Global existence of weak solutions for the Navier–Stokes equations with capillarity on the half-line

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Abstract. In this paper, we construct the global weak solutions to the initial-boundary problem for the Navier–Stokes system with capillarity in the half space $\mathbb{R}_+^1$. The result extends Eugene Tsyganov’s existence theorem which considered the problem in the finite region published in J. Differential Equations 245:3936–3955, 2008.

Mathematics Subject Classification (2000). 35A01, 35A24, 35D30, 74G25.

Keywords. Navier–Stokes equations, Half space problem, Global weak solution, Capillarity.

1. Introduction

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by Van der Waals and Korteweg [13] and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [11, 14, 15].

Korteweg-type models are based on an extended version of non-equilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but also on the gradient of the density. Let us now consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}^n$, we are now interested in the following compressible capillary fluid model, which can be derived from a Cahn–Hilliard like free energy (see the pioneering work by Dunn and Serrin [7] and also in [1, 5, 8]). The conservation of mass and of momentum write:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + pI) = \operatorname{div}(K + S) \end{cases}$$

where the viscous stress tensor S and the Korteweg stress tensor K are given by

$$\begin{cases} S = \lambda \operatorname{div}(uI) + \mu(\partial_i u_j + \partial_j u_i) \\ K = (\kappa \rho \Delta \rho + \frac{1}{2}(\kappa' \rho + \kappa)|\nabla \rho|^2)I - \kappa \nabla \rho \otimes \nabla \rho \end{cases}$$

and where κ is the coefficient of capillarity and is a regular function. The term $\operatorname{div} K$ allows to describe the variation of density at the inter faces between two phases, generally a mixture liquid-vapor. p is a general pressure term. μ and λ are the Lamé viscosity coefficients depending on the density ρ and satisfying $\mu > 0$ and $2\mu + n\lambda \geq 0$.

We now mention some results concerning the Korteweg type system. Danchin, R., Desjardins study the well-posedness of the model in spaces with critical regularity indices with respect to the scaling of the associated equations. And in a functional setting as close as possible to the physical energy spaces, they prove global existence of solutions close to a stable equilibrium, and local in time existence for solutions when the pressure law may present spinodal regions. Uniqueness is also obtained in [6].

Bresch et al. [2] consider a global existence result of weak solutions in dimension 2 or 3 using the regularity of the density in a periodic domain $\Omega = T^d$. Next they discuss the case of a smooth bounded domain depending on the choice of the diffusive term and on the boundary conditions, with a positive result in the case $\Omega = (0, 1) \times T^{d-1}$. Third they also indicate some partial results on the associated shallow water equation that means the system corresponding to $\kappa = 0$.

Haspot [4] improve the existence of global weak solution in dimension two for initial data in the energy space, close to a stable equilibrium and with specific choices on the capillary coefficients. In particular they are interested in capillary coefficients approximating a constant capillarity coefficient κ . Third they show the existence of global weak solution in dimension one for a specific type of capillary coefficients $\kappa = \frac{1}{\rho^{2+\varepsilon}} 1_{\{\rho \leq \alpha\}} + \theta(\rho) 1_{\{\alpha < \rho \leq 2\alpha\}} + \kappa 1_{\{\rho > 2\alpha\}}$ with large initial data in the energy space.

In the one-dimensional space with the other specific type of capillary coefficients $\kappa = \rho^{-5}$, Tsyganov [16] proved the global existence and asymptotic convergence of weak solutions on the Lagrangian coordinate by a easier method different with Kato [12] and Bona and Sachs [3]. And Tsyganov study the model in finite interval $[0, 1]$. And we extend the existence result to the half space situation.

In this paper we prove the existence of global weak solutions for the Navier–Stokes equations with capillarity

$$v_t - u_x = 0, \quad x \in [0, \infty), \quad t > 0, \quad (1.1)$$

$$u_t + p(v)_x = \left(\frac{u_x}{v}\right)_x - v_{xxx}, \quad x \in [0, \infty), \quad t > 0, \quad (1.2)$$

with Cauchy data

$$(v(x, 0), u(x, 0)) = (v_0(x), u_0(x)) \quad (1.3)$$

such that

$$(v_0 - v') \in H^1([0, \infty)), \quad u_0 \in L^2([0, \infty))$$

and the boundary conditions

$$v_x(0, t) = u(0, t) = 0. \quad (1.4)$$

where v , u , and p are the specific volume, velocity, and pressure in the fluid, $v' > 0$ is a constant, and (1.1) and (1.2) are respectively the equations of conservation of mass and balance of forces. We assume throughout that

$$p(v) = v^{-\gamma}, \quad \gamma > 1 \quad (1.5)$$

For convenience, we define a special energy function $\psi(v)$ by

$$\psi(v) := \int_{v'}^v p(v') - p(s) ds \quad (1.6)$$

where $v' > 0$ is fixed. We then have some properties with $\psi(v)$ in following

$$\psi(v) = p(v')(v - v') + \frac{v^{1-\gamma} - (v')^{1-\gamma}}{\gamma - 1} \geq 0 \quad (1.7)$$

as a obvious conclusion of (1.6), we have

$$\psi(v) \sim v^{1-\gamma}, \quad \text{as } v \rightarrow 0; \text{ and } \psi \sim v, \quad \text{as } v \rightarrow \infty \quad (1.8)$$

and when ψ blow-up

$$\lim_{v \rightarrow +0} \psi(v) = +\infty, \quad \lim_{v \rightarrow +\infty} \psi(v) = +\infty \quad (1.9)$$

easy to see that $\psi(v') = \psi'(v') = 0$ and $\psi''(v) = -\psi'(v) > 0$. Thus there exist $C > 0$ depending only on the pressure p ,

$$C^{-1}(v - v')^2 \leq \psi(v) \leq C(v - v')^2, \quad 0 < v \leq v \leq \bar{v}. \quad (1.10)$$

We say that (v, u) is a weak solution of (1.1)–(1.3), provided that $\inf_{x \in [0, \infty)} v(\cdot, t) > 0$ and

$$v(\cdot, t) \in C([0, \infty); H^1([0, \infty))) \cap C((0, \infty); H^2([0, \infty))), \quad (1.11)$$

$$u(\cdot, t) \in C([0, \infty); L^1([0, \infty))) \cap C((0, \infty); H^1([0, \infty))), \quad (1.12)$$

$$v(\cdot, 0) = v_0, \quad u(\cdot, 0) = u_0, \quad v_x(0, t) = u(0, t) = 0. \quad (1.13)$$

and for all test functions $\phi \in C_0^1((0, \infty) \times [0, \infty))$, $\varphi \in C_0^3((0, \infty) \times [0, \infty))$, and all times $t_2 > t_1 \geq 0$.

$$\int_0^\infty v\phi dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^\infty (v\phi_t - u\phi_x) dx dt = 0, \quad (1.14)$$

$$\begin{aligned} & \int_0^\infty u\varphi dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^\infty (u\varphi_t + p(v)\varphi_x) dx dt \\ &= - \int_{t_1}^{t_2} \int_0^\infty \left(\frac{u_x}{v} \varphi_x - v\varphi_{xxx} \right) dx dt. \end{aligned} \quad (1.15)$$

