Blow-up profile to solutions of NLS with oscillating nonlinearities

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Abstract. This paper is concerned with the blow-up solutions of nonlinear Schrödinger equation (NLS) with oscillating nonlinearities. The limiting profiles of the blow-up solutions $u(t, x)$ with initial data $\|u_0\|_{L^2} = \|Q\|_{L^2}$ are obtained. It reads that $|u(t, x)|^2 \rightarrow \|Q\|_{L^2}^2 \delta_{x=y_1}$ (Dirac function), as $t \rightarrow T$, and that $u(t, x)$ converges strongly to $Q(x)$ in the energy space $\Sigma = \{u \in H^1; \int |x|^2 |u|^2 dx < \infty\}$ up to scaling and phase parameters and also translation in the nonradial case.

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1. Introduction

In this paper, we study the Cauchy problem of the following critical nonlinear Schrödinger equation with multiplicative oscillating term

$$iu_t + \Delta u + \cos^2(\omega t)|u|^{\frac{4}{N}}u = 0, \quad t \in [t_0, T), \quad x \in \mathbb{R}^N, \quad (1.1)$$

$$u(t_0, x) = u_0, \quad (1.2)$$

where ω is a positive parameter; $i = \sqrt{-1}$; $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^N$; $u = u(t, x): [t_0, T) \times \mathbb{R}^N \rightarrow \mathbb{C}$ is the complex valued function and $t_0 < T \leq \infty$; N is the space dimension. Equation (1.1) is also called nonlinear Schrödinger equation with time dependent potential, which may model a laser beam that propagates in a medium with variable indices of refraction and has received a lot of attentions recently (see [10–12, 20, 26]). In Physics, the blow-up of solutions corresponds to the wave's collapse, physicists are very interested in the elaborate descriptions of the blow-up solutions.

We recall the classical nonlinear Schrödinger equation without any potential. Ginibre and Velo [5] showed the local well-posedness in $H^1(R^N)$. Glassey [6], Ogawa and Tsutsumi [18] showed the existence of blow-up solutions (see Cazenave [2] for a review). Weinstein [21] and Zhang [23] obtained the sharp conditions of the blow-up and global existence. Weinstein [22] showed the structure and formation of singularity to the blow-up solution with critical mass ($\|Q\|_{L^2}$), and it reads that the blow-up solution is close to $Q(x)$ in $H^1(R^N)$ up to scaling and phase parameters, and also translation in the non-radial case. Based on this property, Merle and Tsutsumi [13] showed a new phenomenon for the blow-up solutions which is the L^2 -concentration. Merle [14] obtained the exact blow-up solution with critical mass by the conformal invariance and compact results. Recently, using the variational methods, Merle and Raphaël [15] improved the Weinstein's [22] results and obtained the sharp decomposition of blow-up solutions with small super-critical mass. Thus, basing on this sharp decomposition, Merle and Raphaël [15–17] obtained a large body of breakthrough work with the help of the Spectral Properties, such as blow-up rates, profiles of blow-up solutions, etc. Hmidi and Keraani [8] used a profile decomposition of bounded sequences in $H^1(R^N)$ proposed by Gérard [7] to obtain a refined version of compactness lemma, and gave a direct proof of the results obtained by Weinstein [22].

For Eq. (1.1), Cazenave [2] established the local well-posedness in the corresponding energy field. Few results about the blow-up solutions of the Cauchy problem (1.1)–(1.2) are known. Damergi and Goubet [4] showed the existence of blow-up solutions for some initial data. Sulem [20] obtained some numerical observations. It is obvious that conservation laws are vital in studying the blow-up solution of the nonlinear Schrodinger equation, but there is no conservation of energy for the Eq. (1.1) because of the effects of the oscillating Nonlinearities. Therefore, How to deal with the loss of conservation of energy for the Cauchy problem (1.1)–(1.2) is of particular interest. Motivated by these problems, we seek to answer the following question: what is the limiting profile of the blow-up solutions for the Cauchy problem (1.1)–(1.2)?

In this paper, in terms of Zhang's [24] ideas, we choose the ground state solution of the nonlinear Schrödinger equation without any potential as the limiting profiles to describe the blow-up solutions of Cauchy problem (1.1)–(1.2). Thus, using the compactness lemma proposed by Hmidi and Keraani [8] and the variational characterization of the corresponding ground state solution, we obtain the shape of the blow-up solutions for the initial data $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$. Specifically, suppose that the initial data $u_0 \in \Sigma$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, and the corresponding solution $u(t, x)$ blows up in finite time $T < \infty$. Then there exists an $y_1 \in R^N$ such that

$$|u(t, x)|^2 \rightarrow \|Q(x)\|_{L^2}^2 \delta_{x=y_1}, \text{ as } t \rightarrow T. \quad (1.3)$$

Applying these properties, we obtain the limiting profiles of blow-up solutions in the energy space Σ . There are functions $y(t) \in R^N$, $\gamma(t) \in R$ such that

$$\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)(x + y(t)))e^{i\gamma(t)} \rightarrow Q(x) \text{ strongly in } \Sigma \text{ as } t \rightarrow T, \quad (1.4)$$

where $\lambda(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u\|_{L^2}}$ and $Q(x)$ is the unique ground state solution of the following equation (see [9])

$$-\Delta Q + Q - |Q|^{\frac{4}{N}}Q = 0, \quad Q \in H^1(R^N). \quad (1.5)$$

Moreover, we extend the results in (1.4) to the case: $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha'$.

