The fast diffusion equation with measure data

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Abstract. We show the existence of solutions to the fast diffusion equation with a general finite and positive Borel measure as the right hand side source term.

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1. Introduction

The fast diffusion equation (FDE for short)

$$u_t - \Delta u^m = 0, \quad 0 < m < 1,$$

is an important prototype of a singular parabolic equation. This equation arises in a number of applications in physics and mathematics, and has been studied quite extensively, see for instance [6, 10, 11, 13, 15, 24]. We refer to the monographs [14, 27] for the basic theory, as well as more details about the applications and further references.

We study the existence of solutions to the FDE when the right hand side is a measure. More specifically, we consider the problem of finding solutions to

$$u_t - \Delta u^m = \mu, \tag{1.1}$$

where μ is a general finite, positive Borel measure. Our main result, Theorem 3.6, states that a solution to (1.1) exists, and provides a sharp estimate in a parabolic Sobolev space. The power m is allowed to take values in the supercritical range:

$$m_c < m < 1, \quad \text{where} \quad m_c = (n - 2)^+ / n.$$

The fundamental difficulty in existence results of this kind is that general measures do not belong to the dual of the natural function space for weak solutions. An explicit illustration of this is provided by the celebrated Barenblatt solution $\mathcal{B}_m$, given by (3.7) below. The Barenblatt solution satisfies (1.1)

with the right hand side given by Dirac's delta. However, the weak gradient of $\mathcal{B}_m^m$ is not an L^2 function, so the usual Sobolev class of weak solutions is not large enough to admit solutions with general right hand side measures.

The procedure used to deal with this situation is to approximate the given measure μ by more regular ones, prove a priori estimates, and then use compactness arguments to pass to the limit. See e.g. [7, 8, 16] for some implementations of this for various nonlinear elliptic and parabolic equations. In the particular case of the fast diffusion equation, the estimates derived in [21] for the porous medium equation (PME), where $m > 1$, continue to hold. However, in the compactness arguments, one encounters the following difficulty for both the FDE and the PME: estimates for the spatial gradient control only ∇u^m , not ∇u , and estimates for the time derivative control only u_t , not $(u^m)_t$. This means that compactness results [26] cannot be applied to neither u nor u^m . The main technical contribution of the paper is dealing with this difficulty in the case $m_c < m < 1$. This is accomplished by exploiting the properties of weak supersolutions.

Usually the next step after an existence result is to consider the question of uniqueness. While uniqueness with a sufficiently regular right hand side is standard, the case of general measures seems to be a delicate matter. As far as we know, the general case is still open, even for nonlinear elliptic equations; for instance, the results of [4, 5, 17, 25] exclude Dirac's delta. Further, it seems that the question of whether the Barenblatt solution is the unique solution to the FDE with right hand side Dirac's delta is also open. Hence we won't discuss the uniqueness question here. However, we note the fact that for the FDE and the PME, there are uniqueness results when the *initial trace* is a measure, see [12, 13, 15, 24] and the monograph [14].

2. Weak supersolutions

In this section, we discuss weak supersolutions and their properties. We shall consider only positive solutions in this paper, and assume throughout that $m_c < m < 1$. We take Ω to be a bounded domain in $\mathbb{R}^n$, and use the notation $\Omega_T = \Omega \times (0, T)$, $U_{t_1, t_2} = U \times (t_1, t_2)$.

Definition 2.1. A nonnegative function u is a weak supersolution of the fast diffusion equation in Ω_T , if $u^m \in L_{loc}^2(0, T; H_{loc}^1(\Omega))$, and

$$\int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \, dx \, dt \geq 0$$

for all $\varphi \in C_0^\infty(\Omega_T)$ such that $\varphi \geq 0$.

u is a weak solution, if the above holds with an equality for all test functions φ .

For our purposes, a key fact about weak supersolutions is the following observation: if u is a weak supersolution, then there is a Radon measure ν such that

$$\int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_T} \varphi \, d\nu(x, t) \quad (2.2)$$

for all smooth compactly supported test functions φ . By appealing to the definition of weak supersolutions, the existence of such a measure follows at once from the Riesz representation theorem.