Our existence result is formulated in the following theorem:

Theorem 1. *Assume that the initial data (v_0, u_0) satisfy*

$$0 < \underline{v} \leq v_0(x) \leq \bar{v}, \quad (1.16)$$

$$\int_0^\infty \left(\frac{1}{2} u_0^2(x) + \psi(v_0(x)) + \frac{1}{2} v_0^2(x) \right) dx = C_0, \quad (1.17)$$

for some positive constants $\underline{v}$, $\bar{v}$, C_0 . Then there is a weak solution (v, u) of (1.1)–(1.3) and positive constant $C(T)$ depending only on $\underline{v}$, $\bar{v}$, C_0 and T , such that

$$0 < C(T)^{-1} \leq v(x, t), \quad 0 \leq t \leq T \quad (1.18)$$

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\int_0^\infty (v(x, t) - v')^2 dx + \int_0^\infty v_x(x, t)^2 dx + \int_0^\infty u(x, t)^2 dx \right. \\ & \quad \left. + t \int_0^\infty u_x(x, t)^2 dx \right) \leq C(T). \end{aligned} \quad (1.19)$$

2. Difference approximations

In this section we construct approximate solutions to the system (1.1)–(1.5) and show that they satisfy various regularity estimates. We employ the method of semi-discrete difference schemes which can be considered standard for Navier–Stokes equations in Lagrangean coordinates. In fact, our analysis shares many ideas with the one in Hoff [9, 10].

Let h be an increment in x , which be small enough. We set $x_k = kh$ for $k \in \mathbb{Z}_+ \cup \{0\}$, and $x_j = jh, j + 1/2 \in \mathbb{Z}_+ \cup \{0\}$. Define the difference operator δ by

$$\delta w_l := \frac{w_{l+1/2} - w_{l-1/2}}{h}$$

for $l \in \mathbb{Z} \cup \mathbb{Z} + \frac{1}{2}$. We then use the following semi-discrete scheme to generate approximations $v_j(t)$ and $u_k(t)$:

$$\dot{v}_j = \delta u_j, \quad j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \quad (2.1)$$

$$\dot{u}_k + \delta p(v)_k = \delta \left(\frac{\delta u}{v} \right)_k - \delta(\delta(\delta v))_k, \quad k = 0, 1, 2, 3, \dots \quad (2.2)$$

$$(v_j(0) - v'), \quad (u_k(0)) \in l^2 \cap l^\infty, \quad (2.3)$$

where $v' > 0$ be a positive constant, denote $\dot{w} := \partial_t w$. It is implied that v_j and u_k in the system (2.1)–(2.3) depend also on h , but we will drop the superscript h throughout the section.

Lemma 2.1. *Assume that the equations (2.1)–(2.3) have a solution defined up to time T with $v_j(t) > 0$ for $0 \leq t \leq T$, Then for $0 \leq t \leq T$:*

$$\begin{aligned} I(t) := & \frac{1}{2} \sum_{k=0}^{+\infty} u_k^2(t)h + \sum_{j=\frac{1}{2}}^{+\infty} \psi(v_j(t))h + \frac{1}{2} \sum_{k=0}^{+\infty} \delta v_k^2(t)h \\ & + \int_0^t \sum_{j=\frac{1}{2}}^{+\infty} \frac{\delta u_j^2}{v_j} h = I(0) = C \end{aligned} \quad (2.4)$$

Proof. Differentiating $I(t)$ with respect to t , rearranging, and applying equations (2.1) to (2.3)

$$\begin{aligned} \dot{I}(t) &= \sum u_k \dot{u}_k h + \sum \delta v_k \delta \dot{v}_k h + \sum (p(v') - p(v_j)) \dot{v}_j h + \sum \frac{\delta u_j^2}{v_j} h \\ &= \sum u_k \dot{u}_k h - \sum \delta (\delta v)_j \delta u_j h + \sum p(v') \delta u_j h + \sum u_k \delta p_k h \\ &\quad - \sum u_k \left(\frac{\delta u}{v} \right)_k h \\ &= \sum u_k \dot{u}_k h + \sum \delta (\delta (\delta v))_k u_k h + \sum p(v') \delta u_j h + \sum u_k \delta p_k h \\ &\quad - \sum u_k \left(\frac{\delta u}{v} \right)_k h \\ &= \sum u_k (\dot{u}_k + \delta p(v)_k - \delta \left(\frac{\delta u}{v} \right)_k + \delta (\delta (\delta v))_k) h + p(v') \sum \delta u_j h \\ &= 0 \end{aligned}$$

This proved (2.4). □

From Lemma 2.1, for fixed h , and the property (1.8) of ψ , we obtain

$$|u_k| \leq C(h), 0 < C^{-1}(h) \leq v_j \leq C(h), \text{ uniformly for all } k, j \text{ and } t \geq 0$$

and so the ode system (2.1) to (2.3) can be solved globally. Next we'll prove that there is a boundary of v_j , which is independent on h .

Lemma 2.2. *There is a solution $(v_j(t), u_k(t))$ of (2.1)–(2.3), defined for all $t > 0$, satisfying*

$$\sum_j (v_j(t) - v')^2 \leq C, \quad \text{and} \quad 0 < \frac{1}{C(T)} \leq v_j(t), \quad 0 \leq t \leq T \quad (2.5)$$

Proof. The first term of (2.5) is a direct conclusion of (2.4) applying (1.9), and we'll prove the last term of (2.5) by three claims below:

Claim 2.2.1. For fix j ($(j+1/2) \in \mathbb{Z}_+$), then for each $t \geq 0$, there is an integer $j' = j'(j, t)$ depending measurably on t , such that:

$$\sum_{l=j+1}^{j'(j,t)} v_l(t)h \geq 1 \quad (2.6)$$

$$v_{j'(j,t)} \geq \frac{1}{C} > 0 \quad (2.7)$$

$$|x_{j'} - x_j| \leq C. \quad (2.8)$$

Proof of claim 2.2.1. Fix $t_0 > 0$ choose first $j'' > j$ such that:

$$\sum_{l=j+1}^{j''-1} v_l(t_0)h \leq 1 < \sum_{l=j+1}^{j''} v_l(t_0)h$$

then choose $j' \geq j''$ be most close to j'' , so that: $v_{j'}(t_0) \geq \min\left(\frac{1}{2}, \frac{v'}{2}\right)$. We will get an upper bounded for $\sum_{l=j+1}^{j'} v_l(t_0)h$. First:

$$\sum_{l=j''}^{j'-1} v_l(t_0)h \leq \sum_{l=j''}^{j'-1} \frac{1}{2}h = \frac{1}{2} |x_{j'-1} - x_{j''}| \leq \frac{1}{2} |x_{j'} - x_j|$$

next: let $l = j'$ or j'' , if $v_l \leq v' + 1$ then $v_l h \leq C$, otherwise if $v_l > v' + 1$ then by the definition of $\psi(v)$ and Lemma 2.1

$$v_l h \leq C\psi(v_l)h \leq C$$

therefore:

$$1 < \sum_{l=j+1}^{j'(j,t)} v_l(t)h \leq C + \frac{1}{2} |x_{j'} - x_j| \quad (2.9)$$

(2.9) together with $v_{j'}(t_0) \geq \min\left(\frac{1}{2}, \frac{v'}{2}\right) \geq \frac{1}{C} > 0$ enable us to define j' as a measurable function of t .

