Structure of the solution set to impulsive functional differential inclusions on the half-line

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Abstract. A topological structure of the set of solutions to impulsive functional differential inclusions on the half-line is investigated. It is shown that the solution set is nonempty, compact and, moreover, an R_δ -set. It is proved on compact intervals and then, using the inverse limit method, obtained on the half-line.

Mathematics Subject Classification (2010). Primary 34A37; Secondary 34A60, 34K45.

Keywords. Solution set, Impulsive functional differential inclusion, Inverse systems, R_δ -set, Topological structure.

1. Introduction

It is our purpose to study a topological structure of the solution set to the impulsive Cauchy problem governed by a semilinear differential inclusion on noncompact intervals.

For a fixed $\tau > 0$ and a given piecewise continuous function $x : [-\tau, 0] \rightarrow E$, where E is a Banach space, the problem we deal with is

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + F(t, y_t), & \text{for a.a. } t \in [0, \infty), t \neq t_k, k \in \mathbb{N}, \\ y(t) = x(t), & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & \text{for } k \in \mathbb{N}^+, \end{cases} \quad (1)$$

where $\{A(t)\}_{t \in [0, \infty)}$ is a family of linear operators in E generating an evolution operator; F is an upper-Carathéodory map; $y_t(\theta) = y(t + \theta)$, $\theta \in [-\tau, 0]$; I_k are impulse functions, $k \in \mathbb{N}$, $y(t^+) = \lim_{s \rightarrow t^+} y(s)$ and the time sequence $(t_k)_{k \in \mathbb{N}}$ is an increasing sequence of given points in $[0, \infty)$ without accumulation points. Hence $y_t(\cdot)$ represents the history of the state from $t - \tau$ to the present time t .

For the first time differential equations with impulses were investigated by Milman and Myshkis [1]. Impulsive differential equations and inclusions have applications in biology, economics, medicine, physics and other fields. These problems often describe phenomenas in which states are changing rapidly. One of the example is the motion of an elastic ball bouncing vertically on a surface. The moments of the impulses are in the time when the ball touches the surface and rapidly its velocity is changed. The moments of the impulse effects for the impulsive problems can be chosen in various ways: randomly, fixed beforehand, determined by the state of a system. For some recent works on impulsive differential problems, concerning the aspects we deal with, we refer to [2–6].

The solution sets for differential problems often correspond with fixed point sets of multivalued operators in function spaces. In this paper we use the inverse system method, which, in studying the topological structure of fixed point sets of operators in function spaces, was initiated in [7]. This method was developed in [8] and then also in [9]. It is observed that differential problems on noncompact intervals can be reformulated as fixed point problems in Fréchet (function) spaces which are inverse limits of Banach spaces that appear when we consider these differential problems on compact intervals. Some interesting properties of fixed point sets of limit maps become very useful.

The existence of mild solutions for problem (1) has been obtained in [2]. We state and prove the compactness of the solution set for this problem. Next we prove that the set of solutions to problem (1) is an R_δ -set.

This gives an important information from the topological point of view. The translation operator along trajectories which is often used to detect, for instance, periodic solutions, being R_δ -valued can be checked to be an admissible (in the sense of Górniewicz) multivalued operator, and the fixed point theory methods can be applied.

The paper is organized as follows. In Sect. 2 we recall useful definitions and preliminary theorems. In the main Sect. 3 we obtain new results. In Theorem 3.4 the compactness of the solution set is proved on the half-line. The result improves Theorem 4.2 in [2], where only the existence of solutions was shown. Our proof is essentially shorter and it shows how one can effectively use the inverse systems technique. Theorems 3.5 and 3.7 are the main results of the paper. We prove the R_δ -structure of the solution set. Note that in [4] it was shown that the solution set for the impulsive problem on compact intervals is an R_δ -set if F is a σ -Ca-selectionable multivalued map and $A(t) = A$ is the infinitesimal generator of a C_0 -semigroup. The problem is that it is not clear if a sufficiently good σ -Ca-selectionability is possible in infinite dimensional spaces. In fact, as we show in the proof of Theorem 3.5, we can approximate the right-hand side of the inclusion by maps which have noncompact values and which are not k -set contractions. Therefore, we propose different arguments to avoid the obstacles and to prove the R_δ -structure on compact intervals. Finally, we combine the information on a topological structure of solution sets on compact intervals with the inverse systems technique to obtain an R_δ -structure on the half-line in Theorem 3.7. In this way we essentially

develop some recent results in [5], where an R_δ -structure of the solution set for the multivalued impulsive differential inclusion on the half-line is shown only in a finite dimensional case, where the compactness properties become much easier, and for the problem without any retard.

2. Preliminaries

Let X, Y be two topological vector spaces. We denote by $P(Y)$ the family of all nonempty subsets of Y and put $P_{cl}(Y) = \{A \in P(Y), \text{closed}\}$, $P_{cl,cv}(Y) = \{A \in P(Y), \text{closed and convex}\}$, $P_{cp}(Y) = \{A \in P(Y), \text{compact}\}$, $P_{cp,cv}(Y) = \{A \in P(Y), \text{compact and convex}\}$.

A multivalued map $F : X \rightarrow P(Y)$ is said to be *upper semicontinuous* (for short u.s.c.) if $F^{-1}(V) = \{x \in X \mid F(x) \subset V\}$ is an open subset of X for every open $V \subseteq Y$. A multivalued map $F : X \rightarrow P(Y)$ is said to be *lower semicontinuous* (for short l.s.c.) if $F_+^{-1}(V) = \{x \in X \mid F(x) \cap V \neq \emptyset\}$ is an open subset of X for every open $V \subseteq Y$. We say that a multivalued map $F : X \rightarrow P(Y)$ is *continuous* provided it is both u.s.c. and l.s.c.

Let (X, d) be a metric space and $BC(X)$ denote the family of all nonempty closed bounded subsets of X . For given $A, B \in BC(X)$ let:

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset O_\epsilon(B) \text{ and } B \subset O_\epsilon(A)\},$$

where $O_\epsilon(A) = \{x \in X \mid \text{dist}(x, A) < \epsilon\}$. Observe that

$$d_H(A, B) = \max\{\sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A)\}.$$

The function $d_H : BC(X) \times BC(X) \rightarrow \mathbb{R}_+$ is a metric on $BC(X)$ and is called the *Hausdorff distance*.

Let E be a Banach space and $B(E)$ the family of all bounded subsets of E . Then the function $\beta : B(E) \rightarrow \mathbb{R}_+$ defined by:

$$\beta(A) = \inf\{r > 0 \mid A \text{ can be covered by finitely many balls of radius } r\}$$

is called the (Hausdorff) *measure of noncompactness*. It is monotone, nonsingular, real and regular (see [10]).

Let $I \subset \mathbb{R}$ be a compact interval, μ be a Lebesgue measure on I and E be a Banach space. A multivalued map $F : I \rightarrow P_{cp}(E)$ is said to be *measurable* (resp. *weakly measurable*) if for every open (resp. closed) subset $V \subset E$ the set $F^{-1}(V)$ is measurable.

A multivalued map $F : I \rightarrow P_{cp}(E)$ is said to be *strongly measurable* if there exists a sequence $\{F_n\}_{n=1}^\infty$ of step multivalued maps such that $d_H(F_n(t), F(t)) \rightarrow 0$ as $n \rightarrow \infty$ for μ -a.e. $t \in I$.

