The curve shortening problem under Robin boundary condition

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Abstract. The curve shortening problem for a graph under Robin boundary condition is studied in this paper. The large time behavior of the global solution is shown to depend critically on the parameters in the boundary condition. Some asymptotic behavior of the solution is also discussed.

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1. Introduction

In the curve shortening problem one studies the motion of a curve driven along its normal direction by its curvature. Since the pioneering works of Gage–Hamilton [10] and Grayson [12], there have been many results for this problem and one may consult the books Cao [6], Chou–Zhu [9], Giga [11] and Zhu [23].

When the curve is expressible as the graph of a function, the curve shortening problem can be reformulated as the following quasilinear parabolic equation

$$u_t = \frac{u_{xx}}{1 + u_x^2}, \quad t > 0, \quad (1.1)$$

where the curve at time t is given by the graph of $u(., t)$. The boundary value problem for this equation was studied in Huisken [13] together with the mean curvature flow. For the Dirichlet problem it is shown that any global solution of (1.1) converges to the linear function satisfying the boundary values at the two endpoints, and for the homogeneous Neumann problem, the solution converges to a constant as time goes to infinity. The non-homogeneous Neumann problem was subsequently studied in Altschuler–Wu [1] where the asymptotic behavior of the solution is determined (see below). One may consult [19] for

FIGURE 1. The division of (α_1, α_2) .

results on other boundary value problems for the curve shortening problems. In connection to its applications in image analysis [20], the curve shortening problem is regarded as the geometric invariant replacement of the heat equation. In view of its potential applications in other areas, it is desirable to undertake a thorough investigation of (1), where results already obtained for the heat equation may be raised as questions for the curve shortening problem. Among many well-studied boundary conditions, the Robin condition is next common to the Dirichlet and Neumann conditions. It is the purpose of this paper to study (1) in $[0, 1] \times (0, \infty)$ under Robin boundary condition

$$\begin{aligned} u_x(0, t) &= \alpha_1 u(0, t) + \beta_1, \\ u_x(1, t) &= -\alpha_2 u(1, t) + \beta_2, \end{aligned} \quad (1.2)$$

where $\alpha = (\alpha_1, \alpha_2) \neq (0, 0)$ and $\beta = (\beta_1, \beta_2)$ are given. One can show that for any smooth initial function satisfying (1.2) problem (1.1) under (1.2) has a global solution, see Sect. 1. Our goal is to study the large time behavior of the global solution. It turns out that it depends critically on the parameters α and β . To formulate it, we divide $\mathbb{R}^2 \setminus \{(0, 0)\}$ into the disjoint union of the following sets: $D_1 = \{\alpha : \delta > 0, \alpha_1 \text{ or } \alpha_2 > 0\}$, $D_2 = \{\alpha : \delta < 0\}$, $D_3 = \{\alpha : \delta > 0, \alpha_1, \alpha_2 < 0\}$, $\Gamma_1 = \{\alpha : \delta = 0, \alpha_1 \alpha_2 < 0\}$, and $\Gamma_2 = \{\alpha : \delta = 0, \alpha_1, \alpha_2 < 0\}$, where $\delta = \alpha_1 + \alpha_2 + \alpha_1 \alpha_2$. See Fig. 1.

This division is partly due to the steady states for the problem. It is clear that any steady state of (1.1) is a linear function. When δ is non-zero, there is always a unique linear function satisfying (1.2), hence the steady state is unique. On the other hand, when $\delta = 0$, steady states exist if and only if the compatibility condition: $\beta_2 - (1 + \alpha_2)\beta_1 = 0$, that's, $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$, holds. When this happens, there are infinitely many steady states satisfying (1.2).

It is not hard to show that there are only four types of possible large time behavior (see Sect. 2) for any global solution of (1.1) under (1.2):

- Case 1. The solution converges uniformly to a steady state.
- Case 2. The solution diverges to infinity uniformly, that's, $\min u(., t)$ tends to ∞ as t tends to ∞ .
- Case 3. The solution diverges to negative infinity uniformly, that's, $\max u(., t)$ tends to $-\infty$ as t tends to ∞ .
- Case 4. The solution becomes unbounded from both sides, that is, $\max u(., t)$ tends to ∞ and $\min u(., t)$ tends to $-\infty$ as t tends to ∞ .

Our main result shows that the large time behavior becomes more diverse as the parameters α varies from D_1, Γ_1, D_2 , and Γ_2 to D_3 . Specifically we state

Theorem 1.1. *Concerning the large time behavior of a global solution of (1.1) under (1.2), we have*

- (1) *For $\alpha \in D_1$, only Case 1 holds.*
- (2) *For $\alpha \in \Gamma_1$, (a) when $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$, only Case 1 holds; (b) when $\alpha_1\beta_2 + \alpha_2\beta_1 < 0$, only Case 2 holds; and (c) when $\alpha_1\beta_2 + \alpha_2\beta_1 > 0$, only Case 3 holds.*
- (3) *For $\alpha \in D_2$, Cases 1 to 3 are possible.*
- (4) *For $\alpha \in \Gamma_2$, (a) when $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$, Cases 1 to 3 are possible, and (b) when $\alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$, Cases 2 to 4 are possible.*
- (5) *For $\alpha \in D_3$, all four cases are possible.*

Here a case is possible means that it can be realized by some global solution.

When α_1 and α_2 are zero, problem (1.1) under (1.2) becomes Neumann problem and is discussed in [1], where it is shown that for any global solution only Case 1 holds when $\beta_1 = \beta_2$, only Case 2 holds when $\beta_2 > \beta_1$, and only Case 3 holds when $\beta_2 < \beta_1$.

Naturally one is led to the study of the asymptotic behavior of the unbounded solutions. Here we give some preliminary results for those solutions whose slopes at endpoints become very large and are of the same sign eventually. We formulate them in two propositions.

Proposition 1.2. *Let α be in $\Gamma_1 \cup (D_2 \cap \{\alpha : \alpha_1\alpha_2 < 0\})$. For any convex (resp. concave) solution u of (1.1) under (1.2), if $\min_x u(x, t) \rightarrow \infty$ (resp. $\max_x u(x, t) \rightarrow -\infty$) as $t \rightarrow \infty$, then for any $\rho \in (0, 1/2)$, the curvature of the graph of $u(., t)$ inside the set $[\rho, 1 - \rho] \times \mathbb{R}$ tends to zero uniformly as t tends to ∞ .*

Proposition 1.3. *Let α be in $\Gamma_2 \cup D_3$. Assume that u is a solution of (1.1) under (1.2) which is unbounded from both sides. Then for any $c > 0$, the curvature of the graph of $u(., t)$ inside the set $[0, 1] \times [-c, c]$ tends to zero uniformly as t tends to ∞ .*

We leave out the asymptotic behavior of those solutions with α belonging to $D_2 \cap \{\alpha : \alpha_1\alpha_2 \geq 0\}$, Γ_2 or D_3 which diverge to infinity or negative infinity uniformly. For Neumann problem, it is known that any unbounded solution is

asymptotic to a portion of a grim reaper [1]. By analogous thinking we conjecture that in this case, the solution is asymptotic to the grim reaper over $(0, 1)$ when its slopes go to infinity at both endpoints, or to a portion of a grim reaper when its slope tends to infinity only at one endpoint.

For related works, we point out the papers of Kohsaka [15] and Chern–Guo–Lo [7] on free boundary problems for (1.1), and Altschuler–Wu [2], Stahl [21] and Huisken [14] for the mean curvature flow under a geometric Neumann condition.

2. Global solutions

Proposition 2.1. *Let u_0 be a function in $C^{2,\sigma}([0, 1])$ for some σ in $(0, 1)$ satisfying (1.2). The problem (1.1) under (1.2) has a unique global solution u satisfying $u(\cdot, 0) = u_0$ which belongs to the parabolic Hölder space $\tilde{C}^{2,\sigma}([0, 1] \times [0, T])$ for every $T > 0$.*

Proof. According to Theorem 8.9 in Lieberman [17] the existence of a solution u of (1.1) and (1.2) in $[0, T]$ follows from an a priori estimate of the form

$$\|u\|_{\tilde{C}^{2,\sigma}} \leq C(\alpha, \beta, T, \|u_0\|_{C^{2,\sigma}})$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ are specified in (1.2). From the structure of (1.1), clearly it suffices to derive a weaker estimate, namely,

$$\|u\|_{\tilde{C}^{1,\sigma'}} \leq C(\alpha, \beta, T, \|u_0\|_{C^{2,\sigma}}) \quad (2.1)$$

for some $\sigma' \in (0, 1)$. Then the desired second order estimate follows from Schauder estimate for linear parabolic equations.

We proceed to derive (2.1). In the following u is a $\tilde{C}^{2,\sigma}$ -solution of (1.1) under (1.2) in $[0, 1] \times [0, T]$.

Step 1 (Uniform estimate) We note that (1.1) admits a traveling wave solution called the grim reaper

$$g(x, t) = \frac{1}{c} \log \sec \left(c \left(x - \frac{1}{2} \right) \right) + ct + d, \\ (x, t) \in \left(\frac{1}{2} - \frac{\pi}{2c}, \frac{1}{2} + \frac{\pi}{2c} \right) \times (-\infty, \infty),$$

where $c \in (0, \pi)$ and $d \in \mathbb{R}$ are two parameters. We observe that g is well-defined on $[0, 1]$ and blows up to ∞ at $1/2 \pm \pi/2c$. When c is very closed to π , the graph of g becomes very steep at 0 and 1.

We shall choose d and c so that $g(x, t) \geq u(x, t)$ for $(x, t) \in [0, 1] \times [0, T]$. First, fix a large d such that $g(x, 0) > u_0(x)$. Next, choose c sufficiently close to π such that

$$g_x(0, t) < -|\alpha_1|g(0, t) + \beta_1, \\ g_x(1, t) > |\alpha_2|g(1, t) + \beta_2,$$

for all t in $[0, T]$. This is possible because

$$-g_x(0, t) = g_x(1, t) = \frac{\sin(c/2)}{\cos(c/2)}$$

tend to ∞ as c approaches π much faster than

$$\begin{aligned} g(0, t) &= g(1, t) = \frac{1}{c} \log \sec \frac{c}{2} + ct + d \\ &\leq -\frac{1}{c} \log \cos \frac{c}{2} + cT + d \end{aligned}$$

($z/\log z \rightarrow \infty$ as $z \rightarrow \infty$). We claim $g > u$ in $[0, T]$. Clearly this is true at $t = 0$. If g touches u at some finite time t_1 , by the strong maximum principle it must occur at $x = 0$ or 1 . Let us assume it is at $x = 0$. Then, $u(0, t_1) = g(0, t_1)$, and $(g - u)_x(0, t_1) > 0$. However,

$$\begin{aligned} g_x(0, t_1) &< -|\alpha_1|g(0, t_1) + \beta_1 \\ &\leq -\alpha_1 u(0, t_1) + \beta_1 \\ &= u_x(0, t_1), \end{aligned}$$

contradiction holds. We have obtained an upper bound for u . A similar lower bound can be obtained by considering a grim reaper with negative c .

Step 2 (Gradient estimate) The function $v = u_x$ satisfies

$$v_t = \frac{v_{xx}}{1 + u_x^2} - \frac{2u_x u_{xx}}{(1 + u_x^2)^2} v_x,$$

and

$$\begin{aligned} v(0, t) &= \alpha_1 u(0, t) + \beta_1, \\ v(1, t) &= -\alpha_2 u(1, t) + \beta_2. \end{aligned}$$

(Note that by parabolic regularity the solution is in $\tilde{C}^{4,\sigma}$ locally in $(0, 1) \times (0, T]$.) Using Step 1 and the maximum principle we conclude

$$\|u_x\|_{L^\infty} = \|v\|_{L^\infty} \leq C(\alpha, \beta, T, \|u_0\|_{C^1}).$$

Step 3 (Hölder estimate in time) From Step 2 we get

$$|u(x, t) - u(y, t)| \leq C_1 |x - y|, \quad x, y \in [0, 1], t \in [0, T]. \quad (2.2)$$

We shall derive

$$|u(x, s_1) - u(x, s_2)| \leq C_2 |s_1 - s_2|^{1/2}, \quad s_1, s_2 \in [0, T], x \in [0, 1]. \quad (2.3)$$

(2.2) and (2.3) together imply a Hölder estimate for u for any exponent in $(0, 1)$.

We choose two test functions as follows. Let ξ_0 be a non-negative function on $[0, \infty)$ which equals to 1 near 0, becomes 0 for all $x \geq 1$ and satisfies $\int_0^1 \xi_0(x) dx = 1$. We set

$$\xi(x) = \xi_0 \left(\frac{|x - x_0|}{|s_2 - s_1|^{1/2}} \right)$$

where $x_0 \in [0, 1]$ and $0 < s_1 < s_2 \leq T$ are fixed. For small $\delta_0 > 0$, let

$$\theta_{\delta_0}(t) = \int_{-\infty}^t \eta_{\delta_0}(\tau) d\tau$$

where $\eta_{\delta_0}(\tau) = (-\chi_{[s_1-\delta_0/2, s_1+\delta_0/2]}(\tau) + \chi_{[s_2-\delta_0/2, s_2+\delta_0/2]}(\tau))/\delta_0$. We write (1.1) in the form

$$u_t = (\arctan u_x)_x.$$

Multiplying both sides of this equation by $\xi(x)\theta_{\delta_0}(t)$ we have

$$\int_0^1 \int_0^T u_t(x, t) \xi(x) \theta_{\delta_0}(t) dt dx = \int_0^1 \int_0^T (\arctan u_x)_x(x, t) \xi(x) \theta_{\delta_0}(t) dt dx.$$

Perform integration by parts on the left hand side of this identity to get

$$\begin{aligned} \int_0^1 \int_0^T u_t(x, t) \xi(x) \theta_{\delta_0}(t) dt dx &= - \int_0^1 \int_0^T u(x, t) \xi(x) \eta_{\delta_0}(t) dt dx \\ &= \frac{1}{\delta_0} \int_0^1 \int_{s_1-\delta_0/2}^{s_1+\delta_0/2} u(x, t) \xi(x) dt dx - \frac{1}{\delta_0} \int_0^1 \int_{s_2-\delta_0/2}^{s_2+\delta_0/2} u(x, t) \xi(x) dt dx \\ &\rightarrow \int_0^1 (u(x, s_1) - u(x, s_2)) \xi(x) dx \end{aligned} \quad (2.4)$$

as $\delta_0 \rightarrow 0$. On the other hand,

$$\begin{aligned} \int_0^T \int_0^1 (\arctan u_x)_x(x, t) \xi(x) \theta_{\delta_0}(t) dx dt \\ &= - \int_0^T \int_0^1 (\arctan u_x)(x, t) \xi'(x) \theta_{\delta_0}(t) dx dt \\ &\quad + \int_0^T (\arctan u_x)(x, t) \xi(x) \theta_{\delta_0}(t) \Big|_0^1 dt. \end{aligned}$$

Using $|\arctan z| \leq \pi/2$, and the form of ξ , we have

$$\begin{aligned} \left| \int_0^T \int_0^1 (\arctan u_x)(x, t) \xi'(x) \theta_{\delta_0}(t) dx dt \right| \\ \leq \pi \|\xi'_0\|_{L^\infty} \times \frac{1}{(s_2 - s_1)^{1/2}} \times (s_2 - s_1)^{1/2} \times |s_2 - s_1 + 2\delta_0| \\ = \pi \|\xi'_0\|_{L^\infty} |s_2 - s_1 + 2\delta_0|, \end{aligned}$$

and

$$\left| \int_0^T (\arctan u_x)(x, t) \xi(x) \theta_{\delta_0}(t) \Big|_0^1 dt \right| \leq \frac{\pi}{2} (s_2 - s_1 + 2\delta_0).$$

Combining with (2.4), we have

$$\left| \int_0^1 (u(x, s_1) - u(x, s_2)) \xi(x) dx \right| \leq C |s_1 - s_2|. \quad (2.5)$$

Now,

$$\begin{aligned} |u(x_0, s_1) - u(x_0, s_2)| &\leq |u(x_0, s_1) - u(x, s_1)| + |u(x, s_1) - u(x, s_2)| \\ &\quad + |u(x, s_2) - u(x_0, s_2)| \\ &\leq 2C_1 |x - x_0| + |u(x, s_1) - u(x, s_2)|. \end{aligned}$$

Therefore,

$$\begin{aligned} |u(x_0, s_1) - u(x_0, s_2)| |s_1 - s_2|^{1/2} &\leq \int_0^1 |u(x_0, s_1) - u(x_0, s_2)| \xi(x) dx \\ &\leq 2C_1 \int_0^1 |x - x_0| \xi(x) dx + \int_0^1 |u(x, s_1) - u(x, s_2)| \xi(x) dx \\ &\leq C |s_1 - s_2|, \end{aligned}$$

after using (2.5), and (2.3) follows.

Step 4 (Gradient Hölder estimate) The function $v = u_x$ satisfies

$$v_t = \left(\frac{v_x}{1 + v^2} \right)_x, \quad (2.6)$$

and

$$\begin{aligned} v(0, t) &= \alpha_1 u(0, t) + \beta_1, \\ v(1, t) &= -\alpha_2 u(1, t) + \beta_2. \end{aligned}$$

This equation can be viewed as a linear parabolic equation in divergence form with a bounded coefficient $(1 + v^2)^{-1}$ whose solution v satisfies a Hölder condition at the boundary from Step 3. Using standard regularity theory [LSU] we conclude that v belongs to $\tilde{C}^{\sigma'}([0, 1] \times [0, T])$ for some $\sigma' \in (0, 1)$, so (2.1) holds. $\square$

3. Some lemmas

Lemma 3.1. *Any solution u of (1.1) under (1.2) satisfies $u_t > 0$ (resp. < 0) for all $t \geq 0$ if $u_t(x, 0) > 0$ (resp. < 0) on $[0, 1]$.*

Proof. By differentiating (1.1) and (1.2) in time, we see that u_t satisfies a linear parabolic equation together with the boundary condition:

$$\begin{aligned} (u_t)_x(0, t) &= \alpha_1 u_t(0, t), \\ (u_t)_x(1, t) &= -\alpha_2 u_t(1, t). \end{aligned} \quad (3.1)$$

Now suppose there is a first time $t_1 > 0$ such that $u_t = 0$. By the strong maximum principle, $\min u_t(x, t_1)$ attains at $x = 0$ or $x = 1$, so $u_t(0, t_1) = 0$ or $u_t(1, t_1) = 0$. However, by the parabolic Hopf's lemma we have $u_{tx}(0, t_1) > 0$ or $u_{tx}(1, t_1) < 0$, contradicting (3.1). So u_t must remain positive or negative all time. $\square$

Lemma 3.2. *For any global solution u of (1.1) under (1.2), there exists some $t_0 \geq 0$ such that $u_t(x, t)$ does not change sign at $x = 0$ and 1 for all $t \geq t_0$.*

Proof. See Lemma 1 in Brunovský, et al [5]. $\square$

Lemma 3.3. *Any global unbounded solution u must fall into one of the following cases: (a) $\min_x u(x, t) \rightarrow -\infty$, (b) $\max_x u(x, t) \rightarrow -\infty$, and (c) $\max_x u(x, t) \rightarrow \infty$ and $\min_x u(x, t) \rightarrow -\infty$.*

Proof. Let u be a global unbounded solution of (1.1) under (1.2). By replacing u by $-u$ if necessary, we may assume without loss of generality that $\max_x u(x, t) \rightarrow \infty$ as $t \rightarrow \infty$. By the strong maximum principle, the maximum of $u(x, t)$ cannot attain in the interior of $[0, 1]$, so it attains either at $x = 0$ or $x = 1$. If it is the latter case, we may replace $u(x)$ by $u(1 - x)$, hence it suffices to assume $u(0, t) \rightarrow \infty$ by Lemma 3.2. By the same lemma, if u does not tend to ∞ or $-\infty$ at $x = 1$, we can find a number M such that $M \leq u(1, t) \leq M + 1$ for all large time. Then

$$u_x(1, t) = -\alpha_2 u(1, t) + \beta_2 \geq -M'$$

is bounded for all large t . Now, fix a grim reaper $g_0(\cdot, 0)$ over $[0, 1]$ whose slope at $x = 1$ is less than $-M'$ and blows up at $x = -1$, say. Let $L = g_0(0, 0) - g_0(1, 0)$. As $u(0, t) \rightarrow \infty$, we can find a T such that $u(0, t) > L + |M| + 1$ for all $t \geq T$. Let $g_1 = g_0 - d$ where d is a large number so that $u(x, T) > g_1(x)$ on $[0, 1]$. Then by the comparison principle $u(\cdot, t)$ and $g_1(\cdot, t)$ cannot touch at $x = 0$ and 1 and so $u(x, t) > g_1(x, t)$ for all $t \geq T$. Contradiction holds because $\min_x g_1(x, t) \rightarrow \infty$. We conclude that $\min_x u(1, t)$ must tend to ∞ or $-\infty$ as $t \rightarrow \infty$. $\square$

4. Large time behavior

In this section we prove Theorem 1.1. We shall need the following auxiliary functions. First, let

$$w(x) = a + (\alpha_1 a + \beta_1)x + \left(\frac{-\delta a + \beta_2 - \beta_1 - \alpha_2 \beta_1}{2m + \alpha_2} \right) x^{2m}, \quad (4.1)$$

where m is a positive number chosen so that both $2m - 2$ and $2m + \alpha_2$ are positive and $w(0) = a$ is a free parameter to be specified later. For each fixed a , w satisfies the Robin boundary condition (1.2). Next, let

$$l(x) = a(1 + \alpha_1 x) + \beta_1 x. \quad (4.2)$$

This linear function satisfies half of (1.2) at $x = 0$. It also satisfies (1.2) at $x = 1$ (and hence is a steady state) if and only if $\delta a = \beta_2 - (1 + \alpha_2)\beta_1$, where $\delta = \alpha_1 + \alpha_2 + \alpha_1 \alpha_2$. Similarly, the linear function

$$\tilde{l}(x) = a[1 + \alpha_2(1 - x)] + \beta_2(x - 1) \quad (4.3)$$

satisfies half of (1.2) at $x = 1$.

Now, we start the proof of Theorem 1.1.

First, consider $\alpha \in D_1$. In this case the problem has a unique steady state l_s . We claim that any global solution u will converge to l_s uniformly as t tends to infinity. To this end it suffices to show every global solution is bounded uniformly in time, for then the general theorem of Zelenjak [22] yields the desired conclusion.

To show a uniform bound for u we make use of the function w in (4.1). As δ is positive, we see immediately from (4.1) that w is a concave (resp. convex) function with $\min_x w(x) \rightarrow \infty$ ($\max_x w(x) \rightarrow -\infty$) as $a \rightarrow \infty$ (resp. $a \rightarrow -\infty$).

Therefore, for any initial function u_0 we can fix a large a so that $w_{-|a|} < u_0 < w_a$. It follows from Lemma 3.1 and the comparison principle that u is bounded between $w_{-|a|}$ and w_a for all time.

Second, consider α in Γ_1 . Let $\gamma \equiv \beta_2 - (1 + \alpha_2)\beta_1 = (\alpha_1\beta_2 - \alpha_2\beta_1)/\alpha_1$. When γ is equal to zero, this problem has infinitely many steady states given by (4.2). As $|a| \rightarrow \infty$, the zero of l approaches the number $-1/\alpha_1$, which does not belong to $(0, 1)$ (noting that $\alpha_1 + 1 > 0$). Therefore, for large a , $l(x) \rightarrow \infty$ (resp. $-\infty$) uniformly as $a \rightarrow \infty$ (resp. $-\infty$). It follows that for any initial u_0 , we can find a large a such that $l_{-|a|} < u_0 < l_a$. By Lemma 3.1 and the comparison principle again, the solution u is pinched between l_a and $l_{-|a|}$ for all time, and we conclude the uniform convergence as in the previous case.

When γ is not equal to zero, we consider the case where $\alpha_1 < 0$ and $\alpha_2 > 0$ first. For $\gamma > 0$, we claim that all global solutions diverge to ∞ as $t \rightarrow \infty$. To see this, take l in (4.2) with a very negative a so that $l < u_0$ in $[0, 1]$. We claim that $l(x) < u(x, t)$ for all t . For, if u touches l at some first time t_1 , by the strong maximum principle it must touch it at $x = 0$ or 1 . If it is at $x = 0$, by the parabolic Hopf's lemma

$$(u(x, t_1) - l(x))_x > 0 \quad \text{at } x = 0.$$

However, since $u(0, t_1) = l(0)$ and l satisfies (1.2) at $x = 0$, we have $u_x(0, t_1) - l_x(0) = 0$, contradiction holds. Therefore, u must touch l at $x = 1$. At $x = 1$, we have $u(1, t_1) = l(1)$. We have, by the parabolic Hopf's lemma,

$$\begin{aligned} 0 &> u_x(1, t_1) - l_x(1) = -\alpha_2 u(1, t_1) + \beta_2 - (a\alpha_1 + \beta_1) \\ &= -\alpha_2 l(1) + \beta_2 - (a\alpha_1 + \beta_1), \\ &= \beta_2 - (1 + \alpha_2)\beta_1 > 0, \end{aligned}$$

contradiction holds again. We conclude that $u(x, t) > l(x)$ for all time. As u must be unbounded, by Lemma 3.3, $\min_x u(x, t) \rightarrow \infty$. When $\gamma < 0$, a similar argument using (4.3) instead of (4.2) shows that $\max_x u(x, t) \rightarrow -\infty$ as $t \rightarrow \infty$.

For the case where $\alpha_1 > 0$ and $\alpha_2 < 0$, the function $v(x, t) \equiv u(1 - x, t)$ satisfies (1.1) with boundary condition

$$\begin{aligned} v_x(0, t) &= \alpha_2 v(0, t) - \beta_2, \\ v_x(1, t) &= -\alpha_1 v(1, t) - \beta_1. \end{aligned}$$

Applying the previous arguments to v , we conclude that

$$\begin{aligned} \min_x u(x, t) &\rightarrow \infty \quad \text{when } \gamma > 0, \text{ and} \\ \max_x u(x, t) &\rightarrow -\infty \quad \text{when } \gamma < 0, \end{aligned}$$

when $\alpha_1 > 0$ and $\alpha_2 < 0$. We have settled the case for α in Γ_1 .

Third, consider α in D_2 . In this case δ is negative. For large $a > 0$ (resp. < 0), the function w in (4.1) becomes convex (resp. concave) and tends to ∞ (resp. $-\infty$) as $a \rightarrow \infty$ (resp. $-\infty$). As there is only one steady state in this case, for all sufficiently large a , the global solutions starting at w will diverge to ∞ or $-\infty$ as $t \rightarrow \infty$ depending on whether a is positive or negative by Lemmas 3.1 and 3.3.

To complete the proof it remains to exclude Case 4. To achieve this we assume such solution exists and draw a contradiction. Let u be such a solution. By the strong maximum principle and Lemma 3.2, we have either $u(0, t) \rightarrow \infty$ and $u(1, t) \rightarrow -\infty$ or $u(0, t) \rightarrow -\infty$ and $u(1, t) \rightarrow \infty$. Replacing u by $v(x, t) = u(1 - x, t)$ if necessary, we may assume that $u(0, t) \rightarrow \infty$ and $u(1, t) \rightarrow -\infty$. By the mean value theorem $\min_x u_x(x, t) \rightarrow -\infty$ as $t \rightarrow \infty$. We can find a large, positive t_0 such that $\min_x u_x(x, t) \geq \min_x u_x(x, t_0)$ for all $t \leq t_0$. Since u_x satisfies a linear parabolic equation, by the strong maximum principle either (a) $\min_x u_x(x, t_0) = u_x(0, t_0)$ or (b) $\min_x u_x(x, t_0) = u_x(1, t_0)$. We first show that (b) is impossible. For, suppose it holds. Consider in (4.3) the linear function $\tilde{l}$ which satisfies $\tilde{l}(1) = u(1, t_0)$ and $\tilde{l}_x(1) = u_x(1, t_0)$. From (b) we see that $\tilde{l}(x) > u(x, t_0)$ for all $x \in [0, 1)$. As $u_t(1, t) < 0$ for all $t \geq t_0$ (Lemma 3.2), u can never touch $\tilde{l}$ again at $x = 1$. In case there is a first time $t_1 > t_0$ u touches $\tilde{l}$ again, it must occur at $x = 0$. At this time, $u(0, t_1) = \tilde{l}(0) = a(1 + \alpha_2) - \beta_2$, so by parabolic Hopf's lemma,

$$\begin{aligned} 0 &< (\tilde{l} - u)_x(0, t) = -a\alpha_2 + \beta_2 - (\alpha_1 u(0, t_1) + \beta_1) \\ &= -a\delta + (1 + \alpha_1)\beta_2 - \beta_1. \end{aligned} \quad (4.4)$$

However, as $a = \tilde{l}(1) = u(1, t_0)$ is very negative and $\delta < 0$ the term $-a\delta + (1 + \alpha_1)\beta_2 - \beta_1$ is negative, contradiction holds. Therefore, $\min_x u_x$ must occur at $x = 0$ at t_0 . But then consider l in (4.2) which satisfies $a = l(0) = u(0, t_0)$. It lies below $u(\cdot, t_0)$. As $u_t(0, t) > 0$ for all $t \geq t_0$, u and l can only touch at a first time $t_1 > t_0$ at $x = 1$. However, then $u(1, t_1) = l(1) = a(1 + \alpha_1) + \beta_1$,

$$\begin{aligned} 0 &> (u - l)_x(1, t) = -\alpha_2 u(1, t_1) + \beta_2 - (a\alpha_1 + \beta_1) \\ &= -\delta a + \gamma. \end{aligned} \quad (4.5)$$

As a is large and $\delta < 0$, $-\delta a + \gamma > 0$, contradiction holds. Therefore, $u > l$ for all time, and so the minimum of this solution tends to infinity. We have shown that there are no unbounded solutions which are unbounded from both sides.

We now come to α in Γ_2 . In this case, the straight lines given by the graphs of l in (4.2) and $\tilde{l}$ in (4.3) both have zeros approaching $-1/\alpha_1 \in (0, 1)$ as $|a| \rightarrow \infty$. For a large $a > 0$, $\max\{l, \tilde{l}\}$ is a convex function whose boundary values are very large and satisfies (1.2). We may smooth it out at their intersection to get a smooth convex function u_0 . By solving (1.1) under (1.2) with this u_0 as initial function we obtain an unbounded solution which diverges to ∞ uniformly as $t \rightarrow \infty$. By smoothing out $\min\{l, \tilde{l}\}$ for a large, negative a , we obtain a concave solution which diverges to $-\infty$ uniformly.

When $\gamma = 0$, the same arguments above involving (4.4) and (4.5) show that there is no global solution which tends to ∞ and $-\infty$ simultaneously.

When $\gamma \neq 0$, such solution exists by an indirect argument as shown below.

First, fix a convex u_{10} and a concave u_{20} , $u_{10} > u_{20}$, so that the global solution u_1 and u_2 starting at u_{10} and u_{20} tend to ∞ and $-\infty$ respectively as $t \rightarrow \infty$. Let u_λ be the solution starting at $(1 - \lambda)u_{10} + \lambda u_{20}$. We claim that

the non-empty set $\{\lambda \in [0, 1] : \min u_\lambda(\cdot, t) \rightarrow \infty \text{ as } t \rightarrow \infty\}$ is open in $[0, 1]$. For, if λ_1 is in this set, the corresponding solution $u_{\lambda_1}(x, t_1) > u_{10}$ at some t_1 . By the continuity dependence of initial data, for all λ near λ_1 the corresponding solution u_λ will satisfy $u_\lambda(x, t_1) > u_{10}$. Since u_{10} is convex, by the comparison principle and Lemma 3.1 $u_\lambda(x, t) \rightarrow \infty$ as $t \rightarrow \infty$. In other words, λ also belongs to this set. Similarly, one can also prove that the non-empty set $\{\lambda \in [0, 1] : \max u_\lambda(\cdot, t) \rightarrow -\infty \text{ as } t \rightarrow \infty\}$ is open. Therefore, the number $\lambda^* = \sup\{\lambda \in [0, 1] : \min u_\lambda(\cdot, t) \rightarrow \infty\}$ is strictly less than one and the solution u_{λ^*} must tend to ∞ and $-\infty$ as $t \rightarrow \infty$ because there are no steady states.

Finally, let's consider α in D_3 . In this case there is a unique steady state, and as in the previous case, there are large convex (resp. concave) solutions which tend to ∞ (resp. $-\infty$) as $t \rightarrow \infty$ respectively. We need to show the existence of solution which tends to ∞ and $-\infty$ simultaneously. The arguments in the last paragraph do produce a solution u_{λ^*} which does not diverge to ∞ or $-\infty$ uniformly. However, since it may tend to the steady state, we cannot conclude there is a solution unbounded from both sides. To establish this fact, we need a more delicate argument.

Consider the functional

$$I(u) = \frac{1}{2} \int_0^1 u_x^2 dx + \frac{\alpha_2}{2} u^2(1) + \frac{\alpha_1}{2} u^2(0)$$

for any α . By standard arguments, the minimization problem

$$\lambda_1 = \inf\{2I(u) : \|u\|_{L^2} = 1, u \in H^1([0, 1])\}$$

has a minimizer φ_1 satisfying

$$-\varphi_{1xx} = \lambda_1 \varphi_1,$$

with $\varphi_{1x}(0) = \alpha_1 \varphi_1(0)$ and $\varphi_{1x}(1) = -\alpha_2 \varphi_1(1)$. It is well-known that φ_1 does not change sign on $[0, 1]$. We shall assume that it is positive in below. Also,

$$\lambda_2 = \inf\{2I(u) : \|u\|_{L^2} = 1, u \in H^1([0, 1]), \langle u, \varphi_1 \rangle = 0\}$$

where

$$\langle u, v \rangle = \int_0^1 u_x v_x dx + \alpha_2 u(1)v(1) + \alpha_1 u(0)v(0)$$

is attained at some φ_2 which satisfies $-\varphi_{2xx} = \lambda_2 \varphi_2$ with the same boundary conditions as φ_1 . We have

$$I(\varphi_1) = \frac{\lambda_1}{2},$$

$$I(\varphi_2) = \frac{\lambda_2}{2},$$

and, by $\langle \varphi_1, \varphi_2 \rangle = 0$,

$$I((1-\lambda)\varphi_1 + \lambda\varphi_2) = \frac{(1-\lambda)^2\lambda_1 + \lambda^2\lambda_2}{2}.$$

We claim that λ_1 and λ_2 are negative by explicitly considering the eigenvalue problem

$$-\varphi_{xx} = \lambda \varphi$$

subject to $\varphi_x(0) = \alpha_1\varphi(0)$ and $\varphi_x(1) = -\alpha_2\varphi(1)$. For any negative λ the solution is of the form

$$\varphi(x) = ae^{\mu x} + be^{-\mu x}, \quad \mu \equiv \sqrt{-\lambda}.$$

In order to fulfill the boundary condition we require the linear system

$$\begin{cases} (\mu - \alpha_1)a - (\mu + \alpha_1)b = 0, \\ (\mu + \alpha_2)e^\mu a + (-\mu + \alpha_2)e^{-\mu}b = 0 \end{cases}$$

to have non-trivial solution (a, b) , and this is true if and only if $(\mu - \alpha_1)(-\mu + \alpha_2)e^{-\mu} + (\mu + \alpha_2)(\mu + \alpha_1)e^\mu = 0$, that's,

$$\alpha_1\alpha_2 - (\alpha_1 + \alpha_2)\mu + \mu^2 = (\alpha_1\alpha_2 + (\alpha_1 + \alpha_2)\mu + \mu^2)e^{2\mu}.$$

Denote the left hand side of the above expression by $f(\mu)$ and the right hand side by $g(\mu)$. It is readily checked that $f(0) = g(0)$ and $g'(0) > f'(0)$, so the graph of g lies above the graph of f for μ near 0. By solving the root for f , its discriminant shows that f is positive in $[0, \infty)$ while g vanishes at $-\alpha_1$ and $-\alpha_2$, which are both positive when α belongs to D_3 . As f/g tends to zero at ∞ , the graphs of f and g must intersect at least twice. It follows that there are two choices μ_1 and μ_2 to make φ a non-zero function. We conclude that the eigenvalue problem has at least two negative eigenvalues, in other words, λ_1 and λ_2 are negative.

Let's take $u_{01} = a\varphi_1 + l_s$ and $u_{02} = a\varphi_2 + l_s$ where l_s is the unique steady state for (1.1) under (1.2). Both u_{01} and u_{02} satisfy (1.2). Consider the functional

$$J(u) = \frac{1}{2} \int_0^1 u_x^2 dx + \frac{1}{2} \alpha_2 u^2(1) + \frac{1}{2} \alpha_1 u^2(0) - \beta_2 u(1) + \beta_1 u(0),$$

which is decreasing along (1.1) and (1.2). Letting $\varphi_\lambda = (1 - \lambda)a\varphi_1 + \lambda a\varphi_2 + l_s$, we have

$$\begin{aligned} J((1 - \lambda)u_{01} + \lambda u_{02}) &= J(\varphi_\lambda) \\ &= I(\varphi_\lambda) - \beta_2 \varphi_\lambda(1) + \beta_1 \varphi_\lambda(0) \\ &= \frac{1}{2} [\lambda_1(1 - \lambda)^2 + \lambda_2 \lambda^2] a^2 + \frac{a}{2} \int_0^1 [(1 - \lambda)\lambda_1 \varphi_1 + \lambda \lambda_2 \varphi_2] l_s dx \\ &\quad + \frac{\beta_1}{2} \varphi_\lambda(0) - \frac{\beta_2}{2} \varphi_\lambda(1) \\ &< 0, \end{aligned}$$

provided $|a|$ is sufficiently large. Observing that φ_1 is a positive convex function, the solution u_1 starting at $a\varphi_1 + l_s$ tends to ∞ (resp. $-\infty$) when a is positive (resp. negative) and large. Suppose there are no unbounded solutions which tend to ∞ and $-\infty$ simultaneously. Consider the solution u_2 starting at $a\varphi_2 + l_s$, $a > 0$ large. Since it does not go unbounded from both sides by assumption nor tends to the steady state as $J(a\varphi_2 + l_s) < 0$ and $J(l_s) = 0$, it must diverge to ∞ or $-\infty$. Let us assume it is the former. Now, we can argue as in the previous case by considering the sets $\{\lambda : \max u_\lambda(., t) \rightarrow -\infty \text{ as } t \rightarrow \infty\}$ and $\{\lambda : \min u_\lambda(., t) \rightarrow \infty \text{ as } t \rightarrow \infty\}$ where u_λ is the solution starting at $(1 - \lambda)$

$(-a\varphi_1) + \lambda a\varphi_2 + l_s$ (denoted by $\tilde{\varphi}_\lambda$) and a is a fixed large positive number. Both sets are non-empty and open. As above, we have $J(\tilde{\varphi}_\lambda) < 0$. The inequality

$$J(u_\lambda(x, t)) \leq J(\tilde{\varphi}_\lambda) < 0$$

shows that u_λ cannot converge to l_s , so it must be unbounded. It follows from Lemma 3.3 that u_{λ^*} , where $\lambda^* = \sup\{\lambda : u_\lambda \rightarrow -\infty \text{ as } t \rightarrow \infty\}$, must tend to ∞ and $-\infty$ simultaneously, contradiction holds. We conclude that there are solutions which are unbounded from both sides.

The proof of Theorem 1.1 is completed.

5. Asymptotic behavior

Now we prove Proposition 1.2. We shall assume α belongs to Γ_1 or D_2 where $\alpha_1 < 0$ and $\alpha_2 > 0$. The other case $\alpha_1 > 0$ and $\alpha_2 < 0$ can be treated similarly. We would like to show that for a convex global solution which tends to infinity, the curvature of its graph decays to zero in the part sitting inside the set $[\rho, 1 - \rho] \times \mathbb{R}$ for every $\rho \in (0, 1/2)$.

As the solution u diverges to ∞ , by Lemma 3.2 we may assume both $a(t) \equiv u(1, t)$ and $b(t) \equiv u(0, t)$ are increasing to ∞ . By convexity, $u_x(x, t) \leq u_x(1, t) = -\alpha_2 u(1, t) + \beta_2$, so

$$\max_{x \in [0, 1]} u_x(x, t) \rightarrow -\infty, \quad \text{as } t \rightarrow \infty. \quad (5.1)$$

For any $\rho \in (0, 1/2)$ we fix ρ' and ρ'' with $0 < \rho'' < \rho' < \rho$. From (5.1) we can find some t_1 such that

$$u(\rho', t) \geq u(\rho, t) + 3, \quad \forall t \geq t_1. \quad (5.2)$$

From

$$\begin{aligned} \frac{d}{dt} \int_{\rho''}^{\rho'} u \, dx &= \int_{\rho''}^{\rho'} u_t \, dx \\ &= \arctan(u_x(\rho', t)) - \arctan(u_x(\rho'', t)), \end{aligned}$$

which tends to 0 as $t \rightarrow \infty$, for any given ε , there is some $t_2 \geq t_1$ such that

$$\left| \frac{d}{dt} \int_{\rho''}^{\rho'} u \, dx \right| \leq \varepsilon, \quad \forall t \geq t_2.$$

By the mean value theorem there exists $\rho(t) \in (\rho'', \rho')$ such that

$$u(\rho(t), t) = \frac{\int_{\rho''}^{\rho'} u(x, t) \, dx}{\rho' - \rho''}, \quad \forall t \geq 0.$$

Therefore, for $t, t' \in (t_2, \infty)$,

$$\begin{aligned} |u(\rho(t), t) - u(\rho(t'), t')| &= \frac{1}{\rho' - \rho''} \left| \int_{\rho''}^{\rho'} u(x, t) dx - \int_{\rho''}^{\rho'} u(x, t') dx \right| \\ &\leq \frac{\varepsilon |t - t'|}{\rho' - \rho''}. \end{aligned}$$

It follows that there is $t_3 \geq t_2$ such that

$$|u(\rho(t), t) - u(\rho(t'), t')| \leq 1, \quad \forall t \in [t' - 1, t'], \quad \forall t' \geq t_3.$$

Then, for $t \in [t' - 1, t']$ with $t' \geq t_3$ and $x \in [\rho, 1 - \rho]$,

$$\begin{aligned} u(\rho(t), t) &\geq u(\rho(t'), t') - 1 \\ &\geq u(\rho', t') - 1 \quad (\text{as } u \text{ is decreasing}) \\ &\geq u(\rho, t') + 2 \quad (\text{by (5.2)}) \\ &\geq u(x, t') + 2. \end{aligned}$$

Since $b(t) \geq u(\rho(t), t)$, we have

$$b(t) - 1 \geq u(x, t') + 1, \quad \forall t \in [t' - 1, t'], \quad \forall t' \geq t_3,$$

where $x \in [\rho, 1 - \rho]$. By a similar consideration, there is some large time still denoted by t_3 such that

$$\begin{aligned} a(t) + 1 &\leq u(x, t') - 1, \\ \forall t &\in [t' - 1, t'], \quad x \in [\rho, 1 - \rho], \quad \forall t' \geq t_3. \end{aligned}$$

We have shown that for any $x \in [\rho, 1 - \rho]$,

$$\begin{aligned} [u(x, t') - 1, u(x, t') + 1] &\subset (a(t), b(t)), \\ \forall t &\in [t' - 1, t'], \forall t' \geq t_3. \end{aligned}$$

This fact means that for any given $t' \geq t_3$, during the time $[t' - 1, t']$, the inverse function $x = x(u, t)$ of $u(\cdot, t)$ is a graph over the interval $[u(x, t') - 1, u(x, t') + 1]$ with $x \in [\rho, 1 - \rho]$. Moreover, by the geometric invariance of (1.1) it satisfies the same equation

$$x_t = \frac{x_{uu}}{1 + x_u^2}, \quad (5.3)$$

in the domain $(u(x, t') - 1, u(x, t') + 1) \times (t' - 1, t']$.

Now we claim that $\sup_{[\rho, 1-\rho]} |k(x, t)| \rightarrow 0$ as $t \rightarrow \infty$ where $k(x, t)$ is the curvature of the graph of $(x, u(x, t))$. For if not, there exist a sequence $\{(x_j, t_j)\}_{j=1}^\infty$ and a positive constant C_0 such that

$$(x_j, t_j) \in [\rho, 1 - \rho] \times (t_3, \infty), \quad t_j \rightarrow \infty,$$

and

$$k(x_j, t_j) \geq C_0 > 0.$$

On the other hand, observe that $\varphi_j(u, t) \equiv x(u + u(x_j, t_j), t + t_j)$ satisfies

$$\varphi_{j,t} = \frac{\varphi_{j,uu}}{1 + \varphi_{j,u}^2}, \quad (u, t) \in (-1, 1) \times (-1, 0]. \quad (5.4)$$

From (5.1) we see that φ_{ju} is uniformly bounded in $(u, t) \in (-1, 1) \times (-1, 0]$. We may apply the interior estimate for quasi-linear parabolic equation (see, chapter 5 in [16]) to (5.4) in divergence form to conclude that the following estimate

$$\|\varphi_j\|_{\tilde{C}^{1,\sigma}([-1/2, 1/2] \times [-1/2, 0])} \leq C_1,$$

holds for some constant C_1 and $\sigma \in (0, 1)$. It implies that we can view (5.4) as a linear parabolic equation for φ_j with Hölder coefficient $1/(1 + \varphi_{ju}^2)$. By parabolic Schauder theory the following higher order estimate

$$\|\varphi_j\|_{\tilde{C}^{2,\sigma}([-1/4, 1/4] \times [-1/4, 0])} \leq C_2,$$

holds for some C_2 . Consequently, there exist a subsequence φ_{j_k} and a function φ such that

$$\varphi_{j_k}(u, 0) \rightarrow \varphi \text{ in } C^2([-1/4, 1/4]), \quad k \rightarrow \infty.$$

By (5.1) we have

$$\sup_{u \in [a(t), b(t)]} |x_u(u, t)| \rightarrow 0, \quad t \rightarrow \infty,$$

which implies that $\sup_{u \in [-1/4, 1/4]} |(\varphi_j)_u(u, 0)| \rightarrow 0$ as $j \rightarrow \infty$. So $\varphi_u \equiv 0$ and $\varphi_{uu} \equiv 0$. In other words, $\varphi_{j_k uu}(0, 0) \rightarrow 0$ as $k \rightarrow \infty$. Noting that

$$\varphi_{j_k uu}(0, 0) = x_{uu}(u(x_{j_k}, t_{j_k}), t_{j_k}),$$

we conclude that $k(x_{j_k}, t_{j_k}) = \varphi_{j_k uu}(0, 0)/(1 + \varphi_{j_k u}^2(0, 0))^{3/2} \rightarrow 0$ as $k \rightarrow \infty$, contradiction holds. Hence the curvature must decay to zero as time approaches infinity.

The proof of Proposition 1.2 is completed.

Next we prove Proposition 1.3. For any global solution which becomes unbounded from both sides we would like to establish the decay of the curvature of its graph in $[0, 1] \times [-c, c]$ for any $c > 0$. We know that such solution u is possible only for Γ_2 or D_3 where both α_1 and α_2 are negative. As $\min u(., t)$ and $\max u(., t)$ become unbounded, by the strong maximum principle either $u(0, t)$ tends to ∞ and $u(1, t)$ to $-\infty$ or the other way around. For definiteness let us assume it is the former case. In fact, the latter case can be reduced to the former by looking at $u(1 - x, t)$.

First we argue that the solution $u(., t)$ becomes strictly decreasing after some time. To see this we observe that both $u(0, t) = \max u(., t)$ and $-u(1, t) = -\min u(., t)$ become very large after some time, and the maximum and minimum are not attained away from the endpoints. Therefore, we can find some t_1 and constants $c_1 < u(0, t_1)$ and $c_2 > u(1, t_1)$ so that c_1 is greater than any interior maximal value of $u(., t_1)$ and c_2 is less than any interior minimal value of $u(., t_1)$. The horizontal lines (x, c_1) and (x, c_2) intersect the graph of $u(., t_1)$ at exactly one point transversally, and $u(., t_1)$ is strictly decreasing in the set $\{x : u(x, t_1) > c_1 \text{ or } u(x, t_1) < c_2\}$. By Sturm comparison theorem (Angenent [3]), the graph of $u(., t)$ is strictly decreasing in the set $\{x : u(x, t) > c_1 \text{ or } u(x, t) < c_2\}$ for all $t \geq t_1$. For the portion of the graph inside the rectangle $[0, 1] \times [c_2, c_1]$ we can use the method of foliation to show that it

eventually becomes strictly decreasing after some $t_2 \geq t_1$, see Angenent [4] or Chou–Zhu [8] for a similar argument.

After showing that $u(x, t)$ becomes strictly decreasing for $t \geq t_2$, the inverse function $x(u, t)$ of $u(x, t)$ satisfies (5.3) in $[a(t), b(t)] \times [t_2, \infty)$ where $a(t) = u(1, t)$ and $b(t) = u(0, t)$ tend to $-\infty$ and ∞ respectively in a monotonic way. Since $x_u(a(t), t)$ and $x_u(b(t), t)$ both tend to zero as t tends to ∞ , without loss of generality we may assume the minimum of x_u along the parabolic boundary of $(a(t), b(t)) \times [t_2, \infty)$ is attained at t_2 . By applying the maximum principle to the equation satisfied by x_u , we have

$$0 > x_u(u, t) \geq -\delta_1,$$

where $1/\delta_1 = -\min u_x(., t_2)$. For any $a > 0$, by applying the interior estimate to the quasi-linear equation (5.3) (expressed in divergence form) in the domain $[-a-2, a+2] \times [t_3, \infty)$, we find a constant C_1 such that

$$\|x_u\|_{\tilde{C}^\sigma([-a-1, a+1] \times [t_3+1, \infty))} \leq C_1,$$

see, chapter 5 in [16]. Here $t_3 \geq t_2$ so that $[-a-2, a+2] \subset [a(t_3), b(t_3)]$. It follows that we can view (5.3) as linear parabolic equation with Hölder coefficient $1/(1+x_u^2)$. By the parabolic Schauder theory, we readily obtain the higher order interior estimate, namely, there exists a constant C_2 (depending on a) such that

$$\|x\|_{\tilde{C}^{3,\sigma}([-a,a] \times [t_3+2, \infty))} \leq C_2. \quad (5.5)$$

For any positive c , we now claim that x_u decays to zero uniformly on $[-c, c]$ as t tends to ∞ . For if not, there is some positive C_3 such that

$$x_u(u_k, t_k) \leq -C_3,$$

where $u_k \in [-c, c]$ and $t_k \rightarrow \infty$. Let $\varphi_k(u, t) = x_u(u + u_k, t + t_k)$. By the above estimate (5.5), we can extract a diagonal subsequence of $\{\varphi_k\}$ which C^2 -converges compactly to a solution φ of the equation

$$\varphi_t = \left(\frac{\varphi_u}{1 + \varphi^2} \right)_u, \quad (u, t) \in (-\infty, \infty) \times (-\infty, \infty).$$

In particular, $\varphi(0, 0) = \lim_{k \rightarrow \infty} \varphi_k(0, 0) \leq -C_3$. However, on the other hand, noting that φ is uniformly bounded, we conclude from Liouville theorem (Moser [18]) that φ is a constant $\varphi(0, 0)$. As

$$1 > \left| \int_{-a}^a \varphi_k(u, t) du \right| \rightarrow \left| \int_{-a}^a \varphi(u, t) du \right| = 2a|\varphi(0, 0)|,$$

for any positive a as $k \rightarrow \infty$, one must have $\varphi(0, 0) = 0$, contradiction holds. Therefore x_u must tend to zero uniformly on each bounded interval of u . By combining with the higher order estimate (5.5), we conclude that x_{uu} also tends to zero on any bounded interval, so does the curvature of the graph of $x(u, t)$.

We have completed the proof of Proposition 1.3.

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