The definition of a weak supersolution u does not include any time derivative for u . Nevertheless, we would like to use test functions depending on u , and hence the forbidden quantity $\frac{\partial u}{\partial t}$ inevitably appears. This defect is usually handled by employing a mollification procedure in the time direction, for instance Steklov averages or convolution with a standard mollifier. We will see that the mollifier

$$u^*(x, t) = \frac{1}{\sigma} \int_0^t e^{(s-t)/\sigma} u(x, s) \, ds \quad (2.3)$$

is convenient. The aim is to obtain estimates independent of the time derivative of u^* , and then pass to the limit $\sigma \rightarrow 0$.

The basic properties of the mollification (2.3) are given in the following lemma. See [23]. Note in particular the expression (2.5) for the time derivative of u^* .

Lemma 2.4. 1. If $u \in L^p(\Omega_T)$, then

$$\begin{aligned} \|u^*\|_{L^p(\Omega_T)} &\leq \|u\|_{L^p(\Omega_T)}, \\ \frac{\partial u^*}{\partial t} &= \frac{u - u^*}{\sigma}, \end{aligned} \quad (2.5)$$

and $u^* \rightarrow u$ in $L^p(\Omega_T)$ as $\sigma \rightarrow 0$.

2. If $\nabla u \in L^p(\Omega_T)$, then $\nabla(u^*) = (\nabla u)^*$,

$$\|\nabla u^*\|_{L^p(\Omega_T)} \leq \|\nabla u\|_{L^p(\Omega_T)},$$

and $\nabla u^* \rightarrow \nabla u$ in $L^p(\Omega_T)$ as $\sigma \rightarrow 0$.

3. If $u_k \rightarrow u$ in $L^p(\Omega_T)$, then also

$$u_k^* \rightarrow u^* \text{ and } \frac{\partial u_k^*}{\partial t} \rightarrow \frac{\partial u^*}{\partial t}$$

in $L^p(\Omega_T)$.

4. If $\nabla u_k \rightarrow \nabla u$ in $L^p(\Omega_T)$, then $\nabla u_k^* \rightarrow \nabla u^*$ in $L^p(\Omega_T)$.

5. Similar results hold for weak convergence in $L^p(\Omega_T)$.

6. If $\varphi \in C(\overline{\Omega_T})$, then

$$\varphi^*(x, t) + e^{-t/\sigma} \varphi(x, 0) \rightarrow \varphi(x, t)$$

uniformly in Ω_T as $\sigma \rightarrow 0$.

If u is a nonnegative weak supersolution, then its mollification u^* satisfies the inequality

$$\int_{\Omega_T} \frac{\partial u^*}{\partial t} \varphi + \nabla(u^m)^* \cdot \nabla \varphi \, dx \, dt \geq 0. \quad (2.6)$$

for all nonnegative smooth test functions φ vanishing on the lateral boundary of Ω_T . This follows by straightforward manipulations involving a change

of variables, Fubini's theorem, and an application of the positivity of u . By the usual approximation argument, we can allow test functions $\varphi \in L^2(0, T; H_0^1(\Omega))$.

Next we prove that the truncation of a weak supersolution is still a weak supersolution.

Lemma 2.7. *Let u be a weak supersolution in Ω_T . Then*

$$u_k = \min(u, k^{1/m})$$

is also a weak supersolution.

Proof. We set

$$f_l(t) = \min(l(k - t^m)_+, 1),$$

and

$$v_l = f_l(u) = \min(l(k - u^m)_+, 1).$$

The idea here is that v_l is an approximation of the characteristic function $\chi_{\{u^m < k\}}$, and we may use v_l as a test function. The desired conclusion is then reached by letting $l \rightarrow \infty$ in the end.

Let $\eta \in C_0^\infty(\Omega_T)$ be an arbitrary, nonnegative test function. To prove that the inequality of Definition 2.1 holds for η , we test (2.6) with $\varphi = v_l \eta$. We get

$$\int_{\Omega_T} \frac{\partial u}{\partial t} v_l \eta + \nabla(u^m)^* \cdot \nabla v_l \eta + \nabla(u^m)^* \cdot \nabla \eta v_l \, dx \, dt \geq 0.$$

We observe that $\nabla v_l = -l \nabla u^m$ in $\{k - 1/l < u^m < k\}$; thus the limits of the last two terms as $\sigma \rightarrow 0$ are

$$-l \int_{\{k-1/l < u^m < k\}} |\nabla u^m|^2 \, dx \, dt + \int_{\Omega_T} \nabla u^m \cdot \nabla \eta v_l \, dx \, dt.$$

