c > 0$ for $i \in P_k$ and $j \notin P_k$.

Let $k_0 = \min\{i : i \in \cup_k P_k\}$, we introduce the set

$$P' = \{i \in P : \quad M_3 \lambda_i < \lambda_{k_0}\} \quad \text{where } M_3 \text{ is a large positive constant.}$$

We remark that, for $i, j \in P \setminus P'$, we have $|a_i - a_j| \geq c > 0$, and therefore we can make appear the variables $\lambda_i^{-3/2}$ and $\varepsilon_{ij}^{3/2}$ from $\lambda_{k_0}^{-3/2}$. We define also the following vector field

$$\begin{aligned} Z_9 &= \sum_{i \in P \setminus P'} \psi \left(\frac{\lambda_i}{C'} (1 - J_0(u)^2 \alpha_i^2 K(a_i)) \right) \delta_i \quad \text{and} \\ Z_{10} &= \sum_{i \in P'} \left(M'_3 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + Z_i^\alpha \right), \end{aligned}$$

where C' and M'_3 are large positive constants and Z_i^α is defined by (4.2).

Note that, for $i \in P'$, it is easy to obtain that, $|a_i - a_j| \geq c > 0$, for $i \neq j$ and therefore $\varepsilon_{ij} = o(1/\lambda_i)$ for each $j \neq i$. Using the Appendix 2 [Proposition 7.2], we get

$$\begin{aligned} \langle -\nabla J_0(u), Z_{10} \rangle &\geq c \sum_{i \in P'} \left(\frac{1}{\lambda_i} + \sum_{j \leq q} \varepsilon_{ij} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)| \right. \\ &\quad \left. + O \left(\sum_{j > q} \varepsilon_{ij} + \sum_{k \neq r} \varepsilon_{kr}^2 \right) \right). \end{aligned} \quad (4.43)$$

For u_3 , we can use the vector field (denoted here Z_{11}) defined in Proposition 3.1 of [10]. (we will apply Z_{11} only to u_3). Then, we get

$$\begin{aligned} \langle -\nabla J_0(u), Z_{11} \rangle &\geq c \sum_{i \leq q, i \notin (\cup_k L_k)} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{\lambda_i} + |1 - J_0(u)^2 \alpha_i^2 K(a_i)| \right) \\ &\quad + c \sum_{i > q} \left(\sum_{j \notin (\cup_k L_k), j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} + \frac{|\nabla K(a_i)|}{\lambda_i} \right. \\ &\quad \left. + O \left(\sum_{j \in \cup_k L_k} \varepsilon_{ij} \right) \right), \end{aligned} \quad (4.44)$$

where we use $\varepsilon_{ij} = o(\lambda_i^{-1})$ since we have $|a_i - a_j| \geq c > 0$ for $i \leq q, i \notin \cup_k L_k$ and $j \in \cup_k L_k$.

Now, we see that there exists $d_0 > 0$ such that for any a satisfying $d_a = d(a, \partial S_+^4) \leq d_0$, we have

$$H(a, a) \geq M_0 |\Delta K(a)| \quad \text{and} \quad H(a, a) \sim d_a^{-2}/4,$$

where M_0 is a large positive constant. We need to introduce the subset of points which are close to the boundary, for that purpose, let us introduce the following set

$$F_0 := \{i > q / \exists i_1 = i, \dots, i_l, \text{ s.t. } d(a_{i_l}, \partial S_+^4) < d_0/p \text{ and } d(a_{i_j}, a_{i_{j-1}}) < d_0/p, \\ \forall j \leq l\}.$$

It is easy to see that the following claims holds :

- if $i \in F_0, j \notin F_0, j \geq q+1$, we have $d(a_i, a_j) \geq d_0/p$
- for any $i \in F_0$, we have $H(a_i, a_i) \geq M_0 |\Delta K(a_i)|$ and $H(a_i, a_i) \sim 1/(4d_i^2)$
- for any $j \notin F_0$ and $j \geq q+1$, we have $d_j > d_0/p$ and $\varepsilon_{ij} = O(\lambda_i^{-2}) + o(\lambda_j^{-2})$ if $i \leq q$.

