Deterministic homogenization of weakly damped nonlinear hyperbolic-parabolic equations

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Abstract. Deterministic homogenization is studied for nonlinear hyperbolic-parabolic equations with a linear damping term. It is shown by the sigma-convergence method that the sequence of solutions to a class of highly oscillatory nonlinear evolution problems converges to the solution to a homogenized quasilinear hyperbolic-parabolic problem.

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1. Introduction

We study the homogenization (as $0 < \varepsilon \rightarrow 0$) of the initial-boundary value problem

$$\begin{cases} \frac{\partial^2 u_\varepsilon}{\partial t^2} + g^\varepsilon(x, t, \frac{\partial u_\varepsilon}{\partial t}) - \varepsilon \Delta \frac{\partial u_\varepsilon}{\partial t} - \frac{\partial}{\partial x_j} \left(a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \right) = f & \text{in } Q, \\ u_\varepsilon(x, 0) = \hat{u}_0(x) & \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial t}(x, 0) = \hat{u}_1(x) & \text{in } \Omega, \\ u_\varepsilon(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (1.1)$$

with the convention of Einstein, where Ω is a bounded open set in $\mathbb{R}^N$ (the N -dimensional numerical space of variables $x = (x_1, \dots, x_N)$, integer $N \geq 1$), D and Δ denote the usual gradient and Laplacian operators in Ω , T is a positive real number, $Q = \Omega \times (0, T)$, $f \in L^2(Q)$, $\hat{u}_0 \in W_0^{1,p}(\Omega)$, $2 \leq p < \infty$, $\hat{u}_1 \in L^2(\Omega)$, and where $g^\varepsilon(x, t, \zeta) = g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \zeta)$ and $a_{ij}^\varepsilon(x, t, \lambda) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \lambda)$, $\varepsilon > 0$, $\zeta \in \mathbb{R}$, $\lambda \in \mathbb{R}^N$, $(x, t) \in Q$, the mappings $(y, \tau, \zeta) \rightarrow g(y, \tau, \zeta)$ and $(y, \tau, \lambda) \rightarrow a_{ij}(y, \tau, \lambda)$, $i, j = 1, \dots, N$, satisfying the following properties:

- (i) $(y, \tau, \zeta) \rightarrow g(y, \tau, \zeta)$ is a continuous real valued function on $\mathbb{R}_y^N \times \mathbb{R}_\tau \times \mathbb{R}_\zeta$ with

- (i)₁ $g(\cdot, \cdot, \zeta) \in \mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_\tau)$, with $g(\cdot, \cdot, \zeta)\zeta \geq 0$ for all $\zeta \in \mathbb{R}$,
- (i)₂ $g(y, \tau, \cdot)$ is strictly convex for all $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$,
- (i)₃ there are three constants $2 \leq q \leq p$ and $c_0, c_1 > 0$ such that $c_0|\zeta|^q \leq g(y, \tau, \zeta) \leq c_1(1 + |\zeta|^q)$ for all $(y, \tau, \zeta) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$,
- (i)₄ there is a positive differentiable real function σ on $\mathbb{R}_+$ such that

$$\left(g\left(\cdot, \cdot, \frac{d\zeta}{dt}\right) - g\left(\cdot, \cdot, \frac{d\varsigma}{dt}\right) \right) (\zeta - \varsigma) \geq \frac{d}{dt} \sigma(|\zeta - \varsigma|)$$

almost everywhere in $(0, T) \times \mathbb{R}^N \times \mathbb{R}$ for all functions $\zeta = \zeta(t), \varsigma = \varsigma(t)$ admitting distributional derivatives that are measurable functions;

- (ii) $(y, \tau, \lambda) \rightarrow a_{ij}(y, \tau, \lambda), i, j = 1, \dots, N$ are real functions defined on $\mathbb{R}_y^N \times \mathbb{R}_\tau \times \mathbb{R}_\lambda$ with:
 - (ii)₁ for every $(\tau, \lambda) \in \mathbb{R} \times \mathbb{R}^N$ and $i, j = 1, \dots, N$, the function $a_{ij}(\cdot, \tau, \lambda)$ from $\mathbb{R}_y^N$ into $\mathbb{R}$ is measurable,
 - (ii)₂ for almost every $y \in \mathbb{R}^N$ and for all $i, j = 1, \dots, N$, the function $a_{ij}(y, \cdot, \cdot)$ from $\mathbb{R}_\tau \times \mathbb{R}_\lambda$ into $\mathbb{R}$ admits distributional derivatives $\frac{\partial a_{ij}}{\partial \tau}, \frac{\partial a_{ij}}{\partial \lambda_k}, k = 1, \dots, N$, which are measurable functions such that there are $\eta_0 > 0$ and positive decreasing functions c_{ij} on $(0, +\infty)$ that are continuous and $c_{ij}(\theta) \leq \theta^{-2}$ with $\eta \left| \frac{\partial a_{ij}}{\partial \tau}(y, \eta\tau, \lambda) \right| \leq c_{ij}(|\lambda|)$ for all $\eta \geq \eta_0$ and for almost all $(y, \tau, \lambda) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, and there are positive decreasing functions $d_{ij} > 0$ on $(0, +\infty)$ that are continuous and $d_{ij}(\theta) \leq \theta^{-2}$ with $\sum_{\ell=1}^N \left| \frac{\partial \lambda_\ell}{\partial t}(x, t) \frac{\partial a_{ij}}{\partial \lambda_\ell}(y, \tau, \lambda(x, t)) \right| \leq d_{ij}(|\lambda(x, t)|)$ for almost all $(x, t, y, \tau) \in Q \times \mathbb{R}^N \times \mathbb{R}$, for all function $\lambda = \lambda(x, t)$ admitting a distributional time derivative which is a measurable function,
 - (ii)₃ for almost all $(y, \lambda) \in \mathbb{R}^N \times \mathbb{R}^N$, each $a_{ij}(y, 0, \lambda)$ exists, and there are $0 < \alpha \leq 1$ and a constant c_2 such that (with the convention of Einstein)

$$|a_{ij}(y, \tau, \lambda)\lambda_i - a_{ij}(y, \tau, \mu)\mu_i| \leq c_2|\lambda - \mu|^\alpha$$

- a.e. in $\mathbb{R}_y^N \times \mathbb{R}_\tau$ for all $\lambda, \mu \in \mathbb{R}^N$ and for all $j = 1, \dots, N$,
- (ii)₄ $a_{ij}(y, \tau, \lambda) = a_{ji}(y, \tau, \lambda)$ a.e. in $\mathbb{R}_y^N \times \mathbb{R}_\tau$ for all $\lambda \in \mathbb{R}^N$ and for all $i, j = 1, \dots, N$,
- (ii)₅ there exists a constant $c_3 > 0$ such that

$$(a_{ij}(y, \tau, \lambda)\lambda_i - a_{ij}(y, \tau, \mu)\mu_i)(\lambda_j - \mu_j) \geq c_3|\lambda - \mu|^p$$

a.e. in $\mathbb{R}_y^N \times \mathbb{R}_\tau$ for all $\lambda = (\lambda_i), \mu = (\mu_i) \in \mathbb{R}^N$.

The equation of the form

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} + g\left(x, t, \frac{\partial u_\varepsilon}{\partial t}\right) - \varepsilon \Delta \left(\frac{\partial u_\varepsilon}{\partial t} \right) - \frac{\partial}{\partial x_j} \left(a_{ij}(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \right) = f \quad (1.2)$$

was first proposed by Greenberg, MacCamy and Mizel [15] when $N = 1$ in approaching quasilinear wave equations for nonlinear string model

$$\frac{\partial^2 u_0}{\partial t^2} - a \left(\frac{\partial u_0}{\partial x} \right) \frac{\partial^2 u_0}{\partial x^2} = 0$$

where a is a positive smooth function. Several authors have used equations of the form (1.2), precisely when $g^\varepsilon(x, t, \zeta)$ and $a_{ij}^\varepsilon(x, t, \lambda)$ do not depend to ε, x and t , in proving the existence and uniqueness of initial value problems to quasilinear wave equations

$$\begin{cases} \frac{\partial^2 u_0}{\partial t^2} + g \left(\frac{\partial u_0}{\partial t} \right) - a_{ij}(Du_0) \frac{\partial^2 u_0}{\partial x_i^2} = f, & t \geq 0, \\ u_0(x, 0) = \hat{u}_0(x), \\ \frac{\partial u_0}{\partial t}(x, 0) = \hat{u}_1(x), \end{cases} \quad (1.3)$$

see [12, 38, 41–44]. In this case, that is, the dependance to ε of u_ε is just with respect to the strong dissipative term $\varepsilon \Delta \frac{\partial u_\varepsilon}{\partial t}$ which may be regarded as the viscosity term, so that the authors of references cited above employed the viscosity method to get the global existence of solutions to (1.3). The viscosity method consists in passing to the limit when $\varepsilon \rightarrow 0$ in (1.2) and showing that the limiting function of u_ε (as $\varepsilon \rightarrow 0$) is a solution of (1.3). A typical example of (1.3) is the following equation

$$\frac{\partial^2 u_0}{\partial t^2} + c \frac{\partial u_0}{\partial t} - \frac{\partial}{\partial x_i} \left(\frac{1}{\sqrt{1 + |Du_0|^2}} \frac{\partial u_0}{\partial x_i} \right) = f$$

which describes the nonlinear vibration of a string for $N = 1$ and that of a membrane for $N = 2$.

The asymptotic behaviour studied in the present work falls into the branch of *homogenization theory* (see, e.g., [4, 8, 16, 17, 34, 35, 37]). Especially we are concerned with the subbranch of deterministic homogenization theory; we can mention some pioneering works as [2, 21, 23, 25]. In this context, as opposed to what is usually done in the viscosity method, we do not know a priori the limit problem which depends to $g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \frac{\partial u_\varepsilon}{\partial t})$, $a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, Du_\varepsilon)$ and $\varepsilon \Delta \frac{\partial u_\varepsilon}{\partial t}$; and it is well known that the latter is a weakly damping term [26]. A damping term depending on time sometimes plays a crucial role in the general sigma-convergence results. Indeed, the sigma-limit of $\frac{\partial u_\varepsilon}{\partial x_i}$ as $\varepsilon \rightarrow 0$ does not depend on the time variable $\tau \in \mathbb{R}$, as in the classical formulation (see, e.g., [32]). We also refer to [28] where, with the damping term $\Delta \frac{\partial u_\varepsilon}{\partial t}$, the time variable $\tau \in \mathbb{R}$ is ignored by corrector terms. In most of the previous works, ε^q ($0 < q < \infty$) is not a factor of the damping term (see, e.g., [27, 28]). With this in mind, it appears that: firstly test functions are not taken in the natural fashion [18, 19, 24, 25, 32], secondly the deterministic homogenization generalizes the viscosity method.

The main purpose in this paper is that, under various deterministic hypotheses beyond the classical setting using the viscosity method, we study the limiting behaviour, as $\varepsilon \rightarrow 0$, of u_ε [the solution of (1.1)] in the major case

where functions $g^\varepsilon(\cdot, \cdot, \zeta)$ and $a_{ij}^\varepsilon(\cdot, \cdot, \lambda)$, $1 \leq i, j \leq N$, are of the form

$$g^\varepsilon(x, t, \zeta) = g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \zeta\right) \text{ and } a_{ij}^\varepsilon(x, t, \lambda) = a_{ij}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \lambda\right) \quad ((x, t) \in Q).$$

In others words, we investigate the homogenization of (1.1) in a general deterministic setting of the behaviour of functions $(y, \tau) \rightarrow g(y, \tau, \lambda)$ and $(y, \tau) \rightarrow a_{ij}(y, \tau, \lambda)$, $1 \leq i, j \leq N$, including periodicity, almost periodicity and other concrete non-periodic settings. Our point of departure is the sigma-convergence [23], which is a natural generalization of the two-scale convergence [2, 21]. Concerning, many authors studied homogenization problems of quasilinear wave equations, we can mention papers [3, 11, 13, 26, 28, 36, 39] sometimes in the periodic setting. We also refer to [20] who homogenized hyperbolic-parabolic problems in perforated domains.

The study is organized as follows. Section 2 is devoted to some preliminary results and notions. In Sect. 3 we prove a convergence theorem providing the solution of the homogenization problem for (1.1) under abstract deterministic hypotheses on $g(\cdot, \cdot, \zeta)$ and $a_{ij}(\cdot, \cdot, \lambda)$ ($1 \leq i, j \leq N$). In Sect. 4, finally, we give some concrete examples of which the study reduces to the abstract setting of Sect. 3.

2. Preliminaries

2.1. Weak solution of (1.1)

Let $v \in \mathcal{C}(\overline{Q}; \mathbb{R})$ and $\varepsilon > 0$. The convexity of the function $g(y, \tau, \cdot)$ [see (i₂)] combined with (i₃) implies

$$|g(y, \tau, \zeta) - g(y, \tau, \zeta')| \leq c_4(1 + |\zeta|^{q-1} + |\zeta'|^{q-1})|\zeta - \zeta'| \quad (2.1)$$

for all $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$ and all $\zeta, \zeta' \in \mathbb{R}$. Then, taking into account (i₁), the function $(x, t, y, \tau) \rightarrow g(y, \tau, v(x, t))$ of $\overline{Q} \times \mathbb{R}^N \times \mathbb{R}$ into $\mathbb{R}_+$ lies in $\mathcal{C}(\overline{Q}; \mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_\tau; \mathbb{R}))$. Consequently the function $(x, t) \rightarrow g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, v(x, t))$ of $\overline{Q}$ into $\mathbb{R}_+$, denoted by $g^\varepsilon(-, v)$, is well defined as an element of $\mathcal{C}(\overline{Q}; \mathbb{R})$ with

$$\|g^\varepsilon(-, u) - g^\varepsilon(-, v)\|_{L^{q'}(Q; \mathbb{R})} \leq c_4(\|1 + |u| + |v|\|_{L^q(\Omega)}^{q-1})\|u - v\|_{L^q(Q)}$$

and

$$c_0\|v\|_{L^q(Q)}^q \leq \|g^\varepsilon(-, v)\|_{L^1(Q)} \leq c_1 \max(1, |Q|)(1 + \|v\|_{L^q(Q)}^q),$$

where $|Q| = \int_Q dx dt$; and there is no difficulty in proving the following proposition (see [29]).

Proposition 2.1. *The transformation $v \rightarrow g^\varepsilon(-, v)$ of $\mathcal{C}(\overline{Q}; \mathbb{R})$ into $\mathcal{C}(\overline{Q}; \mathbb{R})$ extends by continuity to a mapping, still denoted by $v \rightarrow g^\varepsilon(-, v)$, of $L^q(Q; \mathbb{R})$ into $L^{q'}(Q; \mathbb{R})$ with properties*

$$\|g^\varepsilon(-, u) - g^\varepsilon(-, v)\|_{L^{q'}(Q; \mathbb{R})} \leq c_4(\|1 + |u| + |v|\|_{L^q(\Omega)}^{q-1})\|u - v\|_{L^q(Q)}$$

and

$$c_0 \|v\|_{L^q(Q)}^q \leq \|g^\varepsilon(-, v)\|_{L^1(Q)} \leq c_1 \max(1, |Q|)(1 + \|v\|_{L^q(Q)}^q) \quad (2.2)$$

for all $u, v \in L^q(Q; \mathbb{R})$.

Given $\mathbf{v} = (v_i) \in \mathcal{C}(\overline{Q}; \mathbb{R})^N$ and $1 \leq i, j \leq N$, there is no difficulty, using (ii), to show that the function $(x, t, y, \tau) \rightarrow a_{ij}(y, \tau, \mathbf{v}(x, t))v_i(x, t)$ of $\overline{Q} \times \mathbb{R}^N \times \mathbb{R}$ into $\mathbb{R}$ belongs to $\mathcal{C}(\overline{Q}; L^\infty(\mathbb{R}_y^N \times \mathbb{R}_\tau))$. Hence for fixed $\varepsilon > 0$ the function $(x, t) \rightarrow a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \mathbf{v}(x, t))v_i(x, t)$ of Q into $\mathbb{R}$, denoted by $a_{ij}^\varepsilon(-, \mathbf{v})v_i$, is well defined as a function in $L^\infty(Q; \mathbb{R})$ (see [25]). We have the following

Proposition 2.2. *For each $1 \leq i, j \leq N$, the transformation $\mathbf{v} \rightarrow a_{ij}^\varepsilon(-, \mathbf{v})v_i$ of $\mathcal{C}(\overline{Q}; \mathbb{R})^N$ into $L^\infty(Q; \mathbb{R})$ extends by continuity to a mapping, still denoted by $\mathbf{v} \rightarrow a_{ij}^\varepsilon(-, \mathbf{v})v_i$, of $L^p(Q; \mathbb{R})^N$ into $L^{p'}(Q; \mathbb{R})$ with the property*

$$\|a_{ij}^\varepsilon(-, \mathbf{v})v_i - a_{ij}^\varepsilon(-, \mathbf{w})w_i\|_{L^{p'}(Q; \mathbb{R})} \leq c_2 \|\mathbf{v} - \mathbf{w}\|_{L^p(Q; \mathbb{R})^N}^\alpha \quad (2.3)$$

for all $\mathbf{v}, \mathbf{w} \in L^p(Q; \mathbb{R})^N$ ($1 \leq j \leq N$).

Proof. By applying Lemma 2.2 in [25] with

$$\begin{aligned} u(x, t, y, \tau) &= c_2 |\mathbf{v}(x, t) - \mathbf{w}(x, t)|^\alpha \\ &\quad - |a_{ij}(y, \tau, \mathbf{v}(x, t))v_i(x, t) - a_{ij}(y, \tau, \mathbf{w}(x, t))w_i(x, t)| \end{aligned}$$

$\mathbf{v}, \mathbf{w} \in \mathcal{C}(\overline{Q}; \mathbb{R})^N$, $1 \leq j \leq N$, it follows from (ii₂) that

$$(|a_{ij}^\varepsilon(-, \mathbf{v})v_i - a_{ij}^\varepsilon(-, \mathbf{w})w_i|)^{p'} \leq c_2^{p'} (|\mathbf{v} - \mathbf{w}|^{\frac{\alpha}{p-1}})^p \quad \text{a.e. in } Q.$$

Integrating over Q yields

$$\|a_{ij}^\varepsilon(-, \mathbf{v})v_i - a_{ij}^\varepsilon(-, \mathbf{w})w_i\|_{L^{p'}(Q; \mathbb{R})}^{p'} \leq c_2^{p'} \|\mathbf{v} - \mathbf{w}\|_{L^p(Q; \mathbb{R})^N}^{\frac{p\alpha}{p-1}};$$

hence we get (2.3) for all $\mathbf{v}, \mathbf{w} \in \mathcal{C}(\overline{Q}; \mathbb{R})^N$. Since $\mathcal{C}(\overline{Q}; \mathbb{R})^N$ is dense in $L^p(Q; \mathbb{R})^N$, by continuity it follows (2.3) for all $\mathbf{v}, \mathbf{w} \in L^p(Q; \mathbb{R})^N$. $\square$

As a consequence of Propositions 2.1 and 2.2, we have the following

Corollary 2.1. *Let $\varepsilon > 0$.*

(i) *For all $u \in \mathcal{D}'(Q)$ such that $\frac{\partial u}{\partial t} \in L^q(0, T; L^q(\Omega; \mathbb{R}))$ one has*

$$\left\| g^\varepsilon\left(-, \frac{\partial u}{\partial t}\right) \right\|_{L^{q'}(Q)} \leq c_1 \max(1, |Q|) \left(1 + \left\| \frac{\partial u}{\partial t} \right\|_{L^q(Q)}^q \right). \quad (2.4)$$

(ii) *For all $u, v \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}))$*

$$\left(a_{ij}^\varepsilon(-, Du) \frac{\partial u}{\partial x_i} - a_{ij}^\varepsilon(-, Dv) \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j} \right) \geq c_3 |Du - Dv|^p \quad (2.5)$$

a.e. in Q , and for each $1 \leq j \leq N$,

$$\left\| a_{ij}^\varepsilon(-, Du) \frac{\partial u}{\partial x_i} - a_{ij}^\varepsilon(-, Dv) \frac{\partial v}{\partial x_i} \right\|_{L^{p'}(Q; \mathbb{R})} \leq c_2 \|Du - Dv\|_{L^p(Q; \mathbb{R})^N}^\alpha. \quad (2.6)$$

Remark 2.1. Given $u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}))$, since $a_{ij}^\varepsilon(-, Du) \frac{\partial u}{\partial x_i}$ may be viewed as a function in $L^{p'}(0, T; L^{p'}(\Omega))$ then $\frac{\partial}{\partial x_j}(a_{ij}^\varepsilon(-, Du) \frac{\partial u}{\partial x_i})$ turns out to represent the function $t \rightarrow \frac{\partial}{\partial x_j}(a_{ij}^\varepsilon(\cdot, t, Du(\cdot, t)) \frac{\partial u}{\partial x_i}(\cdot, t))$ of $(0, T)$ into $W^{-1,p'}(\Omega; \mathbb{R})$, which lies in $L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}))$. Hence, if $u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}))$ with $\frac{\partial u}{\partial t} \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}))$ then $u \in \mathcal{C}([0, T]; W^{1,2}(\Omega; \mathbb{R}))$, and according to (2.2) if $\frac{\partial u}{\partial t} \in L^q(0, T; L^q(\Omega))$ it follows that $g^\varepsilon(-, \frac{\partial u}{\partial t})$ may be viewed as a function in $L^{q'}(0, T; L^{q'}(\Omega))$, so that if u is a solution of (1.1) with $\frac{\partial u}{\partial t} \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R})) \cap L^q(0, T; L^q(\Omega))$ then $\frac{\partial^2 u}{\partial t^2}$ lies in $L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}))$.