By the symbol $L^1([a, b], E)$ we denote the space of all Bochner integrable functions. For simplicity of notations, we write $L^1([a, b])$ instead of $L^1([a, b], \mathbb{R})$. Let us denote by $L_{loc}^1([0, \infty), E)$ the set of all Bochner integrable functions on compact subsets of $[0, \infty)$.

Let E be a Banach space and $F : I \rightarrow P(E)$ be a multivalued map, where $I \subset \mathbb{R}$ is a compact interval. It is known that if $F : I \rightarrow P_{cp}(E)$ is strongly

measurable and integrably bounded, i.e., there exists $\alpha \in L^1(I)$ such that

$$\|F(t)\| := \max\{\|y\| \mid y \in F(t)\} \leq \alpha(t)$$

for a.e. $t \in I$, then there exists a Bochner integrable selector f of F , i.e., $f(t) \in F(t)$ for a.e. $t \in I$.

We say that a family $\mathcal{V} \subset L^1(I, E)$ is integrably bounded if $V : I \rightarrow P(E)$ given by $V(t) = \{v(t) \mid v \in \mathcal{V}\}$ is integrably bounded.

Theorem 2.1. (see [10], Proposition 4.2.1.) *Let E be a Banach space and $\mathcal{V} \subset L^1([a, b], E)$ be integrably bounded. Assume that the sets $V(t)$ are relatively compact for a.e. $t \in [a, b]$. Then $\mathcal{V}$ is weakly compact in $L^1([a, b], E)$.*

We denote by $\mathcal{C}([-\tau, 0], E)$ the space of piecewise continuous functions $c : [-\tau, 0] \rightarrow E$ with finite number of discontinuity points $\{t_*\}$ such that $t_* \neq 0$ and all values

$$c(t_*^+) = \lim_{h \rightarrow 0^+} c(t_* + h) \text{ and } c(t_*^-) = \lim_{h \rightarrow 0^-} c(t_* + h)$$

are finite. We consider the space $\mathcal{C}([-\tau, 0], E)$ with the L^1 -norm, i.e.,

$$\|c\|_c = \int_{-\tau}^0 \|c(t)\| dt.$$

We do not consider the space $\mathcal{C}([-\tau, 0], E)$ with the uniform convergence norm, because it creates problems: the function $t \in [0, \infty) \rightarrow y_t$ is not continuous, moreover, it is not necessarily measurable (see Example 3.1, [11]). As a consequence, the multivalued superposition operator, which we will define in Sect. 3, would not be well defined. This space of delays with the integral norm was considered in [12] (see also [6]).

We denote by $PC([a, b], E)$ the space of piecewise continuous functions $c : [a, b] \rightarrow E$ with finite number of discontinuity points $\{t_*\}$ and such that

$$c(t_*^+) = \lim_{h \rightarrow 0^+} c(t_* + h) \text{ and } \lim_{h \rightarrow 0^-} c(t_* + h) = c(t_*)$$

are finite. The space $PC([a, b], E)$ is a Banach space with the norm:

$$\|c\|_{PC} = \sup\{\|c(t)\| \mid t \in [a, b]\}$$

and a space of continuous functions $C([a, b], E)$ is a closed subspace of it.

We denote by $PC([a, \infty), E)$ the space of piecewise continuous functions $c : [a, \infty) \rightarrow E$ with infinite number of discontinuity points $t_1, t_2, \dots$ such that $\lim_{n \rightarrow \infty} t_n = +\infty$. The values $c(t_i^-)$, $c(t_i^+)$ for $i = 1, 2, \dots$ are finite and $c(t_i^-) = c(t_i)$. The space $PC([a, \infty), E)$ is a Fréchet space with the family of seminorms $\{p_n\}$ given by:

$$p_n(c) = \|c|_{[0, t_n]}\|_{PC}$$

and a metric:

$$d(c_1, c_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(c_1 - c_2)}{1 + p_n(c_1 - c_2)}.$$

Let us recall that by a *Fréchet space* we mean a locally convex space which is metrizable and complete. Every Banach space is a Fréchet space.

Recall that a subset $A \subset X$ is called a *retract* of X if there exists a continuous function $r : X \rightarrow A$, such that $r(x) = x$, for every $x \in A$. A is called a *neighborhood retract* of X if there exists an open subset $U \subset X$ such that $A \subset U$ and A is retract of U . If we have two spaces X, Y , then every homeomorphism $h : X \rightarrow Y$ such that $h(X)$ is a closed subset of Y is called an *embedding*. We say that X is an *absolute retract* (is an *absolute neighborhood retract*) if and only if for any space Y and for any embedding $h : X \rightarrow Y$ the set $h(X)$ is a retract of Y ($h(X)$ is a neighborhood retract of Y). We write $X \in AR$ (resp. $X \in ANR$).

A compact nonempty space is called an R_δ -set provided there exists a decreasing sequence $\{A_n\}$ of compact absolute retracts such that:

$$A = \bigcap_{n \geq 1} A_n.$$

Any intersection of decreasing sequence of R_δ -sets is R_δ .

The following characterization of R_δ -sets, which develops the well-known Hyman's theorem [13], was shown by D. Bothe.

Theorem 2.2. (see [14]) *Let X be a complete metric space, β denote the Hausdorff measure of noncompactness in X and let $\emptyset \neq A \subset X$.*

Then the following statements are equivalent:

- (a) *A is an R_δ -set,*
- (b) *A is an intersection of a decreasing sequence $\{A_n\}$ of closed contractible spaces with $\beta(A_n) \rightarrow 0$,*
- (c) *A is compact and absolutely neighborhood contractible, i.e., A is contractible in each neighborhood in $Y \in ANR$, where A is embedded.*

Let us recall that an *inverse system* of topological spaces is a family $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$, where Σ is a directed set ordered by the relation $\leq$, X_α is a topological (Hausdorff) space, for every $\alpha \in \Sigma$, and $\pi_\alpha^\beta : X_\beta \rightarrow X_\alpha$ is a continuous mapping for each two elements $\alpha, \beta \in \Sigma$ such that $\alpha \leq \beta$. Moreover, for each $\alpha \leq \beta \leq \gamma$, the following conditions should hold: $\pi_\alpha^\alpha = id_{X_\alpha}$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$.

A subspace of the product $\prod_{\alpha \in \Sigma} X_\alpha$ is called a *limit of the inverse system* S and it is denoted by $\varprojlim S$ whenever

$$\varprojlim S = \left\{ (x_\alpha) \in \prod_{\alpha \in \Sigma} X_\alpha \mid \pi_\alpha^\beta(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta \right\}.$$

Consider two inverse systems $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$. Let us recall (see [8]) that by a *multivalued map of the system S into the system S'* , we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma : \Sigma' \rightarrow \Sigma$, i.e. $\sigma(\alpha') \leq \sigma(\beta')$ for $\alpha' \leq \beta'$, and of multivalued maps $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')} \multimap Y_{\alpha'}$ with nonempty values, defined for every $\alpha' \in \Sigma'$ and such that

$$\pi_{\alpha'}^{\beta'} \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}^{\sigma(\beta')},$$

for each $\alpha' \leq \beta'$.

A map of systems $\{\sigma, \varphi_{\sigma(\alpha')}\}$ induces a *limit map* $\varphi : \lim_{\leftarrow} S \rightarrow \lim_{\leftarrow} S'$ defined as follows:

$$\varphi(x) = \prod_{\alpha' \in \Sigma'} \varphi_{\sigma(\alpha')}(x_{\sigma(\alpha')}) \cap \lim_{\leftarrow} S'.$$

In other words, a limit map is the one such that

$$\pi_{\alpha'} \varphi = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')},$$

for every $\alpha' \in \Sigma'$.