The aim of this paper is to obtain the blow-up profile to the blow-up solutions of the Cauchy problem (1.1)–(1.2) in the energy space $\Sigma := \{u \in H^1(R^N), |x|u \in L^2(R^N)\}$. Zhu, Zhang and Li [25] obtained the limiting profile of the blow-up solutions for the nonlinear Schrödinger equation with a harmonic potential (space dependent potential). Chen and Guo [3] obtained the blow-up profile to the solutions of the coupled Schrödinger equations. The main idea of this paper is from our previous paper [25], but there are two marked differences between this paper and the previous paper [25]. One is that the equation dealt with in this paper is different from that in the paper [25]. In the paper [25], authors studied the Gross-Pitaevskii equation, which is a class of nonlinear Schrödinger equations with a space dependent potential and there is the conservation of energy. In this paper, we consider the NLS with oscillating nonlinearity, which is a class of nonlinear Schrödinger equation with a time dependent nonlinearity. Because of the time dependent nonlinearity, there is no conservation of energy for the NLS with oscillating nonlinearity, which is one of the main technique challenges in this paper. The other is that the tool used in this paper is different from that in the paper [25]. In the paper [25], authors used the classical concentration compactness principle proposed by Lions. In this paper, we apply the refined compactness lemma proposed by Hmidi and Keraani [8] to study the limiting profile and obtain that the blow-up solution is like a Dirac function (see (1.3)). Furthermore, it is easy to check $\int |x|^2 |u(t, x)|^2 dx$ is bounded, but after scaling $U(t, x) = \lambda^{\frac{N}{2}}(t) u(t, \lambda(t)x)$ it is difficult to control $U(t, x)$ in the weighed Sobolev space Σ , and we need a new estimate to obtain the limiting profile of the blow-up solutions in Σ .

In this paper, we denote $L^q(R^N)$, $\|\cdot\|_{L^q(R^N)}$, $H^s(R^N)$ and $\int_{R^N} \cdot dx$ by L^q , $\|\cdot\|_{L^q}$, H^s and $\int \cdot dx$ respectively, and the various positive constants will be simply denoted by C .

2. Preparation

For the Cauchy problem (1.1)–(1.2), we define the energy space $\Sigma := \{u \in H^1(R^N) \mid |x|u \in L^2(R^N)\}$ with the inner product

$$\langle u, v \rangle = \int (\nabla u \nabla \bar{v} + u \bar{v} + |x|^2 u \bar{v}) dx.$$

The norm of Σ is denoted by $\|\cdot\|_{\Sigma}$. Moreover, we define two functionals on Σ by

$$E(u) := \int |\nabla u|^2 dx - \frac{\cos^2(\omega t)}{1 + \frac{2}{N}} \int |u|^{2 + \frac{4}{N}} dx,$$

$$H(u) := \int |\nabla u|^2 dx - \frac{1}{1 + \frac{2}{N}} \int |u|^{2 + \frac{4}{N}} dx.$$

The functionals E and H are well-defined according to the Sobolev embedding theorem (see [2]). Cazenave [2] showed the local well-posedness for the Cauchy problem (1.1)–(1.2) in Σ , as follows.

Proposition 2.1. (Cazenave [2]) *Let $u_0 \in \Sigma$. There exists a unique solution $u(t, x)$ of the Cauchy problem (1.1)–(1.2) on the maximal time $[t_0, T)$ such that $u(t, x) \in C([t_0, T); \Sigma)$ and either $T = \infty$ (global existence), or else $T < \infty$ and $\lim_{t \rightarrow T} \|u(t, x)\|_{\Sigma} = \infty$ (wave's collapse). Furthermore, for all $t \in [t_0, T)$, $u(t, x)$ satisfies the following conservation law*

$$\|u(t, x)\|_{L^2} = \|u_0\|_{L^2}. \quad (2.1)$$

From now on, we give some important lemmas.

Lemma 2.1. (Strauss [19] Kwong [9]) *There exists a unique ground state solution $Q(x)$ to Eq. (1.5) which is positive and spherically symmetric and exponentially decays at infinity.*

Lemma 2.2. (Weinstein [21]) *Let $u \in H^1$, then*

$$H(u) \geq \left[1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}} \right)^{\frac{4}{N}} \right] \|\nabla u\|_{L^2}^2, \quad (2.2)$$

where Q is the unique ground state solution of Eq. (1.5).

Lemma 2.3. (Hmidi and Keraani [8]) *Let $\{v_n\}$ be a bounded family of H^1 such that*

$$\lim_{n \rightarrow \infty} \sup \|\nabla v_n\|_{L^2} \leq M < \infty \quad (2.3)$$

and

$$\lim_{n \rightarrow \infty} \sup \|v_n\|_{L^{2+\frac{4}{N}}} \geq m > 0. \quad (2.4)$$

Then there exists $\{x_n\} \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(x + x_n) \rightharpoonup V(x), \quad \text{weakly in } H^1 \quad (2.5)$$

with $\|V\|_{L^2} \geq \left(\frac{2N}{N+2}\right)^{\frac{N}{4}} \frac{m^{1+N/2}}{M^{N/2}} \|Q\|_{L^2}$.