In the time term, we write

$$\frac{\partial u^*}{\partial t} v_l \eta = \frac{\partial u^*}{\partial t} \eta f(u^*) + \frac{\partial u^*}{\partial t} (f(u) - f(u^*)) \eta. \quad (2.8)$$

The second term in (2.8) can be written, with the help of (2.5), as

$$\frac{u - u^*}{\sigma} (f_l(u) - f_l(u^*)) \eta \leq 0.$$

To see that the sign is indeed negative, note that f_l is decreasing and $\eta \geq 0$. In the first term of (2.8), we integrate by parts to get rid of the time derivative of u^* , and then we take the limit $\sigma \rightarrow 0$. This gives rise to the term

$$- \int_{\Omega_T} F_l(u) \frac{\partial \eta}{\partial t} \, dx \, dt,$$

where we denoted

$$F_l(t) = \int_0^t f_l(s) \, ds.$$

We put all of the above together, and arrive at

$$\begin{aligned} & \int_{\Omega_T} -F_l(u) \frac{\partial \eta}{\partial t} dx dt + \int_{\Omega_T} \nabla u^m \cdot \nabla \eta v_l dx dt \\ & \geq l \int_{\{k-1/l < u^m < k\}} |\nabla u^m|^2 dx dt \\ & \quad + \liminf_{\sigma \rightarrow 0} \left(- \int_{\Omega_T} \frac{u - u^*}{\sigma} (f_l(u) - f_l(u^*)) dx dt \right) \geq 0. \end{aligned}$$

The last inequality follows from the fact that the two preceding terms are positive. We now let $l \rightarrow \infty$; since then $v_l \rightarrow \chi_{\{u^m < k\}}$, and $F_l(u) \rightarrow \min(u, k^{1/m})$, we obtain

$$\int_{\Omega_T} -u_k \frac{\partial \eta}{\partial t} + \nabla u_k^m \cdot \nabla \eta dx dt \geq 0$$

for all smooth, nonnegative test functions η , as desired. $\square$

Next we prove estimates for bounded weak supersolutions. We begin with a Caccioppoli type inequality.

Lemma 2.9. *Let u be a weak supersolution such that $0 \leq u \leq M < \infty$, where $M \geq 1$, and let η be any nonnegative function in $C_0^\infty(\Omega)$. Then*

$$\int_{\Omega_T} \eta^2 |\nabla u^m|^2 dx dt \leq 2M^{m+1} \int_{\Omega} \eta^2 dx + 16M^{2m} \int_{\Omega_T} |\nabla \eta|^2 dx dt.$$

Proof. We test the regularized equation (2.6) by $\varphi = \eta^2(M^m - u^m) \geq 0$; note that η is independent of time. This yields

$$\begin{aligned} 0 & \leq \int_{\Omega_T} \frac{\partial u^*}{\partial t} \eta^2 (M^m - u^m) dx dt \\ & \quad + \int_{\Omega_T} \nabla (u^m)^* \cdot \nabla (-u^m) \eta^2 dx dt \\ & \quad + \int_{\Omega_T} 2\eta \nabla (u^m)^* \cdot \nabla \eta (M^m - u^m) dx dt. \end{aligned}$$

We move the second term to the left hand side. Letting $\sigma \rightarrow 0$, this term becomes

$$\int_{\Omega_T} \eta^2 |\nabla u^m|^2 dx dt.$$

We need to take care of the first and third terms. To handle the first term, we estimate

$$\begin{aligned}
& \int_{\Omega_T} M^m \frac{\partial u^*}{\partial t} \eta^2 \, dx \, dt - \int_{\Omega_T} \frac{\partial u^*}{\partial t} \eta^2 u^m \, dx \, dt \\
&= M^m \int_{\Omega} u^*(x, T) \eta^2(x) \, dx - \int_{\Omega_T} \eta^2 \frac{\partial u^*}{\partial t} (u^*)^m \, dx \, dt \\
&\quad - \int_{\Omega_T} \eta^2 \frac{\partial u^*}{\partial t} (u^m - (u^*)^m) \, dx \, dt \\
&\leq M^m \int_{\Omega} u^*(x, T) \eta^2(x) \, dx + \frac{1}{m+1} \int_{\Omega} (u^*)^{m+1}(x, 0) \eta^2(x) \, dx \\
&\quad - \frac{1}{m+1} \int_{\Omega} (u^*)^{m+1}(x, T) \eta^2(x) \, dx \\
&\leq M^m \int_{\Omega} u^*(x, T) \eta^2(x) \, dx.
\end{aligned}$$

Here we used (2.5) in the first inequality, and the facts $u^* \geq 0$, $u^*(x, 0) = 0$ in the second. To finish off, we let $\sigma \rightarrow 0$ and use $u \leq M$ to get the bound

$$M^{m+1} \int_{\Omega} \eta^2 \, dx$$

for this term.

For the third term, we first take the limit $\sigma \rightarrow 0$. Due to the positivity of the term we moved to the left hand side, we can then take absolute values. After this, we use the estimate $M^m - u^m \leq M^m$, and apply Young's inequality. This gives

$$\begin{aligned}
& 2M^m \int_{\Omega_T} |\nabla u^m| |\nabla \eta| \eta \, dx \, dt \\
&\leq \varepsilon^2 2M^m \int_{\Omega} |\nabla u^m|^2 \eta^2 \, dx \, dt + \varepsilon^{-2} 2M^m \int_{\Omega_T} |\nabla \eta|^2 \, dx \, dt,
\end{aligned}$$

and putting the above estimates together we arrive at the inequality

$$\begin{aligned}
\int_{\Omega_T} \eta^2 |\nabla u^m|^2 \, dx \, dt &\leq M^{m+1} \int_{\Omega} \eta^2 \, dx + \varepsilon^2 2M^m \int_{\Omega} |\nabla u^m|^2 \eta^2 \, dx \, dt \\
&\quad + \varepsilon^{-2} 2M^m \int_{\Omega_T} |\nabla \eta|^2 \, dx \, dt.
\end{aligned}$$

We now choose $\varepsilon < 1$ such that $\varepsilon^2 2M^m = 1/2$. Then we can absorb the matching term to the left hand side to complete the proof. $\square$

We employ the above estimate to get the following bound.

Lemma 2.10. *Let $0 \leq u \leq M$ be a weak supersolution in Ω_T , and let ν be a measure such that (2.2) holds. Assume that U is an open set compactly contained in Ω , and $0 < t_1 < t_2 < T$. Then*

$$\nu(\overline{U_{t_1, t_2}}) \leq c,$$

where the bound depends on M and $\text{dist}(U_{t_1, t_2}, \partial\Omega_T)$.

Proof. Let $U \Subset U' \Subset \Omega$, $s_1 < t_1$, $s_2 > t_2$, and take a smooth cutoff function η such that $0 \leq \eta \leq 1$, η is compactly supported in U'_{s_1, s_2} and $\eta = 1$ in $\overline{U}_{t_1, t_2}$. Then

$$\begin{aligned} \nu(\overline{U}_{t_1, t_2}) &\leq \int_{U'_{s_1, s_2}} \eta \, d\nu(x, t) \\ &= \left| \int_{U'_{s_1, s_2}} -u \frac{\partial \eta}{\partial t} \, dx \, dt + \int_{U'_{s_1, s_2}} \nabla u^m \cdot \nabla \eta \, dx \, dt \right| \\ &\leq cM + c \left(\int_{U'_{s_1, s_2}} |\nabla u|^2 \, dx \, dt \right)^{1/2}, \end{aligned}$$

and the estimate now follows since the second term can be estimated by Lemma 2.9. $\square$

The next step is to construct suitable approximations for a given weak supersolution u . We start this by employing the obstacle problem: by considering the lower semicontinuous representative of u , we may find a sequence (g_i) of smooth functions increasing to u . Then, solving the associated obstacle problem with the obstacle and boundary values given by g_i , we get a sequence of regular supersolutions increasing to u . See [19, Proof of Theorem 3.2, pp. 148–149] for a more detailed description of the procedure. We record the outcome in the following lemma.

Lemma 2.11. *Let u be a weak supersolution in Ω_T . Then, for each $V_{\tau_1, \tau_2} \Subset \Omega_T$, there is a sequence of solutions u_i to the obstacle problem with smooth obstacle and boundary values g_i increasing to u , such that*

$$\frac{\partial u_i}{\partial t} \in L^2(\tau_1, \tau_2; H^{-1}(V)).$$

Further, if u is Hölder continuous, then we can take the sequence (u_i) to converge locally uniformly.

