Nonlinear elliptic fourth order equations existence and multiplicity results

Mohammed Benalili and Kamel Tahri

Abstract. This paper deals with the existence of solutions to a class of fourth order nonlinear elliptic equations. The technique used relies on critical points theory. The solutions appeared as critical points of a functional restricted to a suitable manifold. In the case of constant coefficients we obtain the existence of three distinct solutions.

Mathematics Subject Classification (2000). 58J05.

1. Introduction

Let (M, g) be a Riemannian compact smooth n -manifold, $n \geq 5$, with metric g and scalar curvature S_g , we let $H_2^2(M)$ be the standard Sobolev space which is the completion of the space

$$C_2^2(M) = \{u \in C^\infty(M) : \|u\|_{2,2} < +\infty\}$$

with respect to the norm $\|u\|_{2,2} = \sum_{l=0}^2 \|\nabla^l u\|_2$.

In this paper, we investigate solutions of a class of fourth order elliptic equations, on compact n -dimensional Riemannian manifolds, of the form

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + b(x)u = f(x)|u|^{N-2}u + \lambda|u|^{q-2}u \quad (1.1)$$

where a , b , and f are smooth functions on M , $N = \frac{2n}{n-4}$ is the critical exponent, $1 < q < 2$ a real number, $\lambda > 0$ a real parameter.

Consideration for such problem comes from conformal geometry: indeed, in 1983, Paneitz [11] introduced a conformal fourth order operator defined on 4-dimensional Riemannian manifolds which was generalized by Branson [6] to higher dimensions.

$$PB_g(u) = \Delta^2 u + \operatorname{div} \left(-\frac{(n-2)^2 + 4}{2(n-1)(n-2)} S_g \cdot g + \frac{4}{n-2} Ric \right) du + \frac{n-4}{2} Q^n u$$

where $\Delta u = -\operatorname{div}(\nabla u)$, S_g is the scalar curvature, Ric is the Ricci curvature of g , d is the de Rham differential and where

$$Q^n = \frac{1}{2(n-1)} \Delta S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric|^2$$

is associated to the notion of Q -curvature.

We refer to a Paneitz–Branson type operator as an operator of the form

$$P_g u = \Delta^2 u + \nabla^i(a(x)\nabla_i u) + b(x)u.$$

Equation (1.1) is a perturbation of the equation

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + b(x)u = f(x)|u|^{N-2}u. \quad (1.2)$$

Since the embedding $H_2^2 \hookrightarrow H_N^k$, ($k = 0, 1$) fails to be compact, as known, one encounters serious difficulties in solving equations like (1.1).

Since 1990 many results have been established for precise functions a , h and f . Edmunds et al. ([10]) proved for $n \geq 8$ that if $\lambda \in (0, \lambda_1)$, with λ_1 is the first eigenvalue of Δ^2 on the euclidean open ball B , the problem

$$\begin{cases} \Delta^2 u - \lambda u = u|u|^{\frac{8}{n-4}} & \text{in } B \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B \end{cases}$$

has a non trivial solution.

In 1995, Van der Vorst [12] obtained the same results as Edmunds, Fortunato, Jannelli. when applied to the problem

$$\begin{cases} \Delta^2 u - \lambda u = u|u|^{\frac{8}{n-4}} & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded set of R^n and moreover he showed that the solution is positive.

In [8] Caraffa studied the equation

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + b(x)u = f(x)|u|^{N-2}u \quad (1.3)$$

in the case $f(x) = \text{constant}$; and in the particular case where the functions $a(x)$ and $b(x)$ are precise constants she obtained the existence of positive regular solutions. Djadli, Z. et al [9] have studied Paneitz type operators and they gave many interesting applications.

Let f be a C^∞ function on M , $f^- = -\inf(f, 0)$, $f^+ = \sup(f, 0)$ and for a , f , C^∞ -functions on M , we let

$$\lambda_{a,f} = \inf_{u \in A} \frac{\int_M (\Delta u)^2 dv_g - \int_M a |\nabla u|^2 dv_g}{\int_M u^2 dv_g}$$

where $A = \left\{ u \in H_2^2, u \geq 0, u \neq 0 \text{ s. t. } \int_M f^- u dv_g = 0 \right\}$, and

$$\lambda_{a,f} = +\infty \quad \text{if } A = \emptyset$$

the second author establish the following:

Theorem 1. [4] Let a, b be C^∞ functions on M with b negative. For every C^∞ function, f on M with $\int_M f^- dv_g > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^- dv_g}$ such that if f satisfies the following conditions

- (1) $|b(x)| < \lambda_{a,f}$ for any $x \in M$
- (2) $\frac{\sup f^+}{\int f^- dv_g} < C$
- (3) $\sup_M f > 0$,

then the subcritical equation

$$\Delta^2 u + \nabla^i(a \nabla_i u) + bu = f|u|^{q-2}u, \quad q \in]2, N[\quad (1.4)$$

has at least two distinct solutions u and v satisfying $F_q(u) < 0 < F_q(v)$ and of class $C^{4,\theta}$, for some $\theta \in (0, 1)$, where F_q denotes the energy functional associated to Eq. (1.4).

Theorem 2. [4] Let a, b be C^∞ functions on M with b negative. For every C^∞ function f on M with $\int_M f^- dv_g > 0$ there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^- dv_g}$ such that if f satisfies the following conditions

- (1) $|b(x)| < \lambda_{a,f}$ for any $x \in M$
- (2) $\frac{\sup f^+}{\int f^- dv_g} < C$

the critical equation

$$\Delta^2 u + \nabla^i(a \nabla_i u) + bu = f|u|^{N-2}u$$

has a solution of class $C^{4,\theta}$, for some $\theta \in (0, 1)$, with negative energy.

Let

$$\Delta^2 u + \nabla^i(a(x) \nabla_i u) + b(x)u = f(x)|u|^{N-2}u + \lambda|u|^{q-2}u + \epsilon g(x) \quad (1.5)$$

where a, b, f and g are smooth functions on M , $N = \frac{2n}{n-4}$ is the critical exponent, $2 < q < N$ a real number, $\lambda > 0$ a real parameter and $\epsilon > 0$ any small real number the second author obtain

Theorem 3. [5] Let (M, g) be a compact Riemannian n -manifold, $n \geq 6$, a, b, f, g be smooth real functions on M with

- (i) $f(x) > 0$ and $g(x) > 0$ everywhere on M
- (ii) the operator $P(u) = \Delta^2 u + \nabla^i(a(x) \nabla_i u) + b(x)u$ is coercive
- (iii) if $n > 6$, we suppose $-\frac{\Delta f(x_o)}{2f(x_o)} + C_1(n)S_g(x_o) + C_2(n)a(x_o) > 0$ and if $n = 6$, we suppose that $\frac{4}{3n}S_g(x_o) + \frac{1}{(n-4)}a(x_o) > 0$, where $C_1(n) = \frac{5n^2(n-7)+52(n-1)}{6n(n+2)(5n-6)}$ and $C_2(n) = \frac{8(n-1)}{(n+2)(n-6)}$

Then Eq. (1.5) has at least two distinct solutions in H^2_2 .