By the Galerkin procedure combined with Stone–Weierstrass theorem one can prove the following existence and uniqueness result:

Theorem 2.1. *For each given $\varepsilon > 0$, there exists one and only one function u_ε from $[0, T]$ into $W_0^{1,p}(\Omega; \mathbb{R})$ defined by (1.1) and satisfying $u_\varepsilon \in L^\infty(0, T; W_0^{1,p}(\Omega; \mathbb{R}))$ with $\frac{\partial u_\varepsilon}{\partial t} \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R})) \cap L^\infty(0, T; L^2(\Omega; \mathbb{R}))$.*

Proof. cf., e.g., [33]. □

According to Theorem 2.1 combined with Remark 2.1 we may define the space

$$\begin{aligned} V_\varepsilon &= \{v \in L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R})) : \varepsilon v' \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R})) \\ &\quad \text{and } v'' \in L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}))\}, \end{aligned}$$

where first and second time derivatives, $v' = \frac{\partial v}{\partial t}$ and $v'' = \frac{\partial^2 v}{\partial t^2}$, of v are taken in distributional sense, and p and p' are the dual exponents. With the graph norm

$$\|v\|_{V_\varepsilon} = \|v\|_{L^p(0, T; W_0^{1,p}(\Omega))} + \sqrt{\varepsilon} \|v'\|_{L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}))} + \|v''\|_{L^{p'}(0, T; W^{-1,p'}(\Omega))},$$

V_ε is a reflexive Banach space. Let us observe that

$$\mathcal{V}(0, T) = \{v \in \mathcal{C}([0, T]; W_0^{1,2}(\Omega; \mathbb{R})) : v' \in \mathcal{C}([0, T]; L^2(\Omega; \mathbb{R}))\}$$

is a Banach space for the graph norm

$$\|v\|_{\mathcal{V}(0, T)} = \sup_{0 \leq t \leq T} \|v(t)\|_{W_0^{1,2}(\Omega)} + \sup_{0 \leq t \leq T} \|v'(t)\|_{L^2(\Omega)}; \quad (2.7)$$

and the embedding $V_\varepsilon \subset \mathcal{V}(0, T)$ is continuous. Hence the function u_ε ($\varepsilon > 0$) in Theorem 2.1 belongs to V_ε . Precisely the following Lemma will be of great interest in the sequel.

Lemma 2.1. *Let $(u_\varepsilon)_{\varepsilon > 0}$ be the sequence defined by Theorem 2.1. Then there exists $0 < \varepsilon_0 \leq 1$ such that $(\|u_\varepsilon\|_{V_\varepsilon})_{0 < \varepsilon \leq \varepsilon_0}$ is bounded; precisely*

$$\begin{aligned} \sup_{0 < \varepsilon \leq \varepsilon_0} (\|u'_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))} + \|u_\varepsilon\|_{L^\infty(0, T; W_0^{1,p}(\Omega))}) &< +\infty, \\ \sup_{0 < \varepsilon \leq \varepsilon_0} (\sqrt{\varepsilon} \|u'_\varepsilon\|_{L^2(0, T; W_0^{1,2}(\Omega))}) &< +\infty. \end{aligned}$$

Moreover we have

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \|g^\varepsilon(-, u'_\varepsilon)\|_{L^{q'}(Q)} < +\infty \text{ and}$$

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \|a_{ij}^\varepsilon(-, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i}\|_{L^{p'}(Q; \mathbb{R})} < +\infty, \quad j = 1, \dots, N.$$

Proof. With the convention summation, the variational formulation of (1.1) is

$$\begin{cases} u_\varepsilon \in V_\varepsilon : \\ (u_\varepsilon''(t), v) + \int_\Omega g^\varepsilon(x, t, u'_\varepsilon(x, t))v(x) \, dx + \varepsilon \int_\Omega \frac{\partial u'_\varepsilon}{\partial x_i}(x, t) \frac{\partial v}{\partial x_i}(x) \, dx \\ \int_\Omega a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j}(x) \, dx = \int_\Omega f(x, t)v(x) \, dx, \\ \text{for all } v \in W_0^{1,p}(\Omega; \mathbb{R}) (0 < t < T), \end{cases} \quad (2.8)$$

where $\varepsilon > 0$ is arbitrarily fixed, and where $(\cdot, \cdot)$ stands for the duality pairing between $W^{-1,p'}(\Omega; \mathbb{R})$ and $W_0^{1,p}(\Omega; \mathbb{R})$. Taking in (2.8) $v = u'_\varepsilon(t)$ ($0 < t < T$), and using identities

$$\begin{aligned} (u_\varepsilon''(t), u'_\varepsilon(t)) &= \frac{1}{2} \frac{d}{dt} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2, \\ \int_\Omega a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u'_\varepsilon}{\partial x_j} \, dx \\ &= \frac{1}{2} \frac{d}{dt} \int_\Omega a_{ii}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_i} \, dx - \frac{1}{2} \int_\Omega \frac{\partial a_{ii}^\varepsilon(x, t, Du_\varepsilon)}{\partial t} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_i} \, dx \\ &\quad + \frac{d}{dt} \sum_{i \neq j} \int_\Omega a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \, dx - \sum_{i \neq j} \int_\Omega \frac{\partial a_{ij}^\varepsilon(x, t, Du_\varepsilon)}{\partial t} \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \, dx \end{aligned}$$

(where (ii)₄ is taken into account), and

$$\frac{\partial a_{ij}^\varepsilon(x, t, Du_\varepsilon)}{\partial t} = \frac{1}{\varepsilon} \left(\frac{\partial a_{ij}}{\partial \tau} \right)^\varepsilon(x, t, Du_\varepsilon) + \sum_{\ell=1}^N \frac{\partial u'_\varepsilon}{\partial x_\ell} \left(\frac{\partial a_{ij}}{\partial \lambda_\ell} \right)^\varepsilon(x, t, Du_\varepsilon),$$

one has

$$\begin{aligned} &\frac{d}{dt} \left(\frac{1}{2} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_\Omega a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \, dx \right) + \varepsilon \|Du'_\varepsilon(t)\|_{L^2(\Omega)^N}^2 \\ &\quad + \int_\Omega g^\varepsilon(-, u'_\varepsilon(t))u'_\varepsilon(t) \, dx \leq \frac{1}{2} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|f(t)\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{2} \int_\Omega \frac{1}{\varepsilon} \left(\frac{\partial a_{ij}}{\partial \tau} \right)^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \, dx \\ &\quad + \frac{1}{2} \int_\Omega \frac{\partial u'_\varepsilon}{\partial x_\ell} \left(\frac{\partial a_{ij}}{\partial \lambda_\ell} \right)^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} \, dx. \end{aligned}$$

According to (ii)₂, there exists $0 < \varepsilon_0 < 1$ such that when $|Du_\varepsilon(x, t)| > 0$, $\frac{1}{\varepsilon} \left| \left(\frac{\partial a_{ij}}{\partial \tau} \right)^\varepsilon(x, t, Du_\varepsilon) \right| \leq |Du_\varepsilon(x, t)|^{-2}$ and $\left| \frac{\partial u'_\varepsilon}{\partial x_\ell} \left(\frac{\partial a_{ij}}{\partial \lambda_\ell} \right)^\varepsilon(x, t, Du_\varepsilon) \right| \leq |Du_\varepsilon(x, t)|^{-2}$ for all $0 < \varepsilon \leq \varepsilon_0$ and for almost all $(x, t) \in Q$. Hence there exists a constant

$c \geq 1$ which does not depend to ε such that

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} dx \right) + \varepsilon \|Du'_\varepsilon(t)\|_{L^2(\Omega)^N}^2 \\ & + \int_{\Omega} g^\varepsilon(-, u'_\varepsilon(t)) u'_\varepsilon(t) dx \leq c + \|f(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 \\ & + \frac{1}{2} \int_{\Omega} a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} dx + \varepsilon \int_0^t \|Du'_\varepsilon(s)\|_{L^2(\Omega)^N}^2 ds \\ & + \int_0^t \int_{\Omega} g^\varepsilon(-, u'_\varepsilon(s)) u'_\varepsilon(s) dx ds \quad \text{for all } 0 < \varepsilon \leq \varepsilon_0. \end{aligned}$$

We integrate both sides from 0 to t ($0 < t \leq T$) and it follows that [see (ii)₃]

$$\begin{aligned} & \frac{1}{2} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} dx + \varepsilon \int_0^t \|Du'_\varepsilon(\tau)\|_{L^2(\Omega)^N}^2 d\tau \\ & + \int_0^t \int_{\Omega} g^\varepsilon(-, u'_\varepsilon(\tau)) u'_\varepsilon(\tau) dx d\tau \leq cT + \|f\|_{L^2(Q)}^2 + \frac{1}{2} \|\hat{u}_1\|_{L^2(\Omega)}^2 \\ & + \frac{1}{2} \int_{\Omega} a_{ij}^\varepsilon(x, 0, D\hat{u}_0) \frac{\partial \hat{u}_0}{\partial x_i} \frac{\partial \hat{u}_0}{\partial x_j} dx \\ & + \int_0^t \left(\frac{1}{2} \|u'_\varepsilon(\tau)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} a_{ij}^\varepsilon(x, \tau, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} dx \right) d\tau \\ & + \int_0^t \left(\varepsilon \int_0^\tau \|Du'_\varepsilon(\delta)\|_{L^2(\Omega)^N}^2 d\delta + \int_0^\tau \int_{\Omega} g^\varepsilon(-, u'_\varepsilon(\delta)) u'_\varepsilon(\delta) dx d\delta \right) d\tau \end{aligned}$$

for all $0 < \varepsilon \leq \varepsilon_0$; and the Gronwall's inequality implies that

$$\begin{aligned} & \frac{1}{2} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} a_{ij}^\varepsilon(x, t, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \frac{\partial u_\varepsilon}{\partial x_j} dx \\ & + \varepsilon \int_0^T \|Du'_\varepsilon(t)\|_{L^2(\Omega)^N}^2 dt + \int_Q g^\varepsilon(-, u'_\varepsilon(t)) u'_\varepsilon(t) dx dt < +\infty \end{aligned}$$

($0 < t \leq T$) for all $0 < \varepsilon \leq \varepsilon_0$. Consequently, by (ii)₅ we deduce the following estimates:

$$\begin{aligned} & \sup_{0 < \varepsilon \leq \varepsilon_0} (\|u'_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))} + \|u_\varepsilon\|_{L^\infty(0, T; W_0^{1, p}(\Omega))} \\ & + \sqrt{\varepsilon} \|u'_\varepsilon\|_{L^2(0, T; W_0^{1, 2}(\Omega))}) < +\infty \end{aligned} \quad (2.9)$$

and, thanks to (2.4) combined with (2.9) and the fact that $q \geq 2$,

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \|g^\varepsilon(-, u'_\varepsilon)\|_{L^{q'}(Q; \mathbb{R})} < +\infty. \quad (2.10)$$

Finally, using (2.6) combined with (2.9), we get

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \left\| a_{ij}^\varepsilon(-, Du_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \right\|_{L^{p'}(Q; \mathbb{R})} < +\infty, \quad j = 1, \dots, N. \quad (2.11)$$

□

2.2. Fundamentals of homogenization

The starting point of the deterministic homogenization theory is the concept of homogenization algebras. In [23],Nguetseng stated that: *A homogenization algebra (H -algebra in short) is any separable closed subalgebra A of the Banach algebra $\mathcal{B}(\mathbb{R}_z^d)$ (integer $d \geq 1$) that contains the constants, is stable under complex conjugation, is translation invariant, and finally, has the following property: for any $u \in A$, one has $u^\varepsilon \rightarrow M(u)$ in $L^\infty(\mathbb{R}_x^d)$ -weak $*$ as $\varepsilon \rightarrow 0$ ($\varepsilon > 0$), where $M(u) \in \mathbb{C}$, $u^\varepsilon(x) = u(H_\varepsilon(x)) \equiv u(\frac{x_1}{\varepsilon}, \dots, \frac{x_d}{\varepsilon})$ ($x \in \mathbb{R}^d$), $\mathcal{H} = (H_\varepsilon)_{\varepsilon > 0}$ being the action on $\mathbb{R}^d$ defined by*

$$H_\varepsilon(z) = \left(\frac{z_1}{\varepsilon}, \dots, \frac{z_d}{\varepsilon} \right) \quad (z = (z_1, \dots, z_d) \in \mathbb{R}^d), \quad (2.12)$$

and $\mathcal{B}(\mathbb{R}_z^d)$ with the supremum norm is well known as a commutative C^* -algebra (with identity) of bounded continuous complex functions on $\mathbb{R}_z^d$. This concept derived to the following fundamental result of the deterministic homogenization theory: *Any bounded sequence $(u_\varepsilon)_{\varepsilon \in E}$ in $L^p(\Omega)$ ($1 \leq p < \infty$ and E being an ordinary sequence) admits a subsequence which is weakly Σ -convergent (for an H -algebra A).*

The proof of this result took into account the fact that A is separable, as Nguetseng et al. reiterated it in their paper [31, Theorem 3.1].

Recently, using an idea introduced by Zhikov and Krivenko [45] in the framework of the theory of algebras with mean, and developed by Casado-Diaz and Gayte [9, 10], Nguetseng et al. [30] show a new way of looking at the algebras of homogenization, the so-called H -subalgebras on $\mathbb{R}^d$ for $\mathcal{H}$. But this extension does not destroy bases pre-established in [23]; that is why our work in the sequel based on the concept of H -algebras, follows ideas in both [23] and [30].

Assume that A is an H -algebra on $\mathbb{R}_z^d$. We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A . For the benefit of the reader it is worth recalling that $\Delta(A)$ is the set of all nonzero multiplicative linear forms on A , and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u \rangle$ ($s \in \Delta(A)$), where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between A' (the topological dual of A) and A . The appropriate topology on $\Delta(A)$ is the relative weak $*$ topology on A' . So topologized, $\Delta(A)$ is a metrizable compact space, and the Gelfand transformation is an isometric isomorphism of the C^* -algebra A onto the C^* -algebra $\mathcal{C}(\Delta(A))$. The appropriate measure on $\Delta(A)$ is the so-called M -measure, namely the positive Radon measure β (of total mass 1) on $\Delta(A)$ such that $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ for $u \in A$.

The partial derivative of index i ($1 \leq i \leq d$) on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ \frac{\partial}{\partial y_i} \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^1\}$ into $\mathcal{C}(\Delta(A))$, where $A^1 = \{\psi \in \mathcal{C}^1(\mathbb{R}_y^d) : \psi, \frac{\partial \psi}{\partial z_i} \in A \text{ } (1 \leq i \leq d)\}$. Higher order derivatives can be defined analogously. Now, let A^∞ be the space of $\psi \in \mathcal{C}^\infty(\mathbb{R}_z^d)$ such that $D_z^\alpha \psi = \frac{\partial^{|\alpha|} \psi}{\partial z_1^{\alpha_1} \dots \partial z_d^{\alpha_d}} \in A$ for every multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, and let $\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty\}$. Endowed with a suitable locally convex topology, A^∞ (respectively $\mathcal{D}(\Delta(A))$)

is a Fréchet space and further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$ [23].

Let $1 \leq p < \infty$. We recall by $\Xi^p = \Xi^p(\mathbb{R}_z^d)$, the space of those functions $\psi \in L_{loc}^p(\mathbb{R}_z^d)$ for which $\sup_{0 < \varepsilon \leq 1} \int_U |\psi(H_\varepsilon(x))|^p dx < \infty$ for any bounded set U in $\mathbb{R}_x^d$. This is a Banach space for the norm

$$\|u\|_{\Xi^p} = \sup_{0 < \varepsilon \leq 1} \left(\int_{B_d} |\psi(H_\varepsilon(x))|^p dx \right)^{\frac{1}{p}} \quad (\psi \in \Xi^p),$$

B_d being the open unit ball in $\mathbb{R}_x^d$. Let A be a H -algebra on $\mathbb{R}_z^d$ for $\mathcal{H}$. We define $\mathfrak{X}_A^p = \mathfrak{X}_A^p(\mathbb{R}_z^d)$ to be the closure of A in Ξ^p . Provided with the Ξ^p -norm, $\mathfrak{X}_A^p$ is a Banach space [23]. Moreover the Gelfand transformation $\mathcal{G}$ on A is extended to the canonical mapping, still denoted $\mathcal{G}$, of $\mathfrak{X}_A^p$ onto $L^p(\Delta(A)) = \mathcal{G}(\mathfrak{X}_A^p)$; $L^p(\Delta(A))$ being a Banach space under the classical $L^p(\Delta(A))$ -norm (see [23]).

2.2.1. An appropriate representation in $\mathbb{R}_z^d$ of elements in $L^p(\Delta(A))$.

We recall that for any $u \in A$ one has $|u|^p \in A$ and $\mathcal{G}(|u|^p) = |\mathcal{G}(u)|^p$ with

$$M(|u|^p) = \int_{\Delta(A)} \mathcal{G}(|u|^p) d\beta = \int_{\Delta(A)} |\hat{u}|^p d\beta \geq 0, \quad (2.13)$$

where $\hat{u} = \mathcal{G}(u)$. Naturally (2.13) holds true when $u \in \mathfrak{X}_A^p$. According to (2.13), instead of the Ξ^p -norm, we endow $\mathfrak{X}_A^p$ with the seminorm

$$\|u\|_{p,A} = [M(|u|^p)]^{\frac{1}{p}} \quad (u \in \mathfrak{X}_A^p). \quad (2.14)$$

Combining (2.13) with (2.14) yields

$$\|\mathcal{G}(u)\|_{L^p(\Delta(A))} = \|u\|_{p,A} \quad (u \in \mathfrak{X}_A^p). \quad (2.15)$$

Hence, endowed with the seminorm $\|\cdot\|_{p,A}$, $\mathfrak{X}_A^p$ is complete and $\|u\|_{p,A} = 0$ if and only if $\mathcal{G}(u) = 0$. Since the canonical mapping $\mathcal{G} : \mathfrak{X}_A^p \rightarrow L^p(\Delta(A))$ is not injective, we denote the space of (class of) functions by

$$\mathcal{X}_A^p = \mathfrak{X}_A^p / \mathcal{R}, \quad \text{with } \mathcal{R} = \ker \mathcal{G}$$

endowed with the norm

$$\|u + \mathcal{R}\|_{\mathcal{X}_A^p} = \|u\|_{p,A} \quad (u \in \mathfrak{X}_A^p),$$

which makes it a Banach space. Here is the isometric we need.

Theorem 2.2. *The mapping $\mathcal{G}_1 : \mathcal{X}_A^p \rightarrow L^p(\Delta(A))$ defined by*

$$\mathcal{G}_1(u + \mathcal{R}) = \mathcal{G}(u) \quad (u \in \mathfrak{X}_A^p), \quad (2.16)$$

is a topological isometric isomorphism of $\mathcal{X}_A^p$ onto $L^p(\Delta(A))$.

Proof. By Definition $\mathcal{G}$ is surjective, and by (2.15) $\mathcal{G}$ is an isometric of $(\mathfrak{X}_A^p; \|\cdot\|_{p,A})$ onto $L^p(\Delta(A))$. Since $\|\cdot\|_{p,A}$ is a seminorm on $\mathfrak{X}_A^p$, we deduce by the first isomorphism theorem that the mapping $\mathcal{G}_1 : \mathcal{X}_A^p \rightarrow L^p(\Delta(A))$ defined by (2.16) is a topological isometric isomorphism. $\square$

Remark 2.2. It follows that for any $u \in L^p(\Delta(A))$ there exists a unique $u_0 \in \mathcal{X}_A^p$ such that $\mathcal{G}_1(u_0) = u$; that is, u_0 is the representation in $\mathbb{R}_z^d$ of u .

We also extend the mean value on $\mathcal{X}_A^p$ as follows:

$$M_1(u + \mathcal{R}) = M(u) \quad (u \in \mathfrak{X}_A^p).$$

The following result is a consequence of the preceding theorem.

Corollary 2.2. [30] *The following hold true:*

- (i) *The space $\mathcal{X}_A^p$ is reflexive.*
- (ii) *The topological dual of $\mathcal{X}_A^p$ coincides with the space $\mathcal{X}_A^{p'}$, the duality being given by*

$$\langle u + \mathcal{R}, v + \mathcal{R} \rangle_{\mathcal{X}_A^{p'}, \mathcal{X}_A^p} = M(uv) = \int_{\Delta(A)} \mathcal{G}_1(u + \mathcal{R}) \mathcal{G}_1(v + \mathcal{R}) \, d\beta$$

for $u \in \mathcal{X}_A^p$ and $v \in \mathfrak{X}_A^{p'}$.

Remark 2.3. For $u \in \mathfrak{X}_A^p$, we will make the formal integral representation

$$M_1(u + \mathcal{R}) \equiv \mathcal{L}_A \int_{\mathbb{R}_z^d} u(z) \, dz;$$

then

- (1) $\mathcal{L}_A \int_{\mathbb{R}^d} u(z) \, dz = \int_{\Delta(A)} \mathcal{G}_1(u + \mathcal{R}) \, d\beta$ ($u \in \mathfrak{X}_A^p$),
- (2) $\mathcal{L}_A \int_{\mathbb{R}^d} dz = 1$ for all H -algebra A on $\mathbb{R}_z^d$,
- (3) if $A = A_1 \odot A_2$ then $\mathcal{L}_A \int_{\mathbb{R}_z^d} u(z) \, dz = \mathcal{L}_{A_1} \int_{\mathbb{R}^{d_1}} (\mathcal{L}_{A_2} \int_{\mathbb{R}^{d_2}} u(z) \, dz_2) \, dz_1$, where $A_1 \odot A_2$ denotes the closure in $\mathcal{B}(\mathbb{R}^d)$ of the tensor product $A_1 \otimes A_2$, A_1 and A_2 being H -algebras on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ (respectively) with $d_1 + d_2 = d$,
- (4) $\mathcal{L}_{\mathcal{C}_{per}(Z)} \int_{\mathbb{R}^d} u(z) \, dz = \int_Z u(z) \, dz$ ($u \in \mathfrak{X}_{\mathcal{C}_{per}(Z)}^p \equiv L_{per}^p(Z)$, $Z = (0, 1)^d$).

We refer to [1] for classical Sobolev spaces, and to [23, 25, 32] for special Sobolev spaces such as $W^{1,p}(\Delta(A))$ and $W_{\#}^{1,p}(\Delta(A))$ together with the various associated derivative operators.

Let us deduce an isometric isomorphism of $W^{1,p}(\Delta(A))$ onto a subspace of $\mathcal{X}_A^p$ ($1 < p < \infty$). Due to Theorem 2.2, we set $(\mathcal{G}_1)^{-1}(W^{1,p}(\Delta(A))) = W^1 \mathcal{X}_A^p$, that is,

$$W^1 \mathcal{X}_A^p = \{u \in \mathcal{X}_A^p : \partial_i \mathcal{G}_1(u) \in L^p(\Delta(A)), \quad 1 \leq i \leq d\}.$$

Since for any $u \in W^1 \mathcal{X}_A^p$, $\partial_i \mathcal{G}_1(u) \in L^p(\Delta(A))$ ($1 \leq i \leq d$) and, once more taking into account Theorem 2.2, there exists a unique $u_i \in \mathcal{X}_A^p$ such that $\partial_i \mathcal{G}_1(u) = \mathcal{G}_1(u_i)$. This leads to

Definition 2.1. [30] The formal partial derivative on $\mathbb{R}_z^d$ is defined to be the mapping $\frac{\bar{\partial}}{\partial z_i} = (\mathcal{G}_1)^{-1} \circ \partial_i \circ \mathcal{G}_1$ (usual composition) of $W^1\mathcal{X}_A^p$ into $\mathcal{X}_A^p$, where $\mathcal{G}_1$ is the canonical mapping in Theorem 2.2.

Hence the characterization of $W^1\mathcal{X}_A^p$ is

$$W^1\mathcal{X}_A^p = \left\{ u \in \mathcal{X}_A^p : \frac{\bar{\partial}u}{\partial z_i} \in \mathcal{X}_A^p \quad (1 \leq i \leq d) \right\}. \quad (2.17)$$

In order to simplify the presentation, from now on we will use the same letter u (if there is no danger of confusion) to denote the equivalence class $u + \mathcal{R} \in \mathcal{X}_A^p$ and its representative $u \in \mathfrak{X}_A^p$. The symbol ϱ will denote the canonical mapping of $\mathfrak{X}_A^p$ onto $\mathcal{X}_A^p$. This being so, firstly given u in $W^1\mathfrak{X}_A^p = \{u \in \mathfrak{X}_A^p : \frac{\bar{\partial}u}{\partial z_i} \in \mathfrak{X}_A^p \quad (1 \leq i \leq d)\}$ which is naturally a subspace of $\mathfrak{X}_A^p$, we have

$$\mathcal{G}_1 \left(\varrho \left(\frac{\partial u}{\partial z_i} \right) \right) = \mathcal{G} \left(\frac{\partial u}{\partial z_i} \right) = \partial_i \mathcal{G}(u) = \partial_i \mathcal{G}_1(\varrho(u)) = \mathcal{G}_1 \left(\frac{\bar{\partial} \varrho(u)}{\partial z_i} \right);$$

so that

$$\frac{\bar{\partial}}{\partial z_i} \circ \varrho = \varrho \circ \frac{\partial}{\partial z_i} \quad \text{on } W^1\mathfrak{X}_A^p \quad (1 \leq i \leq d) \quad [30, \text{Remark 2.2}],$$

where the derivative $\frac{\partial}{\partial z_i}$ is taken in the distributional sense on $\mathbb{R}_z^d$. Secondly, we endow $W^1\mathcal{X}_A^p$ with the norm

$$\|u\|_{W^1\mathcal{X}_A^p} = \|u\|_{p,A} + \sum_{i=1}^d \left\| \frac{\bar{\partial}u}{\partial z_i} \right\|_{p,A} \quad (\in W^1\mathcal{X}_A^p),$$

which makes it a Banach space. Besides, the canonical mapping $\mathcal{G}_1$ (Theorem 2.2), viewed as a mapping from $W^1\mathcal{X}_A^p$ into $L^p(\Delta(A))$, is an isometric isomorphism of $W^1\mathcal{X}_A^p$ onto $W^{1,p}(\Delta(A))$; in particular

$$\|\mathcal{G}_1(u)\|_{W^{1,p}(\Delta(A))} = \|u\|_{W^1\mathcal{X}_A^p} \quad \text{for any } u \in W^1\mathcal{X}_A^p.$$

Hence, according to Remark 2.2, if $u \in W^{1,p}(\Delta(A))$ there exists a unique $u_0 \in W^1\mathcal{X}_A^p$ such that

$$\mathcal{G}_1(u_0) = u \quad \text{and} \quad \mathcal{G}_1 \left(\frac{\bar{\partial}u_0}{\partial z_i} \right) = \partial_i u \quad (1 \leq i \leq d).$$

Remark 2.4. We have $\mathcal{L}_A \int_{\mathbb{R}^d} \frac{\bar{\partial}u}{\partial z_i} dz = 0$ for all $u \in W^1\mathcal{X}_A^p$ ($1 \leq i \leq d$).

We will be concern with the space

$$W^1\mathcal{X}_A^p / \mathbb{C} = \{u \in W^1\mathcal{X}_A^p : M_1(u) = 0\}$$

equipped with the seminorm

$$\|u\|_{W^1\mathcal{X}_A^p / \mathbb{C}} = \sum_{i=1}^d \left\| \frac{\bar{\partial}u}{\partial z_i} \right\|_{p,A} \quad (u \in W^1\mathcal{X}_A^p / \mathbb{C}),$$

which makes it a locally convex topological space in general nonseparated and noncomplete. We denote by $W_{\#}^1\mathcal{X}_A^p$ the separated completion of $W^1\mathcal{X}_A^p / \mathbb{C}$ (for the above seminorm), and by J_1 the canonical mapping from $W^1\mathcal{X}_A^p / \mathbb{C}$ into

$W_{\#}^1 \mathcal{X}_A^p$. Therefore, since $\mathcal{G}_1$ sends $W^1 \mathcal{X}_A^p / \mathbb{C}$ onto $W^{1,p}(\Delta(A)) / \mathbb{C}$ isometrically and isomorphically, there is a unique isometric isomorphism $\bar{\mathcal{G}}_1$ of $W_{\#}^1 \mathcal{X}_A^p$ onto $W_{\#}^{1,p}(\Delta(A))$; otherwise the restriction $\frac{\bar{\partial}}{\partial z_i} : W^1 \mathcal{X}_A^p / \mathbb{C} \rightarrow \mathcal{X}_A^p$ ($1 \leq i \leq d$) extends by continuity to a continuous linear mapping, still denoted $\frac{\bar{\partial}}{\partial z_i}$, from $W_{\#}^1 \mathcal{X}_A^p$ into $\mathcal{X}_A^p$ with

$$\begin{aligned} \frac{\bar{\partial}}{\partial z_i} \circ J_1 &= \frac{\bar{\partial}}{\partial z_i} \quad \text{in } W^1 \mathcal{X}_A^p / \mathbb{C}, \quad 1 \leq i \leq d, \\ \bar{\mathcal{G}}_1 \circ J_1 &= J \circ \mathcal{G}_1 \quad \text{in } W^1 \mathcal{X}_A^p / \mathbb{C}, \\ \partial_i \circ J \circ \mathcal{G}_1 &= \mathcal{G}_1 \circ \frac{\bar{\partial}}{\partial z_i} \quad \text{in } W^1 \mathcal{X}_A^p / \mathbb{C}, \text{ or} \\ \partial_i \circ \bar{\mathcal{G}}_1 &= \mathcal{G}_1 \circ \frac{\bar{\partial}}{\partial z_i} \quad \text{in } W_{\#}^1 \mathcal{X}_A^p, \\ \|J_1(u)\|_{W_{\#}^1 \mathcal{X}_A^p} &= \|u\|_{W^1 \mathcal{X}_A^p / \mathbb{C}} \quad (u \in W^1 \mathcal{X}_A^p / \mathbb{C}), \text{ and} \\ \|u\|_{W_{\#}^1 \mathcal{X}_A^p} &= \sum_{i=1}^d \left\| \frac{\bar{\partial} u}{\partial z_i} \right\|_{p,A} \quad (u \in W_{\#}^1 \mathcal{X}_A^p), \end{aligned} \quad (2.18)$$

[where J denotes the classical canonical mapping of $W^{1,p}(\Delta(A)) / \mathbb{C}$ into $W_{\#}^{1,p}(\Delta(A))$].

Remark 2.5. If $u \in W_{\#}^{1,p}(\Delta(A))$ there is a unique $u_0 \in W_{\#}^1 \mathcal{X}_A^p$ such that $\partial_i u = \mathcal{G}_1(\frac{\bar{\partial} u_0}{\partial z_i})$, $1 \leq i \leq d$.

2.2.2. Σ -convergence. Let us recall briefly the concept of Σ -convergence. Let A be a H -algebra on $\mathbb{R}_{y,\tau}^{N+1} = \mathbb{R}_y^N \times \mathbb{R}_\tau$. Let Q be a bounded open set in $\mathbb{R}_{x,t}^{N+1}$. The letter E throughout will denote a family of positive real numbers admitting 0 as an accumulation point. Attention is drawn to the especial case where $E = (\varepsilon_n)$ (integers $n \geq 0$) with $\varepsilon_n > 0$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$; E is then referred to as a *fundamental sequence*.

Definition 2.2. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(Q)$ ($1 \leq p < \infty$) is said to:

(i) *weakly Σ -converge in $L^p(Q)$* to some $u_0 \in L^p(Q; \mathcal{X}_A^p)$ if as $E \ni \varepsilon \rightarrow 0$, we have

$$\int_Q u_\varepsilon v^\varepsilon \, dx \rightarrow \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} u_0 v \, dy \, d\tau \, dx \, dt \quad (2.19)$$

for every $v \in L^{p'}(Q; A)$ ($\frac{1}{p'} = 1 - \frac{1}{p}$), where $v^\varepsilon(x, t) = v(H_\varepsilon(x, t)) = v(\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$;

(ii) *strongly Σ -converge in $L^p(Q)$* to some $u_0 \in L^p(Q; \mathcal{X}_A^p)$ if the following condition is fulfilled:

(SC) Given $\eta > 0$ and $v \in L^p(Q; A)$ with $\|u_0 - v\|_{L^p(Q; \mathcal{X}_A^p)} \leq \frac{\eta}{2}$, there is some $\alpha > 0$ such that $\|u_\varepsilon - v^\varepsilon\|_{L^p(Q)} \leq \eta$ provided $E \ni \varepsilon \leq \alpha$.

We express this by writing $u_\varepsilon \rightarrow u_0$ in $L^p(Q)$ -weak Σ in case (i), and $u_\varepsilon \rightarrow u_0$ in $L^p(Q)$ -strong Σ in case (ii).

Remark 2.6. (i) If $u \in L^p(Q; A)$, then we have $u^\varepsilon \rightarrow \widehat{u}$ in $L^p(Q)$ -strong Σ .
(ii) If $u_\varepsilon \rightarrow u$ in $L^p(Q)$ as $E \ni \varepsilon \rightarrow 0$, then $u_\varepsilon \rightarrow u$ in $L^p(Q)$ -strong Σ .

The results of the Σ -convergence setting [23, 31] carry over mutatis mutandis, together with their proofs, to the present setting. Let us recall the following

Proposition 2.3. *Let E be a fundamental sequence, and suppose the sequence $(v_\varepsilon)_{\varepsilon \in E} \subset L^p(Q)$ is bounded in $L^p(Q)$. Then, a subsequence E' can be extracted from E such that the sequence $(v_\varepsilon)_{\varepsilon \in E'}$ is weakly Σ -convergent in $L^p(Q)$.*

Proof. cf., e.g., [23, 31]. □

2.2.3. Σ -quasi-compactness of $L^p(0, T; W_0^{1,p}(\Omega))$ for A . In what follows, we are concerned with three specific H -algebras: the H -algebra A_y on $\mathbb{R}_y^N$ [for $\mathcal{H}^1 = (H_\varepsilon^1)_{\varepsilon > 0}$ given by $H_\varepsilon^1(x) = \frac{x}{\varepsilon}$ ($x \in \mathbb{R}^N$)], the H -algebra A_τ on $\mathbb{R}_\tau$ [for $\mathcal{H}^2 = (H_\varepsilon^2)_{\varepsilon > 0}$ given by $H_\varepsilon^2(t) = \frac{t}{\varepsilon}$ ($t \in \mathbb{R}$)], and

$$A = A_y \odot A_\tau = \text{closure in } \mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_\tau) \text{ of the tensor product } A_y \otimes A_\tau, \quad (2.20)$$

which is an H -algebra on $\mathbb{R}^N \times \mathbb{R}$ (see [23]) for the action $\mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2$ on $\mathbb{R}^N \times \mathbb{R}$ given by $\mathcal{H} = (H_\varepsilon)_{\varepsilon > 0}$, $H_\varepsilon = H_\varepsilon^1 \times H_\varepsilon^2$ (direct product), i.e.,

$$H_\varepsilon(x, t) = \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

The following lemmas will be of great interest in the sequel. Let us begin by recalling that a sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^1(\Omega)$ is said to be uniformly integrable if $(u_\varepsilon)_{\varepsilon \in E}$ is bounded in $L^1(\Omega)$ and further $\sup_{\varepsilon \in E} \int_\omega |u_\varepsilon| dx \rightarrow 0$ as $|\omega| \rightarrow 0$ ($\omega \subset \Omega$, the characteristic function χ_ω being integrable with $|\omega| = \int \chi_\omega dx$).

Lemma 2.2. *Let A be an H -algebra on $\mathbb{R}_y^N \times \mathbb{R}$. Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^1(Q)$ such that for almost all t in $(0, T)$ the sequence $(u_\varepsilon(\cdot, t))_{\varepsilon \in E}$ is uniformly integrable in $L^1(\Omega)$. Set*

$$v_{r,\varepsilon}(x, t) = \int_{B_r} \widetilde{u}_\varepsilon(x + \varepsilon\rho, t) d\rho \quad ((x, t) \in Q),$$

where $\widetilde{u}_\varepsilon$ is the zero-extension of u_ε off Q , and $B_r = r^{-1}B_N$ is the open ball (centered at the origin in $\mathbb{R}^N$) with radius $r^{-1} > 0$. Then there are a subsequence E' extracted from E , and $u_0 \in L^1(Q; \mathcal{X}_A^1)$ such that, as $E' \ni \varepsilon \rightarrow 0$,

$$v_{r,\varepsilon} \rightarrow v_{r,0} \quad \text{in } L^1(Q)\text{-weak } \Sigma, \quad (2.21)$$

where $v_{r,0}(x, t, \cdot, \cdot) = \int_{B_r} u_0(x, t, \cdot + \rho, \cdot) d\rho$.

Proof. cf. [30]. □

Lemma 2.3. *For every $u \in \mathfrak{X}_A^p$ we have*

$$\lim_{r \rightarrow 0} \left\| \frac{1}{|B_r|} \int_{B_r} u(\cdot + \rho, \cdot) d\rho - M(u) \right\|_{p,A} = 0.$$

Proof. cf. [30]. □

We are now in position to prove the following Σ -quasi-compactness result for A .

Theorem 2.3. *Let $Q = \Omega \times (0, T)$ be an open set in $\mathbb{R}^{N+1}$. Let $1 < p < \infty$. Let A_y be a H -algebra on $\mathbb{R}_y^N$ of class C^∞ and p -total, and let A_τ be a H -algebra on $\mathbb{R}_\tau$ of class C^∞ . Let $A = A_y \odot A_\tau$, and let E be a fundamental sequence. Then, for any sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(0, T; W_0^{1,p}(\Omega))$ such that*

$$\sup_{\varepsilon \in E} (\|u_\varepsilon\|_{L^p(0, T; W_0^{1,p}(\Omega))}) < \infty, \quad (2.22)$$

there are a subsequence E' from E and a couple $(u_0, u_1) \in L^p(0, T; W_0^{1,p}(\Omega)) \times L^p(Q; \mathcal{X}_{A_\tau}^p(\mathbb{R}; W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N)))$ such that, as $E' \ni \varepsilon \rightarrow 0$,

$$u_\varepsilon \rightarrow u_0 \quad \text{in } L^p(0, T; W_0^{1,p}(\Omega))\text{-weak}, \quad (2.23)$$

and

$$\frac{\partial u_\varepsilon}{\partial x_i} \rightarrow \frac{\partial u_0}{\partial x_i} + \frac{\bar{\partial} u_1}{\partial y_i} \quad \text{in } L^p(Q)\text{-weak } \Sigma \quad (1 \leq i \leq N). \quad (2.24)$$

Proof. We follow [9]. Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(0, T; W_0^{1,p}(\Omega))$ with (2.22). By the classical reflexivity of $L^p(0, T; W_0^{1,p}(\Omega))$ combined with Proposition 2.3, there are a subsequence E' extracted from E , a function $u_0 \in L^p(0, T; W_0^{1,p}(\Omega))$, and a vector function $w = (w_i) \in L^p(Q; \mathcal{X}_A^p(\mathbb{R}^{N+1}))^N$ with $A = A_y \odot A_\tau$ such that, as $E' \ni \varepsilon \rightarrow 0$, we have (2.23) and $\frac{\partial u_\varepsilon}{\partial x_i} \rightarrow w_i$ in $L^p(Q)$ -weak Σ ($1 \leq i \leq N$). The rest of the proof is devoted to check that there exists a function $u_1 \in L^p(Q; \mathcal{X}_{A_\tau}^p(\mathbb{R}; W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N)))$ such that $w_i = \frac{\partial u_0}{\partial x_i} + \frac{\bar{\partial} u_1}{\partial y_i}$ ($1 \leq i \leq N$). Let $r > 0$ be freely fixed, and let $\varepsilon \in E'$. Recalling that $\tilde{u}_\varepsilon$ and $\frac{\partial u_\varepsilon}{\partial x_i}$ are extensions of u_ε and $\frac{\partial u_\varepsilon}{\partial x_i}$ by 0 off Q with $\frac{\partial \tilde{u}_\varepsilon}{\partial x_i} = \frac{\partial u_\varepsilon}{\partial x_i}$ [since $u_\varepsilon \in L^p(0, T; W_0^{1,p}(\Omega))$], and that $B_{\frac{r}{\varepsilon}} = \varepsilon r^{-1} B_N$ denotes the open ball in $\mathbb{R}^N$ with radius $\frac{\varepsilon}{r}$ centered at the origin, by equalities

$$\begin{aligned} & \frac{1}{\varepsilon} \left(\tilde{u}_\varepsilon(x, t) - \frac{1}{|B_{\frac{r}{\varepsilon}}|} \int_{B_{\frac{r}{\varepsilon}}} \tilde{u}_\varepsilon(x + \rho, t) \, d\rho \right) \\ &= \frac{1}{\varepsilon |B_{\frac{r}{\varepsilon}}|} \int_{B_{\frac{r}{\varepsilon}}} (\tilde{u}_\varepsilon(x, t) - \tilde{u}_\varepsilon(x + \rho, t)) \, d\rho \\ &= \frac{1}{\varepsilon^{N+1} |B_r|} \int_{B_{\frac{r}{\varepsilon}}} (\tilde{u}_\varepsilon(x, t) - \tilde{u}_\varepsilon(x + \rho, t)) \, d\rho \\ &= \frac{1}{|B_r|} \int_{B_r} \left(\frac{\tilde{u}_\varepsilon(x, t) - \tilde{u}_\varepsilon(x + \varepsilon \rho, t)}{\varepsilon} \right) \, d\rho \\ &= -\frac{1}{|B_r|} \int_{B_r} \left(\int_0^1 \widetilde{D} u_\varepsilon(x + \varepsilon s \rho, t) \cdot \rho \, ds \right) \, d\rho, \end{aligned} \quad (2.25)$$

where $\widetilde{Du}_\varepsilon = (\frac{\partial \widetilde{u}_\varepsilon}{\partial x_i})_{1 \leq i \leq N}$ and the dot "·" is the usual Euclidean inner product in $\mathbb{R}^N$, we deduce from the boundedness of $(\widetilde{Du}_\varepsilon)_{\varepsilon \in E'}$ in $L^p(\mathbb{R}^{N+1})^N$ that the sequence $(w_{r,\varepsilon})_{\varepsilon \in E'}$ defined by

$$w_{r,\varepsilon}(x, t) = \frac{1}{\varepsilon} \left(\widetilde{u}_\varepsilon(x, t) - \frac{1}{|B_{\frac{r}{\varepsilon}}|} \int_{B_{\frac{r}{\varepsilon}}} \widetilde{u}_\varepsilon(x + \rho, t) \, d\rho \right) \quad (\text{a.e. in } Q, \varepsilon \in E'),$$

is bounded in $L^p(Q)$. Hence, according to Proposition 2.3, there are a subsequence extracted from E' , still denoted E' , and a function $w_r \in L^p(Q; \mathcal{X}_A^p(\mathbb{R}^{N+1}))$ such that, as $E' \ni \varepsilon \rightarrow 0$,

$$w_{r,\varepsilon} \rightarrow w_r \quad \text{in } L^p(Q)\text{-weak } \Sigma. \quad (2.26)$$

This being so, for $\varphi \in \mathcal{D}(Q)$ and $\psi \in A$ we have

$$\begin{aligned} & \int_Q \left(\frac{\partial \widetilde{u}_\varepsilon}{\partial x_i}(x, t) - \frac{1}{|B_{\frac{r}{\varepsilon}}|} \int_{B_{\frac{r}{\varepsilon}}} \frac{\partial \widetilde{u}_\varepsilon}{\partial x_i}(x + \rho, t) \, d\rho \right) \varphi(x, t) \psi \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \, dx \, dt \\ &= \int_Q \frac{\partial}{\partial x_i} \left(\widetilde{u}_\varepsilon(x, t) - \frac{1}{|B_{\frac{r}{\varepsilon}}|} \int_{B_{\frac{r}{\varepsilon}}} \widetilde{u}_\varepsilon(x + \rho, t) \, d\rho \right) \varphi(x, t) \psi \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \, dx \, dt \\ &= -\varepsilon \int_Q w_{r,\varepsilon}(x, t) \left(\frac{\partial \varphi}{\partial x_i} \right)(x, t) \psi \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \, dx \, dt \\ &\quad - \int_Q w_{r,\varepsilon}(x, t) \varphi(x, t) \left(\frac{\partial \psi}{\partial y_i} \right) \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \, dx \, dt. \end{aligned}$$

Since $\frac{1}{|B_{\frac{r}{\varepsilon}}|} \int_{B_{\frac{r}{\varepsilon}}} \frac{\partial \widetilde{u}_\varepsilon}{\partial x_i}(x + \rho, t) \, d\rho = \frac{1}{|B_r|} \int_{B_r} \frac{\partial \widetilde{u}_\varepsilon}{\partial x_i}(x + \varepsilon \rho, t) \, d\rho$, and according to Lemma 2.2, $\frac{1}{|B_r|} \int_{B_r} \frac{\partial \widetilde{u}_\varepsilon}{\partial x_i}(\cdot + \varepsilon \rho, \cdot) \, d\rho \rightarrow \frac{1}{|B_r|} \int_{B_r} w_i(\cdot, \cdot + \rho, \cdot, \cdot) \, d\rho$ in $L^1(Q)$ -weak Σ , passing to the limit, as $E' \ni \varepsilon \rightarrow 0$, and taking into account (2.26) one gets

$$\begin{aligned} & \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} \left(w_i(x, t, \cdot, \cdot) - \frac{1}{|B_r|} \int_{B_r} w_i(x, t, \cdot + \rho, \cdot) \, d\rho \right) \varphi(x, t) \psi \, dy \, d\tau \, dx \, dt \\ &= - \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} w_r(x, t, \cdot, \cdot) \varphi(x, t) \frac{\partial \psi}{\partial y_i} \, dy \, d\tau \, dx \, dt. \end{aligned}$$

Therefore, the arbitrariness of φ and ψ leads us to

$$\frac{\overline{\partial} w_r(x, t, \cdot, \cdot)}{\partial y_i} = w_i(x, t, \cdot, \cdot) - \frac{1}{|B_r|} \int_{B_r} w_i(x, t, \cdot + \rho, \cdot) \, d\rho \quad \text{a.e. in } Q \quad (2.27)$$

($1 \leq i \leq N$). Set $v_r = w_r - M_1(w_r) = w_r - M(w_r)$, then $v_r(x, t, \cdot, \cdot)$ viewed as its representative in $\mathfrak{X}_A^p$ lies in $\mathfrak{X}_A^p / \mathbb{C}$. Since $\frac{\overline{\partial} v_r(x, t, \cdot, \cdot)}{\partial y_i} = \frac{\overline{\partial} w_r(x, t, \cdot, \cdot)}{\partial y_i}$ a.e. in Q ($1 \leq i \leq N$), it follows by (2.27) that $v_r(x, t, \cdot, \cdot) \in \mathcal{X}_{A_\tau}^p(\mathbb{R}; W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N))$ with

$$\frac{\overline{\partial} v_r(x, t, \cdot, \cdot)}{\partial y_i} = w_i(x, t, \cdot, \cdot) - \frac{1}{|B_r|} \int_{B_r} w_i(x, t, \cdot + \rho, \cdot) \, d\rho \quad \text{a.e. in } Q \quad (2.28)$$

($1 \leq i \leq N$). With this in mind, we have

$$\begin{aligned}
 & \| \overline{D}_y v_r(x, t, \cdot, \cdot) - \overline{D}_y v_{r'}(x, t, \cdot, \cdot) \|_{p,A} \\
 &= \| \overline{D}_y v_r(x, t, \cdot, \cdot) - w + M(w) + w - M(w) - \overline{D}_y v_{r'}(x, t, \cdot, \cdot) \|_{p,A} \\
 &\leq \| \overline{D}_y v_r(x, t, \cdot, \cdot) - w + M(w) \|_{p,A} + \| w - M(w) - \overline{D}_y v_{r'}(x, t, \cdot, \cdot) \|_{p,A} \\
 &= \left\| \frac{1}{|B_r|} \int_{B_r} w(x, t, \cdot + \rho, \cdot) \, d\rho + M(w(x, t, \cdot, \cdot)) \right\|_{p,A} \\
 &\quad + \left\| \frac{1}{|B_{r'}|} \int_{B_{r'}} w(x, t, \cdot + \rho, \cdot) \, d\rho + M(w(x, t, \cdot, \cdot)) \right\|_{p,A}.
 \end{aligned}$$

Therefore, it follows by Lemma 2.3 that the right-hand side of the above inequality goes to 0 when $r, r' \rightarrow 0$. Thus $(\overline{D}_y v_r(x, t, \cdot, \cdot))_{r>0}$ is a Cauchy sequence in the Banach space $\mathcal{X}_A^p$. Whence there exists a unique $v(x, t, \cdot, \cdot) \in \mathcal{X}_{A_\tau}^p(\mathbb{R}; W_{\#}^1 \mathcal{X}_{A_y}^p)$ such that $v_r(x, t, \cdot, \cdot) \rightarrow v(x, t, \cdot, \cdot)$ in $\mathcal{X}_{A_\tau}^p(\mathbb{R}; W_{\#}^1 \mathcal{X}_{A_y}^p)$ as $r \rightarrow 0$; that is,

$$\frac{\overline{\partial} v_r(x, t, \cdot, \cdot)}{\partial y_i} \rightarrow \frac{\overline{\partial} v(x, t, \cdot, \cdot)}{\partial y_i} \quad \text{in } \mathcal{X}_A^p (1 \leq i \leq N) \text{ as } r \rightarrow 0.$$

Once more, by Lemma 2.3, (2.28) leads us to $\frac{\overline{\partial} v_r(x, t, \cdot, \cdot)}{\partial y_i} \rightarrow w_i(x, t, \cdot, \cdot) - M(w_i(x, t, \cdot, \cdot))$ in $\mathcal{X}_A^p$ ($1 \leq i \leq N$) as $r \rightarrow 0$; hence the uniqueness of the limit implies

$$\frac{\overline{\partial} v(x, t, \cdot, \cdot)}{\partial y_i} = w_i(x, t, \cdot, \cdot) - M(w_i(x, t, \cdot, \cdot)) \quad \text{for each } 1 \leq i \leq N.$$

Clearly, there exists a unique function $u_1 \in L^p(Q; \mathcal{X}_{A_\tau}^p(\mathbb{R}; W_{\#}^1 \mathcal{X}_{A_y}^p))$ satisfying

$$w_i = M(w_i) + \frac{\overline{\partial} u_1}{\partial y_i} \quad \text{for each } 1 \leq i \leq N. \quad (2.29)$$

Since the weakly Σ -convergence in $L^p(Q)$ implies the usual weakly convergence in $L^p(Q)$, recalling that $\frac{\partial u_\varepsilon}{\partial x_i} \rightarrow w_i$ in $L^p(Q)$ -weak Σ , one gets $\frac{\partial u_\varepsilon}{\partial x_i} \rightarrow M(w_i)$ in $L^p(Q)$ -weak ($1 \leq i \leq N$). The uniqueness of the limit combined with (2.23) and (2.29) complete the proof. $\square$

Remark 2.7. According to Theorem 2.3, we will also say that the H -algebra $A = A_y \odot A_\tau$ on $\mathbb{R}_y^N \times \mathbb{R}_\tau$ is p -quasi-pseudoproper.

It is well known that if a H -algebra $A = A_y \odot A_\tau$ on $\mathbb{R}_y^N \times \mathbb{R}_\tau$ is p -quasi-pseudoproper, then in the practice of homogenization theory test functions are taken in the natural fashion $\phi_\varepsilon(x, t) = \psi_0(x, t) + \varepsilon \psi_1(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon})$, $(x, t) \in Q$, $\varepsilon > 0$. The following corollary permits to decompose the corrector u_1 in such a way that one can write formally $u_\varepsilon(x, t) \approx u_0(x, t) + \varepsilon v_1(x, t, \frac{x}{\varepsilon}) + \varepsilon \chi(x, t, \frac{t}{\varepsilon})$. So that we cannot choose test functions as in the classical case.

Corollary 2.3. Let the hypotheses be as in Theorem 2.3 with (2.22), and let us assume further that each u'_ε lies in $L^2(0, T; W^{1,2}(\Omega))$ and

$$\sup_{\varepsilon \in E} (\sqrt{\varepsilon} \|Du'_\varepsilon\|_{L^2(Q)}) < \infty. \quad (2.30)$$

Then $u_1 \in L^p(Q; W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N)) + L^p(Q; \mathcal{X}_{A_\tau}^p(\mathbb{R})/\mathbb{C})$, where $\mathcal{X}_{A_\tau}^p(\mathbb{R})/\mathbb{C} = \{u \in \mathcal{X}_{A_\tau}^p(\mathbb{R}) : \mathcal{L}_{A_\tau} \int_{\mathbb{R}} u \, d\tau = 0\}$.

Proof. For $\varphi \in \mathcal{D}(Q)$, $\psi \in A_y$, $\phi \in A_\tau^\infty$, and $1 \leq i \leq N$ one has

$$\begin{aligned} & \varepsilon \int_Q \frac{\partial^2 u_\varepsilon}{\partial t \partial x_i}(x, t) \varphi(x, t) \psi\left(\frac{x}{\varepsilon}\right) \phi\left(\frac{t}{\varepsilon}\right) dx \, dt \\ & + \varepsilon \int_Q \frac{\partial u_\varepsilon}{\partial x_i}(x, t) \frac{\partial \varphi}{\partial t}(x, t) \psi\left(\frac{x}{\varepsilon}\right) \phi\left(\frac{t}{\varepsilon}\right) dx \, dt \\ & = - \int_Q \frac{\partial u_\varepsilon}{\partial x_i}(x, t) \varphi(x, t) \psi\left(\frac{x}{\varepsilon}\right) \frac{d\phi}{d\tau}\left(\frac{t}{\varepsilon}\right) dx \, dt. \end{aligned}$$

Passing to the limit, as $E' \ni \varepsilon \rightarrow 0$ (the subsequence in Theorem 2.3), and using the boundedness of $(Du_\varepsilon)_{\varepsilon \in E}$ and $(\sqrt{\varepsilon} Du'_\varepsilon)_{\varepsilon \in E}$ in $L^2(Q)$ [see (2.22) and (2.30) respectively], the left hand-side of the above equality goes to zero while, thanks to Theorem 2.3, the right hand-side tends to $-\int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} (\frac{\partial u_0}{\partial x_i} + \frac{\bar{\partial} u_1}{\partial y_i}) \varphi \psi \frac{d\phi}{d\tau} dy d\tau dx dt$. Therefore, by use of the arbitrariness of φ and ψ combined with Remarks 2.3 and 2.4, we are led to

$$\mathcal{L}_{A_\tau} \int_{\mathbb{R}} \frac{\bar{\partial} u_1}{\partial y_i} \frac{d\phi}{d\tau} d\tau = 0 \text{ in } Q \times \mathbb{R}_y^N \quad \text{for all } \phi \in A_\tau^\infty (1 \leq i \leq N).$$

Hence $\bar{D}u_1 = (\frac{\bar{\partial} u_1}{\partial y_1}, \dots, \frac{\bar{\partial} u_1}{\partial y_N})$ does not depend to τ ; which yields

$$\bar{D}v_1 = \bar{D}u_1 \text{ in } Q \times \mathbb{R}_y^N \times \mathbb{R}_\tau, \quad (2.31)$$

where $v_1 = \mathcal{L}_{A_\tau} \int_{\mathbb{R}} u_1 \, d\tau$. It follows from (2.31) that $u_1 - v_1 = \chi$ does not depend to y and $\mathcal{L}_{A_\tau} \int_{\mathbb{R}} \chi \, d\tau = 0$, that is, $u_1 = v_1 + \chi \in L^p(Q; W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N)) + L^p(Q; \mathcal{X}_{A_\tau}^p(\mathbb{R})/\mathbb{C})$. $\square$

3. The abstract homogenization problem

3.1. Setting of the abstract problem and preliminary results

Throughout the rest of this section it is assumed that the H -algebra A in (2.20) is quasi-pseudoproper for $p \geq 2$. Our main goal is to investigate the limiting behaviour, as $\varepsilon \rightarrow 0$, of u_ε [the solution of (1.1) for fixed $\varepsilon > 0$] under the abstract structure hypotheses

$$g(-, \zeta) \in A_{\mathbb{R}} = A \cap \mathcal{C}(\mathbb{R}_{y,\tau}^{N+1}; \mathbb{R}) \quad \text{for all } \zeta \in \mathbb{R}, \quad (3.1)$$

and

$$a_{ij}(-, \lambda) \in \mathfrak{X}_A^{p'}(\mathbb{R}_{y,\tau}^{N+1}; \mathbb{R}) \quad \text{for all } \lambda \in \mathbb{R}^N (1 \leq i, j \leq N) \quad (3.2)$$

with

$$2 \leq p < \infty \quad \text{and} \quad p' = \frac{p}{p-1}.$$

This is referred to as the *homogenization problem for (1.1) in the abstract deterministic setting associated with the H -algebra $A = A_y \odot A_\tau$* .