Let

$$Z_{12} = - \sum_{i \in F_0} \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} 2^i.$$

Using the Appendix 2 [Proposition 7.3], we get

$$\langle -\nabla J_0(u), Z_{12} \rangle \geq c \sum_{i \in F_0} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} \right) + O \left(\sum_{k \neq r} \varepsilon_{kr}^2 \right). \quad (4.45)$$

Our vector field W_3 will be the following

$W_3 = Z_8 + Z_9 + Z_{10} + Z_{11} + M_4 Z_{12}$, where M_4 is a large positive constant.

Combining (4.42)–(4.45) and (4.23), we obtain the estimate of Claim (i) for W_3 instead of W .

Finally, we use a convex combination of W_1 , W_2 and W_3 to build our global vector field W in $V(p, \varepsilon)$ and the estimate (i) follows.

We can obtain the estimate (ii), arguing as in Appendix B of [7], using the estimates (i), $\|\nabla J_0(u + \bar{v})\|$ and $\|\bar{v}\|$.

Observe that in $V_2(p, \varepsilon)$ and $V_3(p, \varepsilon)$ the maximum of λ_i 's is always a decreasing function along the flow lines. In $V_1(p, \varepsilon)$, we observe also that the only case where the maximum of the λ_i 's is not bounded is when each point a_i of the boundary is close to a critical point z_{j_i} of K_1 with $(\partial K / \partial \nu)(z_{j_i}) > 0$ and $z_{j_i} \neq z_{j_k}$ for $i \neq k$, and each point a_i in S_+^4 is close to a critical point y_{j_i} of K with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{j_1}, \dots, y_{j_s}) > 0$, where $s = p - q$ and $\rho(y_{j_1}, \dots, y_{j_s})$ denotes the least eigenvalue of $M(y_{j_1}, \dots, y_{j_s})$. Hence our proposition follows. $\square$

5. Proof of our results

Proof of Proposition 1.3. From Proposition 4.2, we know that the only case where the maximum of the λ_i 's is not bounded is when each point a_i of the boundary is close to a critical point z_{j_i} of K_1 with $\partial K / \partial \nu(z_{j_i}) > 0$ and $z_{j_i} \neq z_{j_k}$ for $i \neq k$, and each point a_i in S_+^4 is close to a critical point y_{j_i} of K with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{j_1}, \dots, y_{j_s}) > 0$, where $\rho(y_{j_1}, \dots, y_{j_s})$ denotes the least eigenvalue of $M(y_{j_1}, \dots, y_{j_s})$ (we remark that all the points can be in the boundary or in S_+^4). Arguing as in [3] and [7], we can find a change of variable

$$(a_1, \dots, a_p, \lambda_1, \dots, \lambda_p) \rightarrow (\tilde{a}_1, \dots, \tilde{a}_p, \tilde{\lambda}_1, \dots, \tilde{\lambda}_p) := (\tilde{a}, \tilde{\lambda})$$

such that

$$\begin{aligned} J_0 & \left(\sum_{i=1}^q \alpha_i \delta_{(a_i, \lambda_i)} + \sum_{i=q+1}^p \alpha_i P \delta_{(a_i, \lambda_i)} + \bar{v} \right) \\ &= \frac{(S_4/2)^{1/2} \left(\sum_{i=1}^q \alpha_i^2 + 2 \sum_{i=q+1}^p \alpha_i^2 \right)}{\left(\sum_{i=1}^q \alpha_i^4 K(\tilde{a}_i) + 2 \sum_{i=q+1}^p \alpha_i^4 K(\tilde{a}_i) \right)^{1/2}} \\ & \quad \times \left[1 + A \left((c - \eta) \sum_{i=1}^q \frac{1}{K(z_{j_i})^2} \frac{1}{\lambda_i} \frac{\partial K}{\partial \nu}(z_{j_i}) + (c' - \eta)^t \Lambda M(\tau_s) \Lambda \right) \right] \end{aligned}$$

where c, c' are two positive constants, η is a small positive constant, ${}^t\Lambda = (\tilde{\lambda}_{q+1}^{-1}, \dots, \tilde{\lambda}_p^{-1})$ and $2A^{-1} = S_4(\sum_{i=1}^q K(z_{j_i})^{-1} + 2 \sum_{i=1}^s K(y_{j_i})^{-1})$.