Our main results in this paper state as follows

Theorem 4. Let (M, g) be an n -dimensional compact Riemannian manifold with $n \geq 6$ and f a positive smooth function on M . Assume that the operator $P(u) = \Delta^2 u + \nabla^i(a(x) \nabla_i u) + b(x)u$ is coercive.

If $n > 6$ and at the point x_o where the function f achieves its maximum the following condition is satisfied

$$\frac{n(n^2 + 4n - 20)}{2(n-6)(n^2-4)} S_g(x_o) + \frac{n(n-1)}{(n-6)(n^2-4)} a(x_o) - \frac{n}{8(n-2)} \frac{\Delta f(x_o)}{f(x_o)} > 0$$

then there exists a $\Lambda > 0$ such that for any $\lambda \in (0, \Lambda)$, Eq. (1.1) have a non trivial solution of class $C^{4,\theta}(M)$, $\theta \in (0, 1)$.

If $n = 6$ and the condition $S_g(x_o) > -3a(x_o)$ is satisfied, then Eq. (1.1) has a solution u of class $C^{4,\theta}(M)$ provided that $\lambda \in (0, \Lambda)$.

In case of constant coefficients, Eq. (1.1) reduces to

$$\Delta^2 u + \alpha \Delta u + \beta u = f(x)|u|^{N-2}u + \lambda|u|^{q-2}u \quad (1.6)$$

where α and β are real constants; we get the existence of three solutions.

Theorem 5. Suppose all the conditions of Theorem 4 are satisfied and moreover $\frac{\alpha^2}{4} > \beta$ with $\alpha > 0$. Then Eq. (1.6) has beside a positive u^+ and a negative u^- smooth solutions a third solution w distinct from u^+ and u^- .

The technique relies on critical points theory. To find solutions we use a method developed in [1]. The solutions appeared as critical points restricted to a suitable manifold. In the case of constant coefficients we obtain the existence of two solutions.

Consider the functionals J_λ , J_λ^+ and J_λ^- defined on H_2^2 by

$$J_\lambda(u) = \frac{1}{2} \left(\|\Delta u\|_2^2 - \int_M a(x)|\nabla u|^2 dv_g + \int_M b(x)u^2 dv_g \right) - \frac{\lambda}{q} \int_M |u|^q dv_g - \frac{1}{N} \int_M f(x)|u|^N dv_g, \quad (1.7)$$

$$J_\lambda^+(u) = \frac{1}{2} \left(\|\Delta u\|_2^2 - \int_M a(x)|\nabla u|^2 dv_g + \int_M b(x)u^2 dv_g \right) - \frac{\lambda}{q} \int_M (u^+)^q dv_g - \frac{1}{N} \int_M f(x)(u^+)^N dv_g \quad (1.8)$$

and

$$J_\lambda^-(u) = \frac{1}{2} \left(\|\Delta u\|_2^2 - \int_M a(x)|\nabla u|^2 dv_g + \int_M b(x)u^2 dv_g \right) - \frac{\lambda}{q} \int_M |u^-|^q dv_g - \frac{1}{N} \int_M f(x)|u^-|^N dv_g \quad (1.9)$$

where $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$.

Let

$$Q_\lambda(u) = \langle \nabla J_\lambda(u), u \rangle$$

and

$$Q_\lambda^\pm(u) = \langle \nabla J_\lambda^\pm(u), u \rangle$$

where $\langle \nabla J_\lambda(u), v \rangle$ denotes the value of $\nabla J_\lambda(u)$ at v .

We consider also the set

$$M_\lambda = \{u \in H_2^2 : Q_\lambda(u) = 0 \text{ and } \|u\| \geq \rho > 0\}$$

and

$$M_\lambda^\pm = \{u \in H_2^2 : Q_\lambda^\pm(u) = 0 \text{ and } \|u\| \geq \rho > 0\}.$$

Along this paper the functions a and b are taken such that

$$\|u\|^2 = \|\Delta u\|_2^2 - \int_M a(x)|\nabla u|^2 dv_g + \int_M b(x)u^2 dv_g$$

is a norm on $H_2^2(M)$ equivalent to the usual one: for example by letting $\max_{x \in M} a(x) < 0$ and $\min_{x \in M} b(x) > 0$ which is equivalent to assume that the operator $P_g(u) = \Delta^2 u + a(x)\Delta u + b(x)u$ is coercive.

First we ensure that the manifold M_λ (respectively $M_\lambda^\pm$) is not empty.

Lemma 1. *There is a real $\lambda_o > 0$ such that the set M_λ is non empty for any $\lambda \in (0, \lambda_o)$.*

Proof. For $u \in H_2^2(M)$ with $\|u\|_{2,2} \geq \rho > 0$ and $t > 0$,

$$Q_\lambda(tu) = t^2\|u\|^2 - \lambda t^q\|u\|_q^q - t^N \int_M f(x)|u|^N dv_g.$$

Put

$$\alpha(t) = \|u\|^2 - t^{N-2} \int_M f(x)|u|^N dv_g$$

and

$$\beta(t) = \lambda t^{q-2}\|u\|_q^q.$$

The Sobolev inequality leads to

$$\alpha(t) \geq \|u\|^2 - \max_{x \in M} f(x) (\max((1+\epsilon)K_o, A_\epsilon))^N \|u\|_{H_2^2(M)}^N t^{N-2}$$

where

$$\frac{1}{K_o} = \inf_{u \in H_2^2(\mathbb{R}^n) - \{0\}} \frac{\|\Delta u\|_2^2}{\|u\|_N^2}$$

is the best constant in the Sobolev's embedding $H_2^2(\mathbb{R}^n) \subset L^N(\mathbb{R}^n)$ (see Aubin [3]) and A_ϵ is a positive constant depending on ϵ , and since the norms $\|\cdot\|$ and $\|\cdot\|_{H_2^2(M)}$ are equivalent, there is $\Lambda > 0$ such that

$$\alpha(t) \geq \|u\|^2 - \Lambda^{-\frac{N}{2}} \max_{x \in M} f(x) (\max((1+\epsilon)K_o, A_\epsilon))^N \|u\|^N t^{N-2}. \quad (1.10)$$

Combining the Hölder and the Sobolev inequalities and taking account of the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{H_2^2(M)}$, we get

$$\beta(t) \leq \lambda V(M)^{1-\frac{2}{N}} (\max((1+\epsilon)K_o, A))^{\frac{N}{2}} \|u\|^q t^{q-2}. \quad (1.11)$$

Let $\alpha_1(t)$ and $\beta_1(t)$ denote respectively right hand sides of inequalities (1.10) and (1.11); $\alpha_1(t)$ vanishes for

$$t_o = \frac{1}{\|u\| (\max_{x \in M} f(x))^{\frac{1}{N-2}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{N}{2(N-2)}}}. \quad (1.12)$$

So if we choose

$$\|u\| = \frac{1}{(\max_{x \in M} f(x))^{\frac{1}{N-2}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{N}{2(N-2)}}$$

$t_o = 1$.

Now since $\alpha_1(t)$ and $\beta_1(t)$ are both decreasing functions, we get

$$\begin{aligned} \min_{t \in (0, \frac{1}{2})} \alpha_1(t) &= \alpha_1\left(\frac{1}{2}\right) = \frac{(1 - 2^{2-N})}{(\max_{x \in M} f(x))^{\frac{2}{N-2}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{N}{N-2}}} \\ &\geq (1 - 2^{2-N}) \rho^2 > 0 \end{aligned}$$

and

$$\min_{t \in (0, \frac{1}{2})} \beta_1(t) = \beta_1\left(\frac{1}{2}\right) = \frac{2^{2-q} \lambda V(M)^{(1-\frac{2}{N})}}{(\max_{x \in M} f(x))^{\frac{q}{N-2}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{q-2+N}{N-2}}} > 0.$$

The equation $Q_\lambda(u) = 0$ admits a solution if $\min_{t \in (0, \frac{1}{2})} \alpha_1(t) \geq \min_{t \in (0, \frac{1}{2})} \beta_1(t)$ that is to say if

$$0 < \lambda < \lambda_o = \frac{(2^{q-2} - 2^{q-N}) V(M)^{(1-\frac{2}{N})}}{(\max_{x \in M} f(x))^{\frac{q-2}{N-2}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{q-2}{N-2}}}.$$

Indeed, putting

$$\gamma(t) = \beta(t) - \alpha(t)$$

we get

$$\gamma(t) \leq \beta_1(t) - \alpha_1(t)$$

and

$$\gamma\left(\frac{1}{2}\right) \leq \beta_1\left(\frac{1}{2}\right) - \alpha_1\left(\frac{1}{2}\right) \leq 0.$$

If $\gamma\left(\frac{1}{2}\right) = 0$,

$$Q_\lambda\left(\frac{1}{2}u\right) = \frac{1}{4}\gamma\left(\frac{1}{2}\right) = 0$$

in case $\gamma\left(\frac{1}{2}\right) < 0$, the continuous function changes sign in the interval $(0, \frac{1}{2})$, then there is $t_1 \in (0, \frac{1}{2})$ such that

$$Q_\lambda(t_1 u) = t_1^2 \gamma(t_1) = 0.$$

Hence the set M_λ is nonempty for any $\lambda \in (0, \lambda_o)$.

Since $Q_\lambda^\pm(u) \geq Q_\lambda(u)$ the same calculations lead to the same conclusion. $\square$

2. Palais–Smale conditions

Lemma 2. *There is $A > 0$ such that $J_\lambda(u) \geq A > 0$ (resp. $J_\lambda^\pm(u) \geq A > 0$) for any $u \in M_\lambda$ with $\lambda \in (0, \min(\lambda_o, \lambda_1))$ where $\lambda_1 = \frac{\frac{(N-2)\Lambda^{-\frac{q}{2}}}{2(N-q)}}{V(M)^{1-\frac{2}{N}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{q}{2}} \rho^{q-2}}$ and λ_o is as in Lemma 1.1.*

Proof. Let $u \in M_\lambda$, then

$$\begin{aligned} \langle \nabla J_\lambda, u \rangle &= \|\Delta u\|_2^2 - \int_M a(x) |\nabla u|^2 dv_g + \int_M b(x) u^2 dv_g \\ &\quad - \lambda \int_M |u|^q dv_g - \int_M f(x) |u|^N dv_g = 0 \end{aligned}$$

so

$$\begin{aligned} \|u\|^2 &= \|\Delta u\|_2^2 - \int_M a(x) |\nabla u|^2 dv_g + \int_M b(x) u^2 dv_g \\ &= \lambda \int_M |u|^q dv_g + \int_M f(x) |u|^N dv_g \end{aligned}$$

hence on M_λ the functional J_λ writes

$$J_\lambda(u) = \frac{N-2}{2N} \|u\|^2 - \lambda \frac{N-q}{Nq} \int_M |u|^q dv_g.$$

Combining Hölder and Sobolev inequalities we obtain

$$J_\lambda(u) \geq \frac{N-2}{2N} \|u\|^2 - \lambda \frac{N-q}{Nq} V(M)^{1-\frac{2}{N}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{q}{2}} \|u\|_{H_2^2(M)}^q$$

and still taking account of the equivalence of the norms $\|\cdot\|_{H_2^2(M)}$ and $\|\cdot\|$, we obtain

$$J_\lambda(u) \geq \left(\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{2}{N}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{q}{2}} \rho^{q-2} \right) \|u\|^2$$

where $\Lambda > 0$ is a constant.

Hence if

$$0 < \lambda < \frac{\frac{N-2}{2(N-q)} \Lambda^{-\frac{q}{2}}}{V(M)^{1-\frac{2}{N}} (\max((1+\epsilon)K_o, A_\epsilon))^{\frac{q}{2}} \rho^{q-2}}$$

then

$$J_\lambda(u) > 0$$

for any $u \in M_\lambda$.

Since $J_\lambda^\pm(u) \geq J_\lambda(u)$ we have the conclusion for $J_\lambda^\pm(u)$. $\square$

Lemma 3. *Let (M, g) be a n -Riemannian manifold, $n \geq 5$. The following assertions are true*

- (i) $(\nabla Q_\lambda(u), u) \leq -A < 0$ (resp. $(\nabla Q_\lambda^\pm(u), u) \leq -A < 0$), for $u \in M_\lambda$ and any $\lambda \in (0, \min(\lambda_o, \lambda_1))$
- (ii) *The critical points of J_λ (resp. $J_\lambda^\pm$) are points of M_λ (resp. $M_\lambda^\pm$).*

Proof. (i) Let $u \in M_\lambda$, then

$$\|u\|^2 = \lambda \int_M |u|^q dv_g + \int_M f(x) |u|^N dv_g$$

and

$$\begin{aligned} \langle \nabla Q_\lambda(u), u \rangle &= 2\|u\|^2 - \lambda q \int_M |u|^q dv_g - N \int_M f(x) |u|^N dv_g \\ &= 2\|u\|^2 - \lambda q \int_M |u|^q dv_g - N \left(\|u\|^2 - \lambda \int_M |u|^q dv_g \right) \\ &= (2 - N) \|u\|^2 + \lambda (N - q) \|u\|_q^q. \end{aligned}$$

The combination of Hölder and Sobolev inequalities allows us to write

$$\begin{aligned} \langle \nabla Q_\lambda(u), u \rangle &\leq (2 - N) \|u\|^2 + \lambda (N - q) V(M)^{1 - \frac{2}{N}} (\max(1 + \epsilon) K_o, A_\epsilon)^{\frac{q}{2}} \\ &\quad \times \|u\|_{H_2^2(M)}^q \end{aligned}$$

and since the norms $\|\cdot\|$ and $\|\cdot\|_{H_2^2(M)}$ are equivalent, we get

$$\begin{aligned} \langle \nabla Q_\lambda(u), u \rangle &\leq \left((2 - N) + \lambda (N - q) V(M)^{1 - \frac{2}{N}} \Lambda^{q-2} (\max(1 + \epsilon) K_o, A_\epsilon)^{\frac{q}{2}} \rho^{q-2} \right) \\ &\quad \times \|u\|^2. \end{aligned}$$

Hence if

$$0 < \lambda < \lambda_1 = \frac{\frac{N-2}{2(N-q)} \Lambda^{-\frac{q}{2}}}{V(M)^{1 - \frac{2}{N}} (\max(1 + \epsilon) K_o, A_\epsilon)^{\frac{q}{2}} \rho^{q-2}}$$

then for any $u \in M_\lambda$.

$$\langle \nabla Q_\lambda(u), u \rangle < 0$$

(ii) By the Lagrange multipliers theorem we get the existence of a real number μ such that for any $u \in M_\lambda$

$$\nabla J_\lambda(u) = \mu \nabla Q_\lambda(u)$$

and by testing at the point $u \in M_\lambda$, we obtain

$$Q_\lambda(u) = \langle \nabla J_\lambda(u), u \rangle = \mu \langle \nabla Q_\lambda(u), u \rangle$$

and since $\langle \nabla Q_\lambda(u), u \rangle < 0$, we get necessarily that $\mu = 0$;

Hence for any $u \in M_\lambda$

$$\nabla J_\lambda(u) = 0.$$

The same computations are carried to conclude for $\langle \nabla Q_\lambda^\pm(u), u \rangle$ and $J_\lambda^\pm$. $\square$

Lemma 4. Let $(u_m)_m$ be a sequence in M_λ (resp. $M_\lambda^\pm$) such that

$$J_\lambda(u_m) \leq c \text{ (resp. } J_\lambda^\pm(u_m) \leq c)$$

and

$$\nabla J_\lambda(u_m) - \mu_m \nabla Q_\lambda(u_m) \rightarrow 0 \text{ (resp. } \nabla J_\lambda^\pm(u_m) - \mu_m \nabla Q_\lambda^\pm(u_m) \rightarrow 0).$$

Suppose that

$$c < \frac{2}{nK_o^{\frac{n}{4}} \max_{x \in M} f(x)^{\frac{n}{4}-1}}$$

then there is a subsequence of $(u_m)_m$ converging strongly in $H_2^2(M)$.

Proof. Let $(u_m)_m \subset M_\lambda$

$$J_\lambda(u_m) = \frac{N-2}{2N} \|u_m\|^2 - \lambda \frac{N-q}{Nq} \int_M |u_m|^q dv(g)$$

We have

$$J_\lambda(u_m) \geq \frac{N-2}{2N} \|u_m\|^2 - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}} \|u_m\|^q$$

$$J_\lambda(u_m) \geq \|u_m\|^2 \left(\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}} \right. \\ \left. \times \|u_m\|^{q-2} \right)$$

$$\text{with } 0 < \lambda < \frac{\frac{(N-2)q}{2(N-q)} \Lambda^{-\frac{q}{2}}}{V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}} \|u\|^{q-2}}.$$

On the other hand, we have

$$c \geq J_\lambda(u_m) \\ \geq \left[\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}} \|u_m\|^{q-2} \right] \|u_m\|^2 > 0$$

hence

$$0 \leq \|u_m\|^2 \leq \frac{c}{\frac{N-2}{2N} - \lambda \frac{N-q}{Nq} \Lambda^{-\frac{q}{2}} V(M)^{1-\frac{q}{N}} (\max((1+\varepsilon)K_o, A_\varepsilon))^{\frac{q}{2}} \rho^{q-2}} < +\infty.$$

So $(u_m)_m$ is bounded in $H_2^2(M)$. Since $H_2^2(M)$ is reflexive and the embedding $H_2^2(M) \subset H_p^k(M)$ ($k = 0, 1$; $p < N$) is compact and we have

- . $u_m \rightarrow u$ weakly in $H_2^2(M)$.
- . $u_m \rightarrow u$ strongly in $H_p^k(M)$; $p < N$.
- . $u_m \rightarrow u$ a.e. in M .

The Brezis–Lieb lemma [7] allows us to write

$$\int_M |\Delta u_m|^2 dv_g = \int_M |\Delta u|^2 dv_g + \int_M |\Delta(u_m - u)|^2 dv_g + o(1)$$

and also

$$\int_M f(x) |u_m|^N dv_g = \int_M f(x) |u|^N dv_g + \int_M f(x) |u_m - u|^N dv_g + o(1).$$

We claim that $u \in M_\lambda$. Indeed since

$u_m \rightarrow u$ weakly in $H_2^2(M)$, we have for any $\phi \in H_2^2(M)$,

$$\int_M (\Delta u_m \Delta \phi - a(x) \langle \nabla u_m, \nabla \phi \rangle + b(x) u_m \phi) dv_g \\ = \int_M (\Delta u \Delta \phi - a(x) \langle \nabla u, \nabla \phi \rangle + b(x) u \phi) dv_g + o(1)$$

and in particular if we let $\phi = u$,

$$\int_M (\Delta u_m \Delta u - a(x) \langle \nabla u_m, \nabla u \rangle + b(x) u_m u) dv_g = \|u\|^2 + o(1)$$

and also if we put $\phi = u_m$, we obtain

$$\int_M (\Delta u_m \Delta u_m - a(x) \langle \nabla u_m, \nabla u_m \rangle + b(x) u_m^2) dv_g = \|u_m\|^2 + o(1).$$

Now since $(u_m)_m$ belongs to M_λ , we get

$$\int_M (\lambda |u_m|^{q-2} u_m u + f(x) |u_m|^{N-2} u_m u) dv(g) = \|u\|^2 + o(1)$$

and by letting $m \rightarrow +\infty$

$$\int_M (\lambda |u_m|^{q-2} u_m u + f(x) |u_m|^{N-2} u_m u) dv(g) \rightarrow \int_M (\lambda |u|^q + f(x) |u|^N) dv(g).$$

Hence

$$\Phi_\lambda(u_m) = \Phi_\lambda(u) = \|u\|^2 - \lambda \int_M |u|^q dv(g) - \int_M f(x) |u|^N dv(g) = 0$$

and we have

$$\|u\| + o(1) = \|u_m\| \geq \rho.$$

Consequently $u \in M_\lambda$.

Also we claim that $\mu_m \rightarrow 0$ as $n \rightarrow +\infty$ in fact testing with u_m , we get

$$\begin{aligned} \langle \nabla J_\lambda(u_m) - \mu_m \nabla \Phi_\lambda(u_m), u_m \rangle &= o(1) \\ &= \underbrace{\langle \nabla J_\lambda(u_m), u_m \rangle}_{=0} - \mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1). \end{aligned}$$

hence

$$\mu_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle = o(1)$$

and by Lemma 3, we have

$$\limsup_m \langle \nabla \Phi_\lambda(u_m), u_m \rangle < 0$$

so $\mu_m \rightarrow 0$ as $m \rightarrow +\infty$.

We are going to show now that $u_m \rightarrow u$ converges strongly in $H_2^2(M)$. First we have

$$\begin{aligned} J_\lambda(u_m) - J_\lambda(u) &= \frac{1}{2} \int_M (\Delta(u_m - u))^2 dv_g - \frac{1}{N} \int_M f(x) |u_m - u|^N dv_g + o(1) \quad (2.1) \end{aligned}$$

and since $u_m - u \rightarrow 0$ converges weakly in $H_2^2(M)$, by testing $\nabla J_\lambda(u_m) - \nabla J_\lambda(u)$, we get

$$\begin{aligned} \langle \nabla J_\lambda(u_m) - \nabla J_\lambda(u), u_m - u \rangle &= o(1) \\ &= \int_M (\Delta(u_m - u))^2 dv_g - \int_M f(x) |u_m - u|^N dv_g = o(1) \end{aligned}$$

that is to say

$$\int_M (\Delta_g(u_m - u))^2 dv_g = \int_M f(x)|u_m - u|^N dv_g + o(1) \quad (2.2)$$

hence taking account of (2.1), we obtain

$$J_\lambda(u_m) - J_\lambda(u) = \frac{2}{n} \int_M (\Delta(u_m - u))^2 dv_g + o(1).$$

The Sobolev inequality allows us to write

$$\|u_m - u\|_N^2 \leq (1 + \varepsilon) K_\circ \int_M (\Delta(u_m - u))^2 dv_g + o(1)$$

so

$$\int_M f(x)|u_m - u|^N dv_g \leq (1 + \varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta(u_m - u)\|_2^N + o(1). \quad (2.3)$$

Taking account of equality (2.2), one writes

$$\begin{aligned} o(1) &\geq \|\Delta(u_m - u)\|_2^2 - (1 + \varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta(u_m - u)\|_2^N + o(1) \\ &\geq \|\Delta(u_m - u)\|_2^2 \left(1 - (1 + \varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta(u_m - u)\|_2^{N-2} \right) + o(1). \end{aligned}$$

Hence if

$$\limsup_m \|\Delta(u_m - u)\|_2^{N-2} < \frac{1}{((1 + \varepsilon) K_\circ)^{\frac{n}{n-4}} \max_{x \in M} f(x)}$$

we get

$$\frac{2}{n} \int_M |\Delta_g(u_m - u)|^2 dv(g) < c.$$

Since

$$c < \frac{2}{n K_\circ^{\frac{n}{4}} (\max_{x \in M} f(x))^{\frac{n-4}{4}}}$$

it follows that

$$\int_M (\Delta(u_m - u))^2 dv_g < \frac{1}{K_\circ^{\frac{n}{4}} (\max_{x \in M} f(x))^{\frac{n-4}{4}}}.$$

Consequently

$$o(1) \geq \underbrace{\|\Delta(u_m - u)\|_2^2 (1 - (1 + \varepsilon)^{\frac{n}{n-4}} \max_{x \in M} f(x) K_\circ^{\frac{n}{n-4}} \|\Delta(u_m - u)\|_2^{N-2})}_{>0} + o(1)$$

which shows that

$$\|\Delta(u_m - u)\|_2^2 = o(1)$$

and $u_m \rightarrow u$ converges strongly in $H_2^2(M)$. □

Theorem 6. *Let (M, g) be an n -dimensional compact Riemannian manifold with $n \geq 6$ and f be a smooth positive function. Assume that the operator $P_g(u) = \Delta^2 u + \nabla^i (a(x) \nabla_i u) + b(x)u$ is coercive and*

$$c < \frac{2}{nK_o^{\frac{n}{4}} \max_{x \in M} f(x)^{\frac{n}{4}-1}}$$

Then there exists a $\Lambda > 0$ such that for any $\lambda \in (0, \Lambda)$, Eq. (1.1) have a non trivial weak solution.

Proof. According to Lemmas 2, 3 and 4, we infer the existence of $v \in M_\lambda$ such that $J_\lambda(v) = \min_{u \in M_\lambda} J_\lambda$. So there is a real μ such that

$$\nabla J_\lambda(v) = \mu \nabla Q_\lambda(v)$$

and multiplying by v and taking account of Lemma 3 we obtain that $\mu = 0$. Hence v is a non solution of Eq. (1.1). $\square$

3. Multiplicity of solutions in case of constant coefficients

When P_g has constant coefficients, we set

$$\begin{aligned} J_\lambda^+(u) &= \frac{1}{2} \left(\|\Delta u\|_2^2 - \alpha \int_M |\nabla u|^2 dv_g + \beta \int_M u^2 dv_g \right) - \frac{\lambda}{q} \int_M (u^+)^q dv_g \\ &\quad - \frac{1}{N} \int_M f(x) (u^+)^N dv_g \end{aligned}$$

where

$$u^+ = \max(u, 0)$$

Critical points of J_λ^+ are solutions to

$$\Delta^2 u + \alpha \Delta u + \beta u = \lambda \left((u^+)^{q-2} + f (u^+)^{N-2} \right) u^+. \quad (3.1)$$

Similar arguments as the ones used in the precedent sections give that J_λ^+ has a critical point u . Standard arguments show that u is of class $C^{4,\theta}$ with $\theta \in (0, 1)$. If $\alpha^2 - 4\beta > 0$, we let $x_1 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$ and $x_2 = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2}$ and moreover if $\alpha > 0$, then $x_1, x_2 > 0$ and

$$(\Delta + x_1)(\Delta + x_2)u = \Delta^2 u + \alpha \Delta u + \beta u \geq 0.$$

Applying the maximum principle twice, we obtain that u is a positive solution of class $C^{4,\theta}$, where $\theta \in (0, 1)$ of the equation

$$\Delta^2 u + \alpha \Delta u + \beta u = \lambda (u^{q-1} + f u^{N-1}).$$

and standard regularity results give that u is smooth.

In the same manner if we set

$$\begin{aligned} J_\lambda^-(u) &= \frac{1}{2} \left(\|\Delta u\|_2^2 - \alpha \int_M |\nabla u|^2 dv_g + \beta \int_M u^2 dv_g \right) - \frac{\lambda}{q} \int_M |u^-|^q dv_g \\ &\quad - \frac{1}{N} \int_M f(x) |u^-|^N dv_g \end{aligned}$$

where

$$u^- = \min(u, 0)$$

then the critical points of J_λ^- are solutions to

$$\Delta^2 u + \alpha \Delta u + \beta u = \lambda(|u^-|^{q-2} + f|u^-|^{N-2})u^-.$$

By the same argument as above we get that u^- is a negative smooth solution. Similar arguments as the ones we used for J_λ give that J_λ^+ and J_λ^- have critical points M_λ^+ and M_λ^- respectively where

$$M_\lambda^\pm = \{u \in H_2^2 : Q_\lambda^\pm(u) = 0 \text{ and } \|u\| \geq \rho > 0\}$$

and

$$Q_\lambda^\pm(u) = \langle \nabla J_\lambda^\pm(u), u \rangle.$$

Summarizing, we get

Theorem 7. *Let (M, g) be an n -dimensional compact Riemannian manifold with $n \geq 6$. Assume that the operator $P_g(u) = \Delta^2 u + \nabla^i(a(x) \nabla_i u) + b(x)u$ is coercive and*

$$c < \frac{2}{nK_o^{\frac{n}{4}} \max_{x \in M} f(x)^{\frac{n}{4}-1}}.$$

If moreover $\frac{\alpha^2}{4} > \beta$ with $\alpha > 0$. Then Eq. (1.6) has two distinct smooth solutions; one positive and the other negative.

Lemma 5. *For any $\lambda > 0$, sufficiently small, J_λ has two local minima.*

Proof. We follow closely the proof of Lemma 8 in [2]. As a consequence of Lemmas 2, 3 and 5, we infer the existence of $v_1 \in M_\lambda^+$ and a $v_2 \in M_\lambda^-$ such that

$$J_\lambda^+(v_1) = \min_{u \in M_\lambda^+} J_\lambda^+(u)$$

and

$$J_\lambda^-(v_2) = \min_{u \in M_\lambda^-} J_\lambda^-(u).$$

Note that v_1 and v_2 are respectively smooth positive and negative solutions of Eq. (1.6). Indeed by Lagrange multipliers theorem we get that

$$\nabla J_\lambda^+(v_1) = \mu \nabla Q_\lambda^+(v_1)$$

and multiplying by v_1 we deduce that

$$\mu = 0$$

Hence v_1 is a solution of (3.1) and as in Sect. 3 we get that v_1 positive, hence a positive solution of Eq. (1.6). v_2 is actually a negative solution of (1.6). We claim that v_1 and v_2 are local minima of J_λ if it is not the case let $w_n \in M_\lambda$ such that $w_n \rightarrow v_1$ in H_2^2 as $n \rightarrow +\infty$ and

$$J_\lambda(w_n) < J_\lambda^+(v_1) \tag{3.2}$$

We can choose w_n as

$$J_\lambda(w_n) = \inf_{u \in B_n \cap M_\lambda} J_\lambda(u) \quad (3.3)$$

where $B_n = \{u \in \|u - v_1\|_{H_1^2} \leq \frac{1}{n}\}$. There exist parameters λ_n and μ_n such that

$$\nabla J_\lambda(w_n) = \lambda_n \nabla Q_\lambda(w_n) + \mu_n (\Delta^2 w_n + \alpha \Delta w_n + \beta w_n) \quad (3.4)$$

with $\mu_n \leq 0$. Taking the inner product of the latter equality with w_n , we get

$$\lambda_n \langle \nabla Q_\lambda(w_n), w_n \rangle + \mu_n \|w_n\|_{H_1^2}^2 = 0$$

and we infer that $\lambda_n \leq 0$.

Equation reads as

$$\Delta^2 w_n + \alpha \Delta w_n + \beta w_n = \frac{(-\lambda_n - \mu_n)}{(1 - \lambda_n - \mu_n)} f|w_n|^{N-2} w_n.$$

By standard methods, w_n is of class $C^{4,\theta}$, $0 < \theta < 1$. Hence w_n goes to v_1 in the C^2 topology, then $w_n > 0$. So (3.3) is a contradiction with (3.2). Hence v_1 and v_2 are respectively positive and negative solution of Eq. (1.6) of minimal positive energy. $\square$

Next we prove

Theorem 8. *Let (M, g) be an n -dimensional compact Riemannian manifold with $n \geq 6$. Assume that the operator $P(u) = \Delta^2 u + \nabla^i (a(x) \nabla_i u) + b(x)u$ is coercive and*

$$c < \frac{2}{nK_o^{\frac{n}{4}} \max_{x \in M} f(x)^{\frac{n}{4}-1}}.$$

If moreover $\frac{\alpha^2}{4} > \beta$ with $\alpha > 0$. Then Eq. (1.6) has third solution w distinct of u^+ and u^- .

Proof. We can suppose that the minima of J_λ are realized by u^+ and u^- . The geometric conditions of the Mountain pass theorem are satisfied. If Γ denotes the set of paths $\gamma : [0, 1] \rightarrow M_\lambda$ such that $\gamma(0) = u^-$ and $\gamma(1) = u^+$. Let $c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} (J_\lambda(\gamma))$. By Lemma 4, we infer that c_λ is a critical level of the function J_λ with critical value w and by Lemma 1.3 $w \in M_\lambda$. Hence w is solution of Eq. (1.6) different from u^+ and u^- . $\square$

4. Test functions

In this section we give the proof of Theorems 4 and 5.

Let $(y^1, \dots, y^n)$ be normal coordinates centred at the point x_o where the function attains its maximum and $S(r)$ be the geodesic sphere centred at x_o and of radius r ($r < d$ the injectivity radius). Denote by $d\sigma$ the volume element of the $(n-1)$ -dimensional unit S^{n-1} .

Put

$$G(r) = \frac{1}{w_{n-1}} \int \int_{S(r)} \sqrt{|g(x)|} d\sigma$$

where w_{n-1} denotes the area of S^{n-1} and $|g(x)|$ the determinant of the metric g . An expansion of $G(r)$ in a neighborhood of $r = 0$ writes as

$$G(r) = 1 - \frac{S_g(x_o)}{6n} r^2 + o(r^2)$$

where $S_g(x_o)$ denotes the scalar curvature of M at the point x_o .

Let $B(x_o, \delta)$ be the ball centred at x_o and of radius δ with $0 < 2\delta < d$ and let η be a smooth function equals to 1 on $B(x_o, \delta)$ and equals to 0 on $M - B(x_o, 2\delta)$.

Put

$$u_\epsilon(x) = \left(\frac{(n-4)n(n^2-4)\epsilon^4}{f(x_o)} \right)^{\frac{n-4}{8}} \frac{\eta(r)}{(r^2 + \epsilon^2)^{\frac{n-4}{2}}}$$

where

$$f(x_o) = \max_{x \in M} f(x)$$

and $r = d(x_o, \cdot)$ is geodesic distance to the point x_o .

We let, for $p - q > 1$,

$$I_p^q = \int_0^{+\infty} \frac{t^q}{(1+t)^p} dt$$

which fulfills

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q.$$

In the case where the dimension of the manifold $n > 6$, we have

Theorem 9. *Let (M, g) be an n -dimensional compact Riemannian manifold with $n > 6$. If at the point x_o where the function f achieves its maximum*

$$\frac{n(n^2 + 4n - 20)}{2(n-6)(n^2-4)} S_g(x_o) + \frac{n(n-1)}{(n-6)(n^2-4)} a(x_o) - \frac{n}{8(n-2)} \frac{\Delta f(x_o)}{f(x_o)} > 0$$

Eq. (1.1) have a non trivial solution of class $C^{4,\theta}(M)$, $\theta \in (0, 1)$.

Proof. As in [8], we get

$$\int_M f(x) |u_\epsilon(x)|^N dv(g) = \frac{1}{K_o^{\frac{n}{4}}(f(x_o))^{\frac{n-4}{4}}} \left(1 - \left(\frac{\Delta f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-2)} \right) \epsilon^2 + o(\epsilon^2) \right)$$

and also

$$\int_M a(x) |\nabla u_\epsilon|^2 dv(g) = \frac{1}{K_o^{\frac{n}{4}}(f(x_o))^{\frac{n-4}{4}}} \left(\frac{4(n-1)a(x_o)}{(n^2-4)(n-6)} \epsilon^2 + o(\epsilon^2) \right).$$

The computations give

$$\int_M b(x) u_\epsilon^2 dv(g) = o(\epsilon^2)$$

and

$$\int_M |\Delta u_\epsilon|^2 dv(g) = \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \frac{n^2 + 4n - 20}{6(n^2 - 4)(n - 6)} S_g(x_\circ) \epsilon^2 + o(\epsilon^2) \right).$$

Summarizing we obtain

$$\begin{aligned} \int_M |\Delta u_\epsilon|^2 - a(x) |\nabla u_\epsilon|^2 + b(x) u_\epsilon^2 dv(g) &= \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \\ &\left(1 - \left(\frac{n^2 + 4n - 20}{6(n^2 - 4)(n - 6)} S_g(x_\circ) + \frac{4(n - 1)}{(n^2 - 4)(n - 6)} a(x_\circ) \right) \epsilon^2 + o(\epsilon^2) \right). \end{aligned}$$

Taking in mind that

$$J_\lambda(u_\epsilon) = \frac{1}{2} \|u_\epsilon\|^2 - \frac{\lambda}{q} \|u_\epsilon\|_q^q - \frac{1}{N} \int_M f(x) |u_\epsilon(x)|^N dv(g)$$

where

$$\|u_\epsilon\|^2 = \int_M |\Delta u_\epsilon|^2 - a(x) |\nabla u_\epsilon|^2 + b(x) u_\epsilon^2 dv(g)$$

and since $\lambda > 0$, we get

$$\begin{aligned} J_\lambda(u_\epsilon) &\leq \frac{1}{2} \|u_\epsilon\|^2 - \frac{1}{N} \int_M f(x) |u_\epsilon(x)|^N dv(g) \\ &\leq \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \left[\frac{2}{n} - \left(\frac{n^2 + 4n - 20}{(n^2 - 4)(n - 6)} S_g(x_\circ) \right. \right. \\ &\quad \left. \left. + \frac{2(n - 1)}{(n^2 - 4)(n - 6)} a(x_\circ) - \frac{1}{4(n - 2)} \frac{\Delta f(x_\circ)}{f(x_\circ)} \right) \epsilon^2 + o(\epsilon^2) \right] \\ &\leq \frac{2}{n K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \left[1 - \left(\frac{(n^2 + 4n - 20)n}{2(n^2 - 4)(n - 6)} S_g(x_\circ) \right. \right. \\ &\quad \left. \left. + \frac{(n - 1)n}{(n^2 - 4)(n - 6)} a(x_\circ) - \frac{n}{8(n - 2)} \frac{\Delta f(x_\circ)}{f(x_\circ)} \right) \epsilon^2 + o(\epsilon^2) \right]. \end{aligned}$$

So the condition

$$J_\lambda(u_\epsilon) < \frac{2}{n K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}}$$

is fulfilled if

$$\left(\frac{(n^2 + 4n - 20)n}{2(n^2 - 4)(n - 6)} S_g(x_\circ) + \frac{(n - 1)n}{(n^2 - 4)(n - 6)} a(x_\circ) - \frac{n}{8(n - 2)} \frac{\Delta f(x_\circ)}{f(x_\circ)} \right) > 0.$$

In the case $n = 6$, the same calculations as in case $n > 6$ lead to

$$\begin{aligned} \int_M f(x) |u_\epsilon(x)|^N dv(g) &= \frac{1}{K_\circ^{\frac{n}{4}}(f(x_\circ))^{\frac{n-4}{4}}} \left(1 - \left(\frac{\Delta f(x_\circ)}{2(n - 2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n - 2)} \right) \right. \\ &\quad \left. \times \epsilon^2 + o(\epsilon^2) \right). \end{aligned} \quad (4.1)$$

Also the same computations as in [8] with minor modifications allow us to write

$$\int_M a(x) |\nabla u_\epsilon|^2 dv(g) = (n-4)^2 \left(\frac{(n-4)n(n^2-4)}{f(x_o)} \right)^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \left(a(x_o) \epsilon^2 \log \left(\frac{1}{\epsilon^2} \right) + O(\epsilon^2) \right)$$

and

$$\begin{aligned} \int_M |\Delta u_\epsilon|^2 dv(g) &= (n-4)^2 \left(\frac{(n-4)n(n^2-4)}{f(x_o)} \right)^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \\ &\times \left(\frac{n(n+2)(n-2)}{(n-4)} I_n^{\frac{n}{2}-1} - \frac{2}{n} S_g(x_o) \epsilon^2 \log \left(\frac{1}{\epsilon^2} \right) + O(\epsilon^2) \right). \end{aligned}$$

Consequently

$$\begin{aligned} \int_M (\Delta u_\epsilon)^2 - a(x) |\nabla u_\epsilon|^2 + b(x) u_\epsilon^2 dv(g) &= (n-4)^2 \left(\frac{(n-4)n(n^2-4)}{f(x_o)} \right)^{\frac{n-4}{4}} \frac{w_{n-1}}{2} \\ &\times \left[\frac{n(n+2)(n-2)}{(n-4)} I_n^{\frac{n}{2}-1} - \left(\frac{2}{n} S_g(x_o) + a(x_o) \right) \epsilon^2 \log \left(\frac{1}{\epsilon^2} \right) + O(\epsilon^2) \right] \\ &= \frac{1}{K_o^{\frac{n}{4}}(f(x_o))^{\frac{n-4}{4}}} \left(1 - \frac{(n-4)}{n(n^2-4)} I_n^{\frac{n}{2}-1} \left(\frac{2}{n} S_g(x_o) + a(x_o) \right) \epsilon^2 \log \left(\frac{1}{\epsilon^2} \right) \right. \\ &\quad \left. + O(\epsilon^2) \right) \end{aligned}$$

and taking account of (4.1), we obtain

$$\begin{aligned} J_\lambda(u_\epsilon) &\leq \frac{1}{2} \|u_\epsilon\|^2 - \frac{1}{N} \int_M f(x) |u_\epsilon(x)|^N dv(g) \\ &\leq \frac{1}{K_o^{\frac{n}{4}}(f(x_o))^{\frac{n-4}{4}}} \left(\frac{1}{2} - \frac{1}{N} - \frac{(n-4)t_o^2}{2n(n^2-4)I_n^{\frac{n}{2}-1}} \left(\frac{2}{n} S_g(x_o) + a(x_o) \right) \right. \\ &\quad \left. \times \epsilon^2 \log \left(\frac{1}{\epsilon^2} \right) + O(\epsilon^2) \right). \end{aligned}$$

So if in the point x_o where the maximum of the function f is achieved, the condition $\frac{2}{n} S_g(x_o) + a(x_o) > 0$ i.e. since $n = 6$, $S_g(x_o) > -3a(x_o)$ is fulfilled, we get for ϵ sufficiently small

$$J_\lambda(u_\epsilon) < \frac{2}{n K_o^{\frac{n}{4}}(f(x_o))^{\frac{n-4}{4}}}.$$

□

Acknowledgments

The authors are deeply grateful to the referees for the valuable suggestions.

References

- [1] Ambrosetti, A.: Critical points and nonlinear variational problems, vol. 49, Societe mathematique de France (1992) (fascicule 2)
- [2] Ambrosetti, A., Azorero, J.G.: Multiplicity results for nonlinear elliptic equations. *J. Funct. Anal.* **137**, 219–242 (1996)
- [3] Aubin, T.: Some nonlinear problems in Riemannian geometry. Springer, Berlin (1998)
- [4] Benalili, M.: Existence and multiplicity of solutions to elliptic equations of fourth order on compact manifolds. *Dyn. PDE* **6**(3), 203–225 (2009)
- [5] Benalili, M.: Existence and multiplicity of solutions to fourth order elliptic equations with critical exponent on compact manifolds. *Bull. Belg. Math. Soc. Simon Stevin* **17**, 607–622 (2010)
- [6] Branson, T.P.: Group representation arising from Lorentz conformal geometry. *J. Funct. Anal.* **74**, 199–291 (1987)
- [7] Brézis, H., Lieb, E.A.: A relation between pointwise convergence of functions and convergence of functionals. *Proc. Am. Math. Soc.* **88**, 486–490 (1983)
- [8] Caraffa, D.: Equations elliptiques du quatrième ordre avec un exposant critique sur les variétés Riemanniennes compactes. *J. Math. Pure Appl.* **80**(9), 941–960 (2001)
- [9] Djadli, Z., Hebeyand, E., Ledoux, M.: Paneitz-type operators and applications. *Duke. Math. J.* **104**(1), 129–169 (2000)
- [10] Edmunds, D.E., Furtunato, F., Janelli, E.: Critical exponents, critical dimensions and biharmonic operators. *Arch. Ration. Mech. Anal.* **112**(3), 269–289 (1990)
- [11] Paneitz, S.: A quatric conformally covariant differential operator for arbitrary pseudoRiemannian manifolds. *SIGMA* **4** (2008)
- [12] Van der Vorst, R.: Fourth order elliptic equations, with critical growth, C.R. Acad. Sci. Paris serie I **320**, 295–299 (1995)

Mohammed Benalili and Kamel Tahri
Department of Mathematics, Faculty of Sciences,
University Aboubakr Belkaid, Tlemcen, Algeria
e-mail: m_benalili@mail.univ-tlemcen.dz

Received: 6 April 2010.

Accepted: 16 February 2011.