Proof. The arguments in [9] can be used to establish the existence of the time derivative in the right space. The uniform convergence follows from uniform a priori Hölder estimates for obstacle problems similar to those in [20]. $\square$

We need to refine the approximants so that their Riesz measures are represented by functions. This is accomplished in the following lemma.

Lemma 2.12. *Let u be a solution to the obstacle problem in V_{τ_1, τ_2} with a smooth obstacle and boundary value function g .*

For the measure μ satisfying

$$\partial_t u - \Delta u^m = \mu,$$

one has $\mu \in L^2(\tau_1, \tau_2; H^{-1}(V))$.

If f_i are smooth functions such that $f_i \rightarrow \mu$ in $L^2(\tau_1, \tau_2; H^{-1}(V))$, for the unique functions w_i such that

$$\partial_t w_i - \Delta w_i^m = f_i, \quad w_i^m - g \in L^2(\tau_1, \tau_2; H_0^1(V)),$$

one has $w_i^m \rightarrow u^m$ in $L^q(\tau_1, \tau_2; W^{1,q}(V))$ for any $q < 2$, and $w_i \rightarrow u$ locally uniformly.

Proof. Since $\frac{\partial u}{\partial t} \in L^2(\tau_1, \tau_2; H^{-1}(V))$, the fact that $\mu \in L^2(\tau_1, \tau_2; H^{-1}(V))$ can be read off the equation satisfied by u .

The L^2 norm of ∇w_i^m has an estimate in terms of norms of g and $\|\mu\|_{L^2(\tau_1, \tau_2; H^{-1}(V))}$. This can be proved as in [28, Theorem 5.14]. To identify the limit of ∇w_i^m as the correct one, we use the test function due to Oleinik, see [28, Theorem 5.3], and the bound in L^2 for ∇w_i^m . Let

$$\eta(x, t) = \begin{cases} \int_t^{\tau_2} (u^m - w_i^m) \, ds, & \tau_1 < t < \tau_2, \\ 0, & t \geq \tau_2. \end{cases}$$

Since u^m and w_i^m have the same boundary values in Sobolev's sense, η is an admissible test function in the equations for u and w_i . Subtracting the resulting equations, we get

$$\begin{aligned} & \int_{V_{\tau_1, \tau_2}} (u - w_i)(u^m - w_i^m) \, dx \, dt \\ & + \int_{V_{\tau_1, \tau_2}} \nabla(u^m - w_i^m) \int_t^{\tau_2} \nabla(u^m - w_i^m) \, ds \, dx \, dt = \langle \mu - f_i, \eta \rangle, \end{aligned}$$

where the brackets indicate the duality pairing between $L^2(\tau_1, \tau_2; H_0^1(V))$ and $L^2(\tau_1, \tau_2; H^{-1}(V))$. We may write the second term on the left hand side as

$$\frac{1}{2} \int_V \left\{ \int_{\tau_1}^{\tau_2} (\nabla u^m - \nabla w_i^m)^2 \, ds \right\} \, dx$$

by integration. Thus both terms on the left hand side are positive, and we are free to take absolute values on the right hand side. Using the L^2 bound for ∇w_i^m , we get

$$|\langle \mu - f_i, \eta \rangle| \leq c \|\mu - f_i\|_{L^2(\tau_1, \tau_2; H^{-1}(V))} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Hence by an application of Hölder's inequality, $\nabla w_i^m \rightarrow \nabla u^m$ in L^1 , and then also in L^q for any $q < 2$, at least for a subsequence. Since u^m and w_i^m have the same boundary values, it follows that $w_i^m \rightarrow u^m$ in $L^q(\tau_1, \tau_2; W^{1,q}(V))$. The uniform convergence again follows from uniform a priori Hölder estimates: g is smooth, so u^m is Hölder continuous. Thus the counterparts of [18, Lemma 2.1 and Theorem 1.14] for the FDE give the result. $\square$

The following lemma provides the compactness property of weak supersolutions which we will apply in the proof of the main result.

Lemma 2.13. *Let $(u_j), j = 1, 2, 3, \dots$, be Hölder continuous weak supersolutions such that $0 \leq u_j \leq M < \infty$ for all j . Then there is a subsequence and a function u such that $u_j \rightarrow u$ pointwise almost everywhere in Ω_T .*

Proof. Let the spacetime cylinders U_{t_1, t_2} and V_{τ_1, τ_2} be chosen so that $U_{t_1, t_2} \subseteq V_{\tau_1, \tau_2} \subseteq \Omega_T$. Apply Lemma 2.11 in V_{τ_1, τ_2} to get solutions v_i to the obstacle problem such that

$$\|u_i - v_i\|_{L^\infty(U_{t_1, t_2})} \leq 1/i \quad \text{and} \quad \nu_i(U_{t_1, t_2}) \leq 2\mu_i(U_{t_1, t_2}),$$

where ν_i is the Riesz measure of v_i . Then, apply Lemma 2.12 to each v_i to get weak supersolutions w_i such that

$$\begin{aligned}\|v_i^m - w_i^m\|_{L^q(\tau_1, \tau_2; W^{1,q}(V))} &\leq 1/i, \\ \|v_i - w_i\|_{L^\infty(U_{t_1, t_2})} &\leq 1/i, \text{ and} \\ \|f_i\|_{L^1(U_{t_1, t_2})} &\leq 2\nu_i(U_{t_1, t_2}).\end{aligned}$$

Here f_i is the smooth function giving the density of the Riesz measure of w_i . These choices imply

$$0 \leq w_i \leq M + 2 \quad \text{and} \quad \|f_i\|_{L^1(U_{t_1, t_2})} \leq c(M);$$

for the latter, note that we may use Lemma 2.10 to estimate $\mu_i(U_{t_1, t_2})$.

We aim at applying a compactness result [26, Corollary 4] to the functions w_i . First, by Lemma 2.9, we have

$$\begin{aligned}\|w_i^m\|_{L^q(t_1, t_2; W^{1,q}(U))} &\leq \|v_i^m\|_{L^q(t_1, t_2; W^{1,q}(U))} + \|w_i^m - v_i^m\|_{L^q(t_1, t_2; W^{1,q}(U))} \\ &\leq c(M) + 1/i \leq c(M) + 1\end{aligned}$$

for any $q < 2$, since $v_i \leq M$. Then, since $m < 1$, we have

$$|\nabla w_i| = |\nabla(w_i^m)^{1/m}| = |w_i^{1/m-1} \nabla w_i^m| \leq (M+2)^{1/m-1} |\nabla w_i^m|.$$

Thus the sequence (∇w_i) is bounded in $L^q(U_{t_1, t_2})$.

To estimate the time derivative of w_i , we first note that

$$\frac{\partial w_i}{\partial t} \in L^1(U_{t_1, t_2}) + L^q(t_1, t_2; W^{-1,q}(U))$$

with a bound in $L^q(t_1, t_2; W^{-1,q}(U))$ for the second component following from the L^q bound for ∇w_i^m . For the L^1 component, we have

$$\begin{aligned}\left| \int_U f_j(x, t) \varphi(x) \, dx \right| &\leq \|f_j(\cdot, t)\|_{L^1(U)} \|\varphi\|_{L^\infty(U)} \\ &\leq c \|f_j(\cdot, t)\|_{L^1(U)} \|\nabla \varphi\|_{L^{s'}(U)}\end{aligned}$$

by Sobolev imbedding with $s' > n$ for all $\varphi \in C_0^\infty(U)$. Thus

$$\|f_j(\cdot, t)\|_{W^{-1,s}(U)} \leq c \|f_j(\cdot, t)\|_{L^1(U)}$$

for almost all $t, t_1 < t < t_2$. Integrating this inequality in time, we have

$$\|f_j\|_{L^1(t_1, t_2; W^{-1,s}(U))} \leq c \|f_j\|_{L^1(U_{t_1, t_2})}$$

for $1 < s < n/(n-1)$. Putting these estimates together, we get the bound

$$\left\| \frac{\partial w_i}{\partial t} \right\|_{L^1(t_1, t_2; W^{-1,s}(U))} \leq c.$$

We may choose U to be a regular set, for instance a polyhedron. In such a case, $W^{1,q}(U)$ embeds compactly to $L^1(U)$. Further, $L^1(U)$ embeds continuously to $W^{-1,s}(U)$. Thus, Corollary 4 of [26] implies that (w_j) is compact in $L^1(U_{t_1, t_2})$, and we may extract a subsequence such that $w_j \rightarrow u$ in $L^1(U_{t_1, t_2})$.

for some function u . For the same indices, the original sequence (u_i) also converges to u in $L^1(U_{t_1, t_2})$, since

$$\begin{aligned} \|u - u_i\|_{L^1(U_{t_1, t_2})} &\leq \|u - w_i\|_{L^1(U_{t_1, t_2})} \\ &\quad + \|w_i - v_i\|_{L^\infty(U_{t_1, t_2})} + \|u_i - v_i\|_{L^\infty(U_{t_1, t_2})} \\ &\leq \|u - w_i\|_{L^1(U_{t_1, t_2})} + 2/i \rightarrow 0 \end{aligned}$$

as $i \rightarrow \infty$. For a further subsequence, we thus have the pointwise convergence a.e. in U_{t_1, t_2} .

To find a subsequence converging in the whole of Ω_T , we exhaust Ω by regular open sets U^k , and choose nested time intervals $(t_1^1, t_2^1) \subseteq (t_1^2, t_2^2) \subseteq \dots, k = 1, 2, \dots$, such that

$$\Omega_T = \cup_{k=1}^{\infty} U_{t_1^k, t_2^k}^k.$$

First, pick a subsequence (u_j^1) converging a.e. in $U_{t_1^1, t_2^1}^1$ to u^1 . The procedure continues inductively by picking a subsequence (u_j^{k+1}) of (u_j^k) that converges pointwise almost everywhere in $U_{t_1^{k+1}, t_2^{k+1}}^{k+1}$ to a function u^{k+1} . Since limits are unique, $u^l = u^k$ almost everywhere in $U_{t_1^k, t_2^k}^k$ if $l > k$. Thus, if we define u by setting

$$u(x) = \begin{cases} u^1(x), & x \in U_{t_1^1, t_2^1}^1, \\ u^k(x), & x \in U_{t_1^k, t_2^k}^k \setminus U_{t_1^{k-1}, t_2^{k-1}}^{k-1}, \quad k \geq 2, \end{cases}$$

the diagonal sequence (u_k^k) converges pointwise almost everywhere in Ω_T to u . $\square$

Note that by using Lemma 2.9 and the fact that weak limits must coincide with pointwise limits, it is straightforward to show that the function u found in Lemma 2.13 is also a weak supersolution. However, we do not need this fact here.

3. The main theorem

In this section, we prove the main theorem. We begin by discussing the necessary a priori estimates. The arguments used in [21] for the PME work also for the FDE, so we omit the detailed proofs.

We will use the following lemma to approximate general Radon measures. One simple way to construct the required functions is given in Lemma 2.12 of [22].

Lemma 3.1. *Given a nonnegative Radon measure μ such that $\mu(\Omega_T) < \infty$, we can find nonnegative functions $f_j \in L^\infty(\Omega_T), j = 1, 2, \dots$, such that the corresponding measures $d\mu_j = f_j dx dt$ satisfy*

$$\mu_j(\Omega_T) = \int_{\Omega_T} f_j dx dt \leq \mu(\Omega_T),$$

and $\mu_j \rightarrow \mu$ weakly, i.e.

$$\int_{\Omega_T} \varphi \, d\mu_j = \int_{\Omega_T} f_j \varphi \, dx \, dt \rightarrow \int_{\Omega_T} \varphi \, d\mu$$

as $j \rightarrow \infty$, for all smooth compactly supported functions φ .

Plugging the approximants to the right hand side, we get solutions in the following sense. See, e.g., [1, 2].

Definition 3.2. Let $f \in L^\infty(\Omega_T)$ be a nonnegative function. We call a nonnegative function $u : \Omega_T \rightarrow \mathbb{R}$ a solution to the Cauchy–Dirichlet problem

$$\begin{cases} u_t - \Delta u^m = f & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega, \\ u(x, 0) = 0 \end{cases} \quad (3.3)$$

if $u^m \in L^2(0, T; H_0^1(\Omega))$,

$$\int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_T} f \varphi \, dx \, dt$$

for all test functions $\varphi \in C_0^\infty(\Omega_T)$ and

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h \int_{\Omega} u(x, t)^{m+1} \, dx \, dt = 0.$$

To derive our estimates, we need the equation satisfied by the mollification u^* of a solution to (3.3). Manipulations similar to those used to get (2.6) lead to

$$\int_{\Omega_T} \frac{\partial u^*}{\partial t} \varphi + \nabla (u^m)^* \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_T} f^* \varphi \, dx \, dt, \quad (3.4)$$

which is required to hold for all test functions $\varphi \in C_0^\infty(\Omega_T)$. From (3.4), one can proceed as in [21, Section 4], and arrive at the following estimates.

Lemma 3.5. Let u be a solution of (3.3). Then

$$\int_{\Omega_T} u^{m\alpha} \, dx \, dt \leq c,$$

where $0 < \alpha < 1 + \frac{2}{mn}$,

$$\int_{\Omega_T} |\nabla u^m|^\beta \, dx \, dt \leq c,$$

where $1 \leq \beta < 1 + \frac{1}{1+mn}$, and

$$\left\| \frac{\partial u_j}{\partial t} \right\|_{L^1(0, T; W_0^{-1,1}(\Omega))} \leq c.$$

The constants depend only on $|\Omega_T|$, the L^1 norm of f , α , β , n and m .

We are now ready to prove the main result of this paper. The critical exponent m_c arises from the requirement that the sequence (u_i) is bounded in L^r for some $r > 1$. For this to be true, we need to find a value of α which is admissible in Lemma 3.5 such that $m\alpha > 1$. This leads to the condition

$$m + 2/n > 1,$$

which is clearly equivalent to $m > m_c$.

Theorem 3.6. *Let μ be a Radon measure such that $\mu(\Omega_T) < \infty$. Then there exists a function u such that $u^m \in L^q(0, T; W_0^{1,q}(\Omega))$, where q is any number such that*

$$1 \leq q < 1 + \frac{1}{1 + mn},$$

and

$$\frac{\partial u}{\partial t} - \Delta u^m = \mu$$

holds in the weak sense.

Proof. By the estimates of Lemma 3.5, $u_j \rightarrow u$ and $\nabla u_j^m \rightarrow \nabla v$ weakly in $L^r(\Omega_T)$ for some $r > 1$. As discussed above, the first of these follows from the assumption $m > m_c$. With these convergences, we may pass to the limit $j \rightarrow \infty$ and obtain

$$\int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla v \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_T} \varphi \, d\mu(x, t).$$

To complete the proof, we need the fact that $v = u^m$. Since the pointwise limit identifies the weak limit, we can prove this by extracting a pointwise convergent subsequence. We use Lemma 2.13 and a diagonalization argument. Let $u_j^k = \min(u_j, k)$ for $k = 1, 2, 3, \dots$. Apply Lemma 2.13 to (u_j^1) to get a subsequence converging pointwise to a function u^1 . Since weak limits must agree with pointwise limits, $u^1 = u$ almost everywhere in $\{u \leq 1\}$. Next, suppose that we have already picked a subsequence of (u_j^k) converging almost everywhere in $\{u \leq k\}$. We proceed by applying Lemma 2.13 to this subsequence at the truncation level $k + 1$. We get a further subsequence (u_j^{k+1}) converging to a function u^{k+1} . Again we must have $u^{k+1} = u$ in $\{u \leq k + 1\}$ by the weak convergence. This procedure yields a table of indices, with the pair (l, m) , $l, m = 1, 2, \dots$, corresponding to u_m^l . The indices in the diagonal positions (k, k) , $k = 1, 2, \dots$, give us the desired pointwise convergent subsequence. $\square$

The exponent q in Theorem 3.6 is sharp in the following sense. Consider the Barenblatt solution [3, 29]

$$\mathcal{B}_m(x, t) = \begin{cases} t^{-\lambda} \left(C + k \frac{|x|^2}{t^{2\lambda/n}} \right)^{-1/(1-m)}, & t > 0, \\ 0, & t \leq 0, \end{cases} \quad (3.7)$$

where

$$\lambda = \frac{n}{n(m-1)+2} \quad \text{and} \quad k = \frac{\lambda(1-m)}{2mn}.$$

Note that λ and k are strictly positive, since $m_c < m < 1$. The constant C is usually chosen so that

$$\int_{\mathbb{R}^n} \mathcal{B}_m(x, t) \, dx = 1$$

for all $t > 0$. With this normalization, the Barenblatt solution satisfies

$$(\mathcal{B}_m)_t - \Delta \mathcal{B}_m^m = \delta$$

in the weak sense, where δ is Dirac's delta at the origin of $\mathbb{R}^{n+1}$. For $q \geq 1$, a computation shows that

$$\int_0^1 \int_{[-1,1]^n} |\nabla \mathcal{B}_m^m|^q \, dx \, dt < \infty$$

if and only if

$$q < 1 + \frac{1}{1+mn}.$$

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