This yields a split of variables $(\alpha, \tilde{a})$ and $\tilde{\lambda}$. Thus we deduce that if $(\alpha, \tilde{a})$ is equal to a critical point, only $\tilde{\lambda}$ can move. Since $\partial K / \partial \nu(z_{j_i}) > 0$ and $\rho(y_{j_1}, \dots, y_{j_s}) > 0$, in order to decrease the functional J_0 , we have to increase $\tilde{\lambda}$, and we obtain a critical point at infinity only in this case.

We need to compute the Morse index of such a critical point at infinity. For this purpose, we let

$$f(\alpha) = \left(\sum_{i=1}^q \alpha_i^2 + 2 \sum_{i=q+1}^p \alpha_i^2 \right) \left(\sum_{i=1}^q \alpha_i^4 K(a_i) + 2 \sum_{i=q+1}^p \alpha_i^4 K(a_i) \right)^{-1/2}.$$

f is a homogeneous function, we will show that $(1/K(a_1), \dots, 1/K(a_p))$ is a critical point of f and $\tau = (\sum_{i=1}^q K(a_i)^{-1} + 2 \sum_{i=q+1}^p K(a_i)^{-1})^{1/2}$ is its critical value. In fact, let $(1 + h_i) = \alpha_i^2 K(a_i) / \alpha_p^2 K(a_p)$ for $i = 1, \dots, p-1$, h_i goes to 0 and expanding the function around $(0, \dots, 0) \in \mathbb{R}^{p-1}$, we derive that $(0, \dots, 0)$ is a nondegenerate maximum point. Finally, using the Morse lemma for $1/K(a_i)$, the proof of (1.5) follows. $\square$

Proof of Theorem 1.4. For η small enough, we introduce the following neighborhood of Σ^+

$$V_\eta(\Sigma^+) = \{u \in \Sigma / e^{2J_0(u)} J_0(u)^3 |u^-|_{L^4}^2 < \eta\}, \quad \text{where } u^- = \max(0, -u).$$

Our global vector field Z will be built using a convex combination of the vector field W defined in $V(p, \varepsilon)$ for $p \geq 1$ and the vector $Y = -\nabla J_0$ outside $\cup_{p \geq 1} V(p, \varepsilon/2)$. $V_\eta(\Sigma^+)$ is invariant under the flow line generated by Z (see [7]). Since $V_\eta(\Sigma^+)$ is contractible, we have $\chi(V_\eta(\Sigma^+)) = 1$, where χ is the Euler-Poincaré characteristic. The proof will be by contradiction, we assume that J_0 has no critical point in $V_\eta(\Sigma^+)$. From Proposition 1.3, we know that the only critical points at infinity of J_0 in $V_\eta(\Sigma^+)$ are the family defined by (1.4). Using the vector field Z , we have $V_\eta(\Sigma^+)$ retracts by deformation onto $\cup W_u(w_\infty)$ (see sections 7 and 8 of [6]), where $W_u(w_\infty)$ is the unstable manifold at infinity of the critical point at infinity w_∞ . We observe that there are three kind of critical points at infinity w_∞ : on the boundary ($s = 0$), in the interior ($q = 0$) and the mixed-type where $q \neq 0$ and $s \neq 0$ where q and s are defined in Proposition 1.3. Let $\Gamma_q := \{(z_{j_1}, \dots, z_{j_q}) : \nabla K_1(z_{j_i}) = 0, \partial_\nu K(z_{j_i}) > 0 \text{ and } z_{j_i} \neq z_{j_k} \text{ if } i \neq k\}$, where ∂_ν denote the normal derivative of K . Then, we have

$$\begin{aligned} 1 = & \sum_{q=1}^{r_1} \sum_{(j_1, \dots, j_q) \in \Gamma_q} (-1)^{4q-1-\sum_{j=1}^q k_1^{j_i}} + \sum_{s=1}^{r_2} \sum_{\tau_s=(j_1, \dots, j_s)/\rho(\tau_s)>0} (-1)^{5s-1-\sum_{i=1}^s k_{j_i}} \\ & + \sum_{1 \leq q \leq r_1, 1 \leq s \leq r_2} \sum_{\Gamma_q, \rho(\tau_s)>0} (-1)^{5s+4q-1-\sum_{i=1}^q k_1^{j_i}-\sum_{i=1}^s k_{j_i}} \end{aligned}$$

where $k_1^{j_i} = \text{index}(K_1, z_{j_i})$ and $k_{j_i} = \text{index}(K, y)$.

Note that the assumption (H_5) implies that $4q - \sum_{i=1}^q k_1^{j_i}$ is even for all q . Thus

$$(C+1) \left(1 - \sum_{s=1}^{r_2} \sum_{\tau_s=(j_1, \dots, j_s)/\rho(\tau_s)>0} (-1)^{5s-1-\sum_{i=1}^s k_{j_i}} \right) = 0, \quad \text{where } C = \#(\cup_q \Gamma_q)$$

which is a contradiction with the assumption of our theorem. Thus there exists at least a critical point of J_0 in $V_\eta(\Sigma^+)$. Arguing as in [7], we prove that this critical point is positive. $\square$

Proof of Theorem 1.6. For the first condition, the existence is proved in [8].

Thus we focus in the case where the second condition holds. Since the assumption (1.7) is satisfied for y_0 , we derive that, $P\delta_{(y_0, \infty)}$ is not a critical point at infinity. In addition, using the fact that $K_1(z_0) = \max K_1$ and $K(y_0) \geq 2K_1(z_0)$, from Proposition 4.2, we deduce that all the levels of the critical points at infinity are above $\bar{\mu}_0 := (S_4/K(y_0))^{1/2}$. Furthermore, using the expansion of J_0 in Proposition 7.1 and the assumption (1.7), we have for λ_0 large enough $J_0(P\delta_{(y_0, \lambda_0)}) \leq \bar{\mu}_0(1 - c\lambda_0^{-2})$. From another part, using again Proposition 7.1, we derive that $J_0(u) \geq \bar{\mu}_0(1 - c'\varepsilon)$, for each $u \in V(p, \varepsilon)$, $p \geq 1$. Thus, for a fixed λ_0 we can choose ε such that

$$J_0(u) > J_0(P\delta_{(y_0, \lambda_0)}), \quad \text{for each } u \in V(p, \varepsilon), \forall p \geq 1. \quad (5.1)$$

Arguing by contradiction and assuming that, there is no critical point of J_0 under the level $\bar{\mu}_0$.

Let $u(s)$ be the solution of the equation (4.1) with $u_0 = P\delta_{(y_0, \lambda_0)}/\|P\delta_{(y_0, \lambda_0)}\|$. From (5.1), we observe that $u(s) \notin V(p, \varepsilon)$, for each $p \geq 1$. Thus, for each $s \geq 0$, we have $(\partial/\partial s)J_0(u(s)) = -|\nabla J_0(u(s))|^2 \leq -c^2$. This implies that $J_0(u(s))$ goes to $-\infty$ as $s \rightarrow +\infty$, which is a contradiction. Hence, the result follows. $\square$

6. Appendix 1

In this appendix, we collect some expansions of the gradient of the functional I_ε associated to the problem (P_ε) for $\varepsilon > 0$ which are needed in Sect. 2. We start by giving the following lemma which corresponds to [23] and Lemma 2.1 of [10]. (Note that in [23], the Claim is proved for the dimension 3, but the same argument is available in our case).

Lemma 6.1. *Let $\delta_{(a, \lambda)}$ be defined in (1.1) and $d = d(a, \partial S_+^4)$. For $\varepsilon > 0$, we have*

- (i) $\delta_{(a, \lambda)}^{-\varepsilon}(x) = 1 - \varepsilon \log \delta_{(a, \lambda)} + O(\varepsilon^2 \log^2 \lambda)$ in S_+^4 .
- (ii) For $a \in S_+^4$, we have $P\delta_{(a, \lambda)} - \delta_{(a, \lambda)} = c_0 \frac{H(a, \cdot)}{\lambda} + f_{(a, \lambda)}$ where $f_{(a, \lambda)}$ satisfies :

$$|f_{(a, \lambda)}|_{L^\infty} \leq \frac{c}{\lambda^3 d^4} \quad \text{and} \quad \lambda \frac{\partial f}{\partial \lambda} = O\left(\frac{1}{\lambda^3 d^4}\right).$$

Now, explicit computations, using (1.1), Lemma 6.1 and the properties of M_ε , yield the following:

Proposition 6.2. *For $u = \alpha_1 \delta_{(x_1, \lambda_1)} + \alpha_2 P\delta_{(x_2, \lambda_2)} + v$, with $(\alpha, \lambda, x, v) \in M_\varepsilon$, we have:*

$$\begin{aligned} \langle \nabla I_\varepsilon(u), \delta_1 \rangle &= \frac{\alpha_1 S_4}{2} (1 - \alpha_1^{2-\varepsilon} K(x_1)) + O\left(\varepsilon \log \lambda_1 + \frac{1}{\lambda_1} + \varepsilon_{12} + \|v\|^2\right), \\ \langle \nabla I_\varepsilon(u), \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} \rangle &= \alpha_2 (1 - \alpha_2^{2-\varepsilon} K(x_2) - \alpha_1^{2-\varepsilon} K(x_1)) c_2 \lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} \\ &\quad + \alpha_1^{3-\varepsilon} \frac{\varepsilon S_4 K(x_1)}{8} \end{aligned}$$

$$\begin{aligned}
& + \alpha_1^{3-\varepsilon} \frac{2c_3}{\lambda_1} \frac{\partial K}{\partial \nu}(x_1) + O\left(\|v\|^2 + \frac{1}{\lambda_1^2} + \varepsilon^2 \log \lambda_1 + \frac{\varepsilon \log \lambda_1}{\lambda_1}\right) \\
& + O\left(\varepsilon_{12}^2 \log \varepsilon_{12}^{-1} + \frac{\varepsilon_{12}}{\lambda_2} (\log \varepsilon_{12}^{-1})^{1/2}\right),
\end{aligned}$$

$$\begin{aligned}
\left\langle \nabla I_\varepsilon(u), \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1} \right\rangle &= (\alpha_1 c_4 (1 - \alpha_1^{2-\varepsilon} K(x_1)) + \alpha_1^{3-\varepsilon} K(x_1) \varepsilon (c_4 \log \lambda_1 + c_7) \\
&+ 2\alpha_1^{3-\varepsilon} \frac{c_5}{\lambda_1} \frac{\partial K}{\partial \nu}(x_1)) e_4 + \alpha_2 \left(1 - \sum \alpha_i^{2-\varepsilon} K(x_i)\right) \frac{c_2}{\lambda_1} \frac{\partial \varepsilon_{12}}{\partial x_1} \\
&- 2\alpha_1^{3-\varepsilon} c_5 \frac{\nabla_T K(x_1)}{\lambda_1} + O\left(\|v\|^2 + \lambda_2 |x_1 - x_2| \varepsilon_{12}^{\frac{5}{2}}\right) \\
&+ O\left(\varepsilon_{12}^2 \log \varepsilon_{12}^{-1} + \frac{\varepsilon_{12}}{\lambda_2} (\log \varepsilon_{12}^{-1})^{\frac{1}{2}} + \frac{1}{\lambda_1^2} + \varepsilon^2 \log^2 \lambda_1\right),
\end{aligned}$$

where

$$\begin{aligned}
S_4 &= 64 \int_{\mathbb{R}^4} \frac{dx}{(1+|x|^2)^4}, \quad c_2 = 64 \int_{\mathbb{R}^4} \frac{dx}{(1+|x|^2)^3}, \quad c_3 = 64 \int_{\mathbb{R}_+^4} \frac{x_4(|x|^2-1)}{(1+|x|^2)^5} dx \\
c_4 &= 132 \int_{\mathbb{R}_+^4} \frac{x_4}{(1+|x|^2)^5} dx, \quad c_5 = 16 \int_{\mathbb{R}^4} \frac{x_4^2}{(1+|x|^2)^5} dx.
\end{aligned}$$

Proposition 6.3. For $u = \alpha_1 \delta_{(x_1, \lambda_1)} + \alpha_2 P\delta_{(x_2, \lambda_2)} + v$, with $(\alpha, \lambda, x, v) \in M_\varepsilon$, we have:

$$\begin{aligned}
\langle \nabla I_\varepsilon(u), P\delta_2 \rangle &= \alpha_2 S_4 (1 - \alpha_2^{2-\varepsilon} K(x_2)) + O\left(\varepsilon \log \lambda_2 + \frac{1}{\lambda_2^2} + \varepsilon_{12} + \|v\|^2\right), \\
\langle \nabla I_\varepsilon(u), \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} \rangle &= \alpha_1 \left(1 - \sum \alpha_i^{2-\varepsilon} K(x_i)\right) c_2 \lambda_2 \frac{\partial \varepsilon_{12}}{\partial \lambda_2} + \alpha_2^{3-\varepsilon} \frac{\varepsilon S_4 K(x_2)}{4} \\
&+ \alpha_2^{3-\varepsilon} \frac{c_2}{12} \frac{\Delta K(x_2)}{\lambda_2^2} + \alpha_2 c_2 \frac{H(x_2, x_2)}{\lambda_2^2} (2\alpha_2^{2-\varepsilon} K(x_2) - 1) \\
&+ O\left(\varepsilon^2 \log \lambda_2 + \frac{1}{\lambda_2^3} + \|v\|^2 + \varepsilon_{12}^2 \log \varepsilon_{12}^{-1}\right. \\
&\left. + \frac{1}{\lambda_2} \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{\frac{1}{2}}\right), \\
\left\langle \nabla I_\varepsilon(u), \frac{1}{\lambda_2} \frac{\partial P\delta_2}{\partial x_2} \right\rangle &= \alpha_1 \left(1 - \sum \alpha_i^{2-\varepsilon} K(x_i)\right) c_2 \frac{1}{\lambda_2} \frac{\partial \varepsilon_{12}}{\partial x_2} - \alpha_2^{3-\varepsilon} c_8 \frac{\nabla K(x_2)}{\lambda_2} \\
&+ O\left(\frac{\varepsilon \log \lambda_2}{\lambda_2} |\nabla K(x_2)| + \frac{1}{\lambda_2^2} + \|v\|^2 + \lambda_1 |x_1 - x_2| \varepsilon_{12}^{\frac{5}{2}}\right) \\
&+ O\left(\varepsilon \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{\frac{1}{2}} + \varepsilon_{12}^2 \log \varepsilon_{12}^{-1} + \frac{\varepsilon_{12}}{\lambda_1} (\log \varepsilon_{12}^{-1})^{\frac{1}{2}}\right).
\end{aligned}$$

7. Appendix 2

We are going to give the expansion of the Euler-Lagrange functional J_0 associated to the problem (P) and its gradient in $V(p, \varepsilon)$ which are useful in the construction of the pseudo gradient. The following propositions are extracted

from [9] and [10]. The first computations are given in [2]. Note that the constants in this paper seem to be different than [10], but taking account the difference of the notations and the definition of $\delta_{(a,\lambda)}$, we derive that the constants are the same.

Proposition 7.1. For $u = \sum_{i=1}^q \alpha_i \delta_{(a_i, \lambda_i)} + \sum_{i=q+1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$, we have

$$\begin{aligned} J_0(u) = & \frac{(S_4/2)^{1/2} \left(\sum_{i=1}^q \alpha_i^2 + 2 \sum_{i=q+1}^p \alpha_i^2 \right)}{\left(\sum_{i=1}^q \alpha_i^4 K(a_i) + 2 \sum_{i=q+1}^p \alpha_i^4 K(a_i) \right)^{\frac{1}{2}}} \left[1 + \frac{c_1}{\theta} \sum_{i=1}^q \frac{1}{K(a_i)^2} \frac{1}{\lambda_i} \frac{\partial K_1}{\partial \nu}(a_i) \right. \\ & - \frac{c_2}{2\theta} \sum_{i \neq j} \frac{\gamma_{ij} \varepsilon_{ij}}{(K(a_i)K(a_j))^{1/2}} + \frac{c_2}{2\theta} \sum_{i > q} \left(\frac{-\Delta K(a_i)}{3\lambda_i^2 K(a_i)^2} - \frac{4H(a_i, a_i)}{\lambda_i^2 K(a_i)} \right) \\ & \left. - \frac{c_2}{2\theta} \sum_{i, j > q, j \neq i} \frac{4}{(K(a_i)K(a_j))^{\frac{1}{2}}} \frac{H(a_i, a_j)}{\lambda_i \lambda_j} \right] \\ & + o \left(\sum_{r \neq k} \varepsilon_{kr} + \sum_{i \leq q} \frac{1}{\lambda_i} + \sum_{i > q} \frac{1}{(\lambda_i d_i)^2} \right), \end{aligned}$$

where S_4 is defined in Proposition 6.2; $\gamma_{ij} = 1$ if $i, j \leq q$; $\gamma_{ij} = 2$ if not and

$$c_1 = 64 \int_{\mathbb{R}_+^4} \frac{x_4 dx}{(1 + |x|^2)^4}, \quad \theta = \frac{S_4}{2} \left(\sum_{i \leq q} K(a_i)^{-1} + 2 \sum_{i > q} K(a_i)^{-1} \right).$$

Proposition 7.2. For $u = \sum_{i=1}^q \alpha_i \delta_{(a_i, \lambda_i)} + \sum_{i=q+1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$, we have:

$$\begin{aligned} \left\langle \nabla J_0(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = & -c_2 J_0(u) \sum_{j \leq q, j \neq i} \alpha_j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - 4c_3 J_0(u)^3 \frac{\alpha_i^3}{\lambda_i} \frac{\partial K}{\partial \nu}(a_i) \\ & + O \left(\sum_{i \leq q, j > q} \varepsilon_{ij} + \sum_{k, r \leq q} \varepsilon_{kr}^2 + \sum_{i \leq q} \frac{1}{\lambda_i^2} \right), \\ \left\langle \nabla J_0(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} \right\rangle = & 2\alpha_i J_0(u) e_4 \left(c_4 (1 - J_0(u)^2 \alpha_i^2 K(a_i)) + c_5 J_0(u)^2 \frac{\alpha_i^2}{\lambda_i} \frac{\partial K}{\partial \nu}(a_i) \right) \\ & - 8c_5 J_0(u)^3 \frac{\alpha_i^3}{\lambda_i} \nabla_T K(a_i) - c_2 J_0(u) \sum_{j \leq q, j \neq i} \frac{\alpha_j}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} (1 + o(1)) \\ & + O \left(\sum_{i \leq q, j > q} \varepsilon_{ij} + \sum_{k \leq q} \lambda_k |a_i - a_k| \varepsilon_{kr}^{5/2} + \sum_{i \leq q} \frac{1}{\lambda_i^2} \right), \\ \left\langle \nabla J_0(u), \delta_i \right\rangle = & \alpha_i J_0(u) S_4 (1 - J_0(u)^2 \alpha_i^2 K(a_i)) + O \left(\frac{1}{\lambda_i} + \sum \varepsilon_{ij} \right), \end{aligned}$$

where $(e_1, \dots, e_4)$ denotes an orthogonal basis of $T_{a_i} S_+^4$ with $\|e_i\| = 1$.

Proposition 7.3. For $u = \sum_{i=1}^q \alpha_i \delta_{(a_i, \lambda_i)} + \sum_{i=q+1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$, we have:

$$\begin{aligned} \left\langle \nabla J_0(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle &= c_2 J_0(u) \left(-2 \sum_{j \neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + 4 \sum_{j=q+1, j \neq i}^p \alpha_j \frac{H(a_i, a_j)}{\lambda_i \lambda_j} \right. \\ &\quad \left. + \alpha_i \frac{\Delta K(a_i)}{3\lambda_i^2 K(a_i)} + 4\alpha_i \frac{H(a_i, a_i)}{\lambda_i^2} \right) \\ &\quad + o \left(\sum_{i \neq j} \varepsilon_{ij} + \sum_{i > q} \frac{1}{(\lambda_i d_i)^2} \right), \\ \left\langle \nabla J_0(u), \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right\rangle \cdot \nabla K(a_i) &\geq c \frac{|\nabla K(a_i)|^2}{\lambda_i} \\ &\quad + O \left(\left(\frac{1}{(\lambda_i d_i)^2} + \sum_{i \neq j} \varepsilon_{ij} \right) |\nabla K(a_i)| \right). \end{aligned}$$

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