We observe that: $\frac{\gamma}{\gamma-1}$ and γ are conjugate indices. Using holder inequality we have that:

$$|x_{j'} - x_j| = \sum_{l=j+1}^{j'} 1 \cdot h = \sum_{l=j+1}^{j'} v_l^{\frac{\gamma-1}{\gamma}} v_l^{\frac{1-\gamma}{\gamma}} h \leq \left(\sum_{l=j+1}^{j'} v_l h \right)^{\frac{\gamma-1}{\gamma}} \left(\sum_{l=j+1}^{j'} v_l^{1-\gamma} h \right)^{\frac{1}{\gamma}}$$

by the definition of ψ , we know $\psi(v) \sim v^{1-\gamma}$ as $v \rightarrow 0$, so $v^{1-\gamma} \leq C\psi(v) + 1, v > 0$ and then we get:

$$\begin{aligned} |x_{j'} - x_j| &\leq \left(C + \frac{1}{2} |x_{j'} - x_j| \right)^{\frac{\gamma-1}{\gamma}} \left(\sum_{l=j+1}^{j'} (C\psi(v_l) + 1)h \right)^{\frac{1}{\gamma}} \\ &\leq \left(C + \frac{1}{2} |x_{j'} - x_j| \right)^{\frac{\gamma-1}{\gamma}} (C + |x_{j'} - x_j|)^{\frac{1}{\gamma}} \end{aligned}$$

this proves (2.8). $\square$

Claim 2.2.2. For fix j and $t \in [0, T]$ and let $j' = j'(j, t)$ as in claim 2.2.1, then for $l < j'$:

$$v_l(t)A(t) = \frac{v_l(0) + I_l(t)}{v_{j'(j,t)}(0)B_l(t)} \quad (2.10)$$

where

$$\begin{aligned} A(t) &= \frac{\exp\left(\int_0^t p_{j'(j,t)}(\tau) + \delta(\delta v)_{j'(j,t)}(\tau) d\tau\right)}{v_{j'(j,t)}(t)} \\ B_l(t) &= \exp\left(\sum_{l < k < j'(j,t)} (u_k(t) - u_k(0))h\right) \\ I_l(t) &= \int_0^t v_{j'(j,s)}(0)A(s)B_l(s)v_l(s)(p_l(s) + \delta(\delta v)_l(s))ds. \end{aligned}$$

Proof of claim 2.2.2. Denote $E(v) = \ln v$, $E_j = E(v_j)$ and then by the equation (2.2) and (2.1),

$$\begin{aligned} \delta \dot{E}_k &= \delta \left(\frac{\delta u}{v} \right)_k = \dot{u}_k + \delta p_k + \delta(\delta(\delta v))_k \\ \delta E_k(\cdot)|_0^s &= u_k(\cdot)|_0^s + \int_0^s (\delta p_k + \delta(\delta(\delta v))_k)(\tau) d\tau \\ &\Rightarrow (E_{j'(j,s)}(\cdot) - E_l(\cdot))|_0^s = \sum_{l < k < j'(j,s)} \delta E_k(\cdot)|_0^s h \\ &= \sum_{l < k < j'(j,s)} u_k(\cdot)|_0^s h + \int_0^s (p_{j'(j,s)}(\tau) - p_l(\tau)) d\tau \\ &\quad + \int_0^s [\delta(\delta v)_{j'(j,s)}(\tau) - \delta(\delta v)_l(\tau)] d\tau \\ &= \ln B_l(s) + \int_0^s (p_{j'(j,s)}(\tau) - p_l(\tau)) d\tau \\ &\quad + \int_0^s [\delta(\delta v)_{j'(j,s)}(\tau) - \delta(\delta v)_l(\tau)] d\tau \\ &\Rightarrow -E_l(s) + \int_0^s [p_l(\tau) + \delta(\delta v)_l(\tau)] d\tau \\ &= -E_{j'(j,s)}(s) + E_{j'(j,s)}(0) - E_l(0) + B_l(s) \\ &\quad + \int_0^s [p_{j'(j,s)}(\tau) + \delta(\delta v)_{j'(j,s)}(\tau)] d\tau \end{aligned}$$

take exponentials, we then get:

$$\begin{aligned} \frac{\exp\left(\int_0^s p_l(\tau) + \delta(\delta v)_l(\tau) d\tau\right)}{v_l(s)} &= \frac{v_{j'(j,s)}(0) \exp\left(\int_0^s p_{j'(j,s)}(\tau) + \delta(\delta v)_{j'(j,s)}(\tau) d\tau\right)}{v_l(0)v_{j'(j,s)}(s)} B_l(s) \\ &= \frac{v_{j'(j,s)}(0)}{v_l(0)} A(s) B_l(s) \end{aligned} \quad (2.11)$$

multiplying by $v_l(s)(p_l(s) + \delta(\delta v)_l(s))$, we then get:

$$\frac{d}{ds} \exp \left(\int_0^s p_l(\tau) + \delta(\delta v)_l(\tau) d\tau \right) = \frac{v_{j'(j,s)}(0)}{v_l(0)} A(s) B_l(s) v_l(s) (p_l(s) + \delta(\delta v)_l(s))$$

Thus:

$$\exp \left(\int_0^t p_l(s) + \delta(\delta v)_l(s) ds \right) = 1 + \frac{I_l(t)}{v_l(0)} = \frac{v_l(0) + I_l(t)}{v_l(0)}$$

Substituting this into (2.11) and rearranging, we then obtain (2.10). $\square$

Pointwise bounds for $v_j(t)$ will follow easily from (2.10) once we have estimated the quantities $A(t)$ and $B_l(t)$ appeared in claim 2.2.2.

Claim 2.2.3. For $j \leq l \leq j'(j, t)$, there holds:

$$0 < \frac{1}{C} \leq B_l(t) \leq C \quad (2.12)$$

$$A(t) \leq C(t). \quad (2.13)$$

Proof of claim 2.2.3.

$$\begin{aligned} \left| \sum_{l < k < j'(j,t)} (u_k(t) - u_k(0)) h \right| &\leq C \left(\sum_{l < k < j'(j,t)} (u_k(t)^2 + u_k(0)^2) h \right)^{1/2} \left(\sum_{l < k < j'(j,t)} 1^2 h \right)^{1/2} \\ &\leq C \cdot |x_{j'} - x_j|^{1/2} \leq C \end{aligned}$$

this implies (2.12).

