Generalized Moser–Trudinger inequality for unbounded domains and its application

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Abstract. We give a version of the Moser–Trudinger inequality for Orlicz–Sobolev spaces embedded into exponential and multiple exponential spaces on unbounded domains in $\mathbb{R}^n$, $n \geq 2$. Applying this result and the Mountain Pass Theorem we study the existence of non-trivial weak solutions to the problem

$$u \in W^1 L^\Phi(\mathbb{R}^n) \quad \text{and} \\ -\operatorname{div} \left(\Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) + V(x) \Phi'(|u|) \frac{u}{|u|} = f(x, u) \quad \text{in } \mathbb{R}^n,$$

where Φ is a Young function such that the space $W^1 L^\Phi(\mathbb{R}^n)$ is embedded into an Orlicz space of the exponential or multiple exponential type, the nonlinearity $f(x, t)$ has the corresponding critical growth and $V(x)$ is a continuous potential.

Mathematics Subject Classification (2000). 46E35, 46E30, 26D910.

Keywords. Orlicz–Sobolev spaces, Mountain pass theorem, Palais–Smale sequence.

1. Introduction

Throughout the paper Ω is a domain in $\mathbb{R}^n$, $n \geq 2$, and ω_{n-1} denotes the surface of the unit sphere.

If Ω is in addition bounded and $W_0^{1,p}(\Omega)$ denotes the usual completion of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, then it is well known that

$$W_0^{1,p}(\Omega) \subset L^{\frac{np}{n-p}}(\Omega) \quad \text{if } 1 \leq p < n, \\ W_0^{1,p}(\Omega) \subset L^\infty(\Omega) \quad \text{if } n < p.$$

In the borderline case $p = n$ we have

$$\begin{aligned} W_0^{1,n}(\Omega) &\subset L^q(\Omega) \quad \text{for every } q \in [1, \infty), \\ W_0^{1,n}(\Omega) &\not\subset L^\infty(\Omega). \end{aligned}$$

This case is studied more precisely by Trudinger [30] who showed that

$$W_0^{1,n}(\Omega) \subset L^\Phi(\Omega),$$

where $L^\Phi(\Omega)$ is the Orlicz space corresponding to the Young function $\Phi(t) = \exp(t^{\frac{n}{n-1}}) - 1$. Moreover, for the functions from $W_0^{1,n}(\Omega)$ there is the famous Moser–Trudinger inequality [23]

$$\sup_{\|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} \exp(K|u|^{\frac{n}{n-1}}) = \begin{cases} C(n, K, \mathcal{L}_n(\Omega)) & \text{when } K \leq n\omega_{n-1}^{\frac{1}{n-1}} \\ \infty & \text{when } K > n\omega_{n-1}^{\frac{1}{n-1}}. \end{cases} \quad (1.1)$$

An interesting extension of inequality (1.1) is its version for unbounded domains. This problem is studied in [1, 22, 24] and [28]. Such a result states that if we control $\|u\|_{L^n(\Omega)}$ and we subtract a suitable part of the Taylor expansion from the function $\exp$, then we obtain a version of the Moser–Trudinger inequality. More precisely, if $P \geq 0$, then

$$\begin{aligned} &\sup_{\|\nabla u\|_{L^n(\Omega)} \leq 1, \|u\|_{L^n(\Omega)} \leq P} \int_{\Omega} \exp(K|u|^{\frac{n}{n-1}}) - \sum_{j=0}^{n-2} \frac{(K|u|^{\frac{n}{n-1}})^j}{j!} \\ &= \begin{cases} C(n, K, P) & \text{when } K \leq n\omega_{n-1}^{\frac{1}{n-1}} \\ \infty & \text{when } K > n\omega_{n-1}^{\frac{1}{n-1}}. \end{cases} \end{aligned} \quad (1.2)$$

Results (1.1) and (1.2) are often used when proving the existence of a non-trivial weak solution to the n -Laplace equation

$$-\operatorname{div}(|\nabla u|^{n-2} \nabla u) = f(x, u),$$

where the nonlinearity f has the growth of an exponential type (see for example [2, 18, 24, 25]).

The aim of this paper is to obtain an analogy of the inequality (1.2) for Orlicz–Sobolev spaces embedded into exponential and multiple exponential spaces.

If $\ell \in \mathbb{N}$ and $\alpha < n - 1$, we set

$$\begin{aligned} \gamma &= \frac{n}{n-1-\alpha} > 0, \quad B = 1 - \frac{\alpha}{n-1} = \frac{n}{(n-1)\gamma} > 0 \\ \text{and} \quad K_{\ell,n,\alpha} &= \begin{cases} B^{\frac{1}{B}} n\omega_{n-1}^{\frac{\gamma}{n}} & \text{for } \ell = 1 \\ B^{\frac{1}{B}} \omega_{n-1}^{\frac{\gamma}{n}} & \text{for } \ell \geq 2. \end{cases} \end{aligned} \quad (1.3)$$

The following is known, if Ω is bounded. The space $W_0 L^n \log^\alpha L(\Omega)$ of the Sobolev type, modeled on the Zygmund space $L^n \log^\alpha L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large t (see [19] and [12]). Moreover it is shown in [12] (see also [11] and [13]) that in the limiting case $\alpha = n - 1$ we have the embedding into a

double exponential space, i.e. the space $W_0 L^n \log^{n-1} L \log^\alpha \log L(\Omega)$, $\alpha < n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^\gamma))$ for large t . Further in the limiting case $\alpha = n-1$ we have the embedding into triple exponential space and so on. The borderline case is always $\alpha = n-1$ and for $\alpha > n-1$ we have the embedding into $L^\infty(\Omega)$. It is well-known that the Zygmund space $L^n \log^\alpha L(\Omega)$ coincides with the Orlicz space $L^\Phi(\Omega)$, where

$$\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t^n \log^\alpha(t)} = 1,$$

the space $L^n \log^{n-1} L \log^\alpha \log L(\Omega)$ coincides with $L^\Phi(\Omega)$ where

$$\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t^n \log^{n-1}(t) \log^\alpha(\log(t))} = 1,$$

and so on. For other results concerning these spaces we refer the reader to [13–17] and [26].

The following notation enables us to deal with the multiple exponential spaces comfortably. For $k \in \mathbb{N}$, let us write

$$\log_{[k]}(t) = \log(\log_{[k-1]}(t)), \quad \text{where} \quad \log_{[1]}(t) = \log(t)$$

and

$$\exp_{[k]}(t) = \exp(\exp_{[k-1]}(t)), \quad \text{where} \quad \exp_{[1]}(t) = \exp(t).$$

Let $\ell \in \mathbb{N}$ and $\alpha < n-1$. Then we have above mentioned embedding results for any Young function Φ satisfying

$$\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t^n \left(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}(t) \right) \log_{[\ell]}^\alpha(t)} = 1 \quad (1.4)$$

(for $\ell = 1$ we read (1.4) as $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t^n \log_{[1]}^\alpha(t)} = 1$). As Ω is bounded, all Young functions satisfying (1.4) give the same Orlicz–Sobolev space.

In this paper, we are interested in C^1 -Young functions $\Phi : [0, \infty) \mapsto [0, \infty)$ satisfying (1.4) and in addition we suppose that there is $C > 0$ such that

$$\frac{1}{C} t^n \leq \Phi(t) \leq C t^n \quad \text{for } t \in \left[0, \frac{1}{C}\right). \quad (1.5)$$

Let us also give two conditions that are often used when discussing the critical case concerning the generalized Moser–Trudinger inequality (Theorem 1.1 below)

$$\Phi(t) \geq t^n \left(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}(t) \right) \log_{[\ell]}^\alpha(t) \left(1 + \log_{[\ell]}^{-\beta}(t) \right) \quad \text{for } t \in [t_\Phi, \infty) \quad (1.6)$$

for some $\beta \in (0, \min\{1, \frac{1}{\gamma}\})$ and $t_\Phi \geq 1$,

$$\Phi(t) \leq t^n \left(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}(t) \right) \log_{[\ell]}^\alpha(t) \left(1 - \log_{[\ell]}^{-\beta}(t) \right) \quad \text{for } t \in [t_\Phi, \infty) \quad (1.7)$$

for some $\beta \in (0, \min\{1, B\})$ and $t_\Phi \geq 1$. Notice that assumptions (1.4) and (1.5) together with the fact that Φ is a C^1 -Young function imply the existence of $c_\Phi > 0$ such that

$$c_\Phi \Phi'(t)t \leq \Phi(t), \quad t > 0. \quad (1.8)$$

Now, let us recall the generalized Moser–Trudinger inequality for embedding into exponential and multiple exponential spaces in the case of a bounded domain $\Omega \subset \mathbb{R}^n$.

Theorem 1.1. *Let $K \geq 0, \ell \in \mathbb{N}, n \geq 2$ and $\alpha < n - 1$. Let Φ be a Young function satisfying (1.4).*

(i) *If $u \in W_0 L^\Phi(\Omega)$, then*

$$\int_{\Omega} \exp_{[\ell]}(K|u(x)|^\gamma) dx < \infty.$$

(ii) *If $K < K_{\ell, n, \alpha}$ and $u \in W_0 L^\Phi(\Omega)$ with $\|\nabla u\|_{L^\Phi(\Omega)} \leq 1$, then*

$$\int_{\Omega} \exp_{[\ell]}(K|u(x)|^\gamma) dx \leq C(\ell, n, \alpha, \Phi, \mathcal{L}_n(\Omega), K).$$

(iii) *If $K > K_{\ell, n, \alpha}$, then*

$$\sup_{u \in W_0 L^\Phi(\Omega), \|\nabla u\|_{L^\Phi(\Omega)} \leq 1} \int_{\Omega} \exp_{[\ell]}(K|u(x)|^\gamma) dx = \infty.$$

(iv) *If $K = K_{\ell, n, \alpha}$, Φ satisfies (1.6) and $u \in W_0 L^\Phi(\Omega)$ with $\|\nabla u\|_{L^\Phi(\Omega)} \leq 1$, then*

$$\int_{\Omega} \exp_{[\ell]}(K|u(x)|^\gamma) dx \leq C(\ell, n, \alpha, \Phi, \mathcal{L}_n(\Omega)).$$

(v) *If $K = K_{\ell, n, \alpha}$ and Φ satisfies (1.5) and (1.7), then*

$$\sup_{u \in W_0 L^\Phi(\Omega), \|\nabla u\|_{L^\Phi(\Omega)} \leq 1} \int_{\Omega} \exp_{[\ell]}(K|u(x)|^\gamma) dx = \infty.$$

The first assertion follows from [12, Remarks 3.11(iv)]. In the case $k \geq 2$, all four remaining assertions of Theorem 1.1 can be found in [9, Theorem 1.1, Theorem 1.2, Theorem 4.2 and Theorem 4.1]. In the case $k = 1$, assertions (ii), (iii), (iv) follow from [20, Theorem 1.1, Theorem 1.2 and Theorem 4.2] while assertion (v) is given in [6, Example 5.1].

- Remarks 1.2.** (i) Theorem 1.1(iv) and (v) tells us that when $K = K_{\ell, n, \alpha}$, we generally do not know whether the supremum is finite. It depends on the choice of the Young function Φ (compare with the fact that all Young functions satisfying (1.4) give the same Orlicz–Sobolev space).
- (ii) There is a large gap between conditions (1.6) and (1.7). In paper [10] it was possible to remove this gap in the case $\ell = 1$ and $\alpha = 0$ showing that the borderline Young function behaves like $t^n \log^{-1}(t)$ for large t .

- (iii) From the proofs of [20, Theorem 1.1], [20, Theorem 4.2], [9, Theorem 1.1] and [9, Theorem 4.2] one can easily see that for any fixed $\tilde{C} \geq 0$, we have versions of Theorem 1.1(ii) and (iv) with

$$\int_{\Omega} \exp_{[\ell]}(K(\tilde{C} + |u|)^{\gamma}) \leq C(\tilde{C}, \ell, n, \alpha, \mathcal{L}_n(\Omega), K).$$

The main result of this paper concerns a version of Theorem 1.1 for unbounded domains. Let

$$\exp_{[\ell]}(t) = \sum_{j=0}^{\infty} a_j t^j$$

be the Taylor expansion of the function $\exp_{[\ell]}$. We set

$$S_{\ell, n, \alpha}(t) = \sum_{0 \leq j < \frac{n}{\gamma}} a_j t^j.$$

Theorem 1.3. *Let $\ell \in \mathbb{N}$, $n \geq 2$, $\alpha < n - 1$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose that the Young function $\Phi : [0, \infty) \mapsto [0, \infty)$ satisfies (1.4) and (1.5). Let $u \in W_0 L^{\Phi}(\Omega)$.*

- (i) *If $K \geq 0$ then*

$$\int_{\Omega} \exp_{[\ell]}(K|u|^{\gamma}) - S_{\ell, n, \alpha}(K|u|^{\gamma}) < \infty.$$

- (ii) *If $0 \leq K < K_{\ell, n, \alpha}$, $\|\nabla u\|_{L^{\Phi}(\Omega)} \leq 1$ and $\|u\|_{L^{\Phi}(\Omega)} \leq P$ for some $P \geq 0$, then*

$$\int_{\Omega} \exp_{[\ell]}(K|u|^{\gamma}) - S_{\ell, n, \alpha}(K|u|^{\gamma}) \leq C(\ell, n, \alpha, \Phi, P, K).$$

- (iii) *If $K > K_{\ell, n, \alpha}$, then there is a sequence $\{u_k\} \subset W_0 L^{\Phi}(\Omega)$ such that $\|\nabla u_k\|_{L^{\Phi}(\Omega)} \leq 1$, $\|u_k\|_{L^{\Phi}(\Omega)} \rightarrow 0$ and*

$$\int_{\Omega} \exp_{[\ell]}(K|u_k|^{\gamma}) - S_{\ell, n, \alpha}(K|u_k|^{\gamma}) \xrightarrow{k \rightarrow \infty} \infty.$$

- (iv) *If $K = K_{\ell, n, \alpha}$, Φ satisfies (1.6), $\|\nabla u\|_{L^{\Phi}(\Omega)} \leq 1$ and $\|u\|_{L^{\Phi}(\Omega)} \leq P$ for some $P \geq 0$, then*

$$\int_{\Omega} \exp_{[\ell]}(K|u|^{\gamma}) - S_{\ell, n, \alpha}(K|u|^{\gamma}) \leq C(\ell, n, \alpha, \Phi, P).$$

- (v) *If $K = K_{\ell, n, \alpha}$ and Φ satisfies (1.7), then there is a sequence $\{u_k\} \subset W_0 L^{\Phi}(\Omega)$ such that $\|\nabla u_k\|_{L^{\Phi}(\Omega)} \leq 1$, $\|u_k\|_{L^{\Phi}(\Omega)} \rightarrow 0$ and*

$$\int_{\Omega} \exp_{[\ell]}(K|u_k|^{\gamma}) - S_{\ell, n, \alpha}(K|u_k|^{\gamma}) \xrightarrow{k \rightarrow \infty} \infty.$$

Next, we apply above theorem to the differential equation

$$u \in WL^{\Phi}(\mathbb{R}^n) \quad \text{and}$$

$$-\operatorname{div} \left(\Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) + V(x) \Phi'(|u|) \frac{u}{|u|} = f(x, u) \quad \text{in } \mathbb{R}^n. \quad (1.9)$$

The potential $V : \mathbb{R}^n \mapsto \mathbb{R}$ satisfies

$$V \text{ is continuous and } V(x) \geq V_0 > 0 \text{ for all } x \in \mathbb{R}^n, \quad (1.10)$$

$$V(x) \rightarrow \infty \text{ as } |x| \rightarrow \infty. \quad (1.11)$$

The function $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is supposed to satisfy the following conditions:

There are $M > 0, t_M > 0, A \in (0, \min(c_\Phi, \frac{1}{n}))$, $\tilde{C}_b > 0, C_b > 0$ and $b > 0$ satisfying

$$\begin{aligned} f &\text{ is uniformly continuous on } \mathbb{R}^n \times [-t_0, t_0] \text{ for every } t_0 > 0, \\ f(x, 0) &= 0 \text{ and } tf(x, t) > 0 \text{ for all } x \in \mathbb{R}^n \text{ and } t \neq 0, \end{aligned} \quad (1.12)$$

$$\begin{aligned} 0 < F(x, t) &:= \int_0^t f(x, s) ds \leq M|t|^{1-\frac{1}{M}}|f(x, t)| \\ &\text{provided } |t| > t_M \text{ and } x \in \mathbb{R}^n, \end{aligned} \quad (1.13)$$

$$0 < F(x, t) \leq A f(x, t) t \text{ provided } t \neq 0 \text{ and } x \in \mathbb{R}^n, \quad (1.14)$$

$$\begin{aligned} |f(x, t)| &\leq \tilde{C}_b |t|^{n-1} + C_b \left(\exp_{[\ell]}(b|t|^\gamma) - S_{\ell, n, \alpha}(b|t|^\gamma) \right) \\ &\text{for every } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n, \end{aligned} \quad (1.15)$$

$$\limsup_{t \rightarrow 0} \frac{F(x, t)}{C_S \Phi(|t|)} < 1 \text{ uniformly on } \mathbb{R}^n, \quad (1.16)$$

where $C_S \geq V_0$ is the constant from the Sobolev-type inequality

$$C_S \int_{\mathbb{R}^n} \Phi(|u|) \leq \int_{\mathbb{R}^n} \Phi(|\nabla u|) + V(x) \Phi(|u|), \quad u \in WL^\Phi(\mathbb{R}^n), \quad (1.17)$$

and finally

$$\liminf_{t \rightarrow \infty} \frac{tf(x, t)}{\exp_{[\ell]}(b|t|^\gamma)} > 0 \text{ uniformly on } \mathbb{R}^n. \quad (1.18)$$

Variational formulation. Let us define the space

$$X(\mathbb{R}^n) = \left\{ u \in WL^\Phi(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x) \Phi(|u|) dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{X(\mathbb{R}^n)} = \|\nabla u\|_{L^\Phi(\mathbb{R}^n)} + \|u\|_{L^\Phi(\mathbb{R}^n, V(x)dx)}.$$

Hence $X(\mathbb{R}^n)$ is a Banach space satisfying

$$\begin{aligned} X(\mathbb{R}^n) &\subset WL^\Phi(\mathbb{R}^n), \quad X(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad q \in [n, \infty) \text{ and} \\ &X(\mathbb{R}^n) \subset L^\Phi(\mathbb{R}^n), \end{aligned}$$

where the first embedding is obviously continuous and the last two embeddings are compact by Proposition 2.10 below. Moreover $C_0^\infty(\mathbb{R}^n)$ -functions are dense in $X(\mathbb{R}^n)$ by Proposition 2.8.

We define

$$J(u) = \int_{\mathbb{R}^n} \Phi(|\nabla u|) + V(x) \Phi(|u|) - F(x, u) dx, \quad u \in X(\mathbb{R}^n). \quad (1.19)$$

By Proposition 4.1 below, this is a C^1 -functional on $X(\mathbb{R}^n)$ and its Fréchet derivative is

$$\begin{aligned} \langle J'(u), v \rangle &= \int_{\mathbb{R}^n} \Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla v + V(x) \Phi'(|u|) \frac{u}{|u|} v \\ &\quad - f(x, u) v \, dx, \quad u, v \in X(\mathbb{R}^n), \end{aligned} \quad (1.20)$$

where the symbol $\langle J'(u), v \rangle$ denotes the value of the linear functional $J'(u)$ of v .

We say that $u \in X(\mathbb{R}^n)$ is a weak solution to problem (1.9) if

$$\langle J'(u), v \rangle = 0 \quad \text{for every } v \in X(\mathbb{R}^n). \quad (1.21)$$

Theorem 1.4. *Let $\ell \in \mathbb{N}, n \geq 2$ and $\alpha < n - 1$. Suppose that the C^1 -Young function $\Phi : [0, \infty) \mapsto [0, \infty)$ satisfies (1.4), (1.5) and (1.7). Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ satisfy (1.10) and (1.11) and let $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying (1.12), (1.13), (1.14), (1.15), (1.16) and (1.18). Then problem (1.9) has a non-trivial weak solution in $X(\mathbb{R}^n)$.*

The paper is organized as follows. In Sect. 3 we prove Theorem 1.3. We also give some auxiliary versions of Theorem 1.3 that we afterwards use when dealing with problem (1.9). Section 4 is devoted to the proof that J is a C^1 -functional and that it satisfies (1.20). In Sect. 5, we check that J has the Mountain Pass Geometry (it satisfies assumptions (i), (ii) and (iii) of Theorem 2.11). In Sect. 6 we show that the Palais–Smale sequence (corresponding to J) converges (in some sense) to a weak solution of (1.9). The proof of Theorem 1.4 using the preceding results is given in Sect. 7. Finally, in Sect. 8, we discuss the case of a nonlinearity with the sub-critical growth (in the sense of [18]).

2. Preliminaries

The n -dimensional Lebesgue measure is denoted by $\mathcal{L}_n$. By $B(x_0, R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at x_0 with the radius $R > 0$. If $x_0 = 0$ we simply write $B(R)$.

For two functions $g, h : [0, \infty) \mapsto [0, \infty)$ we write $g \lesssim h$, if there is $C > 0$ such that $g(t) \leq Ch(t)$ for every $t \in [0, \infty)$. If u is a measurable function on A , then by $u = 0$ (or $u \neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on A .

By C we denote a generic positive constant which may depend on ℓ, n, α and Φ . This constant may vary from expression to expression as usual.

Properties of $\exp_{[\ell]}$. For given $\ell \in \mathbb{N}, n \in \mathbb{N}, \alpha < n - 1$ and $p \geq 1$, one can easily prove that there is $C \geq 1$ such that

$$\exp_{[\ell]}^p(t) \leq C \exp_{[\ell]}(pt) \quad \text{for all } t \geq 0 \quad (2.1)$$

and

$$\left(\exp_{[\ell]}(t) - S_{\ell, n, \alpha}(t) \right)^p \leq C \left(\exp_{[\ell]}(pt) - S_{\ell, n, \alpha}(pt) \right) \quad \text{for all } t \geq 0. \quad (2.2)$$

Lemma 2.1. *Let $\ell \in \mathbb{N}$. The Taylor coefficients of the function $\exp_{[\ell]}$ satisfy*

$$a_j \geq 0 \quad \text{for each } j \in \mathbb{N}.$$

Proof. We have $\exp(t) = \sum_{j_1=0}^{\infty} \frac{t^{j_1}}{k_1!}$,

$$\exp_{[2]}(t) = \sum_{j_1=0}^{\infty} \frac{\exp^{j_1}(t)}{j_1!} = \sum_{j_1=0}^{\infty} \frac{\exp(j_1 t)}{j_1!} = \sum_{j_1, j_2=0}^{\infty} \frac{j_1^{j_2} t^{j_2}}{j_1! j_2!}$$

and by induction

$$\exp_{[\ell]}(t) = \sum_{j_1, \dots, j_{\ell}=0}^{\infty} \frac{\prod_{i=1}^{\ell-1} j_i^{j_{i+1}}}{\prod_{i=1}^{\ell} j_i!} t^{j_{\ell}}.$$

And the non-negativeness of the coefficients follows. $\square$

Young functions and Orlicz spaces. A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a Young function if Φ is increasing, convex, $\Phi(0) = 0$ and $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = \infty$.

Denote by $L^{\Phi}(A, d\mu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure μ . If $\mu = \mathcal{L}_n$ we simply write $L^{\Phi}(A)$. The space $L^{\Phi}(A, d\mu)$ is equipped with the Luxemburg norm

$$\|u\|_{L^{\Phi}(A, d\mu)} = \inf \left\{ \lambda > 0 : \int_A \Phi \left(\frac{|u(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}. \quad (2.3)$$

Given a differentiable Young function Φ we can define the generalized inverse function to $\phi(y) = \Phi'(y)$ by

$$\psi(s) = \inf \{y : \phi(y) > s\} \quad \text{for } s > 0$$

and further we define the associated Young function Ψ by

$$\Psi(t) = \int_0^t \psi(s) ds \quad \text{for } t \geq 0.$$

The dual space to $L^{\Phi}(A, d\mu)$ can be identified as the Orlicz space $L^{\Psi}(A, d\mu)$. We further have the generalized Hölder's inequality

$$\int_A |u(y)v(y)| d\mu(y) \leq 2 \|u\|_{L^{\Phi}(A, d\mu)} \|v\|_{L^{\Psi}(A, d\mu)}. \quad (2.4)$$

Δ_2 -condition. We say that a function Φ satisfies the Δ_2 -condition, if there is $C_{\Delta} > 1$ such that

$$\Phi(2t) \leq C_{\Delta} \Phi(t) \quad \text{whenever } t \geq 0.$$

It is not difficult to check the Δ_2 -condition for our Young functions satisfying (1.4) and (1.5). Therefore one easily proves

$$\|u_k\|_{L^{\Phi}(A, d\mu)} \xrightarrow{k \rightarrow \infty} 0 \iff \int_A \Phi(|u_k|) d\mu(x) \xrightarrow{k \rightarrow \infty} 0 \quad (2.5)$$

and

$$\|u_k\|_{L^{\Phi}(A, d\mu)} \leq C_1 \iff \int_A \Phi(|u_k|) d\mu(x) \leq C_2 \quad (2.6)$$

We often use the following estimate from [7, Lemma 2.2] together with generalized Hölder's inequality.

Lemma 2.2. *If a Young function Φ satisfies (1.4) and (1.5), then $\Psi(\Phi') \lesssim \Phi$.*

Next, we need to be able to estimate the norm by the modular and vice versa. Some rough estimates are given by the following lemma from [21, Lemma 3.8.4].

Lemma 2.3. *Let $u \in L^\Phi(A)$.*

- (i) *If $\|u\|_{L^\Phi(A)} \leq 1$, then $\int_A \Phi(|u|) \leq \|u\|_{L^\Phi(A)}$.*
- (ii) *If $\|u\|_{L^\Phi(A)} > 1$, then $\int_A \Phi(|u|) \geq \|u\|_{L^\Phi(A)}$.*

More careful estimates use the following lemma from [7, Lemma 2.4] (the original statement comes from [8] and it concerns $\mu = \mathcal{L}_n$ only, but the proof is also valid for a general measure μ).

Lemma 2.4. *For every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\|u\|_{L^\Phi(A, d\mu)}^{n+\varepsilon} \leq \int_A \Phi(|u|) d\mu \leq \|u\|_{L^\Phi(A, d\mu)}^{n-\varepsilon} \quad \text{provided } \|u\|_{L^\Phi(A, d\mu)} < \delta.$$

Orlicz–Sobolev spaces. Let A be a nonempty open set in $\mathbb{R}^n$ and let Φ be a Young function satisfying (1.4). In this subsection we consider Orlicz spaces only with the Lebesgue measure. We define the Orlicz–Sobolev space $WL^\Phi(A)$ as the set

$$WL^\Phi(A) := \{u : u, |\nabla u| \in L^\Phi(A)\}$$

equipped with the norm

$$\|u\|_{WL^\Phi(A)} := \|u\|_{L^\Phi(A)} + \|\nabla u\|_{L^\Phi(A)},$$

where ∇u is the gradient of u and we use its Euclidean norm in $\mathbb{R}^n$.

We put $W_0L^\Phi(A)$ for the closure of $C_0^\infty(A)$ in $WL^\Phi(A)$.

Non-increasing rearrangement. The non-increasing rearrangement u^* of a measurable function u on Ω is

$$u^*(t) = \inf \{s > 0 : \mathcal{L}_n(\{x \in \Omega : |u(x)| > s\}) \leq t\}, \quad t > 0.$$

We also define the non-increasing radially symmetric rearrangement $u^\#$ by

$$u^\#(x) = u^*\left(\frac{\omega_{n-1}}{n}|x|^n\right) \quad \text{for } x \in B(R), \quad \mathcal{L}_n(B(R)) = \mathcal{L}_n(\Omega).$$

For an introduction to these rearrangements see e.g. [29]. We need the following result concerning $u^\#$ (see [29, Theorem 1.C]).

Theorem 2.5. *Let Φ be a Young function and let u be a Lipschitz continuous function decaying at infinity ($\mathcal{L}_n(\{x \in \mathbb{R}^n : |u(x)| > t\}) < \infty$ for all $t > 0$). Then*

$$\int_{\mathbb{R}^n} \Phi(|\nabla u(x)|) dx \geq \int_{\mathbb{R}^n} \Phi(|\nabla u^\#(x)|) dx.$$

Tools from the Measure Theory. We make use of a version of [18, Lemma 2.1].

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $\theta \in [0, 1)$ and let $\{u_k\}$ be a sequence of functions from $L^1(\Omega)$ converging to $u \in L^1(\Omega)$ a.e. in Ω . Let $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a continuous function bounded on $\Omega \times [-t_0, t_0]$ for every $t_0 > 0$. Suppose that $f(x, u_k)|u_k|^\theta$ and $f(x, u)|u|^\theta$ belong to $L^1(\Omega)$ and*

$$\int_{\Omega} |f(x, u_k) u_k| \leq C_1.$$

Then $f(x, u_k)|u_k|^\theta \rightarrow f(x, u)|u|^\theta$ in $L^1(\Omega)$.

Sketch of proof. The difference between our statement and [18, Lemma 2.1] is that the original statement deals with $\theta = 0$ only, but this can be repaired by a minor modification of the original proof. $\square$

We also need the Generalized Lebesgue Dominated Convergence Theorem (see [27, Exercise 5.4.13]).

Proposition 2.7. *Let $\{u_k\}, \{v_k\}$ be sequences of measurable functions on Ω such that $|u_k| \leq v_k$ for all $k \in \mathbb{N}$. Let u and v be measurable functions on Ω such that $u_k \rightarrow u$ a.e. in Ω and $v_k \rightarrow v$ a.e. in Ω . Then*

$$\lim_{k \rightarrow \infty} \int_{\Omega} v_k = \int_{\Omega} v \implies \lim_{k \rightarrow \infty} \int_{\Omega} u_k = \int_{\Omega} u.$$

Some properties of $X(\mathbb{R}^n)$.

Proposition 2.8. *The $C_0^\infty(\mathbb{R}^n)$ -functions are dense in $X(\mathbb{R}^n)$.*

Proof. The proof consists of two steps. First, we approximate a given function $u \in X(\mathbb{R}^n)$ by a function with a compact support, then we approximate the second function by a $C_0^\infty(\mathbb{R}^n)$ -function.

For every $m \in \mathbb{N}$ we can find a smooth function ψ_m such that

$$0 \leq \psi_m \leq 1, \quad \psi_m = 1 \text{ on } B(m), \quad \psi_m = 0 \text{ on } \mathbb{R}^n \setminus B(m+1), \\ |\nabla \psi_m| \leq C \text{ (} C \text{ is independent of } m \text{)}.$$

Fix $u \in X(\mathbb{R}^n)$ and $\varepsilon > 0$. There is $m_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n \setminus B(m_0)} \Phi(|\nabla u|) \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^n \setminus B(m_0)} V(x) \Phi(|u|) \leq \varepsilon.$$

Set $u_0 = \psi_{m_0} u$. Hence $u_0 \in W_0 L^\Phi(B(m_0+1))$. Next, as $V(x)$ is bounded away from zero and also bounded on $B(m_0+1)$, we obtain from above and from the Δ_2 -condition

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(|\nabla(u - u_0)|) &= \int_{B(m_0+1) \setminus B(m_0)} \Phi(|-u \nabla \psi_{m_0} + (1 - \psi_{m_0}) \nabla u|) \\ &\leq C \int_{B(m_0+1) \setminus B(m_0)} \Phi(|u|) + C \int_{B(m_0+1) \setminus B(m_0)} \Phi(|\nabla u|) \leq C\varepsilon \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} V(x) \Phi(u - u_0) &= \int_{B(m+1) \setminus B(m)} V(x) \Phi(|u(1 - \psi_{m_0})|) \\ &\leq \int_{B(m+1) \setminus B(m)} V(x) \Phi(|u|) \leq \varepsilon. \end{aligned}$$

Thus, in view of (2.5) we see that we can make $\|u - u_0\|_{X(\mathbb{R}^n)}$ arbitrarily small.

Finally, since $u_0 \in W_0 L^\Phi(B(m_0 + 1))$, $C_0^\infty(B(m_0 + 1))$ -functions are dense in $W_0 L^\Phi(B(m_0 + 1))$ and $X(\mathbb{R}^n)$ -norm and $W_0 L^\Phi(B(m_0 + 1))$ -norm are equivalent for functions from $W_0 L^\Phi(B(m_0 + 1))$ ($V(x)$ is bounded on $B(m_0 + 1)$), we conclude the proof. $\square$

Next, we need a suitable estimate of a radially symmetric function $u \in L^\Phi(\mathbb{R}^n)$ on large spheres.

Lemma 2.9. *Let $u \in L^\Phi(\mathbb{R}^n)$ with $\|u\|_{L^\Phi(\mathbb{R}^n)} \leq P$, for some $P > 0$. Suppose that u is non-negative, radially symmetric and non-increasing with respect to $|x|$. Then there are $R_s > 0$ and $C_s > 0$ independent of u such that*

$$u(x) \leq C_s P \frac{1}{|x|} \quad \text{for } |x| > R_s.$$

Proof. Let us write $u(x) = g(|x|)$. Since we have $\|u\|_{L^\Phi(\mathbb{R}^n)} \leq P$, we obtain for each $R > 0$

$$\begin{aligned} \omega_{n-1} \frac{R^n}{n} \Phi\left(\frac{g(R)}{P}\right) &= \omega_{n-1} \int_0^R \Phi\left(\frac{g(y)}{P}\right) y^{n-1} dy \\ &\leq \omega_{n-1} \int_0^R \Phi\left(\frac{g(y)}{P}\right) y^{n-1} dy \\ &= \int_{B(R)} \Phi\left(\frac{u(x)}{P}\right) dx \leq \int_{\mathbb{R}^n} \Phi\left(\frac{u(x)}{P}\right) dx \leq 1. \end{aligned}$$

Therefore, (1.5) and the fact that Φ is increasing, imply that there are $R_s > 0$ and $C_s > 0$ such that

$$g(|x|) \leq C_s P \frac{1}{|x|} \quad \text{for } |x| > R_s.$$

$\square$

Proposition 2.10. *Suppose that the Young function Φ satisfies (1.4) and (1.5). Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ satisfy (1.10) and (1.11). Let $\{u_k\} \subset X(\mathbb{R}^n)$ be a bounded sequence. Then, passing to a subsequence we can guarantee that there is $u \in X(\mathbb{R}^n)$ such that*

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{in } WL^\Phi(\mathbb{R}^n) \\ u_k &\rightarrow u \quad \text{in } L^\Phi(\mathbb{R}^n) \\ u_k &\rightarrow u \quad \text{in } L^p(\mathbb{R}^n) \text{ for every } p \in [n, \infty) \\ u_k &\rightarrow u \quad \text{a.e. in } \mathbb{R}^n. \end{aligned}$$

Proof. Let us start showing that it is enough to prove that we can pass to subsequences having each convergence property separately. Indeed, in that case we are able to find $\{u_k^w\}$ converging weakly in $WL^\Phi(\mathbb{R}^n)$. Then from $\{u_k^w\}$ we extract $\{u_k^\Phi\}$ which converges in $L^\Phi(\mathbb{R}^n)$ in addition. Next we pass to $\{u_k^{a.e.}\} \subset \{u_k^\Phi\}$ converging a.e. in $\mathbb{R}^n$. Then there is $\{u_k^n\} \subset \{u_k^{a.e.}\}$ converging in $L^n(\mathbb{R}^n)$, $\{u_k^{n+1}\} \subset \{u_k^n\}$ converging in $L^{n+1}(\mathbb{R}^n)$ and so on. Finally, the diagonal subsequence $\{u_k^k\}_{k=n}^\infty$ has all desired properties (notice that if $\{u_k^k\}$ converges in $L^m(\mathbb{R}^n)$ and in $L^{m+1}(\mathbb{R}^n)$, then $\{u_k^k\}$ converges also in $L^p(\mathbb{R}^n)$, for all $p \in (m, m+1)$).

STEP 1. (weak convergence in $WL^\Phi(\mathbb{R}^n)$).

Since $V(x)$ is bounded away from zero, the boundedness in $X(\mathbb{R}^n)$ easily implies the boundedness in $WL^\Phi(\mathbb{R}^n)$. Now, as the second space is reflexive, the assertion follows.

STEP 2. (strong convergence in $L^\Phi(\mathbb{R}^n)$).

We have

$$\int_{\mathbb{R}^n} \Phi(|\nabla u_k|) \leq C_1 \quad \text{and} \quad \int_{\mathbb{R}^n} V(x)\Phi(|u_k|) \leq C_1. \quad (2.7)$$

For every $\varepsilon > 0$ using (1.11) we can find the radius $r_\varepsilon > 0$ such that

$$V(x) \geq \frac{1}{\varepsilon} \quad \text{for all } x \in \mathbb{R}^n \setminus B(r_\varepsilon). \quad (2.8)$$

From (2.7) and (2.8) we obtain

$$\int_{\mathbb{R}^n \setminus B(r_\varepsilon)} \Phi(|u_k|) \leq \varepsilon \int_{\mathbb{R}^n \setminus B(r_\varepsilon)} V(x)\Phi(|u_k|) \leq C_1 \varepsilon \quad \text{for every } k \in \mathbb{N}. \quad (2.9)$$

We observe from (2.5) that for every fixed $m \in \mathbb{N}$ this can be done so that we have $R_m > 0$ satisfying

$$\|u_k\|_{L^\Phi(\mathbb{R}^n \setminus B(R_m))} < \frac{1}{m}.$$

Further, as $V(x)$ is bounded away from zero, from (2.7) we obtain that $\{u_k\}$ is a bounded sequence in $WL^\Phi(B(R_m))$. Since the last space is reflexive and compactly embedded into $L^\Phi(B(R_m))$, we obtain a subsequence $\{u_k^m\}$ converging to some $\tilde{u}^m$ in $L^\Phi(B(R_m))$. Hence, if we define

$$u^m(x) = \begin{cases} \tilde{u}^m(x) & \text{for } x \in B(R_m) \\ 0 & \text{for } x \in \mathbb{R}^n \setminus B(R_m), \end{cases}$$

then we have

$$\|u_k^m - u^m\|_{L^\Phi(\mathbb{R}^n)} \leq \frac{2}{m}$$

for every k large enough. Passing to a subsequence again we can assume that above estimate holds for all $k \in \mathbb{N}$. This can be done for all $m \in \mathbb{N}$ so that

$$\{u_k^1\} \supset \{u_k^2\} \supset \{u_k^3\} \supset \cdots.$$

Thus, the diagonal subsequence $\{u_k^k\}$ satisfies

$$\|u_k^k - u^m\|_{L^\Phi(\mathbb{R}^n)} \leq \frac{2}{m} \quad \text{for all } k \geq m.$$

Hence we see that $\{u^m\}$ is a Cauchy sequence in $L^\Phi(\mathbb{R}^n)$ and its limit is also the limit of $\{u_k^k\}$. Of course, the limit is the function u from Step 1.

STEP 3. (convergence a.e. in $\mathbb{R}^n$)

Having done Step 2, passing to a subsequence we obtain the convergence a.e. in $\mathbb{R}^n$.

STEP 4. ($u \in X(\mathbb{R}^n)$)

By Step 1, we see that $u \in WL^\Phi(\mathbb{R}^n)$. Next, (2.7), convergence a.e. in $\mathbb{R}^n$, Fatou's lemma and (2.6) give $\|u\|_{L^\Phi(\mathbb{R}^n, V(x)dx)} < \infty$. Hence we have $u \in X(\mathbb{R}^n)$.

STEP 5. (strong convergence in $L^p(\mathbb{R}^n)$ for $p \geq n$ fixed)

Fix $p \in [n, \infty)$ and let us show that $v_k := u_k - u$ converge to zero in $L^p(\mathbb{R}^n)$. By a standard symmetrization argument based on the density of $C_0^\infty(\mathbb{R}^n)$ -functions in $X(\mathbb{R}^n)$ and on Theorem 2.5 we can suppose that v_k are non-negative, radially symmetric, non-increasing with respect to $|x|$. Further we have

$$\int_{\mathbb{R}^n} \Phi(|\nabla v_k|) \leq C_2 \quad \text{and} \quad \int_{\mathbb{R}^n} \Phi(|v_k|) \xrightarrow{k \rightarrow \infty} 0. \quad (2.10)$$

In view of (1.5), (2.10) and Lemma 2.9 we can find $R > 0$ such that $u_k \leq 1$ on $\mathbb{R}^n \setminus B(R)$ and

$$\int_{\mathbb{R}^n \setminus B(R)} |v_k|^p \leq \int_{\mathbb{R}^n \setminus B(R)} |v_k|^n \leq C \int_{\mathbb{R}^n \setminus B(R)} \Phi(|v_k|) \xrightarrow{k \rightarrow \infty} 0.$$

Finally, as $v_k|_{\partial B(R)} = \tilde{C}_k \in [0, 1]$, from Remarks 1.2(iii) we have

$$\int_{B(R)} |v_k|^{p+1} \leq \int_{B(R)} C + \exp_{[\ell]}(K_{\ell, n, \alpha}(1 + (v_k - \tilde{C}_k)^\gamma)) \leq C_3.$$

Thus from Lemma 2.6 (with $\theta = 0$ and $f(x, t) = |t|^p$) we obtain

$$\int_{B(R)} |v_k|^p \xrightarrow{k \rightarrow \infty} 0.$$

□

Tools from the Calculus of Variations. Our key instrument is the following version of the Mountain Pass Theorem by Ambrosetti and Rabinowitz [3].

Theorem 2.11. *Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$. Suppose that there exist a neighborhood U of 0 in X and $\xi \in \mathbb{R}$ satisfying the following conditions:*

- (i) $J(0) < \xi$,
- (ii) $J(u) \geq \xi$ on the boundary of U ,
- (iii) there is $w \notin U$ such that $J(w) < \xi$.

Set

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J(u) \geq \xi,$$

where $\Gamma = \{g \in C([0, 1], X) : g(0) = 0, g(1) = w\}$. Then there is a sequence $\{u_k\} \subset X$ such that

$$J(u_k) \rightarrow c \quad \text{and} \quad J'(u_k) \rightarrow 0 \text{ in } X^*. \quad (2.11)$$

The sequence satisfying (2.11) is called the Palais–Smale sequence and the constant c is a Palais–Smale level. Notice that this version slightly differs from an often used version of the Mountain Pass Theorem which requires the Palais–Smale condition (the Palais–Smale sequence has a subsequence convergent in the norm) and asserts that there is a critical point $u_0 \in X$ satisfying $J(u_0) = c$. We have to use this version of the Mountain Pass Theorem, because functionals corresponding to elliptic equations with critical exponential (or multiple exponential) growth satisfy Palais–Smale condition only for some (sufficiently small) Palais–Smale levels. This phenomenon is related to the loss of the Moser–Trudinger embedding for K equal to the borderline parameter (see Theorem 1.3(v)). Our approach is taken from famous paper [5], where the authors deal with the Laplace equation with nonlinearity having the critical polynomial growth in the dimension $n \geq 3$. Recall that in such a case, the difficulties are caused by the fact that the embedding of $W_0^{1,2}(\Omega)$ into $L^{\frac{2n}{n-2}}(\Omega)$ is not compact.

See [5, p. 459] for the discussion concerning the proof of Theorem 2.11. **Versions of the Moser sequence.** Let $x_0 \in \Omega$ and $R > 0$ be such that $B(x_0, R) \subset \Omega$. We make use of the sequences of $W_0 L^\Phi(B(x_0, R))$ -functions constructed in [6, 9] and [20] yielding the proof of Theorem 1.1(iii) and (v), respectively. These sequences are versions of Moser’s sequence from the proof of inequality (1.1) in the case $K > n\omega_{n-1}^{\frac{1}{n-1}}$.

For $\ell = 1$ we set

$$w_k(x) = g_k(|x - x_0|), \quad \text{where}$$

$$g_k(y) = \begin{cases} 0 & \text{for } y \in [R, \infty) \\ (-\frac{2}{R}y + 2)K_{1,n,\alpha}^{-\frac{1}{\gamma}} n^B \log^B(2) k^{\frac{1}{\gamma}-B} \left(1 + \frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \text{for } y \in [\frac{R}{2}, R] \\ K_{1,n,\alpha}^{-\frac{1}{\gamma}} n^B \log^B(\frac{R}{y}) k^{\frac{1}{\gamma}-B} \left(1 + \frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \text{for } y \in [Re^{-\frac{k}{n}}, \frac{R}{2}] \\ K_{1,n,\alpha}^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}} \left(1 + \frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \text{for } y \in [0, Re^{-\frac{k}{n}}]. \end{cases} \quad (2.12)$$

In case $\ell \geq 2$ we fix $T > \exp_{[\ell]}(1)$ and we define

$$w_k(x) = g_k(|x - x_0|), \quad \text{where}$$

$$g_k(y) = \begin{cases} 0 & \text{for } y \in [R, \infty) \\ (-\frac{2}{R}y + 2)K_{\ell,n,\alpha}^{-\frac{1}{\gamma}} \log_{[\ell]}^B(T+2) k^{\frac{1}{\gamma}-B} & \text{for } y \in [\frac{R}{2}, R] \\ \times \left(1 + \frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \\ K_{\ell,n,\alpha}^{-\frac{1}{\gamma}} \log_{[\ell]}^B(T + \frac{R}{y}) k^{\frac{1}{\gamma}-B} \left(1 + \frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \text{for } y \in [R \exp_{[\ell]}^{-\frac{1}{n}}(k), \frac{R}{2}] \\ K_{\ell,n,\alpha}^{-\frac{1}{\gamma}} \log_{[\ell]}^B\left(T + \exp_{[\ell]}^{\frac{1}{n}}(k)\right) k^{\frac{1}{\gamma}-B} & \text{for } y \in [0, R \exp_{[\ell]}^{-\frac{1}{n}}(k)]. \\ \times \left(1 + \frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \end{cases} \quad (2.13)$$

For given $K > K_{\ell,n,\alpha}$ we fix $A \in (K_{\ell,n,\alpha}, K)$ and we define for $\ell = 1$

$$\begin{aligned} \tilde{w}_k(x) &= g_k(|x - x_0|), \quad \text{where} \\ g_k(y) &= \begin{cases} 0 & \text{for } y \in [R, \infty) \\ (-\frac{2}{R}y + 2)A^{-\frac{1}{\gamma}}n^B \log^B(2)k^{\frac{1}{\gamma}-B} & \text{for } y \in [\frac{R}{2}, R] \\ A^{-\frac{1}{\gamma}}n^B \log^B(\frac{R}{y})k^{\frac{1}{\gamma}-B} & \text{for } y \in [Re^{-\frac{k}{n}}, \frac{R}{2}] \\ A^{-\frac{1}{\gamma}}k^{\frac{1}{\gamma}} & \text{for } y \in [0, Re^{-\frac{k}{n}}]. \end{cases} \end{aligned} \quad (2.14)$$

In case $\ell \geq 2$ we fix $T > \exp_{[\ell]}(1)$ and we define

$$\begin{aligned} \tilde{w}_k(x) &= g_k(|x - x_0|), \quad \text{where} \\ g_k(y) &= \begin{cases} 0 & \text{for } y \in [R, \infty) \\ (-\frac{2}{R}y + 2)A^{-\frac{1}{\gamma}} \log_{[\ell]}^B(T+2)k^{\frac{1}{\gamma}-B} & \text{for } y \in [\frac{R}{2}, R] \\ A^{-\frac{1}{\gamma}} \log_{[\ell]}^B(T + \frac{R}{y})k^{\frac{1}{\gamma}-B} & \text{for } y \in [R \exp_{[\ell]}^{-\frac{1}{n}}(k), \frac{R}{2}] \\ A^{-\frac{1}{\gamma}} \log_{[\ell]}^B\left(T + \exp_{[\ell]}^{\frac{1}{n}}(k)\right)k^{\frac{1}{\gamma}-B} & \text{for } y \in [0, R \exp_{[\ell]}^{-\frac{1}{n}}(k)]. \end{cases} \end{aligned} \quad (2.15)$$

By the proofs of [6, Example 5.1], [20, Theorem 1.2] and [9, Theorem 1.2 and Theorem 4.1] we have the following information:

If Φ satisfies (1.4) and (1.5) then there is $k_0 \in \mathbb{N}$ such that

$$\|\nabla \tilde{w}_k\|_{L^\Phi(B(x_0, R))} \leq 1 \quad \text{for } k \geq k_0. \quad (2.16)$$

Moreover, if Φ satisfies (1.4), (1.5) and (1.7) then there is $k_0 \in \mathbb{N}$ such that

$$\|\nabla w_k\|_{L^\Phi(B(x_0, R))} \leq 1 \quad \text{for } k \geq k_0. \quad (2.17)$$

Further, one easily modifies the proofs of [6, Example 5.1] and [9, Theorem 4.1] so that for every $\theta_0 \geq 1$ and every $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that

$$\int_{\Omega} \Phi(\theta |\nabla w_k|) \leq \theta^n \left(1 - \frac{1}{k^{\beta+\varepsilon}}\right) \quad \text{for every } k \geq k_0 \text{ and } \theta \in [1, \theta_0], \quad (2.18)$$

where β comes from (1.7).

3. Generalized Moser–Trudinger inequality for unbounded domains

Proof of Theorem 1.3(ii). We can suppose that $\Omega = \mathbb{R}^n$, because we can define $u = 0$ in $\mathbb{R}^n \setminus \Omega$ without changing the norms. Moreover, we can suppose that u is radially symmetric, that is $u(x) = g(|x|)$, where g is non-negative and non-increasing on $[0, \infty)$. Indeed, the density of $C_0^\infty(\mathbb{R}^n)$ -functions and Fatou's lemma give that we may suppose that $u \in C_0^\infty(\Omega)$ and thus we may apply Theorem 2.5.

Next, let us set $R_1 = \max\{R_s, C_s P\}$, where C_s and R_s come from Lemma 2.9. Hence from Lemma 2.9 and the Lebesgue Monotone Convergence Theorem

(the Taylor coefficients a_j are non-negative by Lemma 2.1) we obtain

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(R_1)} \sum_{j > \frac{n}{\gamma}} a_j K^j |u|^{\gamma j} dx &\leq \omega_{n-1} \sum_{j > \frac{n}{\gamma}} a_j K^j C_s^{\gamma j} P^{\gamma j} \int_{R_1}^{\infty} y^{n-1-\gamma j} dy \\ &= C \sum_{j > \frac{n}{\gamma}} a_j K^j C_s^{\gamma j} P^{\gamma j} R_1^{n-\gamma j} \\ &\leq C R_1^n \sum_{j=0}^{\infty} a_j K^j = C R_1^n \exp_{[\ell]}(K) = C. \end{aligned}$$

If $\frac{n}{\gamma} \in \mathbb{N}$ we also need to estimate the summand corresponding to this index. By (1.5), Lemma 2.9 and by the Δ_2 -condition there is $R_2 \geq R_1$ such that

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(R_2)} |u|^n &\leq C \int_{\mathbb{R}^n \setminus B(R_2)} \Phi(|u|) \leq C \int_{\mathbb{R}^n} \Phi(|u|) = C \int_{\mathbb{R}^n} \Phi\left(P \frac{|u|}{P}\right) \\ &\leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|u|}{P}\right) = C \end{aligned}$$

and thus

$$\int_{\mathbb{R}^n \setminus B(R_2)} a_{\frac{n}{\gamma}} K^{\frac{n}{\gamma}} |u|^n \leq C K^{\frac{n}{\gamma}} \leq C K_{\ell, n, \alpha}^{\frac{n}{\gamma}} = C.$$

Therefore we have

$$\int_{\mathbb{R}^n \setminus B(R_2)} \exp_{[\ell]}(K|u|^\gamma) - S_{\ell, n, \alpha}(K|u|^\gamma) \leq C. \quad (3.1)$$

It remains to estimate the integral over $B(R_2)$. Let us set $\tilde{u} = u - g(R_2)$. We observe from Lemma 2.9 that

$$0 \leq g(R_2) \leq \frac{C_s P}{R_2}. \quad (3.2)$$

Finally, since $\tilde{u} = 0$ on $\partial B(R_2)$, we can apply the generalized Moser–Trudinger inequality (see the version of Theorem 1.1(ii) given by Remarks 1.2 (iii)) for $\tilde{u}$ on $B(R_2)$. Thus from (3.2) we obtain

$$\begin{aligned} \int_{B(R_2)} \exp_{[\ell]}(K|u|^\gamma) - S_{\ell, n, \alpha}(K|u|^\gamma) &\leq \int_{B(R_2)} \exp_{[\ell]}(K|u|^\gamma) \\ &\leq \int_{B(R_2)} \exp_{[\ell]} \left(K \left(\frac{C_s P}{R_2} + |\tilde{u}| \right)^\gamma \right) \leq C. \end{aligned} \quad (3.3)$$

Now, the proof follows from (3.1) and (3.3). $\square$

Proof of Theorem 1.3(iv). It is enough to use the preceding proof with the version of the Moser–Trudinger inequality given by Theorem 1.1(iv) and Remarks 1.2(iii). $\square$

Sketch of proof of Theorem 1.3(i). In the same way as above we apply the symmetrization and (setting $P = \|u\|_{L^\Phi(\Omega)}$) we find $R_2 > R_s$ such that

$$\int_{\mathbb{R}^n \setminus B(R_2)} \exp_{[\ell]}(K|u|^\gamma) - S_{\ell,n,\alpha}(K|u|^\gamma) < \infty.$$

Fix $\tilde{K} > K$. Then there is $t_0 > 0$ such that

$$K \left(t + \frac{C_s P}{R_2} \right)^\gamma \leq \tilde{K} t^\gamma \quad \text{for } t \geq t_0. \quad (3.4)$$

Next, since $u - g(R_2) = 0$ on $\partial B(R_2)$, we can use Theorem 1.1(i) with $u - g(R_2)$ on $B(R_2)$ and with the constant $\tilde{K}$. Thus from (3.2) and (3.4) we obtain

$$\begin{aligned} \int_{B(R_2)} \exp_{[\ell]}(K|u|^\gamma) - S_{\ell,n,\alpha}(K|u|^\gamma) &\leq \int_{B(R_2)} \exp_{[\ell]}(K|u|^\gamma) \\ &\leq \int_{B(R_2) \cap \{u \leq t_0 + \frac{C_s P}{R_2}\}} + \int_{B(R_2) \cap \{u - g(R_2) \geq t_0\}} \\ &\leq C + \int_{B(R_2)} \exp_{[\ell]}(\tilde{K}|u - g(R_2)|^\gamma) < \infty. \end{aligned}$$

Hence we are done. $\square$

Proof of Theorem 1.3(iii). We use the sequence $\{\tilde{w}_k\}$ defined by (2.14) and (2.15), respectively. By (2.16) we know that $\|\nabla \tilde{w}_k\|_{L^\Phi(\Omega)} \leq 1$ for $k \in \mathbb{N}$ large.

Next, we claim that for each $p > 0$ we have

$$\int_{B(x_0, R)} |\tilde{w}_k|^p \xrightarrow{k \rightarrow \infty} 0. \quad (3.5)$$

Let us prove this claim. Fix $p > 0$. Since $\frac{1}{\gamma} - B < 0$ we can find $q \in (0, 1)$ so small that

$$Bq + \frac{1}{\gamma} - B < 0.$$

Now, we define a sequence of auxiliary radii $R_k = R \exp_{[\ell]}^{-1}(k^q)$, $k \in \mathbb{N}$. Notice that for k large enough we have

$$R \exp_{[\ell]}^{-\frac{1}{n}}(k) \leq R_k \leq \frac{R}{2}.$$

We observe from (2.14) and (2.15) that for k large enough we have

$$\int_{B(x_0, R_k)} |\tilde{w}_k|^p \leq \mathcal{L}_n(B(x_0, R_k)) C k^{\frac{p}{\gamma}} = C \exp_{[\ell]}^{-n}(k^q) k^{\frac{p}{\gamma}} \xrightarrow{k \rightarrow \infty} 0$$

and

$$\begin{aligned} \int_{B(x_0, \frac{R}{2}) \setminus B(x_0, R_k)} |\tilde{w}_k|^p &\leq \mathcal{L}_n \left(B \left(x_0, \frac{R}{2} \right) \right) g_k^p(R_k) \\ &\leq C \log_{[\ell]}^{Bp}(\exp_{[\ell]}(k^q)) k^{(\frac{1}{\gamma} - B)p} \\ &= C k^{Bpq + (\frac{1}{\gamma} - B)p} \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Since on $B(x_0, R) \setminus B(x_0, \frac{R}{2})$ we trivially have $|\tilde{w}_k| \leq Ck^{\frac{1}{\gamma}-B} \rightarrow 0$, our claim follows.

Now, as $\Phi(t) \leq Ct^n + Ct^{n+1}$ by (1.4) and (1.5), from (3.5) we obtain

$$\int_{B(x_0, R)} \Phi(|\tilde{w}_k|) \xrightarrow{k \rightarrow \infty} 0$$

and thus (2.5) implies $\|\tilde{w}_k\|_{L^\Phi(\Omega)} \rightarrow 0$. Finally we see that

$$\begin{aligned} & \int_{B(x_0, R)} \exp_{[\ell]}(K|\tilde{w}_k|^\gamma) - S_{\ell, n, \alpha}(K|\tilde{w}_k|^\gamma) \\ & \geq \int_{B(x_0, R \exp_{[\ell]}^{-\frac{1}{n}}(k))} \exp_{[\ell]}(K|\tilde{w}_k|^\gamma) - S_{\ell, n, \alpha}(K|\tilde{w}_k|^\gamma) = I_1 - I_2. \end{aligned}$$

By (3.5) we have $I_2 \rightarrow 0$. Further if $\ell = 1$, then

$$I_1 = C \exp^{-1}(k) \exp\left(\frac{K}{A}k\right) \xrightarrow{k \rightarrow \infty} \infty.$$

If $\ell \geq 2$ and k is large enough, then we can write

$$\begin{aligned} I_1 & \geq C \exp_{[\ell]}^{-1}(k) \exp_{[\ell]}\left(\frac{K}{A}k \frac{\log_{[\ell]}^{B\gamma}(\exp_{[\ell]}^{\frac{1}{n}}(k))}{k^{B\gamma}}\right) \\ & \geq C \exp_{[\ell]}^{-1}(k) \exp_{[\ell]}\left(\sqrt{\frac{K}{A}}k\right) \xrightarrow{k \rightarrow \infty} \infty. \end{aligned}$$

Hence we are done. $\square$

Proof of Theorem 1.3(v). The proof using the sequence $\{w_k\}$ from (2.12) and (2.13), respectively, is similar as the proof of Theorem 1.3(iii). The only noteworthy difference occurs when estimating I_1 for $\ell \geq 2$, but this is done the same way as in [9, Proof of Theorem 4.1]. $\square$

The rest of this section is devoted to several versions of Theorem 1.1 and Theorem 1.3, respectively. These are used in our application to equation (1.9). The first one is a local version of Theorem 1.1(ii).

Proposition 3.1. *Let $\ell \in \mathbb{N}, n \geq 2$ and $\alpha < n - 1$. Suppose that the Young function $\Phi : [0, \infty) \mapsto [0, \infty)$ satisfies (1.4) and (1.5). Let $\{u_k\} \subset WL^\Phi(\mathbb{R}^n)$ satisfy $\|\nabla u_k\|_{L^\Phi(\mathbb{R}^n)} \leq C_1$ and $\|u_k\|_{L^\Phi(\mathbb{R}^n)} \leq C_2$ for every $k \in \mathbb{N}$. Then for every $q \geq 1$ there is $\tau > 0$ with the following property:*

For every $x \in \mathbb{R}^n$ and $R > 0$ we have

$$\|\nabla u_k\|_{L^\Phi(B(x, R))} < \tau \implies \int_{B(x, R)} \exp_{[\ell]}(q|u_k|^\gamma) \leq C.$$

The constant $C \geq 0$ is independent of $k \in \mathbb{N}$ (it may depend on $C_1, C_2, q, \ell, n, \alpha, \Phi, x$ and R).

Sketch of proof. The proof is obtained modifying the proof of Theorem 1.1(ii) (given in [20, Proof of Theorem 1.2] and [9, Proof of Theorem 1.2]) the same way as in [8, Sketch of proof of Proposition 3.4].

We should also note that in paper [8], the set Ω is bounded. However, the boundedness of Ω was used only when showing that the integral averages of u_k over a given ball $B(x, R)$ are uniformly bounded. In our case, it is ensured by the assumption $\|u_k\|_{L^\Phi(\mathbb{R}^n)} \leq C_2$. $\square$

Next, we need a “modular” version of Theorem 1.1(ii).

Proposition 3.2. *Let $\ell \in \mathbb{N}, n \geq 2, \alpha < n - 1$ and $b > 0$. Suppose that the Young function $\Phi : [0, \infty) \mapsto [0, \infty)$ satisfies (1.4) and (1.5). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\{u_k\} \subset W_0 L^\Phi(\Omega)$ satisfy*

$$\int_{\Omega} \Phi(|\nabla u_k|) \leq c < \left(\frac{K_{\ell, n, \alpha}}{b} \right)^{\frac{n}{\gamma}} \quad \text{for every } k \in \mathbb{N}.$$

Then there are $q > 1$ and $C > 0$ such that

$$\int_{\Omega} \exp_{[\ell]}^q(b|u_k|^\gamma) \leq C \quad \text{for every } k \in \mathbb{N}.$$

Sketch of proof. The proof is obtained by a minor modification of the proof of Theorem 1.1(ii) (given in [20, Proof of Theorem 1.2] and [9, Proof of Theorem 1.2]). For more details see the discussion concerning the proof of [8, Proposition 3.2]. $\square$

Finally, we need a version of Proposition 3.2 for unbounded domains.

Proposition 3.3. *Let $\ell \in \mathbb{N}, n \geq 2, \alpha < n - 1, b > 0$ and $P > 0$. Suppose that the Young function $\Phi : [0, \infty) \mapsto [0, \infty)$ satisfies (1.4) and (1.5). Let $\{u_k\} \subset WL^\Phi(\mathbb{R}^n)$ satisfy $\|u_k\|_{L^\Phi(\mathbb{R}^n)} \leq P$ and*

$$\int_{\mathbb{R}^n} \Phi(|\nabla u_k|) \leq c < \left(\frac{K_{\ell, n, \alpha}}{b} \right)^{\frac{n}{\gamma}} \quad \text{for every } k \in \mathbb{N}.$$

Then there are $q > 1$ and $C > 0$ such that

$$\int_{\mathbb{R}^n} \left(\exp_{[\ell]}(b|u_k|^\gamma) - S_{\ell, n, \alpha}(b|u_k|^\gamma) \right)^q \leq C \quad \text{for every } k \in \mathbb{N}.$$

Sketch of proof. The proof is almost the same as the proof of Theorem 1.3(ii). We use (2.1) several times. When estimating the integral over $B(R_2)$, we use Proposition 3.2 instead of Theorem 1.1(ii) (see also Remark 1.2(iii)). $\square$

4. Functional J is C^1

Proposition 4.1. *For the functional J defined by (1.19) we have $J \in C^1(X(\mathbb{R}^n), \mathbb{R})$ and its Fréchet derivative is (1.20).*

Sketch of proof. We use the approach by [4, Proof of Theorem A.V]. In particular, we show that the functional J has the Gateaux derivative everywhere in $X(\mathbb{R}^n)$ and then we show that $u \mapsto J'(u)$ is continuous. First, it is convenient for us to split J into three functionals

$$J_1(u) = \int_{\mathbb{R}^n} \Phi(|\nabla u|), \quad J_2(u) = \int_{\mathbb{R}^n} V(x)\Phi(|u|) \quad \text{and} \quad J_3(u) = \int_{\mathbb{R}^n} F(x, u).$$

To show that J_1 has the Gateaux derivative everywhere in $X(\mathbb{R}^n)$ and that

$$\langle J'_1(u), \varphi \rangle = \int_{\mathbb{R}^n} \Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla \varphi, \quad u, \varphi \in X(\mathbb{R}^n)$$

is continuous, we can use [8, Proof of Lemma 6.2] with a minor modification. Notice that paper [8] deals with the case $\ell = 1$ only, but we have Lemma 2.2 (a multiple exponential version) and thus we can use generalized Hölder's inequality the same way as in paper [8].

The proof that J_2 has the Gateaux derivative everywhere in $X(\mathbb{R}^n)$ and that

$$\langle J'_2(u), \varphi \rangle = \int_{\mathbb{R}^n} V(x)\Phi'(|u|) \frac{u}{|u|} \varphi, \quad u, \varphi \in X(\mathbb{R}^n)$$

is continuous, is similar as above. This time we use generalized Hölder's inequality for spaces $L^\Phi(\mathbb{R}^n, V(x)dx)$ and $L^\Psi(\mathbb{R}^n, V(x)dx)$, etc.

It remains to deal with the functional J_3 . This time we give a detailed proof that

$$\langle J'_3(u), \varphi \rangle = \int_{\mathbb{R}^n} f(x, u)\varphi, \quad u, \varphi \in X(\mathbb{R}^n)$$

and that $u \mapsto J'_3(u)$ is continuous.

Let us start showing that J_3 is Gateaux differentiable everywhere on $X(\mathbb{R}^n)$. For a.e. $x \in \mathbb{R}^n$ it is easy to see that the point wise limit satisfies

$$\lim_{t \rightarrow 0} \frac{F(x, u(x) + t\varphi(x)) - F(x, u(x))}{t} = f(x, u(x))\varphi(x). \quad (4.1)$$

Moreover, we may use the Mean Value Theorem and (1.15) to obtain

$$\begin{aligned} & \left| \frac{F(x, u + t\varphi) - F(x, u)}{t} \right| \\ & \leq |f(x, u + \xi\varphi)| |\varphi| \leq C \left((|u| + |\varphi|)^{n-1} \right. \\ & \quad \left. + \exp_{[\ell]}(b(|u| + |\varphi|)^\gamma) - S_{\ell, n, \alpha}(b(|u| + |\varphi|)^\gamma) \right) |\varphi| \\ & \leq C \left(|u|^{n-1} + |\varphi|^{n-1} + \left(\exp_{[\ell]}(b2^\gamma |u|^\gamma) - S_{\ell, n, \alpha}(b2^\gamma |u|^\gamma) \right) \right. \\ & \quad \left. + \left(\exp_{[\ell]}(b2^\gamma |\varphi|^\gamma) - S_{\ell, n, \alpha}(b2^\gamma |\varphi|^\gamma) \right) \right) |\varphi|. \end{aligned} \quad (4.2)$$

We know that $u, \varphi \in L^s(\mathbb{R}^n)$, for every $s \geq n$, and thus we may apply Hölder's inequality (with the powers n' and n) on the right-hand side of (4.2) and using (2.2) together with Theorem 1.3(i) we easily obtain that the right-hand side

of (4.2) is integrable. Thus, it follows from the Lebesgue Dominated Convergence Theorem applied to (4.1) that

$$\lim_{t \rightarrow 0} \frac{J_3(u + t\varphi) - J_3(u)}{t} = \int_{\mathbb{R}^n} f(x, u)\varphi \quad \text{for every } u, \varphi \in X(\mathbb{R}^n).$$

This is the Gateaux differentiability everywhere on $X(\mathbb{R}^n)$.

To prove that $J'_3(u)$ is continuous, let $u_k \rightarrow u$ in $X(\mathbb{R}^n)$. Passing to a subsequence, we can suppose that $u_k \rightarrow u$ a.e. in $\mathbb{R}^n$ and moreover that there is a majorant $U \in X(\mathbb{R}^n)$, i.e. $|u_k| \leq U$ for every k . The existence of a common majorant is shown in a standard way dealing with a subsequence (still denoted $\{u_k\}$) satisfying $\|u_k - u\|_{X(\mathbb{R}^n)} \leq 2^{-k}$ and setting $U = |u| + \sum_{k=1}^{\infty} |u_k - u|$.

Next, our aim is to prove that

$$\int_{\mathbb{R}^n} |f(x, u_k) - f(x, u)|^{n'} \xrightarrow{k \rightarrow \infty} 0. \quad (4.3)$$

From (1.15), (2.2), Theorem 1.3(i) and from the continuous embedding of $X(\mathbb{R}^n)$ into $L^n(\mathbb{R}^n)$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x, u_k)|^{n'} &\leq C \int_{\mathbb{R}^n} |u_k|^n + C \int_{\mathbb{R}^n} \exp_{[\ell]}(n'b|u_k|^\gamma) - S_{\ell, n, \alpha}(n'b|u_k|^\gamma) \\ &\leq C \int_{\mathbb{R}^n} |U|^n + C \int_{\mathbb{R}^n} \exp_{[\ell]}(n'b|U|^\gamma) - S_{\ell, n, \alpha}(n'b|U|^\gamma) \\ &= C < \infty. \end{aligned}$$

Similarly we obtain

$$\int_{\mathbb{R}^n} |f(x, u)|^{n'} < \infty.$$

Hence (4.3) follows from the fact that $f(x, u_k) \rightarrow f(x, u)$ a.e. in $\mathbb{R}^n$ and from the Lebesgue Dominated Convergence Theorem with the majorant

$$C|U|^n + C(\exp_{[\ell]}(n'b|U|^\gamma) - S_{\ell, n, \alpha}(n'b|U|^\gamma)) + C|f(x, u)|^{n'}.$$

Finally Hölder's inequality together with (4.3) and the continuous embedding of $X(\mathbb{R}^n)$ into $L^n(\mathbb{R}^n)$ imply

$$\begin{aligned} \|J'_3(u_k) - J'_3(u)\|_{C(X(\mathbb{R}^n), \mathbb{R})} &= \sup_{\|\varphi\|_{X(\mathbb{R}^n)} \leq 1} \left| \int_{\mathbb{R}^n} (f(x, u_k) - f(x, u))\varphi \right| \\ &\leq \|f(x, u_k) - f(x, u)\|_{L^{n'}(\mathbb{R}^n)} \\ &\quad \times \sup_{\|\varphi\|_{X(\mathbb{R}^n)} \leq 1} \|\varphi\|_{L^n(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

□

5. Assumptions of the Mountain Pass Theorem

In this section we check that our functional J has the Mountain Pass Geometry (i.e. assumptions (i), (ii) and (iii) from Theorem 2.11 are satisfied).

Lemma 5.1. *If $u \in X(\mathbb{R}^n)$ has a compact support, $u \geq 0$ and $u \neq 0$, then*

$$J(tu) \xrightarrow{t \rightarrow \infty} -\infty.$$

Proof. Since $u \neq 0$ and $u \geq 0$, there is $\tau > 0$ such that

$$\mathcal{L}_n(\{u \geq \tau\}) \geq \tau.$$

Moreover, we observe that it follows from (1.12) and (1.13) that there is $C_1 > 0$ such that

$$F(x, t) \geq C_1 \exp\left(C_1 |t|^{\frac{1}{M}}\right), \quad |t| \geq t_M.$$

Thus, by (1.12) we have a similar inequality on $[\tau, \infty)$ with a constant $C_2 > 0$

$$F(x, t) \geq C_2 \exp(C_1 t^{\frac{1}{M}}) \quad \text{for } t \in [\tau, \infty).$$

Further, for every $t > 1$ we can find $m \in \mathbb{N}$ such that $2^m < t \leq 2^{m+1}$. Thus using the Δ_2 -condition, above estimates, the fact that Φ and $F(x, \cdot)$ are increasing on $[0, \infty)$, the compactness of $\text{supp } u$ and the continuity of $V(x)$ we obtain

$$\begin{aligned} J(tu) &= \int_{\Omega} \Phi(t|\nabla u|) + V(x)\Phi(t|u|) - F(x, tu) \, dx \\ &\leq \int_{\Omega} \Phi(2^{m+1}|\nabla u|) + C\Phi(2^{m+1}|u|) \, dx - \int_{\{u \geq \tau\}} F(x, 2^m u) \, dx \\ &\leq C_{\Delta}^{m+1} \int_{\Omega} \Phi(|\nabla u|) + C\Phi(|u|) \, dx - \tau C_2 \exp(C_1 2^{\frac{m}{M}} \tau^{\frac{1}{M}}) \xrightarrow{m \rightarrow \infty} -\infty. \end{aligned}$$

□

Lemma 5.2. *There are $\varrho > 0$ and $\xi > 0$ with the following property: If $u \in X(\mathbb{R}^n)$ with $\|u\|_{X(\mathbb{R}^n)} = \varrho$, then $J(u) \geq \xi$.*

Proof. Fix $q > n$. By assumptions (1.13), (1.15) and (1.16) we can find $\eta > 0$ so that

$$\begin{aligned} F(x, t) &\leq (1 - 2\eta)C_S\Phi(|t|) + C\left(\exp_{[\ell]}(b|t|^\gamma) - S_{\ell, n, \alpha}(b|t|^\gamma)\right)|t|^q \\ &= F_1(t) + F_2(t). \end{aligned}$$

By (1.17) we obtain

$$\int_{\mathbb{R}^n} F_1(u) = (1 - 2\eta)C_S \int_{\mathbb{R}^n} \Phi(|u|) \leq (1 - 2\eta) \int_{\mathbb{R}^n} \Phi(|\nabla u|) + V(x)\Phi(|u|). \quad (5.1)$$

Next, fix $p > 1$. If ϱ is so small that $b p \varrho^\gamma < K_{\ell, n, \alpha}$, then from Hölder's inequality, Theorem 1.3(ii) (with $P = \varrho$), (2.2) and from the fact that $X(\mathbb{R}^n)$

is continuously embedded into $L^s(\mathbb{R}^n)$, for every $s \in [n, \infty)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} F_2(u) &= C \int_{\mathbb{R}^n} \left(\exp_{[\ell]}(b|u|^\gamma) - S_{\ell,n,\alpha}(b|u|^\gamma) \right) |u|^q \\ &\leq C \left(\int_{\mathbb{R}^n} \exp_{[\ell]} \left(bp\varrho^\gamma \left(\frac{|u|}{\|u\|_{X(\mathbb{R}^n)}} \right)^\gamma \right) \right. \\ &\quad \left. - S_{\ell,n,\alpha} \left(bp\varrho^\gamma \left(\frac{|u|}{\|u\|_{X(\mathbb{R}^n)}} \right)^\gamma \right) \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |u|^{qp'} \right)^{\frac{1}{p'}} \\ &\leq C \|u\|_{L^{qp'}(\mathbb{R}^n)}^q \leq C \|u\|_{X(\mathbb{R}^n)}^q \leq C \|\nabla u\|_{L^\Phi(\mathbb{R}^n)}^q + C \|u\|_{L^\Phi(\mathbb{R}^n, V(x)dx)}^q. \end{aligned}$$

Hence for $\varrho > 0$ small enough Lemma 2.4 with $\varepsilon \in (0, q - n)$ gives

$$\begin{aligned} \int_{\mathbb{R}^n} F_2(u) &\leq C \|\nabla u\|_{L^\Phi(\mathbb{R}^n)}^{q-n-\varepsilon} \|\nabla u\|_{L^\Phi(\mathbb{R}^n)}^{n+\varepsilon} + C \|u\|_{L^\Phi(\mathbb{R}^n, V(x)dx)}^{q-n-\varepsilon} \|u\|_{L^\Phi(\mathbb{R}^n, V(x)dx)}^{n+\varepsilon} \\ &\leq \eta \int_{\mathbb{R}^n} \Phi(|\nabla u|) + V(x)\Phi(|u|). \end{aligned} \quad (5.2)$$

Thus, we obtain from (5.1) and (5.2)

$$J(u) = \int_{\mathbb{R}^n} \Phi(|\nabla u|) + V(x)\Phi(|u|) - F(x, u) \geq \eta \int_{\mathbb{R}^n} \Phi(|\nabla u|) + V(x)\Phi(|u|).$$

Finally, one can easily see from (2.5) that the last integral is bounded away from zero, if the norm is equal to $\varrho > 0$ and thus we are done. $\square$

In the rest of the section we estimate our Palais–Smale level (let us recall that J satisfies the Palais–Smale condition for sufficiently small levels only).

Lemma 5.3. *There is $w \in X(\mathbb{R}^n)$ such that*

$$J(\theta w) < \left(\frac{K_{\ell,n,\alpha}}{b} \right)^{\frac{n}{\gamma}} \quad \text{for every } \theta \in [0, \infty).$$

Proof. Fix $x_0 \in \mathbb{R}^n$ and $R > 0$. By (1.18) we can find $C > 0$ satisfying

$$\liminf_{t \rightarrow \infty} \frac{tf(x, t)}{\exp_{[\ell]}(b|t|^\gamma)} > C \quad \text{uniformly with respect to } x \in \mathbb{R}^n. \quad (5.3)$$

Our aim is to show that there is $k \in \mathbb{N}$ such that the assertion of the lemma holds for w_k given by (2.12) and (2.13), respectively. For the sake of contradiction suppose that for all $k \in \mathbb{N}$ we have

$$\sup\{J(\theta w_k) : \theta \in [0, \infty)\} \geq \left(\frac{K_{\ell,n,\alpha}}{b} \right)^{\frac{n}{\gamma}}.$$

In view of Lemma 5.1 there are $\theta_k > 0, k \in \mathbb{N}$, such that

$$J(\theta_k w_k) = \max\{J(\theta w_k) : \theta \in [0, \infty)\}.$$

Since F is non-negative (see (1.12)), we arrive at

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(\theta_k |\nabla w_k|) + V(x)\Phi(\theta_k |w_k|) &\geq J(\theta_k w_k) = \max\{J(\theta w_k) : \theta \in [0, \infty)\} \\ &\geq \left(\frac{K_{\ell,n,\alpha}}{b} \right)^{\frac{n}{\gamma}} \end{aligned} \quad (5.4)$$

Next, we claim that for each $\theta_0 > 0$ there is $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and $\theta \in [0, \theta_0]$ we have

$$\int_{\mathbb{R}^n} \Phi(\theta |\nabla w_k|) + V(x) \Phi(\theta |w_k|) \leq \theta^n. \quad (5.5)$$

Let us prove this claim. To the first part of the integral we can apply (2.18), i.e. it is estimated by $(1 - \frac{1}{k^{\beta+\varepsilon}})\theta^n$. Concerning the remaining part of the integral, it is enough to use the same argument as when obtaining (3.5). Indeed, since $V(x)$ is bounded on $B(x_0, R)$, θ is bounded, $\Phi(t) \leq Ct^n + Ct^{n+1}$ (by (1.4) and (1.5)), $B > 0$, $B - \frac{1}{\gamma} > 0$ and $(B - \frac{1}{\gamma})n = B$ (by (1.3)), estimates from the proof of (3.5) yield for arbitrary small $q > 0$

$$\begin{aligned} \int_{B(x_0, R)} V(x) \Phi(\theta |w_k|) &\leq C\theta^n k^{Bnq + (\frac{1}{\gamma} - B)n} + C\theta^{n+1} k^{B(n+1)q + (\frac{1}{\gamma} - B)(n+1)} \\ &\leq C\theta^n k^{B(n+1)q + (\frac{1}{\gamma} - B)n} = C\theta^n k^{B(n+1)q - B}. \end{aligned}$$

Hence, as $\beta < B$ ($\beta \in (0, \min\{1, B\})$) comes from assumption (1.7), (5.5) follows easily providing q and ε are small enough.

Now, we claim that θ_k are bounded away from zero. Indeed, for $k \in \mathbb{N}$ such that $\theta_k \leq 1$ and large enough so that (5.5) holds with $\theta = 1$, we have by (5.4), (5.5) and by the fact that Φ is a Young function (hence $\Phi(ts) \leq t\Phi(s)$ for every $t \in [0, 1]$ and $s \geq 0$)

$$\begin{aligned} \theta_k &\geq \theta_k \int_{\mathbb{R}^n} \Phi(|\nabla w_k|) + V(x) \Phi(|w_k|) \geq \int_{\mathbb{R}^n} \Phi(\theta_k |\nabla w_k|) + V(x) \Phi(\theta_k |w_k|) \\ &\geq \left(\frac{K_{\ell, n, \alpha}}{b} \right)^{\frac{n}{\gamma}}. \end{aligned}$$

Further, as $\frac{d}{d\theta} J(\theta w_k)|_{\theta=\theta_k} = 0$, it follows that

$$\int_{\mathbb{R}^n} \Phi'(\theta_k |\nabla w_k|) |\nabla w_k| + V(x) \Phi'(\theta_k |w_k|) |w_k| = \int_{\mathbb{R}^n} w_k f(x, \theta_k w_k).$$

Multiplying both sides by θ_k , using (1.8), (5.3) (recall that θ_k are bounded away from zero) and the definition of w_k we obtain $k_1 \geq k_0$ such that for $k \geq k_1$ we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(\theta_k |\nabla w_k|) + V(x) \Phi(\theta_k |w_k|) &\geq c_\Phi \int_{\mathbb{R}^n} \Phi'(\theta_k |\nabla w_k|) \theta_k |\nabla w_k| \\ &\quad + V(x) \Phi'(\theta_k |w_k|) \theta_k |w_k| \\ &= c_\Phi \int_{\mathbb{R}^n} \theta_k w_k f(x, \theta_k w_k) \\ &\geq c_\Phi \int_{B(x_0, R \exp_{[\ell]}^{-\frac{1}{n}}(k))} \theta_k w_k f(x, \theta_k w_k) \\ &\geq C \int_{B(x_0, R \exp_{[\ell]}^{-\frac{1}{n}}(k))} \exp_{[\ell]}(b |\theta_k w_k|^\gamma) \\ &= C \exp_{[\ell]}^{-1}(k) \exp_{[\ell]}(b |\theta_k w_k|^\gamma). \end{aligned} \quad (5.6)$$

Next, if $\ell = 1$, then

$$\exp_{[\ell]}(b|\theta_k w_k|^\gamma) = \exp\left(\frac{b\theta_k^\gamma}{K_{1,n,\alpha}}(k + \log(k))\right)$$

and if $\ell \geq 2$, then we have by [9, Proof of Theorem 4.1] for k large enough

$$\left(\frac{\log_{[\ell]}(\exp_{[\ell]}^{\frac{1}{n}}(k))}{k}\right)^{B\gamma} (k + \log(k)) \geq k + \frac{1}{2} \log(k)$$

which implies

$$\exp_{[\ell]}(b|\theta_k w_k|^\gamma) \geq \exp_{[\ell]}\left(\frac{b\theta_k^\gamma}{K_{\ell,n,\alpha}}\left(k + \frac{1}{2} \log(k)\right)\right).$$

Therefore there is $k_2 \geq k_1$ so that for $k \geq k_2$ we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(\theta_k |\nabla w_k|) + V(x) \Phi(\theta_k |w_k|) &\geq C \exp_{[\ell]}^{-1}(k) \\ &\times \exp_{[\ell]}\left(\frac{b\theta_k^\gamma}{K_{\ell,n,\alpha}}\left(k + \frac{1}{2} \log(k)\right)\right). \end{aligned} \quad (5.7)$$

Now, for each $k \in \mathbb{N}$ satisfying $\theta_k \geq 2$ let us find $a_k \in \mathbb{N}$ such that $\theta_k \in [2^{a_k}, 2^{a_k+1})$. Therefore the Δ_2 -condition, (5.5) (for $\theta = 1$) and (5.7) give for every $k \geq k_2$ such that $\theta_k \geq 2$

$$\begin{aligned} CC_{\Delta}^{a_k} &\geq C_{\Delta}^{a_k+1} \geq C_{\Delta}^{a_k+1} \int_{\mathbb{R}^n} \Phi(|\nabla w_k|) + V(x) \Phi(|w_k|) \\ &\geq \int_{\mathbb{R}^n} \Phi(\theta_k |\nabla w_k|) + V(x) \Phi(\theta_k |w_k|) \\ &\geq C \exp_{[\ell]}^{-1}(k) \exp_{[\ell]}\left(\frac{b}{K_{a,n,\alpha}} 2^{a_k \gamma} \left(k + \frac{1}{2} \log(k)\right)\right). \end{aligned}$$

Therefore a_k are bounded and thus there is $\theta_0 \geq 1$ such that $\theta_k \leq \theta_0$ for every $k \in \mathbb{N}$.

Hence there is $k_3 \geq k_2$ so that for every $k \geq k_3$ we have a version of estimate (5.5) with $\theta = \theta_k$. It has the following consequences. First, (5.4) and (5.5) give

$$\theta_k \geq \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{1}{\gamma}} \quad \text{for } k \geq k_3. \quad (5.8)$$

Second, (5.5), (5.7) and (5.8) imply

$$\begin{aligned} C &= \theta_0^n \geq \theta_k^n \geq \int_{\mathbb{R}^n} \Phi(\theta_k |\nabla w_k|) + V(x) \Phi(\theta_k |w_k|) \\ &\geq C \exp_{[\ell]}^{-1}(k) \exp_{[\ell]}\left(k + \frac{1}{2} \log(k)\right) \xrightarrow{k \rightarrow \infty} \infty. \end{aligned}$$

Thus, we have a contradiction and we are done. $\square$

6. Properties of the Palais–Smale sequence

In this section we study the properties of the Palais–Smale sequence. Our aim is to show that it contains a subsequence with the gradients converging a.e. in $\mathbb{R}^n$ (see Lemma 6.2) and that the limit (in the sense of (6.6)) is a weak solution to the problem (1.9) (see Lemma 6.3).

Let $\{u_k\}$ be a Palais–Smale sequence from $X(\mathbb{R}^n)$, that is by (2.11),

$$J(u_k) = \int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) - F(x, u_k) \xrightarrow{k \rightarrow \infty} c, \quad (6.1)$$

and by (1.20) there are $\varepsilon_k \rightarrow 0$ such that for every $v \in X(\mathbb{R}^n)$ we have

$$\begin{aligned} |\langle J'(u_k), v \rangle| &= \left| \int_{\mathbb{R}^n} \Phi'(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla v + V(x)\Phi'(|u_k|) \frac{u_k}{|u_k|} v - f(x, u_k)v \right| \\ &\leq \varepsilon_k \|v\|_{X(\mathbb{R}^n)}. \end{aligned} \quad (6.2)$$

Lemma 6.1. *There is a constant $C > 0$ independent of $k \in \mathbb{N}$ such that*

$$\|\nabla u_k\|_{L^\Phi(\mathbb{R}^n)} \leq C, \quad \int_{\mathbb{R}^n} \Phi(|\nabla u_k|) \leq C, \quad (6.3)$$

$$\|u_k\|_{L^\Phi(\mathbb{R}^n, V(x)dx)} \leq C, \quad \int_{\mathbb{R}^n} V(x)\Phi(|u_k|) \leq C, \quad (6.4)$$

and

$$0 \leq \int_{\mathbb{R}^n} f(x, u_k)u_k \leq C. \quad (6.5)$$

Proof. We obtain from (1.14) and (6.1)

$$\int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) \leq C + \int_{\mathbb{R}^n} F(x, u_k) \leq C + A \int_{\mathbb{R}^n} f(x, u_k)u_k.$$

Hence, using (6.2) with $v = u_k$ and (1.8) we arrive at

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) &\leq C + A \left(\int_{\mathbb{R}^n} \Phi'(|\nabla u_k|)|\nabla u_k| + V(x)\Phi'(|u_k|)|u_k| \right. \\ &\quad \left. + \varepsilon_k \|\nabla u_k\|_{X(\mathbb{R}^n)} \right) \\ &\leq C + \frac{A}{c_\Phi} \int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) \\ &\quad + A\varepsilon_k \|\nabla u_k\|_{X(\mathbb{R}^n)}. \end{aligned}$$

Together with Lemma 2.3(ii), $A < c_\Phi$, $\varepsilon_k \rightarrow 0$ and the definition of the norm on $X(\mathbb{R}^n)$ it implies (6.3) and (6.4). The remaining estimate now follows from (6.2) (with $v = u_k$, see also (1.8)). The integral in (6.5) is non-negative by (1.12). $\square$

By (6.3), (6.4) and Proposition 2.10 there is a function $u \in X(\mathbb{R}^n)$ (passing to a suitable subsequence of $\{u_k\}$ if necessary) such that

$$\begin{aligned} u_k &\rightharpoonup u && \text{in } WL^\Phi(\mathbb{R}^n), \\ u_k &\rightarrow u && \text{in } L^\Phi(\mathbb{R}^n), \\ u_k &\rightarrow u && \text{in } L^r(\mathbb{R}^n) \text{ for every } r \in [n, \infty), \\ u_k &\rightarrow u && \text{a.e. in } \mathbb{R}^n. \end{aligned} \quad (6.6)$$

Lemma 6.2. *Passing to a subsequence we have*

$$\nabla u_k \rightarrow \nabla u \quad \text{a.e. on } \mathbb{R}^n. \quad (6.7)$$

Sketch of proof. The proof is similar to the proof of [8, Lemma 5.2], therefore we sketch it pointing out the main differences.

We use the construction from [8, Proof of Lemma 5.2] with the following changes. When we define Ω_ε and ψ_ε according to the fact that $\mathbb{R}^n$ is not bounded we require

$$\Omega_\varepsilon \subset B(\tfrac{1}{\varepsilon}), \quad \mathcal{L}_n(B(\tfrac{1}{\varepsilon}) \setminus \Omega_\varepsilon) < \varepsilon \quad \text{and} \quad \psi_\varepsilon = 0 \quad \text{on } \mathbb{R}^n \setminus B(\tfrac{2}{\varepsilon}).$$

Estimate [8, (5.11)] turns to

$$\|\nabla u_k\|_{L^\Phi(B(x,R))} < \tau \quad \implies \quad \int_{B(x,R)} \exp_{[\ell]} \left(\frac{n}{n-1} \beta |u_k|^\gamma \right) \leq C. \quad (6.8)$$

Our version of this estimate follows from Proposition 3.1 and Lemma 6.1.

Similarly as in [8, Proof of Lemma 5.2] it can be shown that it remains to prove

$$I := \int_{\mathbb{R}^n} \left(\Phi'(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} - \Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) \cdot (\nabla u_k - \nabla u) \psi_\varepsilon \xrightarrow{k \rightarrow \infty} 0.$$

Using (6.2) with $v = \psi_\varepsilon u_k$ and $v = \psi_\varepsilon u$, respectively, and the definition of I , after some computation (cf. [8, Proof of Lemma 5.2, Step 3]) we arrive at

$$\begin{aligned} I &= \int_{\mathbb{R}^n} \Phi'(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla \psi_\varepsilon (u - u_k) + \int_{\mathbb{R}^n} \psi_\varepsilon \Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot (\nabla u - \nabla u_k) \\ &\quad + \int_{\mathbb{R}^n} \psi_\varepsilon f(x, u_k) (u_k - u) + \varepsilon_k \|\psi_\varepsilon u_k\|_{X(\mathbb{R}^n)} + \varepsilon_k \|\psi_\varepsilon u\|_{X(\mathbb{R}^n)} \\ &\quad + \int_{\mathbb{R}^n} V(x) \Phi'(|u_k|) \frac{u_k}{|u_k|} \psi_\varepsilon (u - u_k) \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Now, we want to prove that all six summands tend to zero as $k \rightarrow \infty$.

The proofs that $I_1 \rightarrow 0$ and $I_2 \rightarrow 0$ are similar as in [8, Proof of Lemma 5.2]. In the proof that $I_3 \rightarrow 0$ we have to use Hölder's inequality with different powers

$$\begin{aligned} |I_3| &\leq \int_{\mathbb{R}^n} |\psi_\varepsilon f(x, u_k) (u_k - u)| \leq \|f(x, u_k)\|_{L^{n'}(B(\frac{2}{\varepsilon}) \setminus B_\varepsilon)} \\ &\quad \|u_k - u\|_{L^n(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0, \end{aligned}$$

(recall that $B_\varepsilon \subset B(\frac{2}{\varepsilon}) \setminus \Omega_\varepsilon$ is a small set containing the bad points) where the estimate

$$\|f(x, u_k)\|_{L^{n'}(B(\frac{2}{\varepsilon}) \setminus B_\varepsilon)} \leq C$$

uses (6.8), (1.15), non-negativity of $S_{\ell, n, \alpha}$ and the compactness of $\bar{B}(\frac{2}{\varepsilon}) \setminus B_\varepsilon$. Next, we have

$$\begin{aligned} |I_4| &\leq \varepsilon_k (\|\psi_\varepsilon \nabla u_k\|_{L^\Phi(\mathbb{R}^n)} + \|u_k \nabla \psi_\varepsilon\|_{L^\Phi(\mathbb{R}^n)} + \|\psi_\varepsilon u_k\|_{L^\Phi(\mathbb{R}^n, V(x)dx)}) \\ &= \varepsilon_k (J_1 + J_2 + J_3). \end{aligned}$$

The term J_1 is bounded by (6.3) and $0 \leq \psi_\varepsilon \leq 1$. The boundedness of J_2 follows from $|\nabla \psi_\varepsilon| \leq C$ on $\mathbb{R}^n$ and (6.4) (recall $V(x) \geq V_0 > 0$). Finally, J_3 is bounded by (6.4) and $0 \leq \psi_\varepsilon \leq 1$. Therefore $\varepsilon_k \rightarrow 0$ imply $I_4 \rightarrow 0$.

In the same way we obtain $I_5 \rightarrow 0$.

When showing that $I_6 \rightarrow 0$, it is enough to use the duality between $L^\Phi(\mathbb{R}^n)$ and $L^\Psi(\mathbb{R}^n)$ (see also Lemma 2.2 and (6.6)) and the fact that $V(x)\psi_\varepsilon$ is bounded on $\mathbb{R}^n$. $\square$

Lemma 6.3. *The function $u \in X(\mathbb{R}^n)$ given by (6.6) is a weak solution to problem (1.9), i.e. we have (1.21).*

Proof. The proof consists of two steps. First, we show (1.21) for the test-functions from $C_0^\infty(\mathbb{R}^n)$ only. In the second step, we use the density of $C_0^\infty(\mathbb{R}^n)$ -functions in $X(\mathbb{R}^n)$.

STEP 1.

We want to prove that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla \psi + \int_{\mathbb{R}^n} V(x) \Phi'(|u|) \frac{u}{|u|} \psi \\ &- \int_{\mathbb{R}^n} f(x, u) \psi = 0 \quad \text{whenever } \psi \in C_0^\infty(\mathbb{R}^n). \end{aligned} \quad (6.9)$$

In view of (6.2) with $v = \psi$ it is enough to prove that for each fixed $\psi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \Phi'(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla \psi \xrightarrow{k \rightarrow \infty} \int_{\mathbb{R}^n} \Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot \nabla \psi, \quad (6.10)$$

$$\int_{\mathbb{R}^n} V(x) \Phi'(|u_k|) \frac{u_k}{|u_k|} \psi \xrightarrow{k \rightarrow \infty} \int_{\mathbb{R}^n} V(x) \Phi'(|u|) \frac{u}{|u|} \psi \quad (6.11)$$

and

$$\int_{\mathbb{R}^n} f(x, u_k) \psi \xrightarrow{k \rightarrow \infty} \int_{\mathbb{R}^n} f(x, u) \psi. \quad (6.12)$$

Let us prove (6.10). By the reflexivity of $L^\Psi(\mathbb{R}^n)$, Lemma 2.2 and (6.3) we can pass to a subsequence to obtain

$$\Phi'(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \rightharpoonup w \quad \text{in } L^\Psi(\mathbb{R}^n, \mathbb{R}^n).$$

However, we know that $\nabla u_k \rightarrow \nabla u$ a.e. in $\mathbb{R}^n$ by (6.7), and thus the continuity of Φ' and the fact that the weak limit has to be the same as the a.e. point wise limit imply

$$\Phi'(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \rightharpoonup \Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \quad \text{in } L^\Psi(\mathbb{R}^n, \mathbb{R}^n).$$

As $\nabla \psi$ can be used as a test-function, we obtain (6.10).

The proof of (6.11) follows easily from the Generalized Lebesgue Dominated Convergence Theorem (Proposition 2.7). Indeed, ψ is bounded, $V(x)$ is bounded on $\text{supp } \psi$, $\Phi'(t) \leq Ct^{n-1} + Ct^n$ (by (1.4), (1.5) and (1.8)), and we have (6.6).

Finally, by (1.15) and Theorem 1.3(i) we have $f(x, u), f(x, u_k) \in L^1(\text{supp } \psi)$. Since we also have (6.5), Lemma 2.6 (with $\theta = 0$ and $\Omega = \text{supp } \psi$) implies (6.12). Thus, we have proved (6.9).

STEP 2.

Fix $v \in X(\mathbb{R}^n)$. By Proposition 2.8 there are $\{\psi_k\} \subset C_0^\infty(\mathbb{R}^n)$ such that $\psi_k \rightarrow v$ in $X(\mathbb{R}^n)$.

By Lemma 2.2 we observe

$$\Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \in L^\Psi(\mathbb{R}^n, \mathbb{R}^n).$$

Thus, as $\psi_k \rightarrow v$ in $X(\mathbb{R}^n)$ plainly implies $\nabla \psi_k \rightarrow \nabla v$ in $L^\Phi(\mathbb{R}^n)$, we obtain from generalized Hölder's inequality (2.4)

$$\int_{\mathbb{R}^n} \Phi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \cdot (\nabla v - \nabla \psi_k) \xrightarrow{k \rightarrow \infty} 0. \quad (6.13)$$

Next, since $\psi_k \rightarrow v$ in $L^\Phi(\mathbb{R}^n, V(x)dx)$ and $\Phi'(|u|) \frac{u}{|u|} \in L^\Psi(\mathbb{R}^n, V(x)dx)$ (by Lemma 2.2 and $u \in X(\mathbb{R}^n)$), generalized Hölder's inequality (2.4) gives

$$\begin{aligned} \left| \int_{\mathbb{R}^n} V(x) \Phi'(|u|) \frac{u}{|u|} (v - \psi_k) \right| &\leq 2 \|v - \psi_k\|_{L^\Phi(\mathbb{R}^n, V(x)dx)} \\ &\quad \times \left\| \Phi'(|u|) \frac{u}{|u|} \right\|_{L^\Psi(\mathbb{R}^n, V(x)dx)} \xrightarrow{k \rightarrow \infty} 0. \end{aligned} \quad (6.14)$$

Finally, by Hölder's inequality, (1.15), (2.2), Theorem 1.3(i) and continuous embedding of $X(\mathbb{R}^n)$ into $L^n(\mathbb{R}^n)$ we easily obtain

$$\left| \int_{\mathbb{R}^n} f(x, u) (v - \psi_k) \right| \leq \|f(x, u)\|_{L^{n'}(\mathbb{R}^n)} \|v - \psi_k\|_{L^n(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0. \quad (6.15)$$

Now, (1.21) follows from (6.9), (6.13), (6.14) and (6.15). $\square$

Lemma 6.4. *The function $u \in X(\mathbb{R}^n)$ given by (6.6) satisfies*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} F(x, u_k) = \int_{\mathbb{R}^n} F(x, u).$$

Proof. Fix $\varepsilon > 0$. As $F(x, u) \in L^1(\mathbb{R}^n)$ (we use suitably (1.14), (1.15), Theorem 1.3(i) and Hölder's inequality), there is $R_1 > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(R)} F(x, u) < \varepsilon \quad \text{whenever } R \geq R_1. \quad (6.16)$$

Next, there is $T > 0$ such that

$$\frac{1}{T^{\frac{1}{M}}} < \varepsilon \quad (6.17)$$

(recall that M comes from (1.13)). In view of (1.5), (1.14), (1.15), (1.16) we can find $L \geq 1$ such that

$$F(x, t) \leq Lt^n \quad \text{for } t \in [0, T]. \quad (6.18)$$

Finally, by (1.11), there is $R_0 \geq R_1$ such that

$$V(x) \geq \frac{L}{\varepsilon} \quad \text{for } |x| \geq R_0. \quad (6.19)$$

Now, we proceed to the final computation. We have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} F(x, u_k) - F(x, u) \right| &\leq \int_{B(R_0)} |F(x, u_k) - F(x, u)| + \int_{\mathbb{R}^n \setminus B(R_0)} F(x, u_k) \\ &\quad + \int_{\mathbb{R}^n \setminus B(R_0)} F(x, u) = I_1 + I_2 + I_3. \end{aligned}$$

By (1.15), Theorem 1.3(i) and Hölder's inequality we have $f(x, u)u \in L^1(\mathbb{R}^n)$. Hence, (6.5) and Lemma 2.6 with $\theta = 1 - \frac{1}{M}$ imply that $f(x, u_k)|u_k|^{1-\frac{1}{M}} \rightarrow f(x, u)|u|^{1-\frac{1}{M}}$ in $L^1(B(R_0))$ and thus by (1.13) and Proposition 2.7 we see that $I_1 \rightarrow 0$.

Next, we estimate I_2 . We have

$$\begin{aligned} |I_2| &\leq \int_{(\mathbb{R}^n \setminus B(R_0)) \cap \{|u_k| > T\}} |F(x, u_k)| \\ &\quad + \int_{(\mathbb{R}^n \setminus B(R_0)) \cap \{|u_k| \leq T\}} |F(x, u_k)| = J_1 + J_2. \end{aligned}$$

By (1.13), (6.5) and (6.17) we obtain

$$\begin{aligned} |J_1| &= \int_{(\mathbb{R}^n \setminus B(R_0)) \cap \{|u_k| > T\}} |F(x, u_k)| \leq \int_{\{|u_k| > T\}} M|u_k|^{1-\frac{1}{M}} |f(x, u_k)| \\ &\leq \frac{M}{T^{\frac{1}{M}}} \int_{\mathbb{R}^n} u_k f(x, u_k) \leq C\varepsilon. \end{aligned}$$

Further, $\Phi(t) \geq Ct^n$ on $[0, T]$ (by (1.4) and the fact that Φ is a Young function), (6.4), (6.18) and (6.19) imply

$$\begin{aligned} |J_2| &= \int_{(\mathbb{R}^n \setminus B(R_0)) \cap \{|u_k| \leq T\}} |F(x, u_k)| \leq \int_{(\mathbb{R}^n \setminus B(R_0)) \cap \{|u_k| \leq T\}} L|u_k|^n \\ &\leq \int_{(\mathbb{R}^n \setminus B(R_0)) \cap \{|u_k| \leq T\}} LC\Phi(|u_k|) \leq C\varepsilon \int_{\mathbb{R}^n \setminus B(R_0)} V(x)\Phi(|u_k|) \leq C\varepsilon. \end{aligned}$$

Hence $|I_2| \leq C\varepsilon$.

Finally, $|I_3| \leq \varepsilon$ by (6.16). Thus, we are done. $\square$

7. Proof of Theorem 1.4

Proof of Theorem 1.4. Since we have $J(0) = 0$, Lemmata 5.1, 5.2 and Proposition 4.1, we can apply the Mountain Pass Theorem (Theorem 2.11) which together with Lemma 5.3 gives us a Palais–Smale sequence $\{u_k\} \subset X(\mathbb{R}^n)$ approaching a Palais–Smale level $0 < c < (\frac{K_{\ell,n,\alpha}}{b})^{\frac{n}{\gamma}}$. Passing to a subsequence we can further suppose that we have (6.6). Next, $u \in X(\mathbb{R}^n)$ given by (6.6) is a weak solution to (1.9) by Lemma 6.3. It remains to show that u is non-trivial.

For the sake of contradiction suppose that we have $u = 0$. From (6.1), $u = 0$, $F(x, 0) = 0$, Lemma 6.4 and from the upper estimate concerning the level c we obtain $\tilde{c} > c$ such that for k sufficiently large we have

$$\int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) \leq \tilde{c} < \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{n}{\gamma}}.$$

Hence, as the second term in the above integral is non-negative, estimate (6.4) (recall that $V(x) \geq V_0 > 0$) together with Proposition 3.3 give us $q > 1$ close to 1 that $q' = \frac{q}{q-1} \geq n$ and

$$\int_{\mathbb{R}^n} \left(\exp_{[\ell]}(b|u_k|^\gamma) - S_{\ell,n,\alpha}(b|u_k|^\gamma) \right)^q \leq C.$$

Now, from above estimate, (1.15), (6.6), $q' \geq n$ and Hölder's inequality we infer

$$\left| \int_{\mathbb{R}^n} f(x, u_k) u_k \right| \leq C \|u_k\|_{L^n(\mathbb{R}^n)}^n + C \|\exp_{[\ell]}(b|u_k|^\gamma) - S_{\ell,n,\alpha}(b|u_k|^\gamma)\|_{L^q(\mathbb{R}^n)} \|u_k\|_{L^{q'}(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0.$$

Therefore (6.2) with $v = u_k$, (6.3) and (6.4) imply

$$\int_{\mathbb{R}^n} \Phi'(|\nabla u_k|) |\nabla u_k| + V(x) \Phi'(|u_k|) |u_k| \xrightarrow{k \rightarrow \infty} 0.$$

Next, as Φ is a Young function, we have $\Phi(t) \leq t\Phi'(t)$ for every $t > 0$ and thus we obtain from above

$$\int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) \xrightarrow{k \rightarrow \infty} 0.$$

However, in view of (6.1), Lemma 6.4 and $u = 0$ this contradicts $c > 0$. Hence u is non-trivial and we are done. $\square$

8. Sub-critical case

We can use our methods to obtain the existence of a non-trivial weak solution to (1.9) also in the sub-critical case. It is, instead of (1.15) we have

$$\begin{aligned} &\text{for every } b > 0 \text{ there is } C_b > 0 \text{ such that} \\ &|f(x, t)| \leq C|t|^{n-1} + C_b \left(\exp_{[\ell]}(b|t|^\gamma) - S_{\ell,n,\alpha}(b|t|^\gamma) \right) \\ &\text{whenever } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n. \end{aligned} \tag{8.1}$$

In this case we do not need to assume (1.7) and (1.18).

Theorem 8.1. *Let $\ell \in \mathbb{N}, n \geq 2$ and $\alpha < n - 1$. Suppose that the C^1 -Young function $\Phi : [0, \infty) \mapsto [0, \infty)$ satisfies (1.4) and (1.5). Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ satisfy (1.10) and (1.11) and let $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying (1.12), (1.13), (1.14), (8.1) and (1.16). Then problem (1.9) has a non-trivial weak solution in $X(\mathbb{R}^n)$.*

Proof. Since assumptions (1.7) and (1.18) were used in the proof of Lemma 5.3 only, we can use all our partial results but Lemma 5.3.

Fix $w \in X(\mathbb{R}^n)$ such that w has a compact support, $w \geq 0$ and $w \neq 0$. By Lemma 5.1 and non-negativity of F (see (1.13)), we observe that

$$\sup_{t \in [0, \infty)} J(tw) < \infty. \quad (8.2)$$

Since we have $J(0) = 0$, Lemmata 5.1, 5.2 and Proposition 4.1, we can apply the Mountain Pass Theorem (Theorem 2.11) which together with (8.2) gives us a Palais–Smale sequence $\{u_k\} \subset X(\mathbb{R}^n)$ approaching a Palais–Smale level $c \in (0, \infty)$.

Moreover, we can find $b_0 > 0$ small enough so that $c \in (0, (\frac{K_{\ell, n, \alpha}}{b_0})^{\frac{n}{\gamma}})$. Finally, since assumption (8.1) implies inequality (1.15) with $b = b_0$, we conclude the proof in the same way as the proof of Theorem 1.4. $\square$

Acknowledgments

The author was supported by the research project MSM 0021620839 of the Czech Ministry MŠMT. The author would like to thank Professors Pavel Drábek, Stanislav Hencl and Dalibor Pražák for fruitful discussions.

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Received: 23 November 2010.

Accepted: 18 October 2011.