The resolution of this problem requires a few preliminaries.

Lemma 3.1. *Let $\psi \in A_{\mathbb{R}}$ and $\Psi \in (A_{\mathbb{R}})^N$.*

- (i) *The (trace) function $(y, \tau) \rightarrow g(y, \tau, \psi(y, \tau))$ from $\mathbb{R}_{y, \tau}^{N+1}$ into $\mathbb{C}$, denoted by $g(-, \psi)$ is well defined and lies in $A_{\mathbb{R}}$.*
- (ii) *The functions $(y, \tau) \rightarrow a_{ij}(y, \tau, \Psi(y, \tau))$, $1 \leq i, j \leq N$, from $\mathbb{R}_{y, \tau}^{N+1}$ into $\mathbb{C}$, denoted by $a_{ij}(-, \Psi)$ (respectively), are well defined with $a_{ij}(-, \Psi) \in \mathfrak{X}_A^{p'}$.*

Proof. (i) Let $\psi \in A_{\mathbb{R}}$, and consider the function $(y, \tau, z, v) \rightarrow g(y, \tau, \psi(z, v))$ from $\mathbb{R}_{y, \tau}^{N+1} \times \mathbb{R}_{z, v}^{N+1}$ into $\mathbb{C}$. By (3.1), for all $(z, v) \in \mathbb{R}_{z, v}^{N+1}$, the function $(y, \tau) \rightarrow g(y, \tau, \psi(z, v))$ from $\mathbb{R}_{y, \tau}^{N+1}$ into $\mathbb{C}$, denoted $g(-, \psi(z, v))$, belongs to $A_{\mathbb{R}}$. Furthermore, by continuity of ψ , it follows once more by (3.1) that

$$\lim_{p \rightarrow \infty} \|g(-, \psi(z_p, v_p))\|_{\infty} = \|g(-, \psi(z, v))\|_{\infty}$$

whenever the sequence $((z_p, v_p))_{p \geq 0}$ converges to (z, v) in $\mathbb{R}_{z, v}^{N+1}$. Hence the function $(z, v) \rightarrow g(-, \psi(z, v))$ from $\mathbb{R}_{z, v}^{N+1}$ into $A_{\mathbb{R}}$ lies in $\mathcal{C}(\mathbb{R}_{z, v}^{N+1}; A_{\mathbb{R}})$. But it is obvious that

$$\sup_{(z, v) \in \mathbb{R}^{N+1}} \|g(-, \psi(z, v))\|_{\infty} \leq \sup_{\zeta \in K} \|g(-, \zeta)\|_{\infty} < +\infty,$$

where K is a compact in $\mathbb{R}$ such that $\psi(z, v) \in K$ for all $(z, v) \in \mathbb{R}^{N+1}$, then the function $(z, v) \rightarrow g(-, \psi(z, v))$ lies in $\mathcal{B}(\mathbb{R}_{z, v}^{N+1}; A_{\mathbb{R}})$, that is, the function $(y, \tau, z, v) \rightarrow g(y, \tau, \psi(z, v))$ from $\mathbb{R}_{y, \tau}^{N+1} \times \mathbb{R}_{z, v}^{N+1}$ into $\mathbb{C}$ belongs to $\mathcal{B}(\mathbb{R}_{z, v}^{N+1}; A_{\mathbb{R}})$. Therefore, the function $(y, \tau) \rightarrow g(y, \tau, \psi(y, \tau))$ from $\mathbb{R}_{y, \tau}^{N+1}$ into $\mathbb{C}$, denoted $g(-, \psi)$, is well defined [as the trace $g(y, \tau, \psi(z, v))|_{z=y, v=\tau}$], and belongs to $\mathcal{B}(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R})$.

Let K be the compact determined above. As in [25, Problems II and IV] and according to (i)₃, the mapping $\zeta \rightarrow g(-, \zeta)$ from K into $A_{\mathbb{R}}$ lies in $\mathcal{C}(K; A_{\mathbb{R}})$. Let us first assume that $g(-, \zeta)$ is of the form

$$g(-, \zeta) = \chi(\zeta)\varphi \quad (\zeta \in K) \text{ with } \chi \in \mathcal{C}(K; \mathbb{R}) \text{ and } \varphi \in A_{\mathbb{R}}.$$

By the Stone–Weierstrass theorem (see [7, chapter X, page 37]) there is a sequence (χ_n) of polynomials in ζ such that $\chi_n \rightarrow \chi$ in $\mathcal{C}(K)$ when $n \rightarrow \infty$. Thus, for all $\psi \in A_{\mathbb{R}}$, $\chi_n(\psi) \rightarrow \chi(\psi)$ in $\mathcal{B}(\mathbb{R}^{N+1})$ as $n \rightarrow \infty$. It follows that $\chi(\psi) \in A_{\mathbb{R}}$ since each $\chi_n(\psi)$ belongs to $A_{\mathbb{R}}$ which is a closed subspace of $\mathcal{B}(\mathbb{R}^{N+1})$. Hence $g(-, \zeta) \in A_{\mathbb{R}}$ [as a product of $\chi(\psi) \in A_{\mathbb{R}}$ and $\varphi \in A_{\mathbb{R}}$]. But then this is still true if

$$g(-, \zeta) = \sum_{\text{finite}} \chi(\zeta)\varphi \quad (\zeta \in K) \text{ with } \chi \in \mathcal{C}(K) \text{ and } \varphi \in A_{\mathbb{R}}.$$

Now, we consider the general case. Thanks to the density of $\mathcal{C}(K) \otimes A_{\mathbb{R}}$ in $\mathcal{C}(K; A_{\mathbb{R}})$ (see, e.g., [6, page 46]), there exists a sequence of functions $g_n \in \mathcal{C}(K) \otimes A_{\mathbb{R}}$, integers $n \geq 1$, such that

$$\sup_{\zeta \in K} \|g_n(-, \zeta) - g(-, \zeta)\|_{\infty} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It follows immediately that $\|g_n(-, \psi) - g(-, \psi)\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$. Hence $g(-, \psi)$ lies in $A_{\mathbb{R}}$ by use of the fact that $g_n(-, \psi) \in A_{\mathbb{R}}$ and that $A_{\mathbb{R}}$ is complete under the supremum norm.

(ii) Similarly if $\Psi \in (A_{\mathbb{R}})^N$ and $1 \leq i, j \leq N$ we prove by (3.2) that the function $(y, \tau) \rightarrow a_{ij}(y, \tau, \Psi(y, \tau))$ from $\mathbb{R}_{y, \tau}^{N+1}$ into $\mathbb{C}$, denoted $a_{ij}(-, \Psi)$, is well defined, and belongs to $\Xi^{p'}(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R})$.

let K be a compact set in $\mathbb{R}^N$ such that $\Psi(y, \tau) \in K$ for all $(y, \tau) \in \mathbb{R}_{y, \tau}^{N+1}$. By (ii)₃ the mapping $\lambda \rightarrow a_{ij}(-, \lambda)$ from K into $\mathfrak{X}_A^{p'}(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R})$ lies in $\mathcal{C}(K; \mathfrak{X}_A^{p'}(\mathbb{R}_{y, \tau}^{N+1}))$. Hence if

$$a_{ij}(-, \lambda) = \chi_{ij}(\lambda) \varphi \quad (\lambda \in K) \text{ with } \chi_{ij} \in \mathcal{C}(K) \text{ and } \varphi \in \mathfrak{X}_A^{p'}(\mathbb{R}_{y, \tau}^{N+1}),$$

there is a sequence (χ_{ij}^n) of polynomials in $\lambda = (\lambda_1, \dots, \lambda_N)$ such that $\chi_{ij}^n \rightarrow \chi_{ij}$ in $\mathcal{C}(K)$ when $n \rightarrow \infty$. Thus $\chi_{ij}^n(\Psi) \rightarrow \chi_{ij}(\Psi)$ in $\mathcal{B}(\mathbb{R}_{y, \tau}^{N+1})$ as $n \rightarrow \infty$. Hence we get $\chi_{ij}(\Psi) \in A_{\mathbb{R}}$, so that $a_{ij}(-, \Psi) \in \mathfrak{X}_A^{p'}(\mathbb{R}_{y, \tau}^{N+1})$. This is still true if

$$a_{ij}(-, \lambda) = \sum_{\text{finite}} \chi_{ij}(\lambda) \varphi \quad (\lambda \in K) \text{ with } \chi \in \mathcal{C}(K) \text{ and } \varphi \in \mathfrak{X}_A^{p'}(\mathbb{R}_{y, \tau}^{N+1}).$$

For the general case, there exists a sequence of functions $a_{ij}^n(-, \lambda) \in \mathcal{C}(K) \otimes \mathfrak{X}_A^{p'}(\mathbb{R}_{y, \tau}^{N+1})$, integers $n \geq 1$, such that

$$\sup_{\lambda \in K} \|a_{ij}^n(-, \lambda) - a_{ij}(-, \lambda)\|_{\Xi^{p'}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence $\|a_{ij}^n(-, \Psi) - a_{ij}(-, \Psi)\|_{\Xi^{p'}} \rightarrow 0$ as $n \rightarrow \infty$ and $a_{ij}(-, \Psi)$ lies in $\mathfrak{X}_A^{p'}(\mathbb{R}_{y, \tau}^{N+1})$. $\square$

Given $\Psi = (\psi_i) \in (A_{\mathbb{R}})^N$ and $1 \leq j \leq N$, it follows from (ii)₁-(ii)₃ in Section 1 combined with Lemma 3.1 that the function $(y, \tau) \rightarrow a_{ij}(y, \tau, \Psi(y, \tau)) \psi_i(y, \tau)$, denoted by $a_{ij}(-, \Psi) \psi_i$ (with the convention summation), belongs to $\mathfrak{X}_A^{p', \infty} = \mathfrak{X}_A^{p'}(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R}) \cap L^\infty(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R})$ ($\mathfrak{X}_A^{p', \infty}$ equipped with the L^∞ -norm). This being so, for every $\psi \in \mathcal{C}(\overline{Q}; A_{\mathbb{R}})$ and $\Psi = (\psi_i) \in \mathcal{C}(\overline{Q}; (A_{\mathbb{R}})^N)$ the functions $(x, t) \rightarrow g(-, \psi(x, t))$ from $\overline{Q}$ to $A_{\mathbb{R}}$ and $(x, t) \rightarrow a_{ij}(-, \Psi(x, t)) \psi_i(x, t)$ from $\overline{Q}$ to $\mathfrak{X}_A^{p', \infty}$ [still denoted by $g(-, \psi)$ and $a_{ij}(-, \Psi) \psi_i, 1 \leq j \leq N$] belong to $\mathcal{C}(\overline{Q}; A_{\mathbb{R}})$ and to $\mathcal{C}(\overline{Q}; \mathfrak{X}_A^{p', \infty})$ (respectively) with

$$|g(-, \psi) - g(-, \phi)| \leq c_4(1 + |\psi|^{q-1} + |\phi|^{q-1})|\psi - \phi|, \quad (3.3)$$

$$|a_{ij}(-, \Psi) \psi_i - a_{ij}(-, \Phi) \phi_i| \leq c_2 |\Psi - \Phi|^\alpha \quad (3.4)$$

and

$$(a_{ij}(-, \Psi) \psi_i - a_{ij}(-, \Phi) \phi_i)(\psi_j - \phi_j) \geq c_3 |\Psi - \Phi|^p \quad (3.5)$$

in $\overline{Q} \times \mathbb{R}_{y, \tau}^{N+1}$ for all $\psi, \phi \in \mathcal{C}(\overline{Q}; A_{\mathbb{R}})$ and all $\Psi, \Phi \in \mathcal{C}(\overline{Q}; (A_{\mathbb{R}})^N)$. Therefore, for fixed $\varepsilon > 0$, one defines the functions $(x, t) \rightarrow g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \psi(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}))$ and $(x, t) \rightarrow a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \Psi(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon})) \psi_i(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon})$ of Q into $\mathbb{R}$ [denoted by $g^\varepsilon(-, \psi^\varepsilon)$ and $a_{ij}^\varepsilon(-, \Psi^\varepsilon) \psi_i^\varepsilon$] as elements of $L_{\mathbb{R}}^\infty(Q)$. For $f \in \mathcal{C}_{\mathbb{R}}(\Delta(A))$ and $\mathbf{f} = (f_i) \in \mathcal{C}_{\mathbb{R}}(\Delta(A))^N$, let

$$\widehat{h}(f) = \mathcal{G}(g(-, \mathcal{G}^{-1}f)) \text{ and } \widehat{b}_{ij}(\mathbf{f}) f_i = \mathcal{G}(a_{ij}(-, \mathcal{G}^{-1}\mathbf{f}) \mathcal{G}^{-1}f_i), \quad (3.6)$$

$1 \leq j \leq N$, where $\mathcal{G}^{-1}\mathbf{f} = (\mathcal{G}^{-1}f_i)_{1 \leq i \leq N}$, hence one defines mappings $f \rightarrow \widehat{h}(f)$ from $\mathcal{C}_{\mathbb{R}}(\Delta(A))$ into $L_{\mathbb{R}}^\infty(\Delta(A))$ and $\mathbf{f} \rightarrow \widehat{b}_{ij}(\mathbf{f}) f_i$ from $\mathcal{C}_{\mathbb{R}}(\Delta(A))^N$ into

$L_{\mathbb{R}}^{\infty}(\Delta(A))$. When $(f, \mathbf{f}) \in \mathcal{C}(\overline{Q}; \mathcal{C}_{\mathbb{R}}(\Delta(A))) \times \mathcal{C}(\overline{Q}; \mathcal{C}_{\mathbb{R}}(\Delta(A)))^N$ we will still denote by $\widehat{h}(f)$ and $\widehat{b}_{ij}(\mathbf{f})f_i$, the mappings $(x, t) \rightarrow \widehat{h}(f(x, t))$ and $(x, t) \rightarrow \widehat{b}_{ij}(\mathbf{f}(x, t))f_i(x, t)$ from $\overline{Q}$ into $L_{\mathbb{R}}^{\infty}(\Delta(A))$. Now, for $u \in A_{\mathbb{R}}$ and $\mathbf{u} = (u_i) \in (A_{\mathbb{R}})^N$, letting

$$h(u) = \widehat{h}(\mathcal{G}u) \text{ and } b_{ij}(\mathbf{u})u_i = \widehat{b}_{ij}(\mathcal{G}\mathbf{u})\mathcal{G}u_i, \quad (3.7)$$

we have the following

Proposition 3.1. *Let $A = A_y \odot A_{\tau}$ be the H -algebra defined by (2.20). Assume A is of class $\mathcal{C}^{\infty}$. Then for every $(\psi, \Psi) \in \mathcal{C}(\overline{Q}; A_{\mathbb{R}}) \times \mathcal{C}(\overline{Q}; A_{\mathbb{R}})^N$, $h(\psi) \in \mathcal{C}(\overline{Q}; \mathfrak{X}_A^{q'}(\mathbb{R}^{N+1}; \mathbb{R}))$, $b_{ij}(\Psi)\psi_i \in \mathcal{C}(\overline{Q}; \mathfrak{X}_A^{p'}(\mathbb{R}^{N+1}; \mathbb{R}))$, $1 \leq j \leq N$, and*

$$\begin{aligned} g^{\varepsilon}(-, \psi^{\varepsilon}) &\rightarrow h(\psi) \text{ in } L^{q'}(Q)\text{-weak } \Sigma, \\ a_{ij}^{\varepsilon}(-, \Psi^{\varepsilon})\psi_i^{\varepsilon} &\rightarrow b_{ij}(\Psi)\psi_i \text{ in } L^{p'}(Q)\text{-weak } \Sigma \end{aligned} \quad \text{when } \varepsilon \rightarrow 0. \quad (3.8)$$

Furthermore, the mappings $f \rightarrow h(f)$ from $\mathcal{C}(\overline{Q}; A_{\mathbb{R}})$ into $L^{q'}(Q; \mathfrak{X}_A^{q'}(\mathbb{R}^{N+1}; \mathbb{R}))$ and $\mathbf{f} \rightarrow b_{ij}(\mathbf{f})f_i$ from $\mathcal{C}(\overline{Q}; A_{\mathbb{R}})^N$ into $L^{p'}(Q; \mathfrak{X}_A^{p'}(\mathbb{R}^{N+1}; \mathbb{R}))$ extend by continuity to mappings, still denoted $f \rightarrow h(f)$ and $\mathbf{f} \rightarrow b_{ij}(\mathbf{f})f_i$, of $L^q(Q; \mathfrak{X}_A^q(\mathbb{R}^{N+1}; \mathbb{R}))$ into $L^{q'}(Q; \mathfrak{X}_A^{q'}(\mathbb{R}^{N+1}; \mathbb{R}))$ and of $L^p(Q; \mathfrak{X}_A^p(\mathbb{R}^{N+1}; \mathbb{R}))^N$ into $L^{p'}(Q; \mathfrak{X}_A^{p'}(\mathbb{R}^{N+1}; \mathbb{R}))$ such that

$$\begin{aligned} \|h(v) - h(w)\|_{L^{q'}(Q; \mathfrak{X}_A^{q'}(\mathbb{R}^{N+1}; \mathbb{R}))} \\ \leq c_4(\|1 + |v| + |w|\|_{L^q(Q; \mathfrak{X}_A^q(\mathbb{R}^{N+1}; \mathbb{R}))}^{q-1})\|v - w\|_{L^q(Q; \mathfrak{X}_A^q(\mathbb{R}^{N+1}; \mathbb{R}))}, \end{aligned} \quad (3.9)$$

$$\|b_{ij}(\mathbf{v})v_i - b_{ij}(\mathbf{w})w_i\|_{L^{p'}(Q; \mathfrak{X}_A^{p'}(\mathbb{R}^{N+1}; \mathbb{R}))} \leq c_2\|\mathbf{v} - \mathbf{w}\|_{L^p(Q; \mathfrak{X}_A^p(\mathbb{R}^{N+1}; \mathbb{R}))}^{\alpha} \quad (3.10)$$

and

$$(b_{ij}(\mathbf{v})v_i - b_{ij}(\mathbf{w})w_i)(v_j - w_j) \geq c_3|\mathbf{v} - \mathbf{w}|^p \quad \text{a.e. in } Q \times \mathbb{R}_{y, \tau}^{N+1} \quad (3.11)$$

for all $v, w \in L^q(Q; \mathfrak{X}_A^q(\mathbb{R}^{N+1}; \mathbb{R}))$ and all $\mathbf{v}, \mathbf{w} \in L^p(Q; \mathfrak{X}_A^p(\mathbb{R}^{N+1}; \mathbb{R}))^N$.

Proof. Let $(\psi, \Psi) \in \mathcal{C}(\overline{Q}; A_{\mathbb{R}}) \times \mathcal{C}(\overline{Q}; A_{\mathbb{R}})^N$. Since $g(-, \psi) \in \mathcal{C}(\overline{Q}; A_{\mathbb{R}})$ and $a_{ij}(-, \Psi)\psi_i$, $1 \leq j \leq N$, belong to $\mathcal{C}(\overline{Q}; \mathfrak{X}_A^{p', \infty})$ [$\mathfrak{X}_A^{p', \infty}$ equipped with the L^{∞} -norm], it follows by (3.3)–(3.4) combined with (3.6)–(3.7) that $h(\psi) \in L_{\mathbb{R}}^{q'}(Q; \mathfrak{X}_A^{q'})$, $b_{ij}(\Psi)\psi_i \in L_{\mathbb{R}}^{p'}(Q; \mathfrak{X}_A^{p'})$, $1 \leq j \leq N$, and thanks to [23, Corollary 4.1], (3.8) holds true. This being so, remarking that (3.3)–(3.5) drive to

$$\begin{aligned} |g^{\varepsilon}(-, \psi^{\varepsilon}) - g^{\varepsilon}(-, \phi^{\varepsilon})| &\leq c_4(1 + |\psi^{\varepsilon}|^{q-1} + |\phi^{\varepsilon}|^{q-1})|\psi^{\varepsilon} - \phi^{\varepsilon}| \\ |a_{ij}^{\varepsilon}(-, \Psi^{\varepsilon})\psi_i^{\varepsilon} - a_{ij}^{\varepsilon}(-, \Phi^{\varepsilon})\phi_i^{\varepsilon}| &\leq c_2|\Psi^{\varepsilon} - \Phi^{\varepsilon}|^{\alpha} \\ (a_{ij}^{\varepsilon}(-, \Psi^{\varepsilon})\psi_i^{\varepsilon} - a_{ij}^{\varepsilon}(-, \Phi^{\varepsilon})\phi_i^{\varepsilon})(\psi_j^{\varepsilon} - \phi_j^{\varepsilon}) &\geq c_3|\Psi^{\varepsilon} - \Phi^{\varepsilon}|^p \end{aligned} \quad (3.12)$$

a.e. in Q , for any $\varepsilon > 0$ (see, e.g., [25, Lemma 2.2]), it follows by [23, Corollary 4.1] and [25, Remark 4.1] that

$$\begin{aligned} |h(\psi) - h(\phi)| &\leq c_4(1 + |\psi|^{q-1} + |\phi|^{q-1})|\psi - \phi| \\ |b_{ij}(\Psi)\psi_i - b_{ij}(\Phi)\phi_i| &\leq c_2|\Psi - \Phi|^{\alpha} \\ (b_{ij}(\Psi)\psi_i - b_{ij}(\Phi)\phi_i)(\psi_j - \phi_j) &\geq c_3|\Psi - \Phi|^p \end{aligned} \quad (3.13)$$

a.e. in $Q \times \mathbb{R}_{y,\tau}^{N+1}$. Therefore we are led to (3.9)–(3.10). Since $\mathcal{C}(\overline{Q}; A_{\mathbb{R}})$ is (identifiable with) a dense subspace of $L_{\mathbb{R}}^r(Q; \mathfrak{X}_A^r)$, $r \geq 1$, the mappings $\psi \rightarrow h(\psi)$ and $\Psi \rightarrow b_{ij}(\Psi)\psi_i$ extend by continuity to mappings, still denoted $v \rightarrow h(v)$ and $\mathbf{v} \rightarrow b_{ij}(\mathbf{v})v_i$, of $L_{\mathbb{R}}^q(Q; \mathfrak{X}_A^q)$ into $L_{\mathbb{R}}^{q'}(Q; \mathfrak{X}_A^{q'})$ and of $L_{\mathbb{R}}^p(Q; \mathfrak{X}_A^p)^N$ into $L_{\mathbb{R}}^{p'}(Q; \mathfrak{X}_A^{p'})$ (respectively) such that (3.9)–(3.10) hold for all $u, v \in L_{\mathbb{R}}^q(Q; \mathfrak{X}_A^q)$ and all $\mathbf{u}, \mathbf{v} \in L_{\mathbb{R}}^p(Q; \mathfrak{X}_A^p)^N$. Finally, (3.11) follows from the third inequality in (3.13) combined with the above density and continuity arguments. $\square$

As a consequence of Proposition 3.1, we have the following corollary.

Corollary 3.1. *Let the hypotheses be those of Proposition 3.1. For each real $\varepsilon > 0$, let $\phi_\varepsilon \in \mathcal{D}_{\mathbb{R}}(Q) = \mathcal{C}_0^\infty(Q; \mathbb{R})$ be given by*

$$\phi_\varepsilon(x, t) = \psi_0(x, t) + \varepsilon\psi_1\left(x, t, \frac{x}{\varepsilon}\right) + \varepsilon^2\psi_2\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \quad (3.14)$$

$(x, t) \in Q$, with $\psi_0 \in \mathcal{D}_{\mathbb{R}}(Q)$, $\psi_1 \in \mathcal{D}_{\mathbb{R}}(Q) \otimes_{\mathbb{R}} A_y^\infty$, $\psi_2 \in \mathcal{D}_{\mathbb{R}}(Q) \otimes_{\mathbb{R}} A_y^\infty \otimes_{\mathbb{R}} A_\tau^\infty$, where ${}_{\mathbb{R}}A_y^\infty = A_y^\infty \cap \mathcal{C}(\mathbb{R}^N; \mathbb{R})$ and ${}_{\mathbb{R}}A_\tau^\infty = A_\tau^\infty \cap \mathcal{C}(\mathbb{R}; \mathbb{R})$. Let the index $1 \leq j \leq N$ be fixed. Then, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} g^\varepsilon\left(-, \frac{\partial \phi_\varepsilon}{\partial t}\right) &\rightarrow h\left(\frac{\partial \psi_0}{\partial t}\right) \quad \text{in } L^{q'}(Q)\text{-weak } \Sigma, \\ a_{ij}^\varepsilon(-, D\phi_\varepsilon) \frac{\partial \phi_\varepsilon}{\partial x_i} &\rightarrow b_{ij}(D\psi_0 + \overline{D}_y\psi_1) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\overline{\partial} \psi_1}{\partial y_i}\right) \quad \text{in } L^{p'}(Q)\text{-weak } \Sigma. \end{aligned}$$

Furthermore, if $(v_\varepsilon)_{\varepsilon \in E}$ is a sequence in $L^q(Q)$ (resp. in $L^p(Q)$) such that $v_\varepsilon \rightarrow v_0$ in $L^q(Q)$ -weak Σ [resp. in $L^p(Q)$ -weak Σ] as $E \ni \varepsilon \rightarrow 0$, then, as $E \ni \varepsilon \rightarrow 0$,

$$\begin{aligned} \int_Q g^\varepsilon\left(-, \frac{\partial \phi_\varepsilon}{\partial t}\right) v_\varepsilon \, dx \, dt &\rightarrow \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h\left(\frac{\partial \psi_0}{\partial t}\right) v_0 \, dy \, d\tau \, dx \, dt \\ \int_Q a_{ij}^\varepsilon(-, D\phi_\varepsilon) \frac{\partial \phi_\varepsilon}{\partial x_i} v_\varepsilon \, dx \, dt &\rightarrow \\ \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(D\psi_0 + \overline{D}_y\psi_1) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\overline{\partial} \psi_1}{\partial y_i}\right) v_0 \, dy \, d\tau \, dx \, dt \end{aligned}$$

$(1 \leq j \leq N)$.

3.2. The abstract homogenization result

Let basic notations and hypotheses be as in the above subsection. We assume that the H -algebra $A = A_y \odot A_\tau$ in (2.20) is quasi-pseudoproper for p in the following sense: A_y and A_τ are of class $\mathcal{C}^\infty$ and A_y is p -total (see Theorem 2.3). According to Corollary 2.3, let

$$\mathbb{V}_0^p = L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R})) \times L^p(Q; W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N; \mathbb{R}))$$

We provide $\mathbb{V}_0^p$ with the norm

$$\|\mathbf{u}\|_{\mathbb{V}_0^p} = \|u_0\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}))} + \|u_1\|_{L^p(Q;W_{\#}^1\mathcal{X}_{A_y}^p(\mathbb{R}^N))}, \quad \mathbf{u} = (u_0, u_1) \in \mathbb{V}_0^p,$$

which makes it a Banach space. The space

$$V_0^\infty = \mathcal{D}_{\mathbb{R}}(Q) \times [\mathcal{D}_{\mathbb{R}}(Q) \otimes (\mathbb{R}A_y^\infty/\mathbb{C})]$$

is dense in $\mathbb{V}_0^p$. We will also need the following lemma.

Lemma 3.2. *The variational problem*

$$\left\{ \begin{array}{l} \mathbf{u} = (u_0, u_1) \in \mathbb{V}_0^p : u_0(0) = \hat{u}_0, u_0'(0) = \hat{u}_1 \text{ and} \\ \int_0^T (u_0''(t), v_0(t)) dt + \int_Q \left(\mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h \left(\frac{\partial u_0}{\partial t} \right) dy d\tau \right) v_0 dx dt \\ + \int_Q \mathcal{L}_{A_y} \int_{\mathbb{R}^N} (\mathcal{L}_{A_\tau} \int_{\mathbb{R}} b_{ij}(\mathbb{D}\mathbf{u}) d\tau) \mathbb{D}_i \mathbf{u} \mathbb{D}_j \mathbf{v} dy dx dt = \int_Q f(t) v_0(t) dx dt \\ \text{for all } \mathbf{v} = (v_0, v_1) \in \mathbb{V}_0^p, \end{array} \right. \quad (3.15)$$

has at most one solution. Here, $(\cdot, \cdot)$ stands for the duality pairing between $W^{-1,p'}(\Omega; \mathbb{R})$ and $W_0^{1,p}(\Omega; \mathbb{R})$, and for each $\mathbf{v} = (v_0, v_1)$ in $\mathbb{V}_0^p$, we denote $\mathbb{D}\mathbf{v} = Dv_0 + \overline{D}_y v_1$.

Proof. According to (3.11) combined with (i)₄, there is at most one $\mathbf{u} = (u_0, u_1)$ verifying (3.15). $\square$

Here is the basic abstract deterministic homogenization theorem for (1.1).

Theorem 3.1. *Let $2 \leq q \leq p < \infty$. Suppose (3.1)–(3.2) hold and $A = A_y \odot A_\tau$ is such that A_y and A_τ are of class C^∞ , A_y is p -total. For each fixed real number $0 < \varepsilon \leq \varepsilon_0$ (see Lemma 2.1), let u_ε be the solution of (1.1). Then, as $\varepsilon \rightarrow 0$,*

$$u_\varepsilon \rightarrow u_0 \quad \text{in } L^p(0, T; W_0^{1,p}(\Omega))\text{-weak}, \quad (3.16)$$

$$\frac{\partial u_\varepsilon}{\partial t} \rightarrow \frac{\partial u_0}{\partial t} \quad \text{in } L^q(Q)\text{-weak}, \quad (3.17)$$

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} \rightarrow \frac{\partial^2 u_0}{\partial t^2} \quad \text{in } L^{p'}(0, T; W^{1,p'}(\Omega))\text{-weak}, \quad (3.18)$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \rightarrow \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \quad \text{in } L^p(Q)\text{-weak } \Sigma, \quad (3.19)$$

$1 \leq i \leq N$, where $\mathbf{u} = (u_0, u_1)$ is uniquely defined by (3.15).

Proof. Let $0 < \varepsilon_0 < 1$ defined in Lemma 2.1. Let E be a fundamental sequence in $(0, \varepsilon_0)$. Since A_y and A_τ are of class C^∞ and A_y is p -total, it follows from Remark 2.7 combined with Theorem 2.3 that A is quasi-pseudoprop-er for p ; so that there are a subsequence E' from E and a couple $(u_0, u_1) \in L^p(0, T; W_0^{1,p}(\Omega)) \times L^p(Q; \mathcal{X}_{A_\tau}^p(\mathbb{R}; W_{\#}^1\mathcal{X}_{A_y}^p(\mathbb{R}^N)))$ with $\frac{\partial u_0}{\partial t} \in L^q(Q)$ such that (3.16), (3.17) and (3.19) hold as $E' \ni \varepsilon \rightarrow 0$. Since $(\sqrt{\varepsilon} Du'_\varepsilon)_{\varepsilon \in E'}$ is bounded in $L^2(Q)$, taking into account Corollary 2.3 [see in particular (2.31)], we may regard u_1 as an element of $L^p(Q; W_{\#}^1\mathcal{X}_{A_y}^p(\mathbb{R}^N))$. Therefore, thanks to Lemma 3.2, Theorem is proved if we can show that $\mathbf{u} = (u_0, u_1)$ verifies the variational equation (3.15) (the conclusive argument is classical).

For this purpose, let us fix freely $\phi = (\psi_0, \psi_1) \in V_0^\infty$ with $\psi_0 \in \mathcal{D}_{\mathbb{R}}(Q)$, $\psi_1 \in \mathcal{D}_{\mathbb{R}}(Q) \otimes (\mathbb{R}A_y^\infty/\mathbb{C})$; and let us attach to ϕ the sequence $(\phi_\varepsilon)_{\varepsilon > 0}$, ϕ_ε given

by the appropriate form (3.14) where ψ_2 is arbitrary fixed in $\mathcal{D}_{\mathbb{R}}(Q) \otimes_{\mathbb{R}} A_y^{\infty} \otimes_{\mathbb{R}} A_{\tau}^{\infty}$. In (2.8), take $v = \phi_{\varepsilon}(t)$ ($0 < t < T$), integrate from 0 to T and then use (2.5) to have

$$\begin{aligned}
& \int_Q f(u_{\varepsilon} - \phi_{\varepsilon}) \, dx \, dt - \int_0^T (u_{\varepsilon}''(t), u_{\varepsilon}(t) - \phi_{\varepsilon}(t)) \, dt \\
& - \int_Q g^{\varepsilon}(-, \phi'_{\varepsilon})(u_{\varepsilon} - \phi_{\varepsilon}) \, dx \, dt - \varepsilon \int_Q \frac{\partial u'_{\varepsilon}}{\partial x_i} \left(\frac{\partial u_{\varepsilon}}{\partial x_i} - \frac{\partial \phi_{\varepsilon}}{\partial x_i} \right) \, dx \, dt \\
& - \int_Q a_{ij}^{\varepsilon}(-, D\phi_{\varepsilon}) \frac{\partial \phi_{\varepsilon}}{\partial x_i} \left(\frac{\partial u_{\varepsilon}}{\partial x_j} - \frac{\partial \phi_{\varepsilon}}{\partial x_j} \right) \, dx \, dt \\
& \geq \int_Q (g^{\varepsilon}(-, u'_{\varepsilon}) - g^{\varepsilon}(-, \phi'_{\varepsilon}))(u_{\varepsilon} - \phi_{\varepsilon}) \, dx \, dt \\
& + \int_Q \left(a_{ij}^{\varepsilon}(-, Du_{\varepsilon}) \frac{\partial u_{\varepsilon}}{\partial x_i} - a_{ij}^{\varepsilon}(-, D\phi_{\varepsilon}) \frac{\partial \phi_{\varepsilon}}{\partial x_i} \right) \\
& \left(\frac{\partial u_{\varepsilon}}{\partial x_j} - \frac{\partial \phi_{\varepsilon}}{\partial x_j} \right) \, dx \, dt \geq 0 \quad (\text{see } (i)_4 \text{ and } (ii)_5).
\end{aligned}$$

Observing that

$$\int_0^T (u_{\varepsilon}''(t), u_{\varepsilon}(t)) \, dt = \int_{\Omega} u'_{\varepsilon}(T) u_{\varepsilon}(T) \, dx - \int_{\Omega} \hat{u}_1 \hat{u}_0 \, dx - \|u'_{\varepsilon}\|_{L^2(Q)}^2$$

and

$$\begin{aligned}
& \int_Q Du'_{\varepsilon} \cdot (Du_{\varepsilon} - D\phi_{\varepsilon}) \, dx \, dt \\
& = \int_Q (Du'_{\varepsilon} - D\phi'_{\varepsilon}) \cdot (Du_{\varepsilon} - D\phi_{\varepsilon}) \, dx \, dt + \int_Q D\phi'_{\varepsilon} \cdot (Du_{\varepsilon} - D\phi_{\varepsilon}) \, dx \, dt \\
& = \frac{1}{2} (\|Du_{\varepsilon}(T)\|_{L^2(\Omega)}^2 - \|Du_{\varepsilon}(0)\|_{L^2(\Omega)}^2) + \int_Q D\phi'_{\varepsilon} \cdot (Du_{\varepsilon} - D\phi_{\varepsilon}) \, dx \, dt,
\end{aligned}$$

the preceding inequality is equivalent to

$$\begin{aligned}
& \int_Q f(u_{\varepsilon} - \phi_{\varepsilon}) \, dx \, dt - \varepsilon \int_Q D\phi'_{\varepsilon} \cdot (Du_{\varepsilon} - D\phi_{\varepsilon}) \, dx \, dt \\
& - \int_Q g^{\varepsilon}(-, \phi'_{\varepsilon})(u_{\varepsilon} - \phi_{\varepsilon}) \, dx \, dt - \int_Q a_{ij}^{\varepsilon}(-, D\phi_{\varepsilon}) \frac{\partial \phi_{\varepsilon}}{\partial x_i} \left(\frac{\partial u_{\varepsilon}}{\partial x_j} - \frac{\partial \phi_{\varepsilon}}{\partial x_j} \right) \, dx \, dt \\
& + \int_0^T (u_{\varepsilon}''(t), \phi_{\varepsilon}(t)) \, dt + \|u'_{\varepsilon}\|_{L^2(Q)}^2 + \int_{\Omega} \hat{u}_1 \hat{u}_0 \, dx - \int_{\Omega} u'_{\varepsilon}(T) u_{\varepsilon}(T) \, dx \\
& + \frac{\varepsilon}{2} \|D\hat{u}_0\|_{L^2(\Omega)}^2 \geq \frac{\varepsilon}{2} \|Du_{\varepsilon}(T)\|_{L^2(\Omega)}^2 \geq 0
\end{aligned} \tag{3.20}$$

for any $0 < \varepsilon \leq \varepsilon_0$. Our goal in this step is to pass to the limit in (2.8) when $E' \ni \varepsilon \rightarrow 0$.

Let us determine the limit of each quantity at the left hand side of inequality (3.20). We need to know that, ϕ_ε being defined by (3.14), as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \phi_\varepsilon &\rightarrow \psi_0 \quad \text{in } L^p(0, T; W_0^{1,p}(\Omega))\text{-weak,} \\ \frac{\partial \phi_\varepsilon}{\partial t} &\rightarrow \frac{\partial \psi_0}{\partial t} \quad \text{and} \quad \frac{\partial^2 \phi_\varepsilon}{\partial t^2} \rightarrow \frac{\partial^2 \psi_0}{\partial t^2} + \frac{\bar{\partial}^2 \psi_2}{\partial \tau^2} \quad \text{in } L^r(Q)\text{-weak } \Sigma, \\ \frac{\partial \phi_\varepsilon}{\partial x_i} &\rightarrow \frac{\partial \psi_0}{\partial x_i} + \frac{\bar{\partial} \psi_1}{\partial y_i} \quad \text{in } L^r(Q)\text{-weak } \Sigma, 1 \leq i \leq N. \end{aligned}$$

Then we have obviously

$$\int_Q f(u_\varepsilon - \phi_\varepsilon) \, dx \, dt \rightarrow \int_Q f(u_0 - \psi_0) \, dx \, dt$$

and, according to Corollary 3.1,

$$\int_Q g^\varepsilon(-, \phi'_\varepsilon)(u_\varepsilon - \phi_\varepsilon) \, dx \, dt \rightarrow \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h\left(\frac{\partial \psi_0}{\partial t}\right) (u_0 - \psi_0) \, dy \, d\tau \, dx \, dt$$

and

$$\begin{aligned} &\int_Q a_{ij}^\varepsilon(-, D\phi_\varepsilon) \frac{\partial \phi_\varepsilon}{\partial x_i} \left(\frac{\partial u_\varepsilon}{\partial x_j} - \frac{\partial \phi_\varepsilon}{\partial x_j} \right) \, dx \, dt \\ &\quad \rightarrow \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(D\psi_0 + \bar{D}_y \psi_1) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\bar{\partial} \psi_1}{\partial y_i} \right) \left(\frac{\partial u_0}{\partial x_j} + \frac{\bar{\partial} u_1}{\partial y_j} \right) \, dy \, d\tau \, dx \, dt \\ &\quad - \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(D\psi_0 + \bar{D}_y \psi_1) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\bar{\partial} \psi_1}{\partial y_i} \right) \left(\frac{\partial \psi_0}{\partial x_j} + \frac{\bar{\partial} \psi_1}{\partial y_j} \right) \, dy \, d\tau \, dx \, dt \end{aligned}$$

as $E' \ni \varepsilon \rightarrow 0$. Next, observe that

$$\int_0^T (u''_\varepsilon(t), \phi_\varepsilon(t)) \, dt = \int_Q u_\varepsilon \phi''_\varepsilon \, dx \, dt.$$

Thanks to the fact that $L^p(0, T; W_0^{1,p}(\Omega))$ is compactly embedded in $L^2(Q)$, we have $u_\varepsilon \rightarrow u_0$ in $L^2(Q)$ as $E' \ni \varepsilon \rightarrow 0$. Since $\frac{\partial^2 \phi_\varepsilon}{\partial t^2} \rightarrow \frac{\partial^2 \psi_0}{\partial t^2} + \frac{\bar{\partial}^2 \psi_2}{\partial \tau^2}$ in $L^2(Q)$ -weak, it follows that, as $E' \ni \varepsilon \rightarrow 0$,

$$\int_0^T (u''_\varepsilon(t), \phi_\varepsilon(t)) \, dt \rightarrow \int_0^T (u''_0(t), \psi_0(t)) \, dt.$$

Now, according to (2.7) combined with Remark 2.1, it follows by Lemma 2.1 that

$$\sup_{\varepsilon \in E'} (\|u'_\varepsilon(t)\|_{L^2(\Omega)} + \|u_\varepsilon(t)\|_{L^2(\Omega)}) < \infty$$

a.e. in $(0, T]$. Then, up to a subsequence still denoted E' , we have $u_\varepsilon(T) \rightarrow v_0$ in $L^2(\Omega)$ and $u'_\varepsilon(T) \rightarrow w_0$ in $L^2(\Omega)$ -weak when $E' \ni \varepsilon \rightarrow 0$. But $v_0 = u_0(T)$ and $w_0 = u'_0(T)$, since the mappings $v \rightarrow v(t)$ and $v \rightarrow v'(t)$ send continuously the space $\{u \in L^\infty(0, T; L^2(\Omega)) : u' \in L^\infty(0, T; L^2(\Omega))\}$ into $L^2(\Omega)$. Hence, as $E' \ni \varepsilon \rightarrow 0$,

$$\int_\Omega u'_\varepsilon(T) u_\varepsilon(T) \, dx \rightarrow \int_\Omega u'_0(T) u_0(T) \, dx.$$

On the other hand, let $(u_n)_{n \geq 1}$ be a sequence in $C^1([0, T]; L^2(\Omega))$ such that $u_n(0) = \hat{u}_0$, $u'_n(0) = \hat{u}_1$, and

$$\lim_{n \rightarrow \infty} \|u_n - u_0\|_{W^{1,2}(0,T;L^2(\Omega))} = 0.$$

By decomposition we have

$$\|u'_\varepsilon\|_{L^2(Q)}^2 = \int_0^T (u'_\varepsilon(t), u'_n(t)) \, dt + \int_0^T (u'_\varepsilon(t), u'_\varepsilon(t) - u'_n(t)) \, dt;$$

but

$$\lim_{n \rightarrow \infty} \lim_{E' \ni \varepsilon \rightarrow 0} \left| \int_0^T (u'_\varepsilon(t), u'_\varepsilon(t) - u'_n(t)) \, dt \right| \leq c \lim_{n \rightarrow \infty} \lim_{E' \ni \varepsilon \rightarrow 0} \|u'_\varepsilon - u'_n\|_{L^2(Q)} = 0,$$

then

$$\lim_{n \rightarrow \infty} \lim_{E' \ni \varepsilon \rightarrow 0} \|u'_\varepsilon\|_{L^2(Q)}^2 = \lim_{n \rightarrow \infty} \lim_{E' \ni \varepsilon \rightarrow 0} \int_0^T (u'_\varepsilon(t), u'_n(t)) \, dt = \|u'_0\|_{L^2(Q)}^2.$$

Finally, observing that

$$\begin{aligned} \varepsilon \int_Q D\phi'_\varepsilon \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt &= \varepsilon \int_Q D\psi'_0 \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt \\ &+ \varepsilon^2 \int_Q (D\psi'_1)^\varepsilon \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt + \varepsilon \int_Q (D_y \psi'_1)^\varepsilon \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt \\ &+ \varepsilon^3 \int_Q (D\psi'_2)^\varepsilon \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt + \varepsilon^2 \int_Q (D_y \psi'_2)^\varepsilon \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt \\ &+ \varepsilon^2 \int_Q \left(D \frac{\partial \psi_2}{\partial \tau} \right)^\varepsilon \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt + \varepsilon \int_Q \left(D_y \frac{\partial \psi_2}{\partial \tau} \right)^\varepsilon \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt \end{aligned}$$

and that all the integral sequences at the right are bounded, it follows that

$$\varepsilon \int_Q D\phi'_\varepsilon \cdot (Du_\varepsilon - D\phi_\varepsilon) \, dx \, dt \rightarrow 0 \quad \text{as } E' \ni \varepsilon \rightarrow 0.$$

Therefore, taking the $\liminf_{E' \ni \varepsilon \rightarrow 0}$ of both sides of (3.20), one drives to

$$\begin{aligned} &\int_Q f(u_0 - \psi_0) \, dx \, dt - \int_Q \left(\mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h \left(\frac{\partial \psi_0}{\partial t} \right) \, dy \, d\tau \right) (u_0 - \psi_0) \, dx \, dt \\ &- \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij} (D\psi_0 + \overline{D}_y \psi_1) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\overline{\partial} \psi_1}{\partial y_i} \right) \left(\frac{\partial u_0}{\partial x_j} + \frac{\overline{\partial} u_1}{\partial y_j} \right) \, dy \, d\tau \, dx \, dt \\ &+ \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij} (D\psi_0 + \overline{D}_y \psi_1) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\overline{\partial} \psi_1}{\partial y_i} \right) \left(\frac{\partial \psi_0}{\partial x_j} + \frac{\overline{\partial} \psi_1}{\partial y_j} \right) \, dy \, d\tau \, dx \, dt \\ &+ \int_0^T (u''_0(t), \psi_0(t)) \, dt + \|u'_0\|_{L^2(Q)}^2 + \int_\Omega \hat{u}_1 \hat{u}_0 \, dx - \int_\Omega u'_0(T) u_0(T) \, dx \geq 0 \end{aligned}$$

that is, putting $\mathbb{D}\mathbf{u} = Du_0 + \overline{D}_y u_1$, $\mathbb{D}\Phi = D\psi_0 + \overline{D}_y \psi_1$, $\mathbb{D}_i \mathbf{u} = \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i}$ and $\mathbb{D}_i \Phi = \frac{\partial \psi_0}{\partial x_i} + \frac{\overline{\partial} \psi_1}{\partial y_i}$, one has

$$\begin{aligned} & \int_Q f(u_0 - \psi_0) \, dx \, dt - \int_0^T (u_0''(t), u_0(t) - \psi_0(t)) \, dt \\ & - \int_Q \left(\mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h \left(\frac{\partial \psi_0}{\partial t} \right) \, dy \, d\tau \right) (u_0 - \psi_0) \, dx \, dt \\ & - \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(\mathbb{D}\Phi) \mathbb{D}_i \Phi \mathbb{D}_j (\mathbf{u} - \Phi) \, dy \, d\tau \, dx \, dt \geq 0. \end{aligned} \quad (3.21)$$

But V_0^∞ is dense in $\mathbb{V}_0^p$, then we still have (3.21) for each $\Phi \in \mathbb{V}_0^p$. Taking in particular $\Phi = \mathbf{u} - \theta \mathbf{v}$ with $\theta > 0$ and $\mathbf{v} = (v_0, v_1)$ in $\mathbb{V}_0^p$, next dividing both sides of inequality by θ , we are led to

$$\begin{aligned} & \int_Q f v_0 \, dx \, dt - \int_0^T (u_0''(t), v_0(t)) \, dt \\ & - \int_Q \left(\mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h \left(\frac{\partial u_0}{\partial t} - \theta \frac{\partial v_0}{\partial t} \right) \, dy \, d\tau \right) v_0 \, dx \, dt \\ & - \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(\mathbb{D}(\mathbf{u} - \theta \mathbf{v})) \mathbb{D}_i (\mathbf{u} - \theta \mathbf{v}) \mathbb{D}_j \mathbf{v} \, dy \, d\tau \, dx \, dt \geq 0. \end{aligned}$$

When $\theta \rightarrow 0$, it results by the continuity of h and b_{ij} (see Proposition 3.1) that

$$\begin{aligned} & \int_Q f v_0 \, dx \, dt - \int_0^T (u_0''(t), v_0(t)) \, dt - \int_Q \left(\mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h \left(\frac{\partial u_0}{\partial t} \right) \, dy \, d\tau \right) v_0 \, dx \, dt \\ & - \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(\mathbb{D}\mathbf{u}) \mathbb{D}_i \mathbf{u} \mathbb{D}_j \mathbf{v} \, dy \, d\tau \, dx \, dt \geq 0. \end{aligned}$$

Therefore, the theorem follows by mere routine (see the proof of [25, Theorem 4.1]). $\square$

The variational equation in (3.15) is equivalent to the system of two equations

$$\begin{aligned} & \int_Q \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(\mathbb{D}\mathbf{u}) \mathbb{D}_i \mathbf{u} \frac{\overline{\partial} v_1}{\partial y_j} \, dy \, d\tau \, dx \, dt = 0 \\ & \text{for all } v_1 \in L^p(Q; W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N; \mathbb{R})) \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} & \int_0^T (u_0''(t), v_0(t)) \, dt + \int_Q \left(\mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h \left(\frac{\partial u_0}{\partial t} \right) \, dy \, d\tau \right) v_0 \, dx \, dt \\ & + \int_Q \left[\mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(\mathbb{D}\mathbf{u}) \mathbb{D}_i \mathbf{u} \, dy \, d\tau \right] \frac{\partial v_0}{\partial x_j} \, dx \, dt = \int_Q f(t) v_0(t) \, dx \, dt \\ & \text{for all } v_0 \in L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R})). \end{aligned} \quad (3.23)$$

Let $r = (r_i) \in \mathbb{R}^N$ be freely fixed. Thanks to (3.10)–(3.11) let $\pi(r)$ be the unique solution of the cell-problem:

$$\begin{cases} \pi(r) \in W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N; \mathbb{R}) \text{ and} \\ \mathcal{L}_{A_y} \int_{\mathbb{R}^N} \mathcal{L}_{A_\tau} \int_{\mathbb{R}} b_{ij}(r + \overline{D}_y \pi(r)) \, d\tau \left(r_i + \frac{\overline{\partial} \pi(r)}{\partial y_i} \right) \frac{\overline{\partial} \theta}{\partial y_j} \, dy = 0 \\ \text{for all } \theta \in W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N; \mathbb{R}). \end{cases}$$

Then, choosing in (3.22) test function $v_1(x, t) = \varphi(x, t)\theta$, $\varphi \in \mathcal{D}(Q, \mathbb{R})$ and $\theta \in W_{\#}^1 \mathcal{X}_{A_y}^p(\mathbb{R}^N; \mathbb{R})$, it follows, almost everywhere in Q ,

$$\mathcal{L}_{A_y} \int_{\mathbb{R}^N} \mathcal{L}_{A_\tau} \int_{\mathbb{R}} b_{ij}(Du_0 + \overline{D}_y u_1) \, d\tau \left(\frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \right) \frac{\overline{\partial} \theta}{\partial y_j} \, dy = 0$$

for all $\theta \in W_{\#}^{1,p}(\Delta(A_y); \mathbb{R})$. Comparing with the above cell-problem for $r = Du_0(x, t)$, the unicity argument yields $u_1 = \pi(Du_0)$. Hence by letting

$$q_{ij}(r)r_i = \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} b_{ij}(r + \overline{D}_y \pi(r)) \left(r_i + \frac{\overline{\partial} \pi(r)}{\partial y_i} \right) \, dy \, d\tau, \quad 1 \leq j \leq N,$$

we see by (3.23) that u_0 is a solution of the macroscopic evolution problem

$$\begin{cases} \frac{\partial^2 u_0}{\partial t^2} + \tilde{h} \left(\frac{\partial u_0}{\partial t} \right) - \frac{\partial}{\partial x_j} \left(q_{ij}(Du_0) \frac{\partial u_0}{\partial x_i} \right) = f & \text{in } Q, \\ u_0(x, 0) = \hat{u}_0(x) & \text{in } \Omega, \\ \frac{\partial u_0}{\partial t}(x, 0) = \hat{u}_1(x) & \text{in } \Omega, \\ u_0(x, t) = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \quad (3.24)$$

(with the convention of Einstein), where $\tilde{h}(\zeta) = \mathcal{L}_A \int_{\mathbb{R}_{y,\tau}^{N+1}} h(\zeta) \, dy \, d\tau$, $\zeta \in \mathbb{R}$.

4. Concrete homogenization problems for (1.1)

In this last section, we consider a few examples of homogenization problems for (1.1) in a concrete setting. We will show how their study leads naturally to the abstract setting of Sect. 3 and so we may conclude by merely applying Theorem 3.1 for suitable p -quasi-pseudoproper H -algebras.

4.1. Problem I (periodic setting)

We assume that functions $(y, \tau) \rightarrow g(y, \tau, \zeta)$, $\zeta \in \mathbb{R}$, and $(y, \tau) \rightarrow a_{ij}(y, \tau, \lambda)$, $1 \leq i, j \leq N$, $\lambda \in \mathbb{R}^N$, satisfy the periodicity hypothesis:

$$\begin{cases} \text{For each } k \in \mathbb{Z}^N \text{ and each } \ell \in \mathbb{Z}, \text{ we have a.e.} \\ \text{in } (y, \tau) \in \mathbb{R}^{N+1} \\ g(y + k, \tau + \ell, \zeta) = g(y, \tau, \zeta) & \text{for all } \zeta \in \mathbb{R} \\ \text{and} \\ a_{ij}(y + k, \tau + \ell, \lambda) = a_{ij}(y, \tau, \lambda) & \text{for all } \lambda \in \mathbb{R}^N. \end{cases} \quad (4.1)$$

One also expresses (4.1) by saying that $g(-, \zeta)$ and $a_{ij}(-, \lambda)$ are Z -periodic, where $Z = Y \times \Theta$, with $Y = (0, 1)^N$ and $\Theta = (0, 1)$. We study the homogenization of (1.1) under the periodicity hypothesis (4.1). Of course the right

H -algebra for this problem is the periodic H -algebra $A = \mathcal{C}_{per}(Y) \odot \mathcal{C}_{per}(\Theta) = \mathcal{C}_{per}(Z)$ (see [23]).

Taking into account (4.1), we deduce by (i)₁, (ii)₁ and (ii)₃ that, for all $(\zeta, \lambda) \in \mathbb{R} \times \mathbb{R}^N$, the functions $(y, \tau) \rightarrow g(y, \tau, \zeta)$ and $(y, \tau) \rightarrow a_{ij}(y, \tau, \lambda)$ of $\mathbb{R}_{y, \tau}^{N+1}$ into $\mathbb{R}$, denoted by $g(-, \zeta)$ and $a_{ij}(-, \lambda)$, belong to $\mathcal{C}_{per}(Z; \mathbb{R})$ and to $L_{per}^\infty(Z; \mathbb{R})$ (respectively). Obviously we have $L_{per}^\infty(Z) \subset L_{per}'(Z)$. Hence (3.2) follows by the fact that $L_{per}'(Y) = \mathfrak{X}_{\mathcal{C}_{per}(Z)}^{p'}$ (topologically). Since $\mathcal{C}_{per}(Z)$ is p -quasi-pseudoproper (this is classical), we may apply Theorem 2.3. Precisely, as $\varepsilon \rightarrow 0$, we have $u_\varepsilon \rightarrow u_0$ in $L^p(0, T; W_0^{1,p}(\Omega))$ -weak, and $\frac{\partial u_\varepsilon}{\partial x_i} \rightarrow \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i}$ in $L^p(Q)$ -weak Σ ($1 \leq i \leq N$), where u_ε is the solution of (1.1) (for fixed $\varepsilon > 0$) and according to Theorem 3.1, $\mathbf{u} = (u_0, u_1)$ is uniquely defined by

$$\left\{ \begin{array}{l} \mathbf{u} = (u_0, u_1) \in \mathbb{V}_0^p : \\ \int_0^T (u_0''(t), v_0(t)) dt + \int_Q \left(\int_Z g(y, \tau, \frac{\partial u_0}{\partial t}) dy d\tau \right) v_0(t) dx dt \\ \quad + \int_{Q \times Y} \left(\int_\Theta a_{ij}(y, \tau, \mathbb{D}\mathbf{u}) d\tau \right) \mathbb{D}_i \mathbf{u} \mathbb{D}_j \mathbf{v} dx dt dy = \int_Q f(t) v_0(t) dx dt \\ \text{for all } \mathbf{v} = (v_0, v_1) \in \mathbb{V}_0^p, \end{array} \right.$$

where $\mathbb{D}\mathbf{u} = Du_0 + Dy u_1$ and $\mathbb{D}_i \mathbf{u} = \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i}$, and

$$\mathbb{V}_0^p = L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R})) \times L^p(Q; W_{\#}^{1,p}(Y; \mathbb{R})).$$

4.2. Problem II (almost periodic setting)

We study the limiting behaviour, as $\varepsilon \rightarrow 0$, of u_ε defined for each $\varepsilon > 0$ by (1.1) under the structure hypothesis

$$\left\{ \begin{array}{l} g(-, \zeta) \in AP(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R}) \quad \text{for all } \zeta \in \mathbb{R} \text{ and} \\ a_{ij}(-, \lambda) \in L_{AP}'(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R}) \quad \text{for all } \lambda \in \mathbb{R}^N \\ (1 \leq i, j \leq N), \end{array} \right. \quad (4.2)$$

where $AP(\mathbb{R}^{N+1})$ is the space of all almost periodic continuous complex functions on $\mathbb{R}^{N+1}$ and $L_{AP}'(\mathbb{R}^{N+1})$ denotes the space of all functions $u \in L_{loc}'(\mathbb{R}^{N+1})$ that are almost periodic in Stepanoff sense, see [5, 14, 22]. It is clearly knowing that $L_{AP}'(\mathbb{R}^{N+1})$ is the closure in $(L^{p'}, \ell^\infty)(\mathbb{R}^{N+1})$ of $T(\mathbb{R}^{N+1})$, the algebra of all functions $u : \mathbb{R}^{N+1} \rightarrow \mathbb{C}$ of the form

$$u = \sum_{k \in I} c_k \gamma_k, c_k \in \mathbb{C}, \gamma_k(z) = e^{i2\pi k \cdot z} \quad (z \in \mathbb{R}^{N+1}),$$

I being a finite subset (depending on u) of $\mathbb{R}^{N+1}$. Consequently $AP(\mathbb{R}^{N+1})$ is dense in $L_{AP}'(\mathbb{R}^{N+1})$. Moreover, given $u \in L_{AP}'(\mathbb{R}^{N+1})$ the spectrum of u is the set $Sp(u)$ defined by

$$Sp(u) = \{k \in \mathbb{R}^{N+1} : M(\bar{\gamma}_k u) \neq 0\},$$

M being the mean value on $AP(\mathbb{R}^{N+1})$ extended by continuity on $L_{AP}'(\mathbb{R}^{N+1})$, for details see [23].

We study the homogenization of (1.1) under the almost periodic hypothesis (4.2). As is well known, this problem reduces to the abstract homogenization result (Theorem 3.1) if, for some p -quasi-pseudoproper H -algebra A , conditions (3.1)–(3.2) are fulfilled.

To this end, we will need the following

Lemma 4.1. *Let $(f_i)_{i \in I}$ be a countable family of functions in $\mathcal{C}(\mathbb{R}^N; L_{AP}^{p'}(\mathbb{R}^{N+1}))$. There exists a countable subgroup R of $\mathbb{R}^{N+1}$ such that $(f_i)_{i \in I} \subset \mathcal{C}(\mathbb{R}^N; L_{AP,R}^{p'}(\mathbb{R}^{N+1}))$ where*

$$L_{AP,R}^{p'}(\mathbb{R}^{N+1}) = \{u \in L_{AP}^{p'}(\mathbb{R}^{N+1}) : Sp(u) \subset R\}.$$

Proof. cf. [32, Corollary 4.1]. □

Let us recall that the vector space $L_{AP,R}^{p'}(\mathbb{R}^{N+1})$ coincides with the closure in $(L^{p'}, \ell^\infty)(\mathbb{R}^{N+1})$ of the almost periodic H -algebra $AP_R(\mathbb{R}^{N+1})$, see [22].

Combined (i)₁, (ii)₁ and (ii)₂ with (4.2) it follows that $g \in \mathcal{C}(\mathbb{R}; AP(\mathbb{R}^{N+1}; \mathbb{R}))$ and $a_{ij} \in \mathcal{C}(\mathbb{R}^N; L_{AP}^{p'}(\mathbb{R}^{N+1}; \mathbb{R}))$, $1 \leq i, j \leq N$. Hence applying Lemma 4.1 there is a countable subgroup R of $\mathbb{R}^{N+1}$ such that $g \in \mathcal{C}(\mathbb{R}; AP_R(\mathbb{R}^{N+1}; \mathbb{R}))$ and $a_{ij} \in \mathcal{C}(\mathbb{R}^N; L_{AP,R}^{p'}(\mathbb{R}^{N+1}; \mathbb{R}))$ ($1 \leq i, j \leq N$). By density, for each $1 \leq i, j \leq N$, there is a sequence $(a_n^{ij})_n \subset \mathcal{C}(\mathbb{R}^N; AP_R(\mathbb{R}^{N+1}; \mathbb{R}))$ such that

$$a_n^{ij} \rightarrow a_{ij} \text{ in } \mathcal{C}(\mathbb{R}^N; (L^{p'}, \ell^\infty)(\mathbb{R}^{N+1})) \text{ as } n \rightarrow \infty.$$

Since $(L^{p'}, \ell^\infty)$ is continuously embedded in $\Xi^{p'}$, we deduce that for all $\lambda \in \mathbb{R}^N$ and all $1 \leq i, j \leq N$

$$\|a_n^{ij}(-, \lambda) - a_{ij}(-, \lambda)\|_{\Xi^{p'}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

But each $a_n^{ij}(-, \lambda)$ lies in AP_R which is dense in $\mathfrak{X}_{AP_R}^{p'}$ for the $\Xi^{p'}$ -norm, then we deduce (3.1) and (3.2) where the corresponding H -algebra A (for this problem) is the almost periodic H -algebra $A = AP_R$ defined above. Observing that $AP_R = AP_{pr_y R} \odot AP_{pr_\tau R}$ where $pr_y R \subset \mathbb{R}_y^N$ and $pr_\tau R \subset \mathbb{R}$ are the projections of R on $\mathbb{R}_y^N$ and on $\mathbb{R}$ (respectively), and that $AP_{pr_y R}$ and $AP_{pr_\tau R}$ are of class $\mathcal{C}^\infty$ and p -total, then AP_R is p -quasi-pseudoproper; so that we can apply Theorem 3.1 when $A = AP_R$.

4.3. Problem III

We study the limiting behaviour, as $\varepsilon \rightarrow 0$, of u_ε defined for each $\varepsilon > 0$ by (1.1) under the structure hypothesis

$$\begin{cases} g(-, \zeta) \in AP(\mathbb{R}_{y,\tau}^{N+1}; \mathbb{R}) + \mathcal{B}_\infty(\mathbb{R}_{y,\tau}^{N+1}; \mathbb{R}) \\ \quad \text{for all } \zeta \in \mathbb{R} \text{ and} \\ a_{ij}(-, \lambda) \in L_{\infty, AP}^{p'}(\mathbb{R}_{y,\tau}^{N+1}; \mathbb{R}) \\ \quad \text{for all } \lambda \in \mathbb{R}^N (1 \leq i, j \leq N), \end{cases} \quad (4.3)$$

where $L_{\infty, AP}^{p'}(\mathbb{R}_{y, \tau}^{N+1})$ denotes the closure in $(L^{p'}, \ell^\infty)(\mathbb{R}_{y, \tau}^{N+1})$ of the space of all finite sums

$$\sum_{\text{finite}} \varphi_i u_i \text{ with } \varphi_i \in \mathcal{B}_\infty(\mathbb{R}_{y, \tau}^{N+1}) \text{ and } u_i \in AP(\mathbb{R}_{y, \tau}^{N+1}). \quad (4.4)$$

Let us recall that the space of all finite sums (4.4) coincides with $\mathcal{B}_\infty(\mathbb{R}_{y, \tau}^{N+1}) + AP(\mathbb{R}_{y, \tau}^{N+1})$.

Proposition 4.1. *Assume (4.3). Then there exists a countable subgroup R of $\mathbb{R}^{N+1}$ such that*

$$\begin{aligned} g(-, \zeta) &\in \mathcal{B}_{\infty, R}(\mathbb{R}^{N+1}) \quad \text{for all } \zeta \in \mathbb{R} \text{ and} \\ a_{ij}(-, \lambda) &\in L_{\infty, R}^{p'}(\mathbb{R}^{N+1}; \mathbb{R}) \quad \text{for all } \lambda \in \mathbb{R}^N (1 \leq i, j \leq N), \end{aligned} \quad (4.5)$$

where $L_{\infty, R}^{p'}(\mathbb{R}^{N+1})$ is the closure of $\mathcal{B}_{\infty, R}(\mathbb{R}^{N+1}) = AP_R(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R}) + \mathcal{B}_\infty(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R})$ in $(L^{p'}, \ell^\infty)(\mathbb{R}^{N+1})$ (see [23]).

Proof. It results from (i)₁, (ii)₁ and (ii)₂ combined with (4.3) that

$$\begin{aligned} g &\in \mathcal{C}(\mathbb{R}; \mathcal{B}_\infty(\mathbb{R}^{N+1}; \mathbb{R}) + AP(\mathbb{R}^{N+1}; \mathbb{R})) \text{ and} \\ a_{ij} &\in \mathcal{C}(\mathbb{R}^N; L_{\infty, AP}^{p'}(\mathbb{R}^{N+1}; \mathbb{R})), \quad 1 \leq i, j \leq N. \end{aligned}$$

By density there is a sequence $(a_n^{ij})_{n \in \mathbb{N}}$ defined by $a_n^{ij} = b_n^{ij} + c_n^{ij}$, $b_n^i \in \mathcal{C}(\mathbb{R}^N; \mathcal{B}_\infty(\mathbb{R}^{N+1}))$ and $c_n^{ij} \in \mathcal{C}(\mathbb{R}^N; AP(\mathbb{R}^{N+1}))$ such that

$$a_n^{ij} \rightarrow a_{ij} \text{ in } \mathcal{C}(\mathbb{R}^N; (L^{p'}, \ell^\infty)(\mathbb{R}^{N+1})) \text{ as } n \rightarrow \infty. \quad (4.6)$$

Since $\{c_n^{ij} : 1 \leq i, j \leq N \text{ and } n \in \mathbb{N}\}$ is a countable subset of $\mathcal{C}(\mathbb{R}^N; AP(\mathbb{R}^{N+1}))$, by virtue of Lemma 4.1 there exists a countable subgroup R of $\mathbb{R}^{N+1}$ such that

$$\{c_n^{ij} : 1 \leq i, j \leq N \text{ and } n \in \mathbb{N}\} \subset \mathcal{C}(\mathbb{R}^N; AP_R(\mathbb{R}^{N+1})).$$

Therefore the sequence $(a_n^{ij})_{n \in \mathbb{N}}$ is a subset of $\mathcal{C}(\mathbb{R}^N; \mathcal{B}_{\infty, R}(\mathbb{R}^{N+1}))$. It then results from (4.6) combined with the completeness of $\mathcal{C}(\mathbb{R}^N; L_{\infty, R}^{p'}(\mathbb{R}^{N+1}))$ that $a_{ij} \in \mathcal{C}(\mathbb{R}^N; L_{\infty, R}^{p'}(\mathbb{R}^{N+1}; \mathbb{R}))$. $\square$

Since the H -algebra $\mathcal{B}_{\infty, R} = AP_{pr_y R}(\mathbb{R}^N) \odot AP_{pr_\tau R}(\mathbb{R}) + \mathcal{B}_\infty(\mathbb{R}^N) \odot \mathcal{B}_\infty(\mathbb{R})$ (see [23] and [40]) is p -quasi-pseudo-proper and is dense in $L_{\infty, R}^{p'} \subset (L^{p'}, \ell^\infty)(\mathbb{R}^{N+1})$ the latter being continuously embedded in $\Xi^{p'}(\mathbb{R}^{N+1})$, then $L_{\infty, R}^{p'} \subset \mathfrak{X}_{\mathcal{B}_{\infty, R}}^{p'}$. Therefore (3.1)–(3.2) follow with $A = \mathcal{B}_{\infty, R}$. This achieves the homogenization of (1.1) by virtue of Theorem 3.1.

Corollary 4.1. *Assume the second assertion in (4.3) is replaced by $a_{ij}(-, \lambda) \in L_{y, \tau}^{p'}(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R}) + L_{AP}^{p'}(\mathbb{R}_{y, \tau}^{N+1}; \mathbb{R})$ for all $\lambda \in \mathbb{R}^N$ ($1 \leq i, j \leq N$). Then (4.5) is fulfilled with $A = \mathcal{B}_{\infty, R}(\mathbb{R}^{N+1})$.*

Proof. Indeed we have the embeddings $L^{p'}(\mathbb{R}^{N+1}) \subset L_{\infty, AP}^{p'}(\mathbb{R}^{N+1})$ and $L_{AP}^{p'}(\mathbb{R}^{N+1}) \subset L_{\infty, AP}^{p'}(\mathbb{R}^{N+1})$; so that (4.3) is satisfied. Thus the corollary follows by applying Proposition 4.1. $\square$

4.4. Problem IV

We recall that $\mathcal{B}_\infty(\mathbb{Z}^N)$ denotes the space of all mappings $\sigma : \mathbb{Z}^N \rightarrow \mathbb{C}$ that converge at infinity, i.e., $\lim_{|k| \rightarrow \infty} \sigma(k) = \varsigma$ ($\varsigma \in \mathbb{C}$). Then $\mathcal{B}_\infty(\mathbb{Z}^N)$ is an essential algebra on $\mathbb{Z}^N$ (see [24, Definition 3.3 and Example 3.1]), and further

$$\mathcal{B}_\infty^0(\mathbb{Z}^N) = \{\sigma \in \mathcal{B}_\infty(\mathbb{Z}^N) : \lim_{|k| \rightarrow \infty} \sigma(k) = 0\}$$

is a subalgebra of $\mathcal{B}_\infty(\mathbb{Z}^N)$. Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R}_y^N \rightarrow \mathbb{C}$ of the form

$$f = \sum_{k \in \mathbb{Z}^N} \sigma(k) \tau_k \varphi \text{ with } \sigma \in \mathcal{B}_\infty(\mathbb{Z}^N) \text{ and } \varphi \in \mathcal{K}(Y)$$

where $Y = (0, 1)^N$, and $\mathcal{K}(Y)$ is identified with the space of those $\varphi \in \mathcal{K}(\mathbb{R}_y^N)$ with compact supports in Y . Let $\mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N)$ be the closure in $\mathcal{B}(\mathbb{R}_y^N)$ of the space of all functions of the form

$$\psi = c + \sum_{\text{finite}} f_i \text{ with } c \in \mathbb{C} \text{ and } f_i \in \mathcal{F}.$$

Then $\mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N)$ is an H -algebra on $\mathbb{R}_y^N$ [24, Proposition 3.3], called the discrete H -algebra on $\mathbb{R}_y^N$ associated to $\mathcal{B}_\infty(\mathbb{Z}^N)$ [24, Definition 3.4]. Moreover $\mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N)$ is of class $\mathcal{C}^\infty$, see the first part of the proof of Proposition 3.5 in [24]. Let $\mathcal{F}_0$ be the set of all functions $f : \mathbb{R}_y^N \rightarrow \mathbb{C}$ of the form

$$f = \sum_{k \in \mathbb{Z}^N} \sigma(k) \tau_k \varphi \text{ with } \sigma \in \mathcal{B}_\infty^0(\mathbb{Z}^N) \text{ and } \varphi \in \mathcal{K}(Y).$$

Denoting by $\langle \mathcal{F} \rangle$ (resp. $\langle \mathcal{F}_0 \rangle$) the set of all finite sums

$$\sum_{\text{finite}} f_i \text{ with } f_i \in \mathcal{F} \text{ (resp. } \mathcal{F}_0),$$

we have the identity $\langle \mathcal{F} \rangle + \mathcal{C}_{per}(Y) = \langle \mathcal{F}_0 \rangle + \mathcal{C}_{per}(Y)$, so that $\langle \mathcal{F}_0 \rangle + \mathcal{C}_{per}(Y)$ is dense vector subspace of the H -algebra $\mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N) \odot \mathcal{C}_{per}(Y)$ [24, Lemma 3.2], which is defined as the closure in $\mathcal{B}(\mathbb{R}_y^N)$ of the space of all finite sums

$$\sum_{\text{finite}} \varphi_i u_i \text{ with } \varphi_i \in \mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N) \text{ and } u_i \in \mathcal{C}_{per}(Y).$$

Proposition 4.2. *The H -algebra $\mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N) \odot \mathcal{C}_{per}(Y)$ is of class $\mathcal{C}^\infty$ and p -total ($p \geq 1$).*

Proof. cf. [24]. □

We study the limiting behaviour, as $\varepsilon \rightarrow 0$, of u_ε defined for each $\varepsilon > 0$ by (1.1) under the hypothesis

$$\begin{cases} g(y, \tau, \lambda) \in \mathcal{C}_{per}(\Theta; \mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N) \odot \mathcal{C}_{per}(Y)) \text{ and} \\ a_{ij}(y, \tau, \lambda) = c_{ij}(k, \tau, \lambda) \text{ a.e. in } y \in k + Y \text{ for all} \\ \lambda \in \mathbb{R}^N (1 \leq i, j \leq N), \end{cases} \quad (4.7)$$

where $(c_{ij})_{1 \leq i, j \leq N}$ is a family of mappings $(k, \tau, \lambda) \rightarrow c_{ij}(k, \tau, \lambda)$ from $\mathbb{Z}^N \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}$ satisfying the structure hypothesis

$$c_{ij}(-, \lambda) \in \mathcal{B}_\infty(\mathbb{Z}^N; \mathcal{C}_{per}(\Theta)) \quad \text{for all } \lambda \in \mathbb{R}^N (1 \leq i, j \leq N) \quad (4.8)$$

Proposition 4.3. *Assume (4.8). Then (3.1)–(3.2) hold true with*

$$A = \mathcal{C}_{per}(\Theta; \mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N) \odot \mathcal{C}_{per}(Y)).$$

Proof. Let $\lambda \in \mathbb{R}^N$, and let $1 \leq i, j \leq N$ be fixed. Let $\mathcal{P} = \mathcal{P}(\lambda, ij)$ denote the set of all functions of the form

$$\psi_{ij}(y, \tau, \lambda) = \sum_{k \in \mathbb{Z}^N} c_{ij}(k, \tau, \lambda) \tau_k \varphi(y), \quad \varphi \in \mathcal{K}(Y).$$

Then $\mathcal{P} \subset \mathcal{C}_{per}(\Theta; \mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N))$. We can assume without loss generality that $\|c_{ij}(-, \lambda)\|_\infty > 0$. Let $\eta > 0$. By density of $\mathcal{K}(Y)$ in $L^{p'}(Y)$, let us fix $\varphi \in \mathcal{K}(Y)$ such that $\|\varphi - 1\|_{L^{p'}(Y)} \leq \frac{\eta}{\|c_{ij}(-, \lambda)\|_\infty}$. Since (4.7) implies

$$a_{ij}(y, \tau, \lambda) = \sum_{k \in \mathbb{Z}^N} c_{ij}(k, \tau, \lambda) \chi_{k+Y}(y)$$

where χ_{k+Y} is the characteristic function of $k + Y$, we have

$$\|a_{ij}(-, \lambda) - \psi_{ij}(-, \lambda)\|_{(L^{p'}, \ell^\infty)} \leq \|c_{ij}(-, \lambda)\|_\infty \|\varphi - 1\|_{L^{p'}(Y)} \leq \eta.$$

Hence $a_{ij}(-, \lambda)$ belongs to the closure in $(L^{p'}, \ell^\infty)(\mathbb{R}_{y, \tau}^{N+1})$ of $\mathcal{C}_{per}(\Theta; \mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N))$. But $(L^{p'}, \ell^\infty)(\mathbb{R}^{N+1})$ is continuously embedded in $\Xi^{p'}(\mathbb{R}^{N+1})$, then $a_{ij}(-, \lambda) \in \mathfrak{X}_{\mathcal{C}_{per}(\Theta; \mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N))}^{p'} \subset \mathfrak{X}_{\mathcal{C}_{per}(\Theta; \mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N) \odot \mathcal{C}_{per}(Y))}^{p'}$. $\square$

Since $A = \mathcal{C}_{per}(\Theta; \mathcal{B}_{\infty, \mathcal{F}}(\mathbb{R}_y^N) \odot \mathcal{C}_{per}(Y))$ is a p -quasi-pseudoproper H -algebra on $\mathbb{R}^{N+1}$, this completes the homogenization of (1.1) by virtue of Theorem 3.1.

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