Next to prove (2.13), by (2.6)

$$\begin{aligned} A(t) &\leq A(t) \sum_{l=j+1}^{j'(j,t)} v_l(t) h = \sum_{l=j+1}^{j'(j,t)} (A(t) v_l(t)) h = \sum_{l=j+1}^{j'(j,t)} \frac{v_l(0) + I_l(t)}{v_{j'(j,t)}(0) B_l(t)} h \\ &\leq C \sum_{l=j+1}^{j'(j,t)} (v_l(0) + I_l(t)) h \end{aligned} \quad (2.14)$$

by the definition of ψ , we know $v \leq C(\psi(v) + 1)$ when $v > 0$, so

$$\sum_{l=j+1}^{j'(j,t)} v_l(0) h \leq C(1 + |x_{j'} - x_j|) \leq C$$

and by the initial condition $v_j(0) \leq \bar{v}$ and (2.12), we get

$$I_l(t) \leq C \int_0^t A(s) v_l(s) (p_l(s) + \delta(\delta v)_l(s)) ds$$

so that (2.14) becomes:

$$A(t) \leq C + C \int_0^t A(s) \left(\sum_{l=j+1}^{j'(j,t)} v_l(s) (p_l(s) + \delta(\delta v)_l(s)) h \right) ds \quad (2.15)$$

and if we know that

$$\sum_{l=j+1}^{j'(j,t)} v_l(s) (p_l(s) + \delta(\delta v)_l(s)) h \leq C$$

then call for the Gronwall's inequality to (2.15) would proves (2.13). And we observe

$$\left| \sum_{1/2}^{\infty} \delta(\delta v(s))_l v_l(s) h \right| = \left| - \sum_0^{\infty} \delta v_k(s)^2 h \right| \leq C$$

by the property of convergent serie we get $\left| \sum_{j+1}^{j'(j,t)} \delta(\delta v(s))_l v_l(s) h \right| \leq C(T)$, so we leave to prove

$$\sum_{1/2}^{\infty} p_j(s) v_j(s) h \leq C \quad (2.16)$$

For this aim, let's multiplying (2.2) by u_k and rearrange

$$\frac{d}{dt} \frac{u_k^2}{2} + u_k \cdot \delta \left(p + \delta(\delta v) - \frac{\delta u}{v} \right)_k = 0$$

summing by parts, we therefore obtain

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \sum u_k^2 h \right) + \frac{d}{dt} \left(\frac{1}{2} \sum \delta v_k^2 h \right) + \sum \frac{\delta u_j^2}{v_j} h - \sum p_j \delta u_j h &= 0 \\ \Rightarrow \int_0^t \sum p_j \delta u_j h &= \int_0^t \left(\frac{d}{ds} \left(\frac{1}{2} \sum u_k^2 h \right) + \frac{d}{ds} \left(\frac{1}{2} \sum \delta v_k^2 h \right) + \sum \frac{\delta u_j^2}{v_j} h \right) \end{aligned}$$

then applying the Lemma 2.1 to obtain

$$\left| \int_0^t \sum p_j \delta u_j h \right| \leq C$$

since $p'(v)v = Cp(v)$ and $\dot{v}_j = \delta u_j$,

$$\begin{aligned} \left| \sum p_j v_j h \right| &= \left| \sum p_j(0) v_j(0) h + \int_0^t \sum \frac{d}{ds} (p_j v_j) h \right| \\ &= \left| \sum p_j(0) v_j(0) h + \int_0^t \sum p_j \dot{v}_j h + \int_0^t \sum \dot{p}_j v_j h \right| \\ &\leq C + \left| \int_0^t \sum p'_j v_j \dot{v}_j h \right| \\ &\leq C + C \left| \int_0^t \sum p_j \delta u_j h \right| \\ &\leq C. \end{aligned}$$

□

In the above process, as a byproduct we obtain: $|I_l(t)| \leq C(t)$. Let's finish the proof of Lemma 2.2. By (2.10), we get

$$v_l(t) = \frac{v_l(0) + I_l(t)}{v_{j'(j,t)}(0)B_l(t)A(t)}$$

combine with $0 < C^{-1}(h) \leq v_j \leq C(h)$, uniformly for all t and j , and (2.12), (2.13)

$$v_l(t) = \frac{v_l(0) + I_l(t)}{v_{j'(j,t)}(0)B_l(t)A(t)} \geq C(t) > 0$$

this proved (2.5). $\square$

Lemma 2.3. *There is a positive constant $C(T)$ depending on C_0 and T but independent of h such that*

$$\int_0^T \sum \delta(\delta v)_j^2 h \leq C(T) \quad (2.17)$$

$$\sup_{0 \leq t \leq T} \left(t \sum \delta u_j^2 h + t \sum \delta(\delta v)_j^2 h \right) + \int_0^T t \sum \delta(\delta u)_k^2 h \leq C(T) \quad (2.18)$$

$$\sup_{0 \leq t \leq T} \left(t^2 \sum \dot{u}_k^2 h + t^2 \sum \delta \dot{v}_k^2 h \right) + \int_0^T t^2 \sum \delta \dot{u}_j^2 h \leq C(T) \quad (2.19)$$

$$\sup_{0 \leq t \leq T} \left(t^2 \sum \delta(\delta(\delta v))_k^2 h \right) \leq C(T) \quad (2.20)$$

$$\sup_{0 \leq t \leq T} \left(t^4 \sum \delta \dot{u}_j^2 h + t^4 \sum \delta(\delta \dot{v})_j^2 h \right) \leq C(T). \quad (2.21)$$

Proof. Multiplying (2.2) by $\delta v_k h$, summing over k , integrating with respect to t , and rearranging, we arrive at

$$\begin{aligned} & \int_0^t \sum \dot{u}_k(s) \delta v_k(s) h ds + \int_0^t \sum \delta p_k \delta v_k h ds \\ &= \int_0^t \sum \delta \left(\frac{\delta u}{v} \right)_k \delta v_k h ds - \int_0^t \sum \delta(\delta(\delta v))_k \delta v_k h ds \\ & \sum u_k \delta v_k(s) h \Big|_0^t - \int_0^t \sum u_k \delta \dot{v}_k h + \int_0^t \sum \delta p_k \delta v_k h \\ &= - \int_0^t \sum \left(\frac{\delta u}{v} \right)_j \delta(\delta v)_j h + \int_0^t \sum \delta(\delta v)_j^2 h \\ & \int_0^t \sum \delta(\delta v)_j^2 h = \int_0^t \sum \frac{\delta u_j \delta(\delta v)_j}{v_j} h + \sum u_k \delta v_k(s) h \Big|_0^t \\ & \quad + \int_0^t \sum \delta u_j^2 h + \int_0^t \sum \delta p_k \delta v_k h \\ & \leq \varepsilon \int_0^t \sum \delta(\delta v)_j^2 h + C(\varepsilon) \int_0^t \sum \frac{\delta u_j^2}{v_j} h + \sum u_k^2(t) h + \sum \delta v_k^2(t) h \\ & \quad + \int_0^t \sum \delta u_j^2 h + C \int_0^t \sum \delta v_k^2 h \leq C(T) \end{aligned}$$

this proved (2.17).

In order to prove (2.18), we multiply both sides of (2.2) by $s\delta(\delta u)_k h$, sum over k , and integrate with respect to s on $[0, t]$. Summing by parts wherever necessary and rearranging terms, we obtain that

$$\begin{aligned}
 & \int_0^t s \sum \dot{u}_k(s) \delta(\delta u)_k(s) h ds + \int_0^t s \sum \delta p_k \delta(\delta u)_k h ds \\
 &= \int_0^t s \sum \delta \left(\frac{\delta u}{v} \right)_k \delta(\delta u)_k h ds - \int_0^t s \sum \delta(\delta(\delta v))_k \delta(\delta u)_k h ds \\
 &\quad - \int_0^t s \sum \delta \dot{u}_j \delta u_j h + \int_0^t s \sum \delta p_k \delta(\delta u)_k h \\
 &= \int_0^t s \sum \left(\frac{\delta u_{k+1/2}}{v_{k+1/2}} - \frac{\delta u_{k-1/2}}{v_{k-1/2}} \right)_k \delta(\delta u)_k - \int_0^t s \sum \delta(\delta(\delta v))_k \delta \dot{v}_k h \\
 &\quad - \int_0^t s \frac{d}{ds} \left(\frac{1}{2} \sum \delta u_j^2 h \right) + \int_0^t s \sum \delta p_k \delta(\delta u)_k h \\
 &= \int_0^t s \sum \frac{\delta u_{k+1/2} v_{k-1/2} - \delta u_{k-1/2} v_{k+1/2}}{v_{k+1/2} v_{k-1/2}} \delta(\delta u)_k + \int_0^t s \frac{d}{ds} \left(\frac{1}{2} \sum \delta(\delta v)_j^2 h \right) \\
 &\quad - \int_0^t s \frac{d}{ds} \left(\frac{1}{2} \sum \delta u_j^2 h \right) + \int_0^t s \sum \delta p_k \delta(\delta u)_k h \\
 &= \int_0^t s \sum \frac{\delta(\delta u)_k^2}{v_{k+1/2}} h - \int_0^t s \sum \frac{\delta u_{k-1/2} \delta v_k \delta(\delta u)_k}{v_{k+1/2} v_{k-1/2}} h + \int_0^t s \frac{d}{ds} \left(\frac{1}{2} \sum \delta(\delta v)_j^2 h \right) \\
 &\quad + \frac{1}{2} t \sum \delta u_j^2 h + \frac{1}{2} t \sum \delta(\delta v)_j^2 h + \int_0^t s \sum \frac{\delta(\delta u)_k^2}{v_{k+1/2}} h \\
 &= \int_0^t \frac{1}{2} \sum \delta u_j^2 h + \int_0^t \frac{1}{2} \sum \delta(\delta v)_j^2 h + \int_0^t s \sum \delta p_k \delta(\delta u)_k h \\
 &\quad + \int_0^t s \sum \frac{\delta u_{k-1/2} \delta v_k \delta(\delta u)_k}{v_{k+1/2} v_{k-1/2}} h \\
 &\leq C(T) + \varepsilon \int_0^t s \sum \delta(\delta u)_k^2 h + C(\varepsilon) \int_0^t s \sum \delta p_k^2 h + \varepsilon \int_0^t s \sum \frac{\delta(\delta u)_k^2}{v_{k+1/2} v_{k-1/2}} h \\
 &\quad + C(\varepsilon) \int_0^t s \sum \frac{\delta u_{k-1/2}^2 \delta v_k^2}{v_{k+1/2} v_{k-1/2}} h
 \end{aligned}$$

which leads to

$$\begin{aligned}
 & \sup_{0 \leq t \leq T} \left(t \sum \delta u_j^2 h + t \sum \delta(\delta v)_j^2 h \right) + \int_0^T t \sum \delta(\delta u)_k^2 h \leq C(T) \\
 & + C(T) \int_0^T t \sum \delta u_{k-1/2}^2 \delta v_k^2 h \leq C(T) + C(T) \sup_{0 \leq t \leq T} (t \max_k \delta v_k^2) \quad (2.22)
 \end{aligned}$$

we estimate $t \max_k \delta v_k^2$ as following

$$\begin{aligned}
 t \max_k \delta v_k^2 &= t \delta v_{k_0}^2 = t \sum_0^{k_0} (\delta v_k^2 - \delta v_{k-1}^2) = t \sum_0^{k_0} (\delta v_k + \delta v_{k-1}) \delta(\delta v)_j h \\
 &\leq t^{1/2} \left(\sum_0^{k_0} (\delta v_k + \delta v_{k-1})^2 h \right)^{1/2} \left(t \sum_0^{k_0} \delta(\delta v)_j^2 h \right)^{1/2} \\
 &\leq C(T)
 \end{aligned}$$

take the last inequality to (2.21), and apply the Young inequality with ε , we obtain (2.18).

We now multiply both sides of (2.2) by $s \dot{u}_k h$, sum by parts, and integrate with respect to t to arrive at

$$\begin{aligned}
 &\int_0^t s \sum \dot{u}_k^2 h ds + \int_0^t s \sum \delta p_k \dot{u}_k h ds \\
 &= \int_0^t s \sum \delta \left(\frac{\delta u}{v} \right)_k \dot{u}_k h ds - \int_0^t s \sum \delta(\delta(\delta v))_k \dot{u}_k h ds \\
 &\int_0^t s \sum \dot{u}_k^2 h = - \int_0^t s \sum \delta p_k \dot{u}_k h + \int_0^t s \sum \delta \left(\frac{\delta u}{v} \right)_k \dot{u}_k h \\
 &\quad - \int_0^t s \sum \delta(\delta(\delta v))_k \dot{u}_k h \\
 &= - \int_0^t s \sum \delta p_k \dot{u}_k h - \int_0^t s \sum \frac{\delta u_j \delta \dot{u}_j}{v} h + \int_0^t s \sum \delta(\delta v)_j \delta \dot{u}_j h \\
 &\leq \left| \int_0^t s \sum \delta p_k \dot{u}_k h \right| + \left| \int_0^t s \sum \frac{\delta u_j \delta \dot{u}_j}{v} h \right| + \left| \int_0^t s \sum \delta(\delta v)_j \delta \dot{u}_j h \right| \\
 &=: J_1 + J_2 + J_3 \tag{2.23}
 \end{aligned}$$

$$\begin{aligned}
 J_1 &= \left| \int_0^t s \sum \delta p_k \dot{u}_k h \right| \leq \varepsilon \int_0^t s \sum \dot{u}_k^2 h + C(\varepsilon) \int_0^t s \sum \delta p_k^2 h \\
 &\leq \varepsilon \int_0^t s \sum \dot{u}_k^2 h + C(\varepsilon) \int_0^t s \sum (p'(\xi) \delta v_k)^2 h \quad \text{where } \xi \in [v_{k-1/2}, v_{k+1/2}] \\
 &\leq \varepsilon \int_0^t s \sum \dot{u}_k^2 h + C(\varepsilon) \cdot P \cdot \int_0^t s \sum \delta v_k^2 h \\
 &\leq \varepsilon \int_0^t s \sum \dot{u}_k^2 h + C(T)
 \end{aligned}$$

where $P = \max(p(v_j), p'(v_j), p''(v_j))$ is arrived by (2.5).

$$\begin{aligned}
J_2 &= \left| \int_0^t s \sum \frac{\delta u_j \delta \dot{u}_j}{v} h \right| = \left| t \sum \frac{\delta u_j^2}{v_j} h - \int_0^t \frac{d}{ds} \left(\sum \frac{\delta u_j}{v_j} \right) \delta u_j h \right| \\
&= \left| \frac{1}{2} t \sum \frac{\delta u_j^2}{v_j} h - \frac{1}{2} \int_0^t \sum \frac{\delta u_j^2}{v_j} h + \frac{1}{2} \int_0^t s \sum \frac{\delta u_j^3}{v_j} h \right| \\
&\leq C(T) + C(T) \int_0^t s \sum |\delta u_j^3| h \\
J_3 &= \left| \int_0^t s \sum \delta(\delta v)_j \delta \dot{u}_j h \right| = \left| \int_0^t s \sum \delta v_k \delta(\delta \dot{u})_k h \right| \\
&= \left| t \sum \delta v_k \delta(\delta u)_k h - \int_0^t \frac{d}{ds} \left(s \sum \delta v_k \right) \delta(\delta u)_k h \right| \\
&= \left| -t \sum \delta(\delta v)_j \delta u_j h + \int_0^t \sum \delta(\delta v)_j \delta u_j h - \int_0^t s \sum \delta(\delta u)_k^2 h \right| \\
&\leq t \sum \delta(\delta v)_j^2 h + t \sum \delta u_j^2 h + \left| \int_0^t \sum \delta(\delta v)_j \dot{v}_j h \right| + \left| \int_0^t s \sum \delta(\delta u)_k^2 h \right| \\
&\leq C(T) + \left| \int_0^t \sum \delta(\delta v)_j \dot{v}_j h \right| \\
&\leq C(T) + \sum \delta v_k^2 h \\
&\leq C(T)
\end{aligned}$$

substituting J_1, J_2, J_3 into (2.23)

$$\begin{aligned}
\int_0^t s \sum \dot{u}_k^2 h &\leq C(T) + C(T) \int_0^t s \sum |\delta u_j^3| h \leq C(T) \\
&\quad + C(T) \sup_{0 \leq t \leq T} (t \max |\delta u_j|)
\end{aligned} \tag{2.24}$$

Next, by differentiating (2.2) with respect to s , multiplying by $s^2 \dot{u}_k h$, summing over k , and integrating, we show that

$$\begin{aligned}
&\int_0^t s^2 \sum \ddot{u}_k \dot{u}_k h ds + \int_0^t s^2 \sum \delta \dot{p}_k \dot{u}_k h ds \\
&= \int_0^t s^2 \sum \frac{d}{ds} \left(\delta \left(\frac{\delta u}{v} \right)_k \right) \dot{u}_k h ds - \int_0^t s^2 \sum \delta(\delta(\delta \dot{v}))_k \dot{u}_k h ds \\
&\frac{1}{2} \int_0^t s^2 \frac{d}{ds} \left(\sum \dot{u}_k^2 h \right) + \frac{1}{2} \int_0^t s^2 \frac{d}{ds} \left(\sum \delta \dot{v}_k^2 h \right) \\
&= - \int_0^t s^2 \sum \frac{d}{ds} \left(\frac{\delta u}{v} \right)_j \delta \dot{u}_j h ds - \int_0^t s^2 \sum \delta \dot{p}_k \dot{u}_k h \\
&= - \int_0^t s^2 \sum \frac{\delta \dot{u}_j v_j - \delta u_j \dot{v}_j}{v_j^2} \delta \dot{u}_j h + \int_0^t s^2 \sum \dot{p}_j \delta \dot{u}_j h \\
&= - \int_0^t s^2 \sum \frac{\delta \dot{u}_j^2}{v_j} h + \int_0^t s^2 \sum \frac{\delta u_j^2 \delta \dot{u}_j}{v_j^2} h + \int_0^t s^2 \sum \dot{p}_j \delta \dot{u}_j h
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}t^2 \sum \dot{u}_k^2 h + \frac{1}{2}t^2 \sum \delta \dot{v}_k^2 h - \int_0^t s \sum \dot{u}_k^2 h - \int_0^t s \sum \delta \dot{v}_k^2 h \\
&= - \int_0^t s^2 \sum \frac{\delta \dot{u}_j^2}{v_j} h + \int_0^t s^2 \sum \frac{\delta u_j^2 \delta \dot{u}_j}{v_j^2} h + \int_0^t s^2 \sum \dot{p}_j \delta \dot{u}_j h \\
& \frac{1}{2}t^2 \sum \dot{u}_k^2 h + \frac{1}{2}t^2 \sum \delta \dot{v}_k^2 h + \int_0^t s^2 \sum \frac{\delta \dot{u}_j^2}{v_j} h \\
&= \int_0^t s \sum \dot{u}_k^2 h + \int_0^t s \sum \delta \dot{v}_k^2 h + \int_0^t s^2 \sum \frac{\delta u_j^2 \delta \dot{u}_j}{v_j^2} h + \int_0^t s^2 \sum \dot{p}_j \delta \dot{u}_j h \\
&=: \int_0^t s \sum \dot{u}_k^2 h + K_1 + K_2 + K_3 \\
K_1 &= \int_0^t s \sum \delta \dot{v}_k^2 h = \int_0^t s \sum \delta \dot{v}_k \delta (\delta u)_k h \leq \varepsilon \int_0^t s \sum \delta \dot{v}_k^2 h + C(\varepsilon) \int_0^t s \sum \delta (\delta u)_k^2 h \\
K_2 &= \int_0^t s^2 \sum \frac{\delta u_j^2 \delta \dot{u}_j}{v_j^2} h \leq \varepsilon \int_0^t s^2 \sum \delta \dot{u}_j^2 h + C(\varepsilon) \int_0^t s^2 \sum \delta u_j^4 h \\
K_3 &= \int_0^t s^2 \sum \dot{p}_j \delta \dot{u}_j h \leq \varepsilon \int_0^t s^2 \sum \delta \dot{u}_j^2 h + C(\varepsilon) \int_0^t s^2 \sum \dot{p}_j^2 h \\
&\leq \varepsilon \int_0^t s^2 \sum \delta \dot{u}_j^2 h + C(\varepsilon) \cdot P \cdot \int_0^t s^2 \sum \dot{v}_j^2 h \\
&\leq \varepsilon \int_0^t s^2 \sum \delta \dot{u}_j^2 h + C(\varepsilon) \int_0^t s^2 \sum \delta u_j^2 h \leq \varepsilon \int_0^t s^2 \sum \delta \dot{u}_j^2 h + C(T)
\end{aligned}$$

then we get

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left(t^2 \sum \dot{u}_k^2 h + t^2 \sum \delta \dot{v}_k^2 h \right) + \int_0^T t^2 \sum \delta \dot{u}_j^2 h \\
& \leq C(T) + C(T) \int_0^T t \sum \dot{u}_k^2 h + C(T) \int_0^T t^2 \sum \delta u_j^4 h \quad (2.25)
\end{aligned}$$

substituting (2.24) into (2.25), we then obtain

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left(t^2 \sum \dot{u}_k^2 h + t^2 \sum \delta \dot{v}_k^2 h \right) + \int_0^T t^2 \sum \delta \dot{u}_j^2 h \leq C(T) \\
& + C(T) \sup_{0 \leq t \leq T} (t^2 \max_j |\delta u_j|^2) \quad (2.26)
\end{aligned}$$

Dealing with $t^2 \max_j |\delta u_j|^2$ similar to $t \max_j |\delta v_k|^2$:

$$\begin{aligned}
t^2 \max_j |\delta u_j|^2 &\leq C t^2 \left(\sum \delta u_j^2 h \right)^{1/2} \left(\sum \delta (\delta u)_k^2 h \right)^{1/2} \\
&\leq C \left(t^2 \sum \delta u_j^2 h + t^2 \sum \delta \dot{v}_k^2 h \right) \leq C(T)
\end{aligned}$$

inserting this inequality into (2.26), we obtain (2.19).

From Eq. (2.2),

$$\begin{aligned} t^2 \sum \delta(\delta(v))_k^2 h &= t^2 \sum \left(\frac{\delta(\delta u)_k}{v_{k+1/2}} - \frac{\delta u_{k-1/2} \delta v_k}{v_{k+1/2} v_{k-1/2}} - \dot{u}_k - p'(\xi) \delta v_k \right)^2 h \\ &\leq C(T) \left(t^2 \sum \dot{u}_k^2 h + t^2 \sum \delta v_k^2 h + t^2 \sum \delta(\delta u)_k^2 h + t^2 \sum \delta u_j^2 \delta v_k^2 h \right) \\ &\leq C(T) \left(1 + t^2 \max_j |\delta u_j|^2 \sum \delta v_k^2 h \right) \\ &\leq C(T). \end{aligned}$$

this proved (2.20).

In order to prove (2.21), we first differentiate (2.2) with respect to s , multiply by $s^3 \delta \dot{v}_k h$, sum by parts, integrate with respect to s to arrive at

$$\begin{aligned} &\int_0^t s^3 \sum \ddot{u}_k \delta \dot{v}_k h + \int_0^t s^3 \sum \delta \dot{p}_k \delta \dot{v}_k h \\ &= \int_0^t s^3 \sum \frac{d}{ds} \left(s \left(\frac{\delta u}{v} \right)_k \right) \delta \dot{v}_k h - \int_0^t s^3 \sum \delta(\delta(\dot{v}))_k \delta \dot{v}_k h \\ &\Rightarrow \int_0^t s^3 \sum \delta(\delta \dot{v})_j^2 h = \int_0^t s^3 \sum \ddot{u}_k \delta \dot{v}_k h + \int_0^t s^3 \sum \delta \dot{p}_k \delta \dot{v}_k h \\ &\quad + \int_0^t s^3 \sum \frac{d}{ds} \left(\frac{\delta u}{v} \right)_j \delta(\delta \dot{v})_j h \tag{2.27} \\ &\int_0^t s^3 \sum \ddot{u}_k \delta \dot{v}_k h = t^3 \sum \dot{u}_k \delta \dot{v}_k h - 3 \int_0^t s^2 \sum \dot{u}_k \delta \dot{v}_k h - \int_0^t s^3 \sum \dot{u}_k \delta \ddot{v}_k h \\ &\leq t^3 \sum \dot{u}_k^2 h + t^3 \sum \delta \dot{v}_k^2 h + 3 \int_0^t s^2 \sum \dot{u}_k^2 h + 3 \int_0^t s^2 \sum \delta \dot{v}_k^2 h \\ &\quad + \int_0^t s^3 \sum \delta \dot{u}_j^2 h \\ &\leq C(T) \\ &\int_0^t s^3 \sum \delta \dot{p}_k \delta \dot{v}_k h \leq (1 + P^2) \int_0^t s^3 \sum \delta \dot{v}_k^2 h \leq C(T) \\ &\int_0^t s^3 \sum \frac{d}{ds} \left(\frac{\delta u}{v} \right)_j \delta(\delta \dot{v})_j h \\ &\leq \varepsilon \int_0^t s^3 \sum \delta(\delta \dot{v})_j^2 h + C(\varepsilon) \int_0^t s^3 \sum \left(\frac{\delta \dot{u}_j}{v_j} - \frac{\delta u_j^2}{v_j^2} \right)^2 h \\ &\leq \varepsilon \int_0^t s^3 \sum \delta(\delta \dot{v})_j^2 h \\ &\quad + C(T) \int_0^t s^2 \sum \delta \dot{u}_j^2 h + C(T) \sup_{0 \leq t \leq T} (t^2 \max_j |\delta u_j|^2) \int_0^t \sum \delta u_j^2 h \\ &\leq C(T) + \varepsilon \int_0^t s^3 \sum \delta(\delta \dot{v})_j^2 h \end{aligned}$$

insert the last three inequality into (2.27), we obtain

$$\int_0^t s^3 \sum \delta(\delta\dot{v})_j^2 h \leq C(T) \quad (2.28)$$

Differentiating (2.2) with respect to s , multiplying by $s^4 \delta(\delta\dot{u})_k h$, integrating, and summing by parts, we can also obtain that

$$\begin{aligned} & \frac{1}{2}t^4 \sum \delta\dot{u}_j^2 h + \frac{1}{2}t^4 \sum \delta(\delta\dot{v})_j^2 h + \int_0^t s^4 \sum \frac{d}{ds} \left(\delta \left(\frac{\delta u}{v} \right)_k \right) \delta(\delta\dot{u})_k h \\ &= 2 \int_0^t s^3 \sum \delta\dot{u}_j^2 h + 2 \int_0^t t^3 \sum \delta(\delta\dot{v})_j^2 h + \int_0^t s^4 \sum \delta\dot{p}_k \delta(\delta\dot{u})_k h \\ &\leq C(T) + \int_0^t s^4 \sum \delta\dot{p}_k \delta(\delta\dot{u})_k h \\ &\int_0^t s^4 \sum \delta\dot{p}_k \delta(\delta\dot{u})_k h = - \int_0^t s^4 \sum \dot{p}_j \delta(\delta\ddot{v})_j h = - \int_0^t s^4 \sum p'(\xi) \dot{v}_j \delta(\delta\ddot{v})_j h \\ &\leq C \left| \int_0^t s^4 \sum \dot{v}_j \delta(\delta\ddot{v})_j h \right| \leq C \left| \int_0^t s^4 \sum \delta\dot{v}_k \delta\ddot{v}_k h \right| \\ &\leq C \left| \int_0^t s^4 \sum \frac{d}{ds} (\delta\dot{v}_k^2) h \right| \\ &\leq Ct^4 \sum \delta\dot{v}_k^2 h + \leq C \int_0^t s^3 \sum \delta\dot{v}_k^2 h \\ &\leq C(T) \end{aligned}$$

insert this into the above inequality to obtain

$$\frac{1}{2}t^4 \sum \delta\dot{u}_j^2 h + \frac{1}{2}t^4 \sum \delta(\delta\dot{v})_j^2 h + \int_0^t s^4 \sum \frac{d}{ds} \left(\delta \left(\frac{\delta u}{v} \right)_k \right) \delta(\delta\dot{u})_k h \leq C(T) \quad (2.29)$$

$$\begin{aligned} & \frac{1}{2}t^4 \sum \delta\dot{u}_j^2 h + \frac{1}{2}t^4 \sum \delta(\delta\dot{v})_j^2 h \\ &+ \int_0^t s^4 \sum \frac{d}{ds} \left(\frac{\delta(\delta u)_k}{v_{k+1/2}} - \frac{\delta u_{k-1/2} \delta v_k}{v_{k+1/2} v_{k-1/2}} \right) \delta(\delta\dot{u})_k h \leq C(T) \\ & \frac{1}{2}t^4 \sum \delta\dot{u}_j^2 h + \frac{1}{2}t^4 \sum \delta(\delta\dot{v})_j^2 h + \int_0^t s^4 \sum \frac{\delta(\delta\dot{u})_k^2}{v_{k+1/2}} h \\ &\leq C(T) + \int_0^t s^4 \sum \frac{\delta(\delta\dot{u})_k}{v_{k+1/2} v_{k-1/2}} \left(\frac{d}{ds} (\delta u_{k-1/2} \delta v_k) \right) h \\ &+ \int_0^t s^4 \sum \frac{\delta(\delta u)_k \delta u_{k+1/2}}{v_{k+1/2}^2} \delta(\delta\dot{u})_k h \\ &- \int_0^t s^4 \sum \frac{\delta(\delta\dot{u})_k (\delta u_{k-1/2} \delta v_k)}{(v_{k+1/2} v_{k-1/2})^2} \frac{d}{ds} (v_{k+1/2} v_{k-1/2}) h \\ &=: C(T) + L_1 + L_2 - L_3 \end{aligned}$$

$$\begin{aligned}
|-L_3| &= \left| - \int_0^t s^4 \sum \frac{\delta(\delta \dot{u})_k (\delta u_{k-1/2} \delta v_k)}{(v_{k+1/2} v_{k-1/2})^2} \frac{d}{ds} (v_{k+1/2} v_{k-1/2}) h \right| \\
&\leq C(T) \left| \int_0^t s^4 \sum \delta v_k \delta(\delta \dot{u})_k h \right| \\
&\leq C(T) \left| \int_0^t s^4 \sum \delta(\delta v)_j \delta \dot{u}_j h \right| \\
&\leq C(T) \int_0^t s^4 \sum \delta(\delta v)_j^2 h + C(T) \int_0^t s^4 \sum \delta \dot{u}_j^2 h \\
&\leq C(T) \\
|L_2| &= \left| \int_0^t s^4 \sum \frac{\delta(\delta u)_k \delta u_{k+1/2}}{v_{k+1/2}^2} \delta(\delta \dot{u})_k h \right| \leq C(T) \left| \int_0^t s^4 \sum \delta(\delta u)_k \delta(\delta \dot{u})_k h \right| \\
&\leq C(T) t^4 \sum \delta(\delta u)_k^2 h + \int_0^t s^3 \sum \delta(\delta u)_k^2 h \leq C(T) \\
|L_1| &= \left| \int_0^t s^4 \sum \frac{\delta(\delta \dot{u})_k}{v_{k+1/2} v_{k-1/2}} \left(\frac{d}{ds} (\delta u_{k-1/2} \delta v_k) \right) h \right| \\
&= \left| \int_0^t s^4 \sum \frac{\delta(\delta \dot{u})_k}{v_{k+1/2} v_{k-1/2}} ((\delta \dot{u}_{k-1/2} \delta v_k) + (\delta u_{k-1/2} \delta \dot{v}_k)) h \right| \\
&\leq C(T) \left| \int_0^t s^3 \sum \frac{\delta \dot{u}_{k-1/2} \delta(\delta \dot{u})_k}{v_{k+1/2} v_{k-1/2}} h \right| + \left| \int_0^t s^4 \sum \frac{\delta(\delta \dot{u})_k (\delta u_{k-1/2} \delta \dot{v}_k)}{v_{k+1/2} v_{k-1/2}} h \right| \\
&\leq C(T, \varepsilon) \int_0^t s^2 \sum \delta \dot{u}_j^2 h + \varepsilon \int_0^t s^4 \sum \frac{\delta(\delta \dot{u})_k^2}{v_{k+1/2}} h + \varepsilon \int_0^t s^4 \sum \frac{\delta(\delta \dot{u})_k^2}{v_{k+1/2}} h \\
&\quad + C(T, \varepsilon) \sup_{0 \leq t \leq T} (t^2 \max_j |\delta u_j|^2) \int_0^t s^2 \sum \delta \dot{v}_k^2 h \\
&\leq C(T) + \varepsilon \int_0^t s^4 \sum \frac{\delta(\delta \dot{u})_k^2}{v_{k+1/2}} h
\end{aligned}$$

and then we get

$$\frac{1}{2} t^4 \sum \delta \dot{u}_j^2 h + \frac{1}{2} t^4 \sum \delta(\delta \dot{v})_j^2 h + \int_0^t s^4 \sum \frac{\delta(\delta \dot{u})_k^2}{v_{k+1/2}} h \leq C(T)$$

this proved (2.21), and we now complete the proof of Lemma 2.3. $\square$

3. Weak solution

We construct approximate solutions v^h, u^h as follows: $v^h(x_j, t) = v_j(t)$ for $j + \frac{1}{2} \in \mathbb{Z}_+$, and $u^h(x_k, t) = u_k(t)$ for $k \in \mathbb{Z}_+ \cup \{0\}$, and take $v^h(\cdot, t)$ and $u^h(\cdot, t)$ to be linear on each interval $[x_j, x_{j+1}]$ and $[x_k, x_{k+1}]$, respectively. Below we state a number of regularity properties of v^h and u^h .

Lemma 3.1. *There are positive constants C , $C(T)$ depending on C_0 and C_0 , T , but independent of h such that*

$$0 < \frac{1}{C(T)} \leq v^h(t), \quad 0 \leq t \leq T \quad (3.1)$$

$$\sup_{0 \leq t \leq T} \left(\int_0^\infty (v^h(x, t) - v')^2 dx + \int_0^\infty v_x^h(x, t)^2 dx + t^2 \int_0^\infty \dot{v}_x^h(x, t)^2 dx \right) \leq C(T) \quad (3.2)$$

$$\sup_{0 \leq t \leq T} \left(\int_0^\infty u^h(x, t)^2 dx + t \int_0^\infty u_x^h(x, t)^2 dx \right) \leq C(T) \quad (3.3)$$

$$\sup_{0 \leq t \leq T} \left(t^2 \int_0^\infty \dot{u}^h(x, t)^2 dx + t^4 \int_0^\infty \dot{u}_x^h(x, t)^2 dx \right) \leq C(T) \quