Mountain pass theorem in ordered Banach spaces and its applications to semilinear elliptic equations

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Abstract. In this paper, we establish a mountain pass theorem in an ordered Banach space, which is a Riesz-Banach space such that the absolute value is continuous. Applying our theorem to semilinear elliptic equations with sign-changing nonlinear terms, we prove the existence of a positive solution.

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1. Introduction

We establish a mountain pass theorem in ordered Banach spaces and apply it to the semilinear elliptic equation

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in order to prove the existence of a positive solution. Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The typical nonlinear term we study is $f(x, u) = a(x)u^p + \lambda g(x, u)$ or more general nonlinear term, where $a(x)$ and $g(x, u)$ may change their signs. To get a positive solution, we define

$$\begin{aligned} I(u) &:= \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - F(x, u) \right) dx, \\ F(x, u) &:= \int_0^u f(x, s) ds. \end{aligned}$$

A critical point of $I(\cdot)$ becomes a solution. The usual mountain pass theorem due to Ambrosetti and Rabinowitz [4] (see Rabinowitz [15] and Struwe [16] also) gives a non-trivial solution but our theorem presents a non-trivial non-negative solution. If $f(x, u)$ is odd with respect to u , the Nehari manifold method (see Willem [18] and Brown and Zhang [8]) is also valid for getting a positive solution. Indeed, we put

$$J(u) := I'(u)u = \int_{\Omega} (|\nabla u|^2 - f(x, u)u) dx,$$

$$N := \{u \in H_0^1(\Omega) \setminus \{0\} : J(u) = 0\}.$$

Here N is called a *Nehari manifold*. Under suitable conditions on $f(x, u)$, $I(u)$ on N is bounded from below and has a minimizer $u_0 \in N$, which becomes a critical point of $I(u)$. If $f(x, u)$ is odd with respect to u , $|u_0|$ also belongs to N and becomes a critical point. Thus one obtains a positive solution. If $f(x, u)$ is not odd, we cannot replace the minimizer u_0 by $|u_0|$. Especially, if $f(x, 0) \neq 0$, the above method is fail.

Another approach to get a positive solution is to use an invariant set of the descending flow. Liu and Sun [14] has applied this method to prove the existence of a positive solution and multiple sign-changing solutions. See also [1, 5–7, 9, 13, 17, 19] and their references.

The purpose of this paper is to prove the existence of a positive solution even when $f(x, u)$ is not odd or $f(x, 0) \neq 0$. To this end, we establish the mountain pass theorem in ordered Banach spaces. Our method in this paper is a little similar to [14] but our abstract theorem is very handy for applying the actual elliptic equation because our main assumption on $I(u)$ is

$$I(u^+) \leq I(u) \quad \text{or} \quad I(|u|) \leq I(u).$$

Here u^+ and $|u|$ denote the positive part and the absolute value of u , respectively, in the Riesz-Banach space, which will be defined in Sect. 2. It is easy to check that $I(u)$ for the actual elliptic equation satisfies these assumptions. The usual mountain pass theorem provides a non-trivial solution and then one proves that it becomes a positive solution with the help of the strong maximum principle. We emphasize that our theorem gives a non-trivial non-negative solution without the help of the weak maximum principle. After we get a non-negative solution, we use the strong maximum principle to prove the strict positivity of solution. The original mountain pass theorem is widely used for getting a non-trivial solution but our theorem is specialized to obtain a positive solution only. However, our theorem covers a wide class of second order elliptic equations.

To establish the result, we organize the paper into four sections. In Sect. 2, we define ordered Banach spaces, state the main theorems and prove them. In Sect. 3, we study a semilinear elliptic equation, which is solved neither by the usual mountain pass theorem nor by the Nehari manifold method. By applying our abstract theorem to it, we prove the existence of a positive solution. In Sect. 4, we study another semilinear elliptic equation.

2. Main results

In this section, we define ordered Banach spaces and state the main results.

Definition 2.1. E is called an *ordered Banach space* if the following conditions (i)–(iv) are all fulfilled:

- (i) $(E, \leq)$ is an ordered set.
- (ii) $(E, \|\cdot\|)$ is a real Banach space.
- (iii) Let $u, v, w \in E$. If $u \leq v$, then $u + w \leq v + w$. If $u \leq v$, then $\lambda u \leq \lambda v$ for any $\lambda \in [0, \infty)$.
- (iv) $E^+ := \{u \in E : u \geq 0\}$ is a closed subset of E .

E^+ is called a *positive cone*.

Definition 2.2. An ordered Banach space E is called a *Riesz-Banach space* if

$$u \vee v := \sup\{u, v\}, \quad u \wedge v := \inf\{u, v\}$$

exist for any $u, v \in E$ (see Aliprantis and Burkinshaw [3]). For a Riesz-Banach space E , we define

$$|u| := u \vee (-u), \quad u^+ := u \vee 0, \quad u^- := (-u) \vee 0.$$

Before going to the main result, we note that in a Riesz-Banach space, the continuities of the mappings $u \mapsto |u|$, $u \mapsto u^+$ and $u \mapsto u^-$ are equivalent to each other. Indeed, it is verified by the relations (see [3, Theorem 1.5])

$$|u| = u^+ + (-u)^+, \quad u^+ = \frac{1}{2}(u + |u|), \quad u^- = (-u)^+.$$

We shall construct our mountain pass theorem in a Riesz-Banach space in which the mapping $u \mapsto |u|$ is continuous. We give examples of such Riesz-Banach spaces.

Example 2.3. The Lebesgue space $L^p(\Omega)$ and the Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are ordered Banach spaces, where we define the order $u \leq v$ if $u(x) \leq v(x)$ a.e. $x \in \Omega$. The $L^p(\Omega)$ and the first order Sobolev spaces $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ are Riesz-Banach spaces. Indeed, if $u \in W^{1,p}(\Omega)$, then $|u| \in W^{1,p}(\Omega)$. For the proof, we refer the readers to Gilbarg and Trudinger [10, Lemma 7.6] or Heinonen, Kilpeläinen and Martio [11, Theorem 1.20]. Therefore for $u, v \in W^{1,p}(\Omega)$, we have

$$\begin{aligned} u \vee v &= \frac{1}{2}(u + v) + \frac{1}{2}|u - v| \in W^{1,p}(\Omega), \\ u \wedge v &= \frac{1}{2}(u + v) - \frac{1}{2}|u - v| \in W^{1,p}(\Omega). \end{aligned}$$

Thus the first order Sobolev spaces become Riesz-Banach spaces. Moreover we have

$$\nabla|u| = \begin{cases} \nabla u & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) = 0, \\ -\nabla u & \text{if } u(x) < 0, \end{cases}$$

$$\|\nabla|u|\|_p = \|\nabla u\|_p \quad \text{for } u \in W^{1,p}(\Omega), \quad (2.1)$$

where $\|u\|_p$ denotes the $L^p(\Omega)$ -norm. Furthermore, the mapping $u \mapsto |u|$ is continuous in $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$ when $1 \leq p < \infty$, that is, if $u_n \rightarrow u$ in $W^{1,p}(\Omega)$ (or $W_0^{1,p}(\Omega)$), then $|u_n| \rightarrow |u|$ in $W^{1,p}(\Omega)$ (or $W_0^{1,p}(\Omega)$). For the proof, see [11, Lemma 1.22]. In $W^{1,\infty}(\Omega)$, such a mapping is not continuous. Indeed, let $N = 1$, $\Omega = (-1, 1)$ and put $u(x) = x$, $u_n(x) = x + 1/n$. Then u_n converges to u in $W^{1,\infty}(-1, 1)$. It is easy to see that $(d/dx)|u| = 1$ for $x > 0$, $(d/dx)|u| = -1$ for $x < 0$, $(d/dx)|u_n| = 1$ for $x > -1/n$ and $(d/dx)|u_n| = -1$ for $x < -1/n$. Therefore

$$(d/dx)|u_n| - (d/dx)|u| = 2 \quad \text{for } -1/n < x < 0,$$

and hence $\|(d/dx)|u_n| - (d/dx)|u|\|_\infty = 2$. Thus the mapping $u \mapsto |u|$ is not continuous in $W^{1,\infty}(-1, 1)$.

The second or more order Sobolev spaces are not Riesz-Banach spaces. To verify it, consider $N = 1$, $\Omega = (-1, 1)$ and put $u(x) = x$. Then the distributional derivative of $|u|$ is computed as

$$\frac{d}{dx}|u(x)| = 2H(x) - 1, \quad \frac{d^2}{dx^2}|u(x)| = 2\delta,$$

where $H(x)$ is the Heaviside function, i.e., $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$ and δ is the Dirac distribution. Therefore $|u| \notin W^{2,p}(-1, 1)$.

We show other examples of Riesz-Banach spaces except for Sobolev spaces. Define l^p with $1 \leq p \leq \infty$ by

$$l^p = \left\{ \xi = (\xi_k)_{k=1}^\infty : \sum_{k=1}^\infty |\xi_k|^p < \infty \right\} \quad (1 \leq p < \infty),$$

$$l^\infty = \{ \xi = (\xi_k)_{k=1}^\infty : \sup_k |\xi_k| < \infty \}.$$

For $\xi = (\xi_k), \eta = (\eta_k) \in l^p$, the order $\xi \leq \eta$ is defined by $\xi_k \leq \eta_k$ for all k . Then l^p is a Riesz-Banach space and the mapping $\xi \mapsto |\xi|$ is continuous.

Let $C(\overline{\Omega})$, $C^\theta(\overline{\Omega})$ and $Lip(\overline{\Omega})$ be the set of all functions on $\overline{\Omega}$ which are continuous, Hölder continuous with exponent θ and Lipschitz continuous, respectively. Here Ω is bounded open in $\mathbb{R}^N$. These are Riesz-Banach spaces. Indeed, using the inequality

$$|\max(t, s) - \max(t', s')| \leq |t - t'| + |s - s'|,$$

we see that for $u, v \in C^\theta(\overline{\Omega})$ (or $Lip(\overline{\Omega})$),

$$u \vee v = \max(u(x), v(x)) \in C^\theta(\overline{\Omega}) \quad (\text{or } Lip(\overline{\Omega})).$$

The mapping $u \mapsto |u|$ is continuous in $C(\overline{\Omega})$, which follows from

$$\||u| - |v|\|_{C(\overline{\Omega})} \leq \|u - v\|_{C(\overline{\Omega})}.$$

Since for $N = 1$, $Lip(-1, 1) = W^{1,\infty}(-1, 1)$ and these norms are equal to each other, the example already stated in $W^{1,\infty}(-1, 1)$ is still valid and the mapping $u \mapsto |u|$ is not continuous in this space.

In $C^\theta(\overline{\Omega})$, the mapping is not continuous. Indeed, let $N = 1$, $\Omega = (-1, 1)$, $u(x) = |x|^\theta$ for $x \geq 0$, $u(x) = -|x|^\theta$ for $x \leq 0$, and $u_n(x) = u(x) + 1/n$. Then

u_n converges to u in $C^\theta[-1, 1]$ as $n \rightarrow \infty$. Putting $x = -(1/n)^{1/\theta}$, $y = 0$, we have

$$\| |u_n| - |u| \|_{C^\theta} \geq \frac{(|u_n(x)| - |u(x)|) - (|u_n(y)| - |u(y)|)}{|x - y|^\theta} = 2.$$

Thus the mapping $u \mapsto |u|$ is not continuous.

Now, we state our mountain pass theorem. Let $I(u)$ be a real valued C^1 functional on a Riesz-Banach space E in which the mapping $u \mapsto |u|$ is continuous. Let $e_0, e_1 \in E^+$. Then we define

$$c := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad (2.2)$$

$$\Gamma := \{\gamma \in C([0, 1], E^+) : \gamma(0) = e_0, \gamma(1) = e_1\}. \quad (2.3)$$

Since E^+ is convex by Definition 2.1, Γ is non-empty. For $\varepsilon > 0$, we set the notations

$$E_\varepsilon^+ := \{u \in E : \text{dist}(u, E^+) \leq \varepsilon\},$$

$$\text{dist}(u, E^+) := \inf\{\|u - v\| : v \in E^+\}.$$

Assumption 2.4. We assume that there exist points $e_0, e_1 \in E^+$ and an open subset U of E satisfying the following conditions:

- (I₁) $e_0 \in E^+ \cap U$, $e_1 \in E^+ \setminus \bar{U}$;
- (I₂) $\max(I(e_0), I(e_1)) < \inf_{\partial U \cap E^+} I(u)$;
- (I₃) if $\{u_n\} \subset E$, $\text{dist}(u_n, E^+)$ converges to zero, $I(u_n)$ converges to c given by (2.2) and $I'(u_n)$ converges to zero, then $\{u_n\}$ has a convergent subsequence;
- (I₄) there exist constants $a \in (-\infty, c)$ and $\varepsilon > 0$ satisfying either (i) or (ii):
 - (i) $I(|u|) \leq I(u)$ if $a < I(u) < c$ and $u \in E_\varepsilon^+$;
 - (ii) $I(u^+) \leq I(u)$ if $a < I(u) < c$ and $u \in E_\varepsilon^+$.

Theorem 2.5. Let E be a Riesz-Banach space such that the mapping $u \mapsto |u|$ is continuous. Let $I \in C^1(E, \mathbb{R})$ satisfy Assumption 2.4. Then there exists a critical point $u \in E^+$ such that $I'(u) = 0$ and $I(u) = c$.

As stated in Example 2.3, the first order Sobolev space $W_0^{1,p}(\Omega)$ with $1 \leq p < \infty$ is a Riesz-Banach space such that the absolute value is continuous. Hence Theorem 2.5 works well in this space and therefore it is applicable to the second order elliptic equations only. However, our theorem covers many types of second order elliptic equations with general nonlinear terms, which will be seen in the later sections.

We say that I satisfies the *Palais-Smale condition at level c* ((PS) _{c} condition for short) if any sequence $\{u_n\}$ in E such that $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence. $I(u)$ is said to satisfy the *Palais-Smale condition* if it satisfies (PS) _{c} for all $c \in \mathbb{R}$. It is clear that (PS) _{c} implies (I₃).

Corollary 2.6. Let E be as in Theorem 2.5 and let $I \in C^1(E, \mathbb{R})$ satisfy (I₁), (I₂), (PS) _{c} and either $I(|u|) \leq I(u)$ for all $u \in E$ or $I(u^+) \leq I(u)$ for all $u \in E$. Then the same conclusion as in Theorem 2.5 holds.

To prove Theorem 2.5, we need the next lemma, which is called a deformation lemma. It can be proved in the same way as in [15, Theorem A.4] or [16, Theorem 3.4].

Lemma 2.7. *Suppose that there exist constants $c \in \mathbb{R}$, $\varepsilon_0, r > 0$ such that if $\text{dist}(u, E^+) \leq \varepsilon_0$ and $|I(u) - c| \leq \varepsilon_0$, then $\|I'(u)\| \geq r$. Then for any $\varepsilon > 0$ and any $b \in (-\infty, c)$, there exist $\delta \in (0, c - b)$ and $\eta \in C([0, 1] \times E, E)$ such that*

- (i) $\eta(0, u) = u$ for $u \in E$,
- (ii) $\|\eta(t, u) - u\| \leq \varepsilon$ for $(t, u) \in [0, 1] \times E$,
- (iii) $I(\eta(t, u)) \leq I(u)$ for $t \in [0, 1]$ and $u \in E$,
- (iv) if $I(u) \leq b$, then $\eta(t, u) = u$ for $t \in [0, 1]$,
- (v) if $u \in E^+$ and $b < I(u) \leq c + \delta$, then $b < I(\eta(1, u)) \leq c - \delta$.

Proof of Theorem 2.5. Suppose on the contrary that there are no points $u \in E^+$ such that $I'(u) = 0$ and $I(u) = c$. Then we claim the existence of constants $r, \varepsilon_0 > 0$ such that if $\text{dist}(u, E^+) < \varepsilon_0$ and $|I(u) - c| < \varepsilon_0$, then $\|I'(u)\| \geq r$. Indeed, if such constants r, ε_0 do not exist, there exists a sequence $\{u_n\}$ such that $\text{dist}(u_n, E^+) \rightarrow 0$, $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$. By (I₃), a subsequence of $\{u_n\}$ converges to a limit u . Hence $u \in E^+$, $I(u) = c$ and $I'(u) = 0$. This contradicts the first assumption of the proof. Thus the constants $r, \varepsilon_0 > 0$ exist. By (I₂) with the definition of c , we see

$$\max(I(e_0), I(e_1)) < \inf_{\partial U \cap E^+} I(u) \leq c.$$

Let a be as in (I₄). Then we choose a constant $b > 0$ such that

$$\max(I(e_0), I(e_1), a) < b < c.$$

For $\varepsilon > 0$ in (I₄) and b as above, we use Lemma 2.7 to get $\delta \in (0, c - b)$ and $\eta \in C([0, 1] \times E, E)$ which satisfy the assertions (i)–(v) of Lemma 2.7. From the definition of c , we choose a $\gamma \in \Gamma$ such that

$$c \leq \max_{0 \leq t \leq 1} I(\gamma(t)) < c + \delta. \quad (2.4)$$

Since $I(e_i) < b$ for $i = 0, 1$, the assertion (iv) of Lemma 2.7 means

$$\eta(1, \gamma(0)) = e_0, \quad \eta(1, \gamma(1)) = e_1.$$

Recall that $\gamma(t) \in E^+$ for $0 \leq t \leq 1$ because $\gamma \in \Gamma$. We first deal with the case (i) of (I₄). Put $\tilde{\gamma}(t) := |\eta(1, \gamma(t))|$. Since the mapping $u \mapsto |u|$ is continuous, so is $\tilde{\gamma}(t)$. Furthermore, $\tilde{\gamma}(t) \in E^+$, $\tilde{\gamma}(0) = e_0$ and $\tilde{\gamma}(1) = e_1$. Thus $\tilde{\gamma} \in \Gamma$. For t satisfying $I(\gamma(t)) \leq b$, it follows from (iv) of Lemma 2.7 that $\eta(1, \gamma(t)) = \gamma(t) \in E^+$, and hence $\tilde{\gamma}(t) = \gamma(t)$. Accordingly, if $I(\gamma(t)) \leq b$, then

$$I(\tilde{\gamma}(t)) \leq b < c - \delta. \quad (2.5)$$

For t satisfying $I(\gamma(t)) > b$, by (v) of Lemma 2.7 with (2.4), we have

$$b < I(\eta(1, \gamma(t))) \leq c - \delta. \quad (2.6)$$

Moreover, we use (ii) of Lemma 2.7 with $\gamma(t) \in E^+$ to get $\eta(1, \gamma(t)) \in E_\varepsilon^+$. By (i) of (I₄) with (2.6), we obtain $I(\tilde{\gamma}(t)) \leq I(\eta(1, \gamma(t)))$. Consequently, if $I(\gamma(t)) > b$, then

$$I(\tilde{\gamma}(t)) \leq I(\eta(1, \gamma(t))) \leq c - \delta. \quad (2.7)$$

By (2.5) and (2.7), we obtain

$$\max_{0 \leq t \leq 1} I(\tilde{\gamma}(t)) \leq c - \delta.$$

This contradicts the definition of c . Consequently, there is a $u \in E^+$ such that $I'(u) = 0$ and $I(u) = c$. In the case (ii) of (I₄), we define $\tilde{\gamma}(t) := \eta(1, \gamma(t))^+$. Then the argument above remains valid and the proof is complete. $\square$

3. First application

In this section, we apply Corollary 2.6 to the semilinear elliptic equation

$$\begin{cases} -\Delta u = a(x)u^p + \lambda g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

to get a positive solution, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Similar problems have been studied by Alama and Tarantello [2] and Li and Wang [12]. However the assumptions below are more general. We suppose

(g0) $g \in C(\overline{\Omega} \times [0, \infty), \mathbb{R})$ and $g(x, 0) \geq 0$ in Ω .

To ensure the Palais-Smale condition, we assume either (g1) or (g2) below:

(g1) There exist constants $q, C > 0$ and $\theta \in [0, 2]$ such that $1 < q < \infty$ if $N = 1, 2$ and $1 < q < (N + 2)/(N - 2)$ if $N \geq 3$ and

$$|g(x, s)| \leq C(|s|^q + 1), \quad (3.2)$$

$$(p + 1)G(x, s) - sg(x, s) \leq C|s|^\theta + C, \quad (3.3)$$

for $s \geq 0$ and $x \in \Omega$, where p is the exponent appeared in (3.1) and G is given by

$$G(x, s) := \int_0^s g(x, t)dt \quad \text{for } s \geq 0. \quad (3.4)$$

(g2) Suppose that $a(x) \geq a_0 > 0$ in Ω with some $a_0 > 0$ and

$$s^{-p}|g(x, s)| \rightarrow 0 \quad \text{uniformly on } \Omega \quad \text{as } s \rightarrow \infty. \quad (3.5)$$

Theorem 3.1. *Let $1 < p < \infty$ if $N = 1, 2$ and $1 < p < (N + 2)/(N - 2)$ if $N \geq 3$. Suppose that $a \in C(\overline{\Omega}, \mathbb{R})$, $a(x_0) > 0$ at some $x_0 \in \Omega$ and $g(x, s)$ satisfies (g0) and either (g1) or (g2). Then there is a $\lambda_0 > 0$ such that (3.1) has a positive solution for $\lambda \in [0, \lambda_0)$.*

Remark 3.2. In general, (3.1) has no positive solutions if $\lambda > 0$ is large. Indeed, let $a \equiv g \equiv 1$ and consider

$$-\Delta u = u^p + \lambda, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.6)$$

Let λ_1 be the first eigenvalue of $-\Delta$ and ϕ be the corresponding eigenfunction such that $\|\phi\|_1 = 1$ and

$$-\Delta\phi = \lambda_1\phi, \quad \phi > 0 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega.$$

Taking the L^2 -inner product of the first equations in (3.6) with ϕ and using Jensen's inequality, we have

$$\lambda_1(u, \phi) \geq (u, \phi)^p + \lambda, \quad (3.7)$$

where (u, ϕ) denotes the L^2 -inner product. Denote the maximum of the function $\lambda_1 t - t^p$ over $t \geq 0$ by L . By (3.7), if $\lambda > L$, then (3.6) has no positive solutions. Therefore the restriction $\lambda \in [0, \lambda_0)$ in Theorem 3.1 cannot be removed.

Example 3.3. We give examples of $g(x, s)$ satisfying the assumptions of Theorem 3.1. In the following, we assume $b, c, d \in C(\bar{\Omega})$ such that $b(x)$ may change its sign but $c(x), d(x) \geq 0$ in Ω . Let $g(x, s) = b(x)s^q + c(x)$ with $0 < q < p$, which satisfies (g0) and (g2). Let

$$g(x, s) = b(x)s^p + c(x)s^q + d(x),$$

with $1 < p < q < (N+2)/(N-2)$. Since $c(x) \geq 0$ and $q > p$, we can easily verify (3.3). Thus $g(x, s)$ satisfies (g0) and (g1).

In both cases of (g1) and (g2), $g(x, s)$ may change its sign and in case of (g1), $a(x)$ also may change its sign. Moreover, we allow the case $g(x, 0) > 0$. Therefore the usual mountain pass theorem and the Nehari manifold method are not applicable to this problem. Indeed, put $f(x, s) = a(x)s^p + \lambda g(x, s)$. If $f(x, 0) = 0$ and f is locally Lipschitz continuous, we replace $f(x, s)$ by

$$\tilde{f}(x, s) = \begin{cases} f(x, s) & \text{for } s \geq 0, \\ 0 & \text{for } s < 0, \end{cases}$$

and define

$$\begin{aligned} \tilde{I}(u) &:= \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \tilde{F}(x, u) \right) dx, \\ \tilde{F}(x, u) &:= \int_0^u \tilde{f}(x, s) ds. \end{aligned}$$

If we could obtain a non-trivial critical point of $\tilde{I}$, it becomes a positive solution of (3.1) by the strong maximum principle. However, we assume neither $f(x, 0) = 0$ nor the Lipschitz continuity of $f(x, s)$. Therefore we cannot apply the trick above. To get a positive solution, we employ Corollary 2.6. Recall that $g(x, s)$ is defined for $s \geq 0$ only. We extend $g(x, s)$ for all $s \in \mathbb{R}$ in the following lemma.

Lemma 3.4. *If $g(x, s)$ satisfies the assumptions of Theorem 3.1, then $g(x, s)$ has an extension on $\Omega \times \mathbb{R}$ which fulfills the same assumptions for $s \in \mathbb{R}$ and satisfies*

$$G(x, s) \leq G(x, |s|) \quad \text{for } x \in \Omega, \quad s \in \mathbb{R}, \quad (3.8)$$

where $G(x, s)$ is defined by (3.4) for all $s \in \mathbb{R}$.

Proof. Put $h(x, s) := g(x, s) - g(x, 0)$ for $s \geq 0$. Then $h(x, 0) = 0$, and hence $h(x, s)$ is extended on $\Omega \times \mathbb{R}$ as an odd function with respect to s . Then we define

$$g(x, s) := h(x, s) + g(x, 0) \quad \text{for } s \in \mathbb{R}. \quad (3.9)$$

Let us show that $g(x, s)$ satisfies the assumptions of Theorem 3.1 for $s \leq 0$ and fulfills (3.8). Put

$$H(x, s) := \int_0^s h(x, t) dt.$$

Integrating (3.9), we have

$$G(x, s) = H(x, s) + g(x, 0)s \quad \text{for } s \in \mathbb{R}. \quad (3.10)$$

Since $h(x, s)$ is odd, $H(x, s)$ is even with respect to s . Since $g(x, 0) \geq 0$ by (g0), we get for $s \geq 0$,

$$G(x, -s) = H(x, -s) - g(x, 0)s \leq H(x, s) + g(x, 0)s = G(x, s).$$

Thus (3.8) holds. Let g satisfy (g1). Substituting (3.9) and (3.10) into (3.3), we have

$$(p+1)H(x, s) \leq sh(x, s) - psg(x, 0) + C|s|^\theta + C \quad \text{for } s \geq 0.$$

Since $H(x, s)$ and $sh(x, s)$ are even with respect to s , the above inequality is equivalent to

$$(p+1)H(x, -s) \leq -sh(x, -s) - psg(x, 0) + C|s|^\theta + C \quad \text{for } s \geq 0.$$

Again, substituting (3.9) and (3.10) into the relation above, we find

$$(p+1)G(x, -s) \leq -sg(x, -s) - 2psg(x, 0) + C|s|^\theta + C,$$

which is estimated as

$$(p+1)G(x, -s) \leq -sg(x, -s) + C'|s| + C|s|^\theta + C, \quad \text{for } s \geq 0,$$

with some $C' > 0$. Thus (3.3) holds for $s \leq 0$. In the same way as above, we see that (3.2) holds for $s \leq 0$ after replacing C by a larger constant. It is clear that if $g(x, s)$ satisfies (g2) as $s \rightarrow \infty$, then it does (g2) as $s \rightarrow -\infty$. $\square$

We define a C^1 functional $I(u)$ on $H_0^1(\Omega)$ by

$$I(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} a(x) |u|^{p+1} - \lambda G(x, u) \right) dx. \quad (3.11)$$

Lemma 3.5. *If g satisfies either (g1) with $\theta < 2$ or (g2), then for any $\lambda > 0$, $I(u)$ satisfies the Palais-Smale condition. If g satisfies (g1) with $\theta = 2$, then $I(u)$ with $\lambda > 0$ small enough satisfies the Palais-Smale condition.*

Proof. The Fréchet derivative of $I(u)$ is calculated as

$$I'(u)v = \int_{\Omega} (\nabla u \nabla v - a(x) |u|^{p-1} uv - \lambda g(x, u)v) dx, \quad (3.12)$$

for $v \in H_0^1(\Omega)$. Let u_n be any sequence in $H_0^1(\Omega)$ such that $I(u_n)$ is bounded and $I'(u_n)$ converges to zero. First, we consider the case where g satisfies (g1). Using (3.11) and (3.12), we get

$$(p+1)I(u_n) - I'(u_n)u_n = \int_{\Omega} \left(\frac{p-1}{2} |\nabla u_n|^2 - \lambda(p+1)G(x, u_n) + \lambda g(x, u_n)u_n \right) dx. \quad (3.13)$$

We define the $H_0^1(\Omega)$ -norm by $\|\nabla u\|_2$ because of Poincaré's inequality. Since $I(u_n)$ is bounded and $I'(u_n)$ converges to zero, we have a $C_1 > 0$ such that

$$|(p+1)I(u_n) - I'(u_n)u_n| \leq C_1 \|\nabla u_n\|_2 + C_1. \quad (3.14)$$

Combining (3.13), (3.14) and (3.3), we obtain

$$\int_{\Omega} \left(\frac{p-1}{2} |\nabla u_n|^2 - \lambda C |u_n|^\theta - \lambda C \right) dx \leq C_1 \|\nabla u_n\|_2 + C_1. \quad (3.15)$$

Using the Sobolev embedding, we get

$$\frac{p-1}{2} \|\nabla u_n\|_2^2 \leq \lambda C_2 \|\nabla u_n\|_2^\theta + \lambda C |\Omega| + C_1 \|\nabla u_n\|_2 + C_1,$$

with a constant $C_2 > 0$, where $|\Omega|$ denotes the volume of Ω . If either $\theta < 2$ or $\theta = 2$ with $\lambda > 0$ small enough, the inequality above shows that $\|\nabla u_n\|_2$ is bounded. Extract a subsequence from $\{u_n\}$ (again denoted by $\{u_n\}$) which weakly converges to a limit $u_0 \in H_0^1(\Omega)$. By the compact embedding, u_n converges to u_0 strongly in $L^r(\Omega)$ for any $r \in [1, \infty)$ if $N = 1, 2$ and $r \in [1, 2N/(N-2))$ if $N \geq 3$, and it converges a.e. in Ω . Using (3.2) (see [15, Proposition B.1]), we can prove

$$\int_{\Omega} g(x, u_n)u_n dx \rightarrow \int_{\Omega} g(x, u_0)u_0 dx, \quad (3.16)$$

$$\int_{\Omega} g(x, u_n)v dx \rightarrow \int_{\Omega} g(x, u_0)v dx, \quad (3.17)$$

as $n \rightarrow \infty$ for any $v \in H_0^1(\Omega)$. Substituting $u = u_n$, $v = u_0$ into (3.12) and letting $n \rightarrow \infty$, we get

$$\int_{\Omega} |\nabla u_0|^2 dx = \int_{\Omega} (a|u_0|^{p+1} + \lambda g(x, u_0)u_0) dx. \quad (3.18)$$

On the other hand, we put $u = v = u_n$ into (3.12) to get

$$\int_{\Omega} |\nabla u_n|^2 dx = I'(u_n)u_n + \int_{\Omega} (a|u_n|^{p+1} + \lambda g(x, u_n)u_n) dx. \quad (3.19)$$

By (3.16), the right-hand side of (3.19) converges to that of (3.18). Thus $\|\nabla u_n\|_2$ converges to $\|\nabla u_0\|_2$. Consequently, $\{u_n\}$ strongly converges to u_0 in $H_0^1(\Omega)$.

Next, we deal with the case where $g(x, s)$ satisfies (g2). Fix $\lambda > 0$ arbitrarily. For any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ by (g2) such that

$$\begin{aligned} \lambda |sg(x, s)| &\leq \varepsilon a(x)|s|^{p+1} + C_\varepsilon, \\ \lambda |G(x, s)| &\leq \varepsilon a(x)|s|^{p+1} + C_\varepsilon, \end{aligned}$$

for $(x, s) \in \Omega \times \mathbb{R}$. Then (3.11) and (3.12) with $u = v = u_n$ are estimated as

$$\begin{aligned} I(u_n) &\geq \int_{\Omega} \left(\frac{1}{2} |\nabla u_n|^2 - ((p+1)^{-1} + \varepsilon) a(x) |u_n|^{p+1} - C_{\varepsilon} \right) dx, \\ I'(u_n) u_n &\leq \int_{\Omega} (|\nabla u_n|^2 - (1 - \varepsilon) a(x) |u_n|^{p+1} + C_{\varepsilon}) dx. \end{aligned}$$

Fix $\varepsilon > 0$ so small that $r := (1 - \varepsilon)/((p+1)^{-1} + \varepsilon) > 2$. Then we have

$$rI(u_n) - I'(u_n)u_n \geq \int_{\Omega} \left(\frac{r-2}{2} |\nabla u_n|^2 - (r+1)C_{\varepsilon} \right) dx.$$

Since $I(u_n)$ and $I'(u_n)$ are bounded, there is a $C_1 > 0$ such that

$$\frac{r-2}{2} \|\nabla u_n\|_2^2 \leq C_1 \|\nabla u_n\|_2 + C_1.$$

Thus $\{u_n\}$ is bounded in $H_0^1(\Omega)$. In the same method as in the case of (g1), we can prove that $\{u_n\}$ has a subsequence converging strongly in $H_0^1(\Omega)$. $\square$

Proof of Theorem 3.1. For $r \in [1, \infty)$ if $N = 1, 2$, and for $r \in [1, 2N/(N-2)]$ if $N \geq 3$, we define a Sobolev embedding constant A_r by

$$\|u\|_r \leq A_r \|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega). \quad (3.20)$$

By (3.2) or (3.5), there exist constants $C_0 > 0$ and $q > 1$ such that $1 < q < (N+2)/(N-2)$ if $N \geq 3$ and

$$|G(x, s)| \leq C_0(|s|^{q+1} + 1) \quad \text{for } (x, s) \in \Omega \times \mathbb{R}. \quad (3.21)$$

In case of (g2), q is equal to p . Put

$$C_1 := (p+1)^{-1} \|a\|_{\infty} A_{p+1}^{p+1}, \quad C_2 := C_0 A_{q+1}^{q+1}. \quad (3.22)$$

Fix $\rho > 0$ so small that

$$C_1 \rho^{p+1} + C_2 \rho^{q+1} \leq \rho^2/4. \quad (3.23)$$

Next, we define $\lambda_1 > 0$ by

$$\lambda_1 C_0 |\Omega| = \rho^2/8 \quad (3.24)$$

Since $a(x_0) > 0$, we choose a $\delta > 0$ so small that $a(x) > 0$ in the ball $B(x_0, \delta)$ centered at x_0 with radius δ . Fix $\phi \in C_0^{\infty}(\Omega)$ such that $\phi \geq 0$, $\phi \not\equiv 0$ and the support of ϕ is in $B(x_0, \delta)$. Then we fix $r > 0$ so large that $r \|\nabla \phi\|_2 > \rho$ and

$$r^2 \|\nabla \phi\|_2^2 < \frac{r^{p+1}}{p+1} \int_{B(x_0, \delta)} a(x) |\phi|^{p+1} dx. \quad (3.25)$$

We define $\lambda_2 > 0$ so small that

$$\frac{r^{p+1}}{2(p+1)} \int_{B(x_0, \delta)} a(x) |\phi|^{p+1} dx + \lambda \int_{B(x_0, \delta)} G(x, r\phi) dx > 0, \quad (3.26)$$

for $\lambda \in [0, \lambda_2]$. If g satisfies (g1) with $\theta = 2$, we define $\lambda_3 > 0$ by Lemma 3.5 such that $I(u)$ with $\lambda \in [0, \lambda_3]$ satisfies the Palais-Smale condition. If g satisfies either (g1) with $\theta < 2$ or (g2), then we put $\lambda_3 = \infty$. Last, we define $\lambda_0 = \min(\lambda_1, \lambda_2, \lambda_3, 1)$.

We put $E := H_0^1(\Omega)$ and fix $\lambda \in [0, \lambda_0]$. We shall verify that $I(u)$ satisfies the assumptions of Corollary 2.6. By (2.1) and (3.8), $I(|u|) \leq I(u)$ for all $u \in E$. It remains to verify (I₁) and (I₂). For ρ given by (3.23), we define

$$U := \{u \in H_0^1(\Omega) : \|\nabla u\|_2 < \rho\}. \quad (3.27)$$

Using (3.20)–(3.24) and noting $\lambda \leq 1$, we estimate $I(u)$ as

$$\begin{aligned} I(u) &\geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p+1} \|a\|_\infty \|u\|_{p+1}^{p+1} - \lambda C_0 (\|u\|_{q+1}^{q+1} + |\Omega|) \\ &\geq \frac{1}{2} \|\nabla u\|_2^2 - C_1 \|\nabla u\|_2^{p+1} - C_2 \|\nabla u\|_2^{q+1} - \lambda C_0 |\Omega| \\ &\geq \rho^2/8, \end{aligned} \quad (3.28)$$

provided that $\|\nabla u\|_2 = \rho$. This shows that

$$\inf_{u \in \partial U} I(u) \geq \rho^2/8. \quad (3.29)$$

Put $e_0 = 0$ and $e_1 = r\phi$. Then $e_0 \in E^+ \cap U$ and $e_1 \in E^+ \setminus \overline{U}$. By (3.25) and (3.26), we have

$$\begin{aligned} I(e_1) &= \frac{r^2}{2} \|\nabla \phi\|_2^2 - \frac{r^{p+1}}{p+1} \int_{B(x_0, \delta)} a|\phi|^{p+1} dx - \lambda \int_{B(x_0, \delta)} G(x, r\phi) dx \\ &\leq -\frac{r^{p+1}}{2(p+1)} \int_{B(x_0, \delta)} a|\phi|^{p+1} dx - \lambda \int_{B(x_0, \delta)} G(x, r\phi) dx < 0. \end{aligned}$$

Consequently, all the assumptions of Corollary 2.6 are fulfilled. Therefore there is a $u \in E^+$ such that $I'(u) = 0$ and $I(u) \geq \rho^2/8$. Thus u is a non-trivial non-negative weak solution of (3.1). The elliptic regularity theorem with (3.2) or (3.5) guarantees that $u \in W^{2,q}(\Omega)$ for any $q \in [1, \infty)$, especially $u \in C^1(\overline{\Omega})$.

We shall show that u is strictly positive when $\lambda > 0$ is small enough. To make a strict discussion, we rewrite the critical value c , the solution u and $I(u)$ by c_λ , u_λ and $I_\lambda(u)$, respectively. We first prove that there is a $C > 0$ independent of λ such that

$$1/C \leq c_\lambda \leq C \quad \text{for } \lambda \in [0, \lambda_0]. \quad (3.30)$$

Indeed, by (3.29), we have the lower estimate $c_\lambda \geq \rho^2/8$. To get the upper bound, we put $\gamma(t) := te_1$ for $0 \leq t \leq 1$. Then $\gamma \in \Gamma$ and we have for $t \in [0, 1]$,

$$\begin{aligned} I_\lambda(te_1) &= \frac{t^2}{2} \|\nabla e_1\|_2^2 - \frac{t^{p+1}}{p+1} \int_{B(x_0, \delta)} a|e_1|^{p+1} dx - \lambda \int_{B(x_0, \delta)} G(x, te_1) dx \\ &\leq \frac{1}{2} \|\nabla e_1\|_2^2 + \lambda_0 \sup_{0 \leq t \leq 1} \int_{B(x_0, \delta)} |G(x, te_1)| dx. \end{aligned}$$

Denote the right-hand side by C . Then we get

$$c_\lambda \leq \max_{0 \leq t \leq 1} I_\lambda(te_1) \leq C.$$

Thus (3.30) holds. Since $I_\lambda(u_\lambda) = c_\lambda$ is bounded for $\lambda \in [0, \lambda_0]$ and $I'_\lambda(u_\lambda) = 0$, we use the same argument as in Lemma 3.5 to prove that $\|\nabla u_\lambda\|_2$ is bounded

for $\lambda \in [0, \lambda_0]$. By the bootstrap argument, we have

$$\sup_{0 \leq \lambda \leq \lambda_0} \|u_\lambda\|_{2,q} < \infty \quad \text{for any } q \in [1, \infty),$$

where $\|u_\lambda\|_{2,q}$ denotes the $W^{2,q}(\Omega)$ -norm of u_λ . We claim that u_λ is strictly positive for $\lambda \in [0, \lambda'_0]$ with a small $\lambda'_0 > 0$. Suppose on the contrary that there exist sequences $\{\lambda_n\}$, $\{u_n\}$ and $\{x_n\}$ such that $\lambda_n \geq 0$, λ_n converges to 0, $u_n \in E^+$, u_n is a critical point corresponding to c_{λ_n} , $x_n \in \Omega$ and $u_n(x_n) = 0$. Since the $W^{2,q}(\Omega)$ -norm of u_n is bounded, there is a subsequence (again denoted by $\{u_n\}$) of $\{u_n\}$ which converges to a limit u_0 weakly in $W^{2,q}(\Omega)$ for all $q \in [1, \infty)$. By the compact embedding, u_n converges to u_0 strongly in $C^1(\bar{\Omega})$. Since λ_n converges to 0, u_0 satisfies

$$-\Delta u_0 = a(x)u_0^p \quad \text{in } \Omega, \quad u_0 = 0 \quad \text{on } \partial\Omega. \quad (3.31)$$

Letting $n \rightarrow \infty$ in the lower estimate of (3.30) with $\lambda = \lambda_n$, we have

$$\int_{\Omega} \left(\frac{1}{2} |\nabla u_0|^2 - \frac{1}{p+1} a(x) |u_0|^{p+1} \right) dx \geq 1/C.$$

Thus $u_0 \not\equiv 0$, and hence u_0 is a non-trivial non-negative solution of (3.31). Putting $M = \|au_0^{p-1}\|_{\infty}$ and adding (3.31) to Mu_0 , we have

$$(M - \Delta)u_0 \geq (M - \|au_0^{p-1}\|_{\infty})u_0 = 0.$$

By Hopf's maximum principle,

$$u_0 > 0 \text{ in } \Omega, \quad \partial u_0 / \partial \nu < 0 \text{ on } \partial\Omega. \quad (3.32)$$

Here $\partial u_0 / \partial \nu$ denotes the outward normal derivative of u_0 . Taking a subsequence again, if necessary, we may assume that x_n converges to a limit x_0 . If $x_0 \in \Omega$, then $u_0(x_0) = 0$. This contradicts (3.32). Suppose $x_0 \in \partial\Omega$. Note that x_n is sufficiently close to $\partial\Omega$ for n large enough. Choose $y_n \in \partial\Omega$ such that $\text{dist}(x_n, \partial\Omega) = |x_n - y_n|$. Then $(y_n - x_n)/|y_n - x_n|$ is the outward normal vector. Since $u_n(x_n) = u_n(y_n) = 0$, by Rolle's theorem there exists a point z_n on the line segment connecting x_n with y_n such that $\partial u_n(z_n) / \partial \nu = 0$. Since x_n and y_n converge to x_0 , so does z_n . Then it holds that $\partial u_0(x_0) / \partial \nu = 0$. This contradicts (3.32). Consequently, there is a $\lambda'_0 > 0$ such that u_λ is strictly positive in Ω for all $\lambda \in [0, \lambda'_0]$. The proof is complete. $\square$

4. Second application

We give another application of our theorem. Consider the equation

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

We assume (f1)–(f3) below:

- (f1) There exist constants $0 \leq \theta < 2$, $\mu > 2$, $C > 0$ and p such that $1 < p < \infty$ if $N = 1, 2$ and $1 < p < (N + 2)/(N - 2)$ if $N \geq 3$ and

$$|f(x, s)| \leq C(|s|^p + 1), \quad (4.2)$$

$$\mu F(x, s) - sf(x, s) \leq C|s|^\theta + C, \quad (4.3)$$

for $s \geq 0$ and $x \in \Omega$, where F is given by

$$F(x, s) := \int_0^s f(x, t) dt. \quad (4.4)$$

- (f2) There exist $x_0 \in \Omega$ and $\delta > 0$ such that

$$\min_{|x-x_0| \leq \delta} \frac{F(x, s)}{s^2} \rightarrow \infty \quad \text{as } s \rightarrow \infty. \quad (4.5)$$

- (f3) There exist $a(x) \in C(\overline{\Omega})$ and a constant $C > 0$ such that $\lambda_1(-\Delta - a)$ is positive and

$$-Cs \leq f(x, s) \leq a(x)s \quad \text{for } x \in \Omega, \quad (4.6)$$

if $s > 0$ is small enough. Here $\lambda_1(-\Delta - a)$ denotes the first eigenvalue of $-\Delta - a(x)$ with the Dirichlet boundary condition.

Theorem 4.1. *Let $f \in C(\overline{\Omega} \times [0, \infty), \mathbb{R})$ satisfy (f1)–(f3). Then (4.1) has a positive solution.*

- Remark 4.2.* (i) The superlinear condition on $f(x, s)$ near $x = x_0$, i.e., the divergence of $f(x, s)/s$ to ∞ uniformly on $B(x_0, \delta)$ as $s \rightarrow \infty$, is a sufficient condition for (f2).
(ii) If $f(x, s)/s$ converges to zero uniformly on Ω as $s \rightarrow 0+$, then (f3) clearly holds.

Example 4.3. An example of the nonlinear term $f(x, s)$ satisfying (f1)–(f3) is $f(x, s) = a(x)s^p - b(x)s^q$ with $1 < q < p < (N + 2)/(N - 2)$, where $a(x_0) > 0$ at some $x_0 \in \Omega$ and either $a(x) \geq 0$ in Ω or $b(x) \geq 0$ in Ω . Indeed, (f2), (f3) and (4.2) are fulfilled because of Remark 4.2. If $b \geq 0$ in Ω , then we put $\mu = p + 1$ to get

$$(p + 1)F(x, s) - sf(x, s) = -\frac{p - q}{q + 1}b(x)s^{q+1} \leq 0.$$

If $a \geq 0$ in Ω , then we put $\mu = q + 1$ to get

$$(q + 1)F(x, s) - sf(x, s) = -\frac{p - q}{p + 1}a(x)s^{p+1} \leq 0.$$

Thus (4.3) holds.

Proof of Theorem 4.1. Since $f(x, 0) = 0$ by (4.6), we extend $f(x, s)$ on $\Omega \times \mathbb{R}$ as an odd function with respect to s . Put $E := H_0^1(\Omega)$ and define

$$I(u) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - F(x, u) \right) dx.$$

We shall show that $I(u)$ satisfies the assumptions of Corollary 2.6. In the same discussion as in Lemma 3.5, we see that $I(u)$ satisfies the Palais-Smale condition. Since $F(x, u)$ is even with respect to u , it holds that $I(|u|) = I(u)$.

We have only to verify (I_1) and (I_2) . From the continuity of the eigenvalue, we take an $\varepsilon_0 > 0$ so small that $\lambda_1(-(1 - \varepsilon_0)\Delta - a) > 0$. This means that

$$\int_{\Omega} (|\nabla u|^2 - au^2) dx \geq \varepsilon_0 \|\nabla u\|_2^2 \quad \text{for } u \in H_0^1(\Omega), \quad (4.7)$$

By (f3), there exists an $s_0 > 0$ such that

$$f(x, s) \leq a(x)s \quad \text{for } 0 \leq s \leq s_0, \quad x \in \Omega.$$

Since $f(x, s)$ is odd in s , the inequality above shows

$$F(x, s) \leq (1/2)a(x)s^2 \quad \text{for } |s| \leq s_0,$$

which with (4.2) implies

$$F(x, s) \leq (1/2)a(x)s^2 + C|s|^{p+1} \quad \text{for } s \in \mathbb{R}.$$

Hereafter C denotes various positive constants. By (3.20) and (4.7), $I(u)$ is estimated as

$$\begin{aligned} I(u) &\geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2} \int_{\Omega} a(x)u^2 dx - C\|u\|_{p+1}^{p+1} \\ &\geq (\varepsilon_0/2) \|\nabla u\|_2^2 - CA_{p+1}^{p+1} \|\nabla u\|_2^{p+1}. \end{aligned}$$

Then we choose $\rho > 0$ so small that

$$I(u) \geq (\varepsilon_0/4) \|\nabla u\|_2^2 = \varepsilon_0 \rho^2/4, \quad (4.8)$$

provided that $\|\nabla u\|_2 = \rho$. Fix $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$, $\phi \not\equiv 0$ and the support of ϕ is in $B(x_0, \delta)$, where x_0 and δ have been given in (f2). Put $\alpha := \|\phi\|_\infty/2$ and define

$$D := \{x \in B(x_0, \delta) : \phi(x) > \alpha\}.$$

By (f2), there is an $S > 0$ such that $F(x, s) > 0$ for $x \in B(x_0, \delta)$ if $s \geq S$. For $r \in (S/\alpha, \infty)$, we put

$$\tilde{D} = \tilde{D}(r) := \{x \in B(x_0, \delta) : r\phi(x) > S\}.$$

Then $D \subset \tilde{D}$ and $F(x, r\phi(x)) > 0$ in $\tilde{D}$. Therefore we get

$$\begin{aligned} - \int_{B(x_0, \delta)} F(x, r\phi) dx &\leq - \int_{\tilde{D}} F(x, r\phi) dx + |\Omega| \max_{0 \leq s \leq S, x \in \bar{\Omega}} |F(x, s)| \\ &\leq - \int_D F(x, r\phi) dx + C. \end{aligned}$$

Hence $I(r\phi)$ is estimated as

$$\begin{aligned} I(r\phi) &= \frac{r^2}{2} \|\nabla \phi\|_2^2 - \int_{B(x_0, \delta)} F(x, r\phi) dx \\ &\leq r^2 \left(\frac{1}{2} \|\nabla \phi\|_2^2 - \alpha^2 \int_D \frac{F(x, r\phi)}{r^2 \phi^2} dx \right) + C. \end{aligned}$$

Since $\alpha \leq \phi(x) \leq \|\phi\|_\infty = 2\alpha$ in D , the integrand $F(x, r\phi)/r^2 \phi^2$ diverges to ∞ uniformly on x as $r \rightarrow \infty$. Therefore $I(r\phi) < 0$ for $r > 0$ large enough. For $\rho > 0$ satisfying (4.8), we define U by (3.27). Fix $r > 0$ so large that $I(r\phi) < 0$ and $r\|\nabla \phi\|_2 > \rho$. Putting $e_0 = 0$ and $e_1 = r\phi$, we obtain (I_1) and (I_2) . Then

Corollary 2.6 gives us a non-negative non-trivial solution u_0 . By (f3), we choose a constant $M > 0$ so large that

$$f(x, s) \geq -Ms \quad \text{for } s \in [0, \|u_0\|_\infty], \quad x \in \Omega.$$

Adding (4.1) with $u = u_0$ to Mu_0 , we get

$$(M - \Delta)u_0 = Mu_0 + f(x, u_0) \geq 0 \quad \text{in } \Omega.$$

By the strong maximum principle, $u_0(x)$ is strictly positive. $\square$

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