Lemma 2.4. *Let $u_0 \in \Sigma$, and $u(t, x) \in C([t_0, T); \Sigma)$ be the solution of the Cauchy problem (1.1)–(1.2). Then, $\forall t \in [t_0, T)$, one has*

$$\frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx = 16E(u(t, x)). \quad (2.6)$$

Moreover, there exists an $y_1 \in \mathbb{R}^N$ such that

$$\lim_{t \rightarrow T} \int |u(t, x)|^2 x dx = - \left(\int |Q(x)|^2 dx \right) y_1, \quad (2.7)$$

where $Q(x)$ is the unique ground state solution of Eq. (1.5).

Proof. Since $u_0 \in \Sigma$ and $u(t, x) \in C([t_0, T]; \Sigma)$ is the solution of the Cauchy problem (1.1)–(1.2), $J(t) := \int |x|^2 |u(t, x)|^2 dx$ is well-defined. By some computations, one has for all $t \in [t_0, T)$

$$\begin{aligned} J'(t) &= 2\Re \int |x|^2 \bar{u} u_t dx \\ &= -2\Im \int |x|^2 \bar{u} (\Delta u + \cos^2(\omega t) |u|^{\frac{4}{N}} u) dx \\ &= 4\Im \int x \cdot \nabla u \bar{u} dx, \end{aligned} \quad (2.8)$$

and

$$J''(t) = -4N\Im \int \bar{u} u_t dx - 8\Im \int x \nabla \bar{u} u_t dx. \quad (2.9)$$

Since $u(t, x)$ is the solution of (1.1), one deduces that

$$\Im \int \bar{u} u_t dx = - \int |\nabla u|^2 dx + \cos^2(\omega t) \int |u|^{2+\frac{4}{N}} dx, \quad (2.10)$$

and

$$\Im \int x \nabla \bar{u} u_t dx = \frac{N-2}{2} \int |\nabla u|^2 dx - \frac{N}{2+\frac{4}{N}} \cos^2(\omega t) \int |u|^{2+\frac{4}{N}} dx. \quad (2.11)$$

By (2.9)–(2.11) and the definition of the energy functional $E(u)$, one has

$$J''(t) = 16E(u), \quad (2.12)$$

which implies (2.6) is true.

On the other hand, for any $x_0 \in R^N$, let $g(t) := \int x_0 \cdot x |u(t, x)|^2 dx$, then

$$g'(t) = 2\Im \int x_0 \cdot \nabla u \bar{u} dx, \quad (2.13)$$

thus, one gets

$$\begin{aligned} g''(t) &= -4\Im \int x_0 \nabla \bar{u} u_t dx \\ &= -4\Im \int x_0 \nabla \bar{u} i(\Delta u + \cos^2(\omega t) |u|^{\frac{4}{N}} u) = 0. \end{aligned} \quad (2.14)$$

Hence, one has

$$g(t) = x_0 \int |u_0|^2 dx + (t - t_0) x_0 \Im \int x \nabla u_0 \bar{u}_0 dx.$$

Since $x_0 \in R^N$ is arbitrary, one deduces that $\forall t \in [t_0, T]$

$$\int x |u(t, x)|^2 dx = \int x |u_0|^2 dx + 2(t - t_0) \Im \int x \nabla u_0 \bar{u}_0 dx. \quad (2.15)$$

Therefore, one can choose a proper $y_1 \in R^N$ such that

$$\lim_{t \rightarrow T} \int x |u(t, x)|^2 dx = - \left(\int |Q(x)|^2 dx \right) y_1.$$

This completes the proof. $\square$

3. Concentration property of blow-up solution

In this section, we further study the concentration properties of blow-up solutions with initial data $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Using the variational characterization of the corresponding ground state, we obtain that if the initial data $u_0 \in \Sigma$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$, then the blow-up solution remains close to Q in $H^1(R^N)$ up to scaling and phase parameters, and also translation in the nonradial case. Moreover, using these properties of blow-up solutions, we obtain that the blow-up solution $|u(t, x)|^2$ like a δ -function as $t \rightarrow T$ at the point $x = y_1$, which implies that the point y_1 concentrates all mass of blow-up solution of the Cauchy problem (1.1)–(1.2).

Proposition 3.1. *Let $u_0 \in \Sigma$ and $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Suppose that $t_0 = \frac{\pi}{2\omega}$ and the corresponding solution of the Cauchy problem (1.1)–(1.2) $u(t, x)$ blows up in finite time $T \leq \frac{\pi}{\omega}$. Then there are functions $y(t) \in R^N$, $\gamma(t) \in R$ such that*

$$\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)(x + y(t)))e^{i\gamma(t)} \rightarrow Q(x), \quad \text{strongly in } H^1 \quad \text{as } t \rightarrow T, \quad (3.1)$$

where $\lambda(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u\|_{L^2}}$ and $Q(x)$ is the unique ground state solution of Eq. (1.5).

Proof. We prove Proposition 3.1 by contradiction. We show that for any sequence $t_k \rightarrow T$, there are y_k and $\gamma(t_k)$ such that

$$\lambda^{\frac{N}{2}}(t_k)v(t_k, \lambda(t_k)(x + y(t_k)))e^{i\gamma(t_k)} \rightarrow Q(x) \quad \text{strongly in } H^1 \quad \text{as } k \rightarrow \infty, \quad (3.2)$$

where $Q(x)$ is the ground state solution of Eq. (1.5). If not, then (3.2) does not hold along some sequence t_k . But then we can find a subsequence of t_k along with (3.2) holds, this is a contradiction. Since t_k is an arbitrary sequence approaching T , and (3.1) follows.

Since $u(t, x)$ is the solution of the Eq. (1.1), multiplying (1.1) with u_t and integrating, one has that

$$\frac{d}{dt} \int |\nabla u|^2 dx - \frac{\cos^2(\omega t)}{1 + \frac{2}{N}} \frac{d}{dt} \int |u|^{2+\frac{4}{N}} dx = 0, \quad (3.3)$$

which implies that

$$\frac{d}{dt} E(u(t)) = \frac{\omega}{1 + \frac{2}{N}} \sin(2\omega t) \int |u|^{2+\frac{4}{N}} dx. \quad (3.4)$$

By $\frac{\pi}{2\omega} = t_0 \leq t < T \leq \frac{\pi}{\omega}$, one has that $\frac{d}{dt} E(u(t)) \leq 0$ and

$$E(u(t)) \leq E(u_0), \quad \text{for } t_0 < t < T. \quad (3.5)$$

Since the solution $u(t, x) \in C([t_0, T]; \Sigma)$ blows up in finite time $T < \frac{\pi}{\omega}$, satisfying $\lim_{t \rightarrow T} \|\nabla u\|_{L^2} = \infty$. For any $t_k \rightarrow T$ as $k \rightarrow \infty$, one takes

$$\frac{1}{\lambda_k^2} = \int |\nabla u(x, t_k)|^2 dx \rightarrow \infty, \quad \text{as } k \rightarrow \infty.$$

Considering $U_k = \lambda_k^{\frac{N}{2}} u(\lambda_k x)$, by direct computations, one has

$$\begin{cases} \|U_k\|_{L^2} = \|u(t_k)\|_{L^2} = \|u_0\|_{L^2} = \|Q(x)\|_{L^2}, \\ \|\nabla U_k\|_{L^2} = \lambda_k \|\nabla u(t_k)\|_{L^2} = 1. \end{cases} \quad (3.6)$$

Therefore, U_k is a uniformly bounded sequence in H^1 . Note that

$$\begin{aligned} H(U_k) &= \int |\nabla U_k|^2 dx - \frac{1}{1 + \frac{2}{N}} \int |U_k|^{2 + \frac{4}{N}} dx \\ &= \lambda_k^2 \int |\nabla u(t_k)|^2 - \frac{1}{1 + \frac{2}{N}} \lambda_k^2 \int |u(t_k)|^{2 + \frac{4}{N}} dx \\ &\leq \lambda_k^2 \int |\nabla u(t_k)|^2 - \frac{\cos^2(\omega t_k)}{1 + \frac{2}{N}} \lambda_k^2 \int |u(t_k)|^{2 + \frac{4}{N}} dx \\ &\leq \lambda_k^2 E(u_0) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (3.7)$$

On the other hand, by (3.6) and Gagliardo-Nirenberg inequality, one has that $H(U_k) \geq 0$ and

$$\lim_{k \rightarrow \infty} H(U_k) = 0. \quad (3.8)$$

Combining (3.8) with (3.7), one has

$$\lim_{k \rightarrow \infty} \|U_k\|_{L^{2 + \frac{4}{N}}}^{2 + \frac{4}{N}} = 1 + \frac{2}{N}.$$

Applying Lemma 2.4 to the sequence U_k ($M = 1, m^{2+4/N} = 1 + \frac{2}{N}$), one has that there exist $\{y_k\} \subset \mathbb{R}^N$ and $U \in H^1$ such that

$$U_k(x + y_k) \rightharpoonup U(x) \quad \text{weakly in } H^1$$

with $\|U\|_{L^2} \geq \|Q\|_{L^2}$. Since $\|U\|_{L^2} \leq \|U_k(x + y_k)\|_{L^2} = \|Q\|_{L^2}$ and the Brézis-Lieb Lemma (see [1]), one has

$$U_k(x + y_k) \rightarrow U(x) \quad \text{strongly in } L^2. \quad (3.9)$$

By the Gagliardo-Nirenberg's inequality (see Lemma 2.3), there exists $\gamma_k \in \mathbb{R}$ such that

$$\|U_k(x + y_k)e^{i\gamma_k} - U\|_{L^{2 + \frac{4}{N}}}^{2 + \frac{4}{N}} \leq C \|U_k(x + y_k)e^{i\gamma_k} - U\|_{L^2}^{\frac{4}{N}} \|\nabla(U_k(x + y_k)e^{i\gamma_k} - U)\|_{L^2}^2.$$

It follows from (3.9) and $\|\nabla U_k(x + y_k)\|_{L^2} \leq C$ that

$$U_k(x + y_k)e^{i\gamma_k} \rightarrow U \quad \text{strongly in } L^{2 + \frac{4}{N}}. \quad (3.10)$$

Next, we shall show that $U_k(x + y_k)e^{i\gamma_k}$ converges to U strongly in H^1 . We need only now show that $\|\nabla U\|_{L^2} = 1$ by the Brézis-Lieb Lemma (see [1]). Note that

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} H(U_k e^{i\gamma_k}) \\ &= 1 - \frac{1}{1 + \frac{2}{N}} \lim_{k \rightarrow \infty} \|U_k\|_{L^{2 + \frac{4}{N}}}^{2 + \frac{4}{N}} \\ &= 1 - \frac{1}{1 + \frac{2}{N}} \|U\|_{L^{2 + \frac{4}{N}}}^{2 + \frac{4}{N}}. \end{aligned} \quad (3.11)$$

Hence, if $\|\nabla U\|_{L^2} < 1$, by (3.11) one has $H(U) < 0$, this is impossible by Lemma 2.3 and the fact $U \not\equiv 0$. On the other hand, if $\|\nabla U\|_{L^2} > 1$, by (3.11) one has $H(U) > 0$. Therefore, we have showed that $\|\nabla U\|_{L^2} = 1$, which implies that $U_k(x + y_k)e^{i\gamma_k}$ converges strongly to U in H^1 .

Applying $\|u_0\|_{L^2} = \|Q\|_{L^2}$, (3.6) and (3.7), one deduces that U_{k_j} is a minimizing sequence of the variational problem

$$I(\|Q\|_{L^2}) \equiv \inf\{H(f) | f \in H^1, \|f\|_{L^2} = \|Q\|_{L^2}\}. \quad (3.12)$$

Moreover, one obtains that U is a minimizer of variational problem (3.12) and $|U|$ is also a minimizer of (3.12). Indeed, since $\|\nabla|U|\|_{L^2} \leq \|\nabla U\|_{L^2}$, one gets

$$H(|U|) \leq H(U) = 0.$$

It is obvious that $\|U\|_{L^2} = \||U|\|_{L^2} = \|Q\|_{L^2}$, and one has $H(|U|) \geq 0$ by Lemma 2.3. Hence, $H(|U|) = 0$.

Therefore, applying the variational characteristic of the ground state of Eq. (1.5), one deduces that, $\exists y \in R^N, \gamma \in R$ such that $U(x) = Q(x + y)e^{i\gamma}$, which implies that

$$\lambda_k^{\frac{N}{2}} u(t_k, \lambda_k(x + y_k))e^{i\gamma_k} \rightarrow Q(x + y)e^{i\gamma} \quad \text{strongly in } H^1 \text{ as } k \rightarrow \infty. \quad (3.13)$$

By redefining the sequences y_k and γ_k , we have (3.2) is true. This completes the proof of Proposition 3.1. $\square$

In the end, we using the Proposition 3.1 to obtain the following concentration properties of the blow-up solutions for the Cauchy problem (1.1)–(1.2): The blow-up solution converges to a Dirac function when t closes to the blow-up time T .

Theorem 3.2. *Let $u_0 \in \Sigma$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Suppose that $t_0 = \frac{\pi}{2\omega}$ and the corresponding solution of the Cauchy problem (1.1)–(1.2) $u(t, x)$ blows up in finite time $T \leq \frac{\pi}{\omega}$. Then one has*

$$|u(t, x)|^2 \rightarrow \left\| Q(x) \right\|_{L^2}^2 \delta_{x=y_1}, \quad \text{as } t \rightarrow T. \quad (3.14)$$

where $\lambda(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u\|_{L^2}}$, $Q(x)$ is the unique ground state solution of Eq. (1.5) and $y_1 \in R^N$ is defined in Lemma 2.5.

Proof. Applying Proposition 3.1, one has that there exist functions $\gamma(t) \in R$, $y(t) \in R^N$ such that

$$e^{i\gamma(t)} \lambda^{\frac{N}{2}}(t) u(t, \lambda(t)(x + y(t))) \rightarrow Q(x) \quad \text{in } H^1 \quad \text{as } t \rightarrow T,$$

where $\lambda(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u\|_{L^2}} \rightarrow 0$, as $t \rightarrow T$.

Therefore,

$$\lambda^N(t) |u(t, \lambda(t)(x + y(t)))|^2 \rightarrow |Q(x)|^2 \text{ in } L^1, \quad \text{as } t \rightarrow T,$$

and

$$|u(t, x + y(t))|^2 \rightarrow \|Q(x)\|_{L^2}^2 \delta_{x=0}, \quad \text{as } t \rightarrow T, \quad (3.15)$$

by the fact $\|\nabla u(t, x)\|_{L^2} \rightarrow \infty$, as $t \rightarrow T$.

Applying (2.6) and (3.5), one has that

$$\frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx \leq 16E(u_0),$$

which implies that there exists a constant $C_2 > 0$ such that

$$\forall t \in [t_0, T), \quad \int |x|^2 |u(t, x)|^2 dx \leq C_2. \quad (3.16)$$

Hence, there exists a $C > 0$ such that

$$C \int |y(t)|^2 |u(t, x)|^2 dx \leq \int |x + y(t)|^2 |u(t, x + y(t))|^2 dx \leq C_2.$$

Hence, one deduces that

$$\overline{\lim}_{t \rightarrow T} |y(t)| \leq \frac{\sqrt{C_2/C}}{\|Q(x)\|_{L^2}}, \quad (3.17)$$

and there exists an $R_0 > 0, \forall t \in [t_0, T)$

$$|y(t)| \leq R_0. \quad (3.18)$$

We claim that for $\forall R > R_0$

$$\lim_{t \rightarrow T} \left\{ \int_{B(0, R)} |u(t, x)|^2 x dx - \left(\int |Q(x)|^2 dx \right) y(t) \right\} = 0, \quad (3.19)$$

where $B(0, R) = \{x \in \mathbb{R}^N \mid |x| \leq R\}$ is the ball of radius R centered at the origin. Indeed,

$$\begin{aligned} \int_{B(0, R)} |u(t, x)|^2 x dx &= \int_{B(0, R)} |u(t, x)|^2 (x - y(t)) dx + \int_{B(0, R)} |u(t, x)|^2 y(t) dx \\ &= \int_{B(-y(t), R)} |u(t, y + y(t))|^2 y dy \\ &\quad + \int_{B(-y(t), R)} |u(t, y + y(t))|^2 y(t) dy \\ &= \int_{B(0, \delta)} |u(t, y + y(t))|^2 y dy \\ &\quad + \int_{B(-y(t), R) \setminus B(0, \delta)} |u(t, y + y(t))|^2 y dy \\ &\quad + \int_{B(-y(t), R)} |u(t, y + y(t))|^2 y(t) dy. \end{aligned}$$

Because of $R > R_0$, there exists a $\delta > 0, B(0, \delta) \subset B(-y(t), R)$, by (3.15), one has

$$\int_{B(0, R)} |u(t, x)|^2 x dx - \|Q(x)\|_{L^2}^2 y(t) \rightarrow 0, \quad \text{as } t \rightarrow T, \quad (3.20)$$

this completes the proof of Claim (3.19).

On the other hand, by (3.16), one has $\forall t \in [t_0, T)$

$$\left| \int_{|x| \geq R} |u(t, x)|^2 x dx \right| \leq \frac{C_2}{R^2}. \quad (3.21)$$

Thus, (3.20) and (3.21) implies that

$$\lim_{t \rightarrow T} \left\{ \int |u(t, x)|^2 x dx - \left(\int |Q(x)|^2 dx \right) y(t) \right\} = 0. \quad (3.22)$$

Since Lemma 2.5, there exists a $y_1 \in R^N$ such that

$$\lim_{t \rightarrow T} \int |u(t, x)|^2 x dx = - \left(\int |Q(x)|^2 dx \right) y_1. \quad (3.23)$$

It follows from (3.22)–(3.23) that

$$y(t) \rightarrow y_1, \quad \text{as } t \rightarrow T,$$

and

$$|u(t, x - y_1)|^2 \rightarrow \|Q(x)\|_{L^2}^2 \delta_{x=0}, \quad \text{as } t \rightarrow T.$$

Therefore, one obtains

$$|u(t, x)|^2 \rightarrow \|Q(x)\|_{L^2}^2 \delta_{x=y_1}, \quad \text{as } t \rightarrow T,$$

which concludes the proof of Theorem 3.2. $\square$

At the end of this paper, we show the L^2 -Concentration properties of blow-up solution by the compactness lemma proposed by Hmidi and Keraani [8].

Theorem 3.3. *Let $u_0 \in \Sigma$ and $t_0 = \frac{\pi}{2\omega}$. Assume the corresponding solution of the Cauchy problem (1.1)–(1.2) $u(t, x)$ blows up in finite time $T \leq \frac{\pi}{\omega}$. Suppose that $a(t) > 0$ is any function such that $a(t)\|\nabla u(t)\|_{L^2} \rightarrow \infty$ as $t \rightarrow T$. Then, there exists $y(t) \in R^N$ such that*

$$\liminf_{t \rightarrow T} \int_{|x - y(t)| \leq a(t)} |u(t, x)|^2 dx \geq \int Q^2 dx, \quad (3.24)$$

where $Q(x)$ is the unique ground state solution of Eq. (1.5).

Proof. Since $u(t, x) \in C([t_0, T]; \Sigma)$ blows up in finite time $T < \frac{\pi}{\omega}$ satisfying $\lim_{t \rightarrow T} \|\nabla u\|_{L^2} = \infty$. For any $t_k \rightarrow T$ as $k \rightarrow \infty$, we take

$$\frac{1}{\lambda_k^2} = \int |\nabla u(x, t_k)|^2 dx \rightarrow \infty, \quad \text{as } k \rightarrow \infty. \quad (3.25)$$

Considering $U_k = \lambda_k^{\frac{N}{2}} u(\lambda_k x)$, by direct computations, one has

$$\begin{cases} \|U_k\|_{L^2} = \|u(t_k)\|_{L^2} = \|u_0\|_{L^2} = \|Q(x)\|_{L^2}, \\ \|\nabla U_k\|_{L^2} = \lambda_k \|\nabla u(t_k)\|_{L^2} = 1. \end{cases} \quad (3.26)$$

Therefore, U_k is a uniformly bounded sequence in H^1 . It follows from (3.5) that

$$\begin{aligned} H(U_k) &= \int |\nabla U_k|^2 dx - \frac{1}{1 + \frac{2}{N}} \int |U_k|^{2 + \frac{4}{N}} dx \\ &\leq \lambda_k^2 \int |\nabla u(t_k)|^2 - \frac{\cos^2(\omega t_k)}{1 + \frac{2}{N}} \lambda_k^2 \int |u(t_k)|^{2 + \frac{4}{N}} dx \\ &\leq \lambda_k^2 E(u_0) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (3.27)$$

On the other hand, by (3.26) and the Gagliardo-Nirenberg inequality, one has $H(U_k) \geq 0$ and

$$\lim_{k \rightarrow \infty} H(U_k) = 0. \quad (3.28)$$

Combining (3.28) with (3.27), one has

$$\lim_{k \rightarrow \infty} \|U_k\|_{L^{2 + \frac{4}{N}}}^{2 + \frac{4}{N}} = 1 + \frac{2}{N}.$$

Applying Lemma 2.4 to the sequence U_k ($M = 1, m^{2+4/N} = 1 + \frac{2}{N}$), one has that there exist $\{y_k\} \subset R^N$ and $U \in H^1$ such that

$$U_k(x + y_k) \rightharpoonup U(x) \quad \text{weakly in } H^1, \quad \text{with } \|U\|_{L^2} \geq \|Q\|_{L^2}.$$

That is,

$$\lambda_k^{\frac{N}{2}} u(t_k, \lambda_k(x + y_k)) \rightharpoonup U(x) \quad \text{weakly in } H^1, \quad (3.29)$$

which implies that for every $A > 0$,

$$\liminf_{k \rightarrow \infty} \int_{|x| \leq A} \lambda_k^N |u(t_k, \lambda_k(x + y_k))|^2 dx \geq \int_{|x| \leq A} |U|^2 dx.$$

Since the assumption $\frac{a(t_k)}{\lambda(t_k)} \rightarrow \infty$, one has

$$\liminf_{k \rightarrow \infty} \sup_{y \in R^N} \int_{|x-y| \leq a(t_k)} |u(t_k, x)|^2 dx \geq \int_{|x| \leq A} |U|^2 dx.$$

For every $A > 0$, one has

$$\liminf_{k \rightarrow \infty} \sup_{y \in R^N} \int_{|x-y| \leq a(t_k)} |u(t_k, x)|^2 dx \geq \int |U|^2 dx \geq \int Q^2 dx. \quad (3.30)$$

Therefore, since the sequence $\{t_k\}$ is arbitrary, one has

$$\liminf_{t \rightarrow T} \sup_{y \in R^N} \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx \geq \int Q^2 dx.$$

On the other hand, for every $t \in [t_0, T)$, the function $y \mapsto \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx$ is continuous and goes to 0 at infinity. It follows that

$$\sup_{y \in R^N} \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx = \int_{|x-y(t)| \leq a(t)} |u(t, x)|^2 dx, \quad (3.31)$$

for some $y(t) \in R^N$. This completes the proof of Theorem 3.1. $\square$

4. Limiting profiles in Σ

In this section, we obtain the limiting profiles of the Cauchy problem (1.1)–(1.2) in the natural energy space Σ . Using the concentration properties obtained in Sect. 3 and the variational characterization of $Q(x)$, we obtain that if the initial data $u_0 \in \Sigma$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$, then the blow-up solution remains close to Q in $H^1(R^N)$ up to scaling and phase parameters, and also translation in the nonradial case. Moreover, we obtain the decomposition of blow-up solutions $u(t, x)$ with initial data $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha'$.

Theorem 4.1. *Let $u_0 \in \Sigma$ and $\|u_0\|_{L^2} = \|Q\|_{L^2}$. Suppose that $t_0 = \frac{\pi}{2\omega}$ and the corresponding solution of the Cauchy problem (1.1)–(1.2) $u(t, x)$ blows up in finite time $T \leq \frac{\pi}{\omega}$. Then there are functions $y(t) \in R^N, \gamma(t) \in R$ such that*

$$\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)(x + y(t)))e^{i\gamma(t)} \rightarrow Q(x) \quad \text{strongly in } \Sigma \quad \text{as } t \rightarrow T, \quad (4.1)$$

where $\lambda(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u\|_{L^2}}$ and $Q(x)$ is the unique ground state solution of Eq. (1.5).

Proof. Applying Proposition 3.1, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. when $|t - T| < \delta$, there are functions $y(t) \in R^N, \gamma(t) \in R$ such that

$$\int |\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)(x + y(t)))e^{i\gamma(t)} - Q(x)|^2 dx < \frac{\varepsilon}{3|R|^2}, \quad (4.2)$$

where the positive constant R is sufficiently large and chosen by later.

By Theorem 3.2, one has that there exists an $y_1 \in R^N$ such that $|y_1| \leq R_0$ and

$$\lim_{t \rightarrow T} y(t) = y_1, \quad (4.3)$$

$$|u(t, x)|^2 \rightarrow \|Q\|_{L^2}^2 \delta_{x=y_1}, \quad \text{as } t \rightarrow T. \quad (4.4)$$

Now, for a sufficiently large $R > R_0 > 0$, one has

$$\begin{aligned} & \int |x|^2 |\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)(x + y(t)))e^{i\gamma(t)} - Q(x)|^2 dx \\ &= \left\{ \int_{|x| \leq R} + \int_{|x| > R} \right\} |x|^2 |\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)(x + y(t)))e^{i\gamma(t)} - Q(x)|^2 dx \\ &\leq |R|^2 \int |\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)(x + y(t)))e^{i\gamma(t)} - Q(x)|^2 dx \\ &\quad + \int_{|x| > R} |x|^2 |\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)(x + y(t)))e^{i\gamma(t)}|^2 dx \\ &\quad + \int_{|x| > R} |x|^2 |Q(x)|^2 dx \\ &= I + II + III. \end{aligned} \quad (4.5)$$

Applying (4.2), one has

$$|I| < \frac{\varepsilon}{3}. \quad (4.6)$$

Since $R > R_0 > 0$, (4.2) and (4.3), one has

$$\begin{aligned}
 |II| &\leq \int_{|x|>R} |x|^2 |\lambda^{\frac{N}{2}}(t) u(t, \lambda(t)(x + y(t)))|^2 dx \\
 &\leq \int_{|z|>\lambda(t)(R+y(t))} \left| \frac{z}{\lambda(t)} - y(t) \right|^2 |u(t, z)|^2 dz \\
 &\leq \int_{|z|>\lambda(t)(R+y_1)} \left| \frac{z}{\lambda(t)} - y(t) \right|^2 \|Q(z)\|_{L^2}^2 \delta_{z=x_1} dz \\
 &\leq \frac{\varepsilon}{3}.
 \end{aligned} \tag{4.7}$$

Since $Q(x)$ is the positive and spherically symmetric solution with exponentially decay at infinity (see Lemma 2.2), one has

$$|III| = \int_{|x|>R} |x|^2 |Q(x)|^2 dx < \frac{\varepsilon}{3}, \tag{4.8}$$

for sufficiently large $R > 0$.

Therefore, by (4.5)–(4.8), one has

$$\left| \int |x|^2 |\lambda^{\frac{N}{2}}(t) u(t, \lambda(t)(x + y(t))) e^{i\gamma(t)} - Q(x)|^2 dx \right| < \varepsilon. \tag{4.9}$$

Combining with Proposition 3.1, we have (4.1) holds in Σ . This completes the proof. $\square$

At the end of this section, using the variational characterization of Eq. (1.5) and scaling technique, we prove that if the initial data $u_0 \in \Sigma$ close to the ground state $Q(x)$, then the blow-up solution $u(t, x)$ remains close to $Q(x)$ in Σ .

Theorem 4.2. *Let $u_0 \in \Sigma$ such that there exists a $\alpha_1 > 0$ such that the following property is true. For all $0 < \alpha' \leq \alpha_1$, there exists $\delta(\alpha')$ with $\delta(\alpha') \rightarrow 0$ as $\alpha' \rightarrow 0$ such that for all $u \in H^1$,*

$$0 < \alpha(u) = \alpha(u_0) = \int |u_0|^2 - \int Q^2 < \alpha'. \tag{4.10}$$

Further assuming $t_0 = \frac{\pi}{2\omega}$ and the corresponding solution of the Cauchy problem (1.1)–(1.2) $u(t, x)$ blows up in finite time $T \leq \frac{\pi}{\omega}$. Then there are functions $y(t) \in R^N$ and $\gamma(t) \in R$ such that

$$\left\| \lambda^{\frac{N}{2}}(t) u(t, \lambda(t)(x + y(t))) e^{i\gamma(t)} - Q(x) \right\|_{\Sigma} \leq \delta(\alpha'), \tag{4.11}$$

where $\lambda(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u\|_{L^2}}$ and $Q(x)$ is the unique ground state solution of Eq. (1.5).

Proof. We prove this conclusion by contradiction. We show that for any sequence $t_k \rightarrow T$, there is a subsequence t_{k_j}, y_{k_j} and $\gamma(t_{k_j})$ such that

$$\left\| \lambda^{\frac{N}{2}}(t_{k_j}) u(\lambda(t_{k_j})(x + y(t_{k_j})), t_{k_j}) e^{i\gamma(t_{k_j})} - Q(x) \right\|_{\Sigma} \leq \delta(\alpha'), \tag{4.12}$$

where $Q(x)$ is the unique ground state solution of Eq. (1.5). If not, then (4.12) does not hold along some sequence t_{k_r} . But then we can find a subsequence

of t_{k_r} along with (4.12) holds, this is a contradiction. Since t_k is an arbitrary sequence approaching T , and (4.11) follows.

Assume that for any $t_n \rightarrow T$, there is a subsequence t_{n_k} such that $u_{n_k} := u(t_{n_k}, x) \in \Sigma$ such that

$$\lim_{k \rightarrow \infty} \int |u_{n_k}|^2 dx = \int Q^2 dx \quad \text{and} \quad \lim_{k \rightarrow \infty} \|\nabla u_{n_k}\|_{L^2}^2 = \infty. \quad (4.13)$$

Consider now $U_{n_k} = \lambda_{n_k}^{\frac{N}{2}} u_{n_k}(\lambda_{n_k} x)$, where $\lambda_{n_k} = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u_{n_k}\|_{L^2}}$. We have the following properties for U_{n_k}

$$\|U_{n_k}\|_{L^2} \rightarrow \|Q\|_{L^2}, \quad (4.14)$$

$$\|\nabla U_{n_k}\|_{L^2} = 1, \quad (4.15)$$

$$\lim_{k \rightarrow \infty} H(U_{n_k}) \leq \lim_{n \rightarrow \infty} \lambda_{n_k}^2 E(u_{n_k}) \leq \lim_{n \rightarrow \infty} \lambda_{n_k}^2 E(u_0) = 0, \quad (4.16)$$

by (3.5).

On the other hand, $H(U_{n_k}) \geq \left[1 - \left(\frac{\|U_{n_k}\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{N}}\right] \|\nabla U_{n_k}\|_{L^2}^2$ by Lemma 2.3, thus $\lim_{k \rightarrow \infty} H(U_{n_k}) = 0$.

Now we take

$$w_{n_k} = \frac{\|Q\|_{L^2}}{\|U_{n_k}\|_{L^2}} U_{n_k}, \quad (4.17)$$

we have $\|w_{n_k}\|_{L^2} = \|Q\|_{L^2}$, $\|\nabla w_{n_k}\|_{L^2} \rightarrow 1$ and $H(w_{n_k}) \rightarrow 0$ as $k \rightarrow \infty$. Using the same procedure in Proposition 3.1 and Theorem 4.1, we have the conclusion, and this completes the proof. $\square$

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