Now we summarize some useful properties of limits of inverse systems.

Proposition 2.3. (see [15]) *Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system. If, for every $\alpha \in \Sigma$, X_α is compact and nonempty, then $\lim_{\leftarrow} S$ is compact and nonempty.*

Proposition 2.4. (see [7]) *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system. If, for every $n \in \mathbb{N}$, X_n is an R_δ -set, then $\lim_{\leftarrow} S$ is R_δ , as well.*

Theorem 2.5. (see [8]) *Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system and $\varphi : \lim_{\leftarrow} S \rightarrow \lim_{\leftarrow} S$ be a limit map induced by a map $\{id, \varphi_\alpha\}$, where $\varphi_\alpha : X_\alpha \rightarrow X_\alpha$. Then the fixed point set of φ is a limit of the inverse system generated by the sets $Fix(\varphi_\alpha)$. In particular, if the sets $Fix(\varphi_\alpha)$ are compact acyclic [resp. R_δ], then it is compact acyclic [resp. R_δ], as well.*

3. Structure of the solution set on noncompact intervals

We consider the set $\Delta_\infty = \{(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid 0 \leq s \leq t\}$ and the evolution system $\{T(t, s)\}_{(t, s) \in \Delta_\infty}$. Let us recall that a two parameter family $\{T(t, s)\}_{(t, s) \in \Delta_\infty}$, where $T(t, s) : E \rightarrow E$ is a bounded linear operator, is an *evolution system* if the following conditions are satisfied:

- (a) $T(t, r)T(r, s) = T(t, s)$, for $0 \leq s \leq r \leq t$
- (b) $T(t, t) = Id$ for $t \in [0, \infty)$,
- (c) $(t, s) \rightarrow T(t, s)$ is strongly continuous on Δ_∞ , i.e. the map $(t, s) \rightarrow T(t, s)x$ is continuous on Δ_∞ for every $x \in E$.

We assume

- (A) $^\infty$ $\{A(t)\}_{t \in [0, \infty)}$ is a family of linear not necessarily bounded operators $A(t) : D(A) \subset E \rightarrow E, t \in [0, \infty)$, $D(A)$ a dense subset of E not depending on t generating an evolution operator $T : \Delta_\infty \rightarrow \mathcal{L}(E)$, where $\mathcal{L}(E)$ is the space of all bounded linear operators in E .

The generalized Cauchy operator $G^k : L^1([t_{k-1}, t_k], E) \rightarrow C([t_{k-1}, t_k], E)$ is defined by

$$G^k f(t) = \int_{t_{k-1}}^t T(t, s) f(s) ds, \quad t \in [t_{k-1}, t_k].$$

Proposition 3.1. (see [3], Theorem 2) *The generalized Cauchy operator G^k satisfies the properties:*

(G1) *there exists $c_k \geq 0$ such that*

$$\|G^k f(t) - G^k g(t)\| \leq c_k \int_{t_{k-1}}^t \|f(s) - g(s)\| ds, \quad t \in [t_{k-1}, t_k]$$

for every $f, g \in L^1([t_{k-1}, t_k], E)$,

(G2) *for any compact $K \subset E$ and sequence $(f_n(t))_{n=1}^\infty, f_n \in L^1([t_{k-1}, t_k], E)$, such that $(f_n(t))_{n=1}^\infty \subset K$ for almost every $t \in [t_{k-1}, t_k]$, the weak convergence $f_n \rightharpoonup f_0$ implies the convergence $G^k f_n \rightarrow G^k f_0$.*

Theorem 3.2. (see [10], Proposition 4.2.2.) *Let the operator G^k satisfy conditions (G1) and (G2) and let the set $\{f_n\}_{n=1}^\infty$ be integrably bounded with the property $\beta(f_n(t))_{n=1}^\infty \leq \eta(t)$ for almost every $t \in [t_{k-1}, t_k]$, where $\eta(\cdot) \in L^1([t_{k-1}, t_k])$. Then*

$$\beta(G^k f_n(t))_{n=1}^\infty \leq 2D \int_{t_{k-1}}^t \eta(s) ds, \quad t \in [t_{k-1}, t_k],$$

where $D \geq 0$ is an upper bound for norms $\|T(t, s)\|$ on $[t_{k-1}, t_k]$.

We consider $F : [0, \infty) \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp, cv}(E)$ such that

(F1) $^\infty$ $F(\cdot, c)$ has a strongly measurable selection for every $c \in \mathcal{C}([-\tau, 0], E)$,

(F2) $^\infty$ $F(t, \cdot)$ is u.s.c. for a.e. $t \in [0, \infty)$,

(F3) $^\infty$ F for almost every $t \in [0, \infty)$ has at most a linear growth, i.e., there exists a function $\alpha \in L^1_{loc}([0, \infty))$ such that

$$\|F(t, c)\| \leq \alpha(t)(1 + \|c\|_C) \text{ for a.e. } t \in [0, \infty),$$

(F4) $^\infty$ There exists a function $\mu \in L^1_{loc}([0, \infty))$ such that

$$\beta(F(t, D)) \leq \mu(t) \sup_{-\tau \leq \theta \leq 0} \beta(D(\theta)) \text{ for a.e. } t \in [0, \infty)$$

and for every bounded $D \subset \mathcal{C}([-\tau, 0], E)$, $D(\theta) = \{c(\theta) | c \in D\}$.

The following problem on a compact interval

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + F(t, y_t), & \text{for a.e. } t \in [0, t_m], t \neq t_k, k < m, \\ y(t) = x(t), & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & \text{for } k < m \end{cases} \quad (2)$$

was considered in [2]. The authors proved the existence of a mild solution and that the solution set S_m for this problem is compact.

A piecewise continuous function $y : [-\tau, t_m] \rightarrow E$ is a *mild solution* for the impulsive Cauchy problem (2) if

- (a) $y(t) = T(t, 0)x(0) + \sum_{0 < t_k < t} T(t, t_k)I_k(y_{t_k}) + \int_0^t T(t, s)f(s)ds, t \in [0, t_m]$, where $f \in L^1([0, t_m], E)$, $f(s) \in F(s, y_s)$ for almost every $s \in [0, t_m]$,
- (b) $y(t) = x(t), t \in [-\tau, 0]$,
- (c) $y(t_k^+) = y(t_k) + I_k(y_{t_k}), k < m$.

Let $[0, t_m]$ be a fixed interval on the real line. Put $\Delta_m = \{(t, s) \in [0, t_m] \times [0, t_m] | s \leq t \leq t_m\}$. The authors of [2] obtained the compactness of the solution set of problem (2) under the following assumptions:

$(A)_m$ $\{A(t)\}_{t \in [0, t_m]}$ is a family of linear (not necessarily bounded) operators $(A(t) : D(A) \subset E \rightarrow E, t \in [0, t_m], D(A)$ a dense subset of E not depending on t) generating an evolution operator $T : \Delta_m \rightarrow \mathcal{L}(E)$,

and the map $F : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp, cv}(E)$ satisfies:

$(F1)_m$ $F(\cdot, c)$ has a strongly measurable selection for every $c \in \mathcal{C}([-\tau, 0], E)$,

$(F2)_m$ $F(t, \cdot)$ is u.s.c. for a.e. $t \in [0, t_m]$,

$(F3)_m$ F for almost every $t \in [0, t_m]$ has at most a linear growth, i.e., there exists a function $\alpha \in L^1([0, t_m])$ such that

$$\|F(t, c)\| \leq \alpha(t)(1 + \|c\|_C) \text{ for a.e. } t \in [0, t_m],$$

$(F4)_m$ There exists a function $\mu \in L^1([0, t_m])$ such that

$$\beta(F(t, D)) \leq \mu(t) \sup_{-\tau \leq \theta \leq 0} \beta(D(\theta)) \text{ for a.e. } t \in [0, t_m]$$

and for every bounded $D \subset \mathcal{C}([-\tau, 0], E)$, $D(\theta) = \{c(\theta) \mid c \in D\}$.

For $z \in PC([0, t_m], E)$ we can define $z_i \in C([t_i, t_{i+1}], E)$, $i = 0, 1, \dots, m-1$ as $z_i(t) = z(t)$ on $(t_i, t_{i+1}]$ and $z_i(t_i) = z(t_i^+)$. For every set $K \subset PC([0, t_m], E)$ we denote by K_i , $i = 0, 1, \dots, m-1$ the set $K_i = \{z_i \mid z \in K\}$. It is easy to see that

Proposition 3.3. *A set $K \in PC([0, t_m], E)$ is relatively compact in $PC([0, t_m], E)$ if and only if each set K_i , $i = 0, 1, \dots, m-1$ is relatively compact in $C([t_i, t_{i+1}], E)$.*

For any $z \in PC([0, t_m], E)$, or $z \in PC([0, \infty), E)$, such that $z(0) = x(0)$ we define the function $z[x] : [-\tau, t_m] \rightarrow E$ as

$$z[x] = \begin{cases} x(t), & t \in [-\tau, 0], \\ z(t), & t \in [0, t_m]. \end{cases}$$

where $x : [-\tau, 0] \rightarrow E$ is the function from the initial condition in (1). We denote $\Omega[x] = \{z[x] \mid z \in \Omega\}$.

For a given multivalued map $F : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp, cv}(E)$ satisfying $(F1)_m - (F4)_m$ we consider the multivalued superposition operator $\mathcal{P}_F : \mathcal{D} \rightarrow P(L^1([0, t_m], E))$ defined as

$$\mathcal{P}_F(z) = \{f \in L^1([0, t_m], E) \mid f(s) \in F(s, z[x]_s) \text{ for a.e. } s \in [0, t_m]\}. \quad (3)$$

This multivalued superposition operator $\mathcal{P}_F$ is well defined (see e.g. [10]). Notice that the function $s \in [0, t_m] \rightarrow z[x]_s \in \mathcal{C}([-\tau, 0], E)$ is continuous.

Now, we prove the compactness of the solution set of problem (1). Note that we simultaneously obtain a nonemptiness of the solution set and our proof is essentially shorter than the one in [2].

Theorem 3.4. *Let hypothesis $(A)^\infty$ hold, let the multivalued map $F : [0, \infty) \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp, cv}(E)$ satisfy conditions $(F1)^\infty - (F4)^\infty$ and maps $I_k : \mathcal{C}([-\tau, 0], E) \rightarrow E$, $k \in \mathbb{N}$, be continuous. Then the solution set for problem (1) is a nonempty and compact subset of $PC([0, \infty), E)[x]$.*

Proof. Besides (1), for every $m \in \mathbb{N}^+$, we consider problem (2) on a compact interval $[0, t_m]$.

Let $C_m = PC([0, t_m], E)[x]$. Consider the sequence of multivalued maps $\phi_m : C_m \multimap C_m$ as follows:

$$\phi_m(y)(t) := \left\{ T(t, 0)x(0) + \sum_{0 < t_k < t} T(t, t_k)I_k(y_{t_k}) + \int_0^t T(t, s)f(s)ds, t \in [0, t_m] \right. \\ \left. \mid f \in L^1([0, t_m], E), f(s) \in F(s, y_s) \text{ for a.e. } s \in [0, t_m] \right\}$$

for $t \in [0, t_m]$ and $\phi_m(y)(t) = x(t)$ for $t \in [-\tau, 0]$. Now we consider the projections $p_m^{m+1} : C_{m+1} \rightarrow C_m$, which are defined as follows $p_m^{m+1}(y) = y|_{[-\tau, t_m]}$.

We have the equalities

$$\phi_m p_m^{m+1}(y)(t) = \left\{ T(t, 0)x(0) + \sum_{0 < t_k < t} T(t, t_k)I_k(y_{t_k}) + \int_0^t T(t, s)f(s)ds \right. \\ \left. \mid t \in [0, t_m], f \in L^1([0, t_m], E), f(s) \in F(s, y_s) \text{ for a.e. } s \in [0, t_m] \right\}$$

$$p_m^{m+1}\phi_{m+1}(y)(t) = \left\{ T(t, 0)x(0) + \sum_{0 < t_k < t} T(t, t_k)I_k(y_{t_k}) + \int_0^t T(t, s)f(s)ds \right. \\ \left. \mid t \in [0, t_m], f \in L^1([0, t_{m+1}], E), f(s) \in F(s, y_s) \text{ for a.e. } s \in [0, t_{m+1}] \right\},$$

and from the observation that

$$\{f \in L^1([0, t_m], E) \mid f(s) \in F(s, y_s) \text{ for a.e. } s \in [0, t_m]\} \\ = \{f|_{[-\tau, t_m]}, f \in L^1([0, t_{m+1}], E) \mid f(s) \in F(s, y_s) \text{ for a.e. } s \in [0, t_{m+1}]\}$$

we obtain that $\phi_m p_m^{m+1} = p_m^{m+1}\phi_{m+1}$, so $\{id, \phi_m\}$ is the map of the inverse system $\{C_m, p_m^n\}$. The map $\{id, \phi_m\}$ induces the limit map $\phi : C \multimap C$, where $C = PC([0, \infty), E)[x]$

$$\phi(y)(t) = \left\{ T(t, 0)x(0) + \sum_{0 < t_k < t} T(t, t_k)I_k(y_{t_k}) + \int_0^t T(t, s)f(s)ds, t \in [0, \infty) \right. \\ \left. \mid f \in L_{loc}^1([0, \infty), E), f(s) \in F(s, y_s) \text{ for a.e. } s \in [0, \infty) \right\}$$

for $t \in [0, t_m]$ and $\phi(y)(t) = x(t)$ for $t \in [-\tau, 0]$. Note that $S := Fix(\phi) = \lim_{\leftarrow} S_m$ is the solution set of problem (1). It is known (see [2], Theorem 3.7) that for every $m \geq 1$ the solution set S_m to (2) is a nonempty and compact subset of $PC([0, t_m], E)[x]$. Using Proposition 2.3 we see that the set S is nonempty and compact. $\square$

Now we prove the results about an R_δ -structure of the solution set. At first we examine the case of problems on compact intervals. In the next step we deal with the problem on the half-line.

Theorem 3.5. *Let E be a Banach space and let hypothesis $(A)_m$ hold. Suppose that the multivalued map $F : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp, cv}(E)$ satisfies conditions $(F1)_m$ – $(F4)_m$. Moreover, assume that the maps $I_k : \mathcal{C}([-\tau, 0], E) \rightarrow E, k \in \mathbb{N}$, are continuous and there exist constants $r_k > 0$ such that,*

(I1)

$$\beta(I_k(D)) \leq r_k \sup_{-\tau \leq \theta \leq 0} \beta(D(\theta)),$$

(I2) *for every bounded $D \subset \mathcal{C}([-\tau, 0], E)$,*

$$\sum_{k=1}^m r_k < \frac{1}{B_m},$$

where $B_m = \sup_{(t,s) \in \Delta_m} \|T(t, s)\|_{\mathcal{L}(E)}$.

Then the solution set for problem (2) is an R_δ -set in $PC([0, t_m], E)[x]$.

Note that, since the evolution operator T is strongly continuous on the compact set Δ_m , the number B_m is finite, that is, $B_m < \infty$.

Proof. We will proceed in several steps.

Step 1. Consider the non-impulsive Cauchy problem

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + F(t, y_t), & \text{for a.a. } t \in [0, t_1], \\ y(t) = x(t), & \text{for } t \in [-\tau, 0]. \end{cases} \quad (4)$$

In [2] it was shown that solutions of (4) are bounded (by some $\bar{K}_1 \geq 0$). We show that solutions on the interval $[t_1, t_2]$ are bounded by $\bar{K}_2$. To do this we consider the non-impulsive Cauchy problems

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + F(t, y_t), & \text{for a.a. } t \in [t_1, t_2], \\ y(t) = z^1(t), & \text{for } t \in [-\tau, t_1], \\ y(t_1^+) = z^1(t_1) + I_1(z_{t_1}^1). \end{cases} \quad (5)$$

Here the function z^1 is any solution on the $[-\tau, t_1]$. The mild solutions for (5) have the forms:

$$y(t) = T(t, t_1)z^1(t_1) + T(t, t_1)I_1(z_{t_1}^1) + \int_{t_1}^t T(t, s)f(s)ds,$$

where $f \in L^1([t_1, t_2], E)$, $f(s) \in F(s, y_s)$, $t \in [t_1, t_2]$.

We have

$$\begin{aligned} \|y(t)\| &\leq B_m \|z^1(t_1)\| + B_m \|I_1(z_{t_1}^1)\| + B_m \int_{t_1}^t \|f(s)\| ds \\ &\leq B_m(\bar{K}_1 + \bar{R}) + B_m \int_{t_1}^t \alpha(s)(1 + \|y_t\|_{\mathcal{C}}) ds \\ &\leq B_m(\bar{K}_1 + \bar{R} + \|\alpha\|_{L^1[t_1, t_2]}) + B_m \int_{t_1}^t \alpha(s) \left(\int_{-\tau}^0 \|y_s(\theta)\| d\theta \right) ds, \end{aligned}$$

where $\bar{R}$ is a common upper bound for $\|I_1(z_{t_1})\|$, where z is any solution on $[-\tau, t_1]$, which exists because the solution set for (4) is compact and I_1 is continuous.

So, we have

$$\begin{aligned} \|y(t)\| &\leq \bar{M}_2 + B_m \int_{t_1}^t \alpha(s) \left(\int_{-\tau}^0 \|y_s(\theta)\| d\theta \right) ds \\ &\leq \bar{M}_2 + B_m \int_{t_1}^t \alpha(s) \tau \cdot \sup_{-\tau \leq \theta \leq 0} \|y_s(\theta)\| ds \\ &\leq \bar{M}_2 + B_m \int_{t_1}^t \alpha(s) \tau \cdot \sup_{-\tau \leq \theta \leq 0} \|y(s+\theta)\| ds \\ &\leq \bar{M}_2 + B_m \int_{t_1}^t \alpha(s) \tau \cdot \sup_{s-\tau \leq \theta \leq s} \|y(\theta)\| ds \\ &\leq \bar{M}_2 + B_m \int_{t_1}^t \alpha(s) \tau \cdot \sup_{-\tau \leq \theta \leq s} \|y(\theta)\| ds \end{aligned}$$

where $\bar{M}_2 \geq B_m(\bar{K}_1 + \bar{R} + \|\alpha\|_{L^1[t_1, t_2]})$.

The right hand side is an increasing function in t , so we have the same estimate for all $t_1 < r \leq t$. Therefore

$$\sup_{t_1 < r \leq t} \|y(r)\| \leq \bar{M}_2 + \tau B_m \int_{t_1}^t \alpha(s) \sup_{-\tau \leq \sigma \leq s} \|y(\sigma)\| ds.$$

Since $\|y(s)\| \leq \bar{M}_2$ for $s < t_1$, we obtain

$$\sup_{-\tau < r \leq t} \|y(r)\| \leq \bar{M}_2 + \tau B_m \int_{t_1}^t \alpha(s) \sup_{-\tau \leq \sigma \leq s} \|y(\sigma)\| ds.$$

The function $\psi_2(t) = \sup_{-\tau < r \leq t} \|y(r)\|$ is piecewise continuous.

For the function

$$v(t) = \int_{t_1}^t \alpha(s) \psi_2(s) ds$$

we have $v(t_1) = 0$ and $v'(t) = \alpha(t) \psi_2(t)$.

We obtained the estimate $\psi_2(t) \leq \bar{M}_2 + \tau B_m v(t)$, so

$$v'(t) \leq \alpha(t) (\bar{M}_2 + \tau B_m v(t)).$$

Next we multiply the above inequality by $e^{-\tau B_m \int_{t_1}^t \alpha(s) ds}$

$$v'(t) e^{-\tau B_m \int_{t_1}^t \alpha(s) ds} \leq e^{-\tau B_m \int_{t_1}^t \alpha(s) ds} \alpha(t) (\bar{M}_2 + \tau B_m v(t)).$$

Hence

$$(v(t) e^{-\tau B_m \int_{t_1}^t \alpha(s) ds})' \leq e^{-\tau B_m \int_{t_1}^t \alpha(s) ds} \alpha(t) \bar{M}_2.$$

We integrate both sides of this inequality from t_1 to t_2 and we obtain

$$v(t_2) e^{-\tau B_m \int_{t_1}^{t_2} \alpha(s) ds} \leq \bar{M}_2 \int_{t_1}^{t_2} \alpha(t) e^{-\tau B_m \int_{t_1}^t \alpha(s) ds} dt,$$

so

$$v(t_2) \leq K_2,$$

where $K_2 = \frac{\bar{M}_2 \int_{t_1}^{t_2} \alpha(t) e^{-\tau B_m \int_{t_1}^t \alpha(s) ds} dt}{e^{-\tau B_m \int_{t_1}^{t_2} \alpha(s) ds}}$. The function v is nondecreasing, so

$$v(t) \leq K_2 \text{ for all } t \in [t_1, t_2],$$

hence

$$\|y(t)\| \leq \bar{M}_2 + \tau B_m K_2 := \bar{K}_2.$$

Without any loss of generality we can assume that $\bar{K}_2 \geq \bar{K}_1$. So, for every $k \geq 1$, every solution of the non-impulsive Cauchy problem

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + F(t, y_t), & \text{for a.a. } t \in [0, t_k], \\ y(t) = x(t), & \text{for } t \in [-\tau, 0] \\ y(t_i^+) = y(t_i) + I_i(y_{t_i}), & \text{for } i < k \end{cases} \quad (6)$$

is bounded by $\bar{K}_k \geq \bar{K}_{k-1}$. We define a mapping $\tilde{F} : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp, cv}(E)$,

$$\tilde{F}(t, c) = \begin{cases} F(t, c), & \text{if } t \in [0, t_m] \text{ and } \|c\|_C \leq \bar{K}_m, \\ F(t, \frac{\bar{K}_m c}{\|c\|_C}), & \text{if } t \in [0, t_m] \text{ and } \|c\|_C > \bar{K}_m. \end{cases} \quad (7)$$

The function $r : \mathcal{C}([-\tau, 0], E) \rightarrow clB(0, \bar{K}_m) \subset \mathcal{C}([-\tau, 0], E)$ given by the formula $r(c) = \frac{\bar{K}_m c}{\|c\|_C}$ for every $c \in \mathcal{C}([-\tau, 0], E)$ with $\|c\|_C > \bar{K}_m$ and $r(c) = c$ for c with $\|c\|_C \leq \bar{K}_m$ is a continuous retraction of $\mathcal{C}([-\tau, 0], E)$ onto a closed ball. Therefore $\tilde{F}(t, c) = F(t, r(c))$ and $\tilde{F}$ has the same measurability and continuity properties as F . For every $m \geq 1$ and $t \in [0, t_m]$ the following inequalities hold

$$\|\tilde{F}(t, c)\| = \|F(t, c)\| \leq \alpha(t)(1 + \|c\|_C) \leq \alpha(t)(1 + \bar{K}_m)$$

for every $\|c\|_C \leq \bar{K}_m$, and

$$\begin{aligned} \|\tilde{F}(t, c)\| &= \left\| F\left(t, \frac{\bar{K}_m c}{\|c\|_C}\right) \right\| \\ &\leq \alpha(t) \left(1 + \left\| \frac{\bar{K}_m c}{\|c\|_C} \right\|_C \right) \leq \alpha(t)(1 + \bar{K}_m) \end{aligned}$$

for every $\|c\|_C > \bar{K}_m$. So we have:

$$\|\tilde{F}(t, c)\| \leq \alpha(t)(1 + \bar{K}_m) \equiv \psi_m(t) \in L^1([0, t_m]).$$

Now we consider an impulsive problem for fixed m with a multivalued map $\tilde{F}$

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + \tilde{F}(t, y_t), & \text{for a.e. } t \in [0, t_m], t \neq t_k, k < m, \\ y(t) = x(t), & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & \text{for } k < m. \end{cases} \quad (8)$$

Let $\tilde{S}_m$ be the solution set of problem (8). If y is a solution of (2), i.e., $y \in S_m$, then $\|y\| \leq \bar{K}_m$. F and $\tilde{F}$ coincide on $clB(0, \bar{K}_m)$, so we have that $y \in \tilde{S}_m$. If $y \in \tilde{S}_m$, then we can easily see that $\|y\| \leq \bar{K}_m$ and $\dot{y}(t) \in A(t)y(t) + F(t, y_t)$

for a.e. $t \in [0, t_m]$, $t \neq t_k$, $k < m$ and that $y(t) = x(t)$ for $t \in [-\tau, 0]$, so $y \in S_m$. We have $S_m = \tilde{S}_m$. Consequently, we can assume from now on, without any loss of generality, that

$$(F3')_m \quad \|F(t, c)\| \leq \psi_m(t) \text{ for every } t \in [0, t_m], \text{ where } \psi_m(t) \in L^1([0, t_m]).$$

Step 2. Now we prove that there exists a sequence of multivalued maps $\{G_n\}_{n=1}^\infty$, $G_n : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cl,cv}(E)$ such that:

- (i) each multivalued map $G_n(t, \cdot) : \mathcal{C}([-\tau, 0], E) \rightarrow P_{cl,cv}(E)$, $n \geq 1$ is continuous for a.e. $t \in [0, t_m]$,
- (ii) $F(t, c) \subset \cdots \subset G_{n+1}(t, c) \subset G_n(t, c) \subset \overline{\text{conv}}F(t, B_{3d_n}(c))$, $n \geq 1$,
- (iii) $F(t, c) = \bigcap_{n \geq 1} G_n(t, c)$,
- (iv) for each $n \geq 1$ there exists a selection $g_n : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow E$ of G_n , such that $g_n(\cdot, c)$ is measurable and $g_n(t, \cdot)$ is locally Lipschitz.

Consider the sequence $d_n = \frac{1}{3^n}$, $n \geq 1$. Let us cover $\mathcal{C}([-\tau, 0], E)$ by the open balls $\{B_{d_n}(c)\}_{c \in PC([-\tau, 0], E)}$. Since the space $\mathcal{C}([-\tau, 0], E)$ is metric, there exists a locally finite refinement $\{V_j\}_{j \in J}$ of the cover $\{B_{d_n}(c)\}_{c \in \mathcal{C}([-\tau, 0], E)}$. Now, we can associate a locally Lipschitz partition of unity $\{p_j\}_{j \in J}$ subordinated to the open covering $\{V_j\}_{j \in J}$. For every $j \in J$ let c_j be such that $V_j \subset B_{d_n}(c_j)$ and define

$$G_n(t, c) = \sum_{j \in J} p_j(c) \cdot \overline{\text{conv}}F(t, B_{2d_n}(c_j)).$$

To prove (ii) and (iii) note that $p_j(c) > 0$ implies that $c \in V_j \subset B_{d_n}(c_j)$, hence $B_{2d_n}(c_j) \subset B_{3d_n}(c)$ and therefore

$$F(t, c) \subset G_n(t, c) \subseteq \overline{\text{conv}}F(t, B_{3d_n}(c)).$$

Now we prove that $F(t, c) \supset \bigcap_{n \geq 1} G_n(t, c)$, because is obvious.

Let U be an open and convex set such that $F(t, c) \subset U$. Then from u.s.c. there exists $\delta > 0$ such that, if $d(c, \bar{c}) < \delta$, then $F(t, \bar{c}) \subset U$. Hence, if $3d_n < \delta$ for almost every n , then

$$F(t, B_{3d_n}(c)) \subset U \implies \overline{\text{conv}}F(t, B_{3d_n}(c)) \subset \overline{\text{conv}}U \subset \bar{U}.$$

Let n_0 be such that $3d_{n_0} < \delta$.

$$\bigcap_{n \geq 1} G_n(t, c) \subset G_{n_0}(t, c) \subseteq \overline{\text{conv}}F(t, B_{3d_{n_0}}(c)) \subset \bar{U}.$$

Since U was arbitrary, we have $\bigcap_{n \geq 1} G_n(t, c) \subset \bigcap_{\mathcal{U}} \bar{U} = F(t, c)$, where $\mathcal{U}$ denotes a family of all open and convex subsets U such that $F(t, c) \subset U$. To prove (iv) we take, for every c_j , $j \in J$, a measurable selection g_j of the multivalued map $F(\cdot, c_j)$ and define $g_n : [0, t_m] \times \mathcal{C}([-\tau, 0], E) \rightarrow E$ as

$$g_n(t, c) = \sum_{j \in J} p_j(c) \cdot g_j(t).$$

Step 3. Now we consider the differential problem:

$$\begin{cases} \dot{y}(t) \in A(t)y(t) + G_n(t, y_t), & \text{for a.e. } t \in [0, t_m], t \neq t_k, k < m, \\ y(t) = x(t), & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & \text{for } k < m. \end{cases} \quad (9)$$

Let S_m^n denote the solution set of problem (9).

We show that each sequence $\{y_n\}$ such that $y_n \in S_m^n$ for all $n \geq 1$ has a convergent subsequence $y_{n_k} \rightarrow y \in S_m$.

At first we notice that $y_n(t) = x(t)$ for every $t \in [-\tau, 0]$ and

$$y_n(t) = T(t, 0)x(0) + \sum_{0 < t_k < t} T(t, t_k)I_k(y_{n_{t_k}}) + \int_0^t T(t, s)f_n(s)ds$$

for $t \in [0, t_m]$, where $f_n \in L^1([0, t_m], E)$ is such that $f_n(s) \in F(s, y_{n_s})$ for almost every $s \in [0, t_m]$.

Let $R > B_m = \sup_{(t,s) \in \Delta_m} \|T(t, s)\|_{\mathcal{L}(E)}$ be such that $\sum_{k=1}^m r_k + \frac{1}{R} < \frac{1}{B_m}$. We know that for every bounded linear operator $S : E \rightarrow E$ we have the property: $\beta(S\Omega) \leq \|S\|\beta(\Omega)$ for every $\Omega \in B(E)$. From this property (here $S = T(t, s)$) we have:

$$\beta(\{T(t, s)f_n(s)\}_{n \geq 1}) \leq B_m \beta(\{f_n(s)\}_{n \geq 1}).$$

For any $p \geq 1$ we obtain

$$\begin{aligned} \beta(\{f_n(s)\}_{n \geq 1}) &= \beta(\{f_n(s)\}_{n \geq p}) \leq \beta[F(\{s\} \times B(\{y_{n_s}\}_{n \geq p}, 3d_p))] \\ &\leq \mu(s) \cdot \sup_{-\tau \leq \theta \leq 0} \beta(B(\{y_n(s + \theta)\}_{n \geq p} + 3d_p)) \\ &\leq \mu(s) \left[\max\left(\sup_{-\tau \leq \sigma \leq 0} \beta(\{x(\sigma)\}), \sup_{0 \leq \sigma \leq s} \beta(\{y_n(\sigma)\}_{n \geq p})\right) + 3d_p \right] \\ &= \mu(s)(\bar{\rho}(s) + 3d_p), \end{aligned}$$

where $\bar{\rho}(s) = \sup_{0 \leq \sigma \leq s} \beta(\{y_n(\sigma)\}_{n \geq 1})$. We have

$$\begin{aligned} \beta\left(\left\{\sum_{0 < t_k < t} T(t, t_k)I_k(y_{n_{t_k}})\right\}_{n \geq 1}\right) &= \beta\left(\left\{\sum_{0 < t_k < t} T(t, t_k)I_k(y_{n_{t_k}})\right\}_{n \geq p}\right) \\ &\leq \sum_{0 < t_k < t} \|T(t, t_k)\| \cdot \beta(\{I_k(y_{n_{t_k}})\}_{n \geq p}) \\ &\leq B_m \sum_{0 < t_k < t} r_k \sup_{-\tau \leq \theta \leq 0} \beta(\{y_{n_{t_k}}(\theta)\}_{n \geq p}) \\ &\leq B_m \sum_{0 < t_k < t} r_k \cdot \max\left(\sup_{-\tau \leq \sigma \leq 0} \beta(\{x(\sigma)\}), \sup_{0 \leq \sigma \leq t_k} \beta(\{y_n(\sigma)\}_{n \geq 1})\right) \\ &= B_m \sum_{0 < t_k < t} r_k \cdot \sup_{0 \leq \sigma \leq t_k} \beta(\{y_n(\sigma)\}_{n \geq 1}) \\ &= B_m \sum_{0 < t_k < t} r_k \cdot \bar{\rho}(t_k). \end{aligned}$$

Now

$$\begin{aligned} & \beta(\{y_n(t)\}_{n \geq 1}) \\ &= \beta \left(\{T(t, 0)x(0) + \sum_{0 < t_k < t} T(t, t_k)I_k(y_{n_{t_k}}) + \int_0^t T(t, s)f_n(s)ds\}_{n \geq 1} \right) \\ &\leq B_m \sum_{0 < t_k < t} r_k \cdot \bar{\rho}(t_k) + 2B_m \int_0^t \mu(s)(\bar{\rho}(s) + 3d_p)ds \end{aligned}$$

for every $p \geq 1$. Since $d_p \searrow 0$ as $p \rightarrow \infty$, we obtain

$$\begin{aligned} \bar{\rho}(t) &\leq B_m \sum_{0 < t_k < t} r_k \cdot \bar{\rho}(t_k) + 2B_m \int_0^t \mu(s)\bar{\rho}(s)ds \\ &\leq \left(1 - \frac{B_m}{R}\right) \bar{\rho}(t) + 2B_m \int_0^t \mu(s)\bar{\rho}(s)ds \end{aligned}$$

and, consequently,

$$\frac{B_m}{R} \bar{\rho}(t) \leq 2B_m \int_0^t \mu(s)\bar{\rho}(s)ds.$$

Thus

$$\bar{\rho}(t) \leq 2R \int_0^t \mu(s)\bar{\rho}(s)ds.$$

By the Gronwall inequality we get $\bar{\rho}(t) = 0$ and, as a consequence,

$$\beta(\{y_n(t)\}_{n \geq 1}) = 0.$$

This also implies that $\beta(\{f_n(s)\}_{n \geq 1}) = 0$.

For $t \leq t'$ in $[0, t_1]$ we have

$$\|y_n(t') - y_n(t)\| \leq \int_t^{t'} B_1 \psi_m(s)ds,$$

so, the sequence $\{y_n\}$ is equicontinuous. This implies the existence of a subsequence $\{y_{n_l}\}$ which is convergent on $[0, t_1]$.

Define $y_{n_l}^1(t) = T(t, 0)x(0) + T(t, t_1)I_1(y_{n_{t_1}}) + \int_0^t T(t, s)f_{n_l}(s)ds$ for $t \geq t_1$. Notice that $y_{n_l}^1(t_1^+) = y_{n_l}(t_1)$ for every $l \geq 1$. For $t \leq t'$ in $[t_1, t_2]$ we have, as before,

$$\|y_{n_l}^1(t') - y_{n_l}^1(t)\| \leq \int_t^{t'} B_2 \psi_m(s)ds,$$

and $\{y_{n_l}^1\}$ is equicontinuous. So, we can choose a convergent subsequence $\{y_{n_{l_s}}^1\}$ on $[t_1, t_2]$. We glue functions

$$y_{n_{l_s}}(t) = \begin{cases} y_{n_l}, & \text{for } t \leq t_1, \\ y_{n_{l_s}}^1, & \text{for } t > t_1. \end{cases}$$

We proceed up to m and find a convergent subsequence of $\{y_n\}$. Denote the limit by y .

Since $\beta(\{f_n(s)\}_{n \geq 1}) = 0$, we can assume, up to subsequence, that $f_n \rightarrow f_0 \in L^1([0, t_m], E)$. Therefore, since the impulse maps are continuous,

$$y(t) = T(t, 0)x(0) + \sum_{0 < t_k < t} T(t, t_k)I_k(y_{t_k}) + \int_0^t T(t, s)f_0(s)ds.$$

To prove that $f_0(s) \in F(s, y_s)$ for a.e. $s \in [0, t_m]$ it is sufficient to use the convexity and closedness of values of F , an upper semicontinuity of $F(t, \cdot)$ and some standard procedures based on the Mazur lemma.

Step 4. From Step 3 it follows that $\sup\{d(v, S_m); v \in S_m^n\} \rightarrow 0$ (an easy proof by contradiction). Therefore $\sup\{d(v, S_m); v \in \overline{S_m^n}\} \rightarrow 0$, as well. Hence, since S_m is compact and $S_m^{n+1} \subset S_m^n$, $\beta(S_m^n) = \beta(\overline{S_m^n}) \searrow 0$ as $n \rightarrow \infty$ and $S_m = \bigcap_{n=1}^{\infty} \overline{S_m^n}$.

Step 5. We show, what is sufficient to finish the proof, that $\overline{S_m^n}$ is contractible for every $n \geq 1$.

Fix $\bar{y} \in \overline{S_m^n}$. We divide the interval $[0, 1]$ on m parts, so we have $0 < \frac{1}{m} < \frac{2}{m} < \dots < \frac{m-1}{m} < 1$. Let $r \in (0, \frac{1}{m}]$. We consider the problem:

$$\begin{cases} \dot{y}(t) = A(t)y(t) + g_n(t, y_t), & \text{for a.e. } t \in [t_m - mr(t_m - t_{m-1}), t_m], \\ y(t) = \bar{y}(t), & \text{for } t \in [-\tau, t_m - mr(t_m - t_{m-1})], \\ y(t_{m-1}) = \bar{y}(t_{m-1}) + I_{m-1}(\bar{y}_{t_{m-1}}). \end{cases} \quad (10)$$

Here g_n is a measurable—locally Lipschitz selection of G_n from Step 2.

Let $\tilde{y}_{n,r}^m$ denote the unique solution of this problem. Then the function $y_{n,r}^m$ defined as:

$$y_{n,r}^m(t) = \begin{cases} \bar{y}(t), & t \in [0, t_m - mr(t_m - t_{m-1})], \\ \tilde{y}_{n,r}^m(t), & t \in (t_m - mr(t_m - t_{m-1}), t_m], \end{cases}$$

satisfies $y_{n,r}^m \in \overline{S_m^n}$.

Next for $r \in (\frac{1}{m}, \frac{2}{m}]$ we consider the problem:

$$\begin{cases} \dot{y}(t) = A(t)y(t) + g_n(t, y_t), & \text{for a.e. } t \in [t_{m-1} - m(r - \frac{1}{m})(t_{m-1} - t_{m-2}), t_m], \\ y(t) = \bar{y}(t), & \text{for } t \in [-\tau, t_{m-1} - m(r - \frac{1}{m})(t_{m-1} - t_{m-2})], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & k = m-1, \\ y(t_{m-2}) = \bar{y}(t_{m-2}) + I_{m-2}(\bar{y}_{t_{m-2}}). \end{cases} \quad (11)$$

Let $\tilde{y}_{n,r}^{m-1}$ denote the unique solution of this problem. Then we have $y_{n,r}^{m-1} \in \overline{S_m^n}$, where:

$$y_{n,r}^{m-1}(t) = \begin{cases} \bar{y}(t), & t \in [0, t_{m-1} - m(r - \frac{1}{m})(t_{m-1} - t_{m-2})], \\ \tilde{y}_{n,r}^{m-1}(t), & t \in (t_{m-1} - m(r - \frac{1}{m})(t_{m-1} - t_{m-2}), t_m]. \end{cases}$$

The last problem we consider is for $r \in (\frac{m-1}{m}, 1]$:

$$\begin{cases} \dot{y}(t) = A(t)y(t) + g_n(t, y_t), & \text{for a.e. } t \in [t_1 - m(r - \frac{m-1}{m})t_1, t_m], \\ y(t) = \bar{y}(t), & \text{for } t \in [-\tau, t_1 - m(r - \frac{m-1}{m})t_1], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}), & \text{for } k \in \{2, 3, \dots, m-1\}, \\ y(t_1) = \bar{y}(t_1) + I_1(\bar{y}_{t_1}). \end{cases} \quad (12)$$

Let $\bar{y}_{n,r}^1$ denote the unique solution of this problem. Then the function $y_{n,r}^1$ defined as:

$$y_{n,r}^1(t) = \begin{cases} \bar{y}(t), & t \in [0, t_1 - m(r - \frac{m-1}{m})t_1], \\ \bar{y}_{n,r}^1(t), & t \in (t_1 - m(r - \frac{m-1}{m})t_1, t_m], \end{cases}$$

also belongs to $\overline{S_m^n}$.

Finally we consider the following function $h_n : [0, 1] \times \overline{S_m^n} \rightarrow \overline{S_m^n}$:

$$h_n(r, \bar{y}) = \begin{cases} \bar{y}, & r = 0, \\ y_{n,r}^m, & r \in (0, \frac{1}{m}], \\ y_{n,r}^{m-1}, & r \in (\frac{1}{m}, \frac{2}{m}], \\ \vdots \\ y_{n,r}^1, & r \in (\frac{m-1}{m}, 1]. \end{cases} \quad (13)$$

Here the functions $y_{n,r}^m, y_{n,r}^{m-1}, \dots, y_{n,r}^1$ are determined by the choice of $\bar{y} \in \overline{S_m^n}$. One can show that the function h_n is continuous applying a standard method, which use a continuous dependence on initial conditions, and remembering that the maps I_k are continuous, when checking a continuity in $r \in \{\frac{i}{m}; i = 1, \dots, m-1\}$. The function h_n , as continuous on $[0, 1] \times \overline{S_m^n}$, is a homotopy. By definition we have $h_n(0, \bar{y}) = \bar{y}$ and $h_n(1, \bar{y}) = y_{n,1}^1$, so $\overline{S_m^n}$ is a contractible set for every $n \in \mathbb{N}$. Therefore, from Theorem 2.2 the set S_m is an R_δ -set. $\square$

Theorem 3.5 enables us to examine a structure of the solution set on the half-line. The assumption (I2) will be replaced by the following:

(I2) $^\infty$

$$\sum_{k=1}^m r_k < \frac{1}{B_m},$$

for every $m \geq 1$, where B_m is defined in (I2) $^\infty$.

Notice that, since $m \mapsto a(m) = \sum_{k=1}^m r_k$ is nondecreasing and $m \mapsto b(m) = \frac{1}{B_m}$ is nonincreasing, (I2) $^\infty$ often means that $r_k = 0$ for every $k \geq 1$. In such a case all impulse maps are completely continuous. In particular, if there is an eigenvalue with a positive real part, then $r_k = 0$ for every $k \geq 1$.

Example 3.6. Let

$$A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad A(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{for every } t \geq 0.$$

Then $B_m = 1$ for every $m \geq 1$, and (I2) $^\infty$ takes the form

$$\sum_{k=1}^m r_k < 1 \quad \text{for every } m \geq 1.$$

Note that assumption (I2) $^\infty$ implies $\lim_{k \rightarrow \infty} r_k = 0$.

Theorem 3.7. *Let E be a Banach space and hypothesis (A) $^\infty$ hold. Suppose that the multivalued map $F : [0, \infty) \times \mathcal{C}([-\tau, 0], E) \rightarrow P_{cp,cv}(E)$ satisfies conditions (F1) $^\infty$ -(F4) $^\infty$. Moreover, assume that the maps $I_k : \mathcal{C}([-\tau, 0], E)$*

$\rightarrow E, k \in \mathbb{N}$, are continuous and satisfy (I1) and (I2) $^\infty$. Then the solution set for problem (1) is an R_δ -set in $PC([0, \infty), E)[x]$.

Proof. We have proved in Theorem 3.5, that solution sets on compact intervals are R_δ sets (that is, for problem (2)). Next we consider an inverse system like in the proof of Theorem 3.4. Using Theorem 2.5 we obtain that the solution set of problem (1) is an R_δ -set. $\square$

Finally we state two open questions:

1. Is Theorem 3.7 true, under sensible assumptions, for k -set contractive (not completely continuous) impulse maps I_k , if $\lim_{m \rightarrow +\infty} B_m = +\infty$?
2. Is Theorem 3.7 true for weakly u.s.c. weakly compact valued perturbations of m -accretive operators?

We believe it is possible to find some interesting positive answers and we leave the problems for further research.

Acknowledgments

The authors are indebted to the referee for his valuable comments and remarks on some very recent papers connected with the material presented above. With his helpful suggestions the paper has become more complete and familiar for the reader.

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Received: 6 July 2011.

Accepted: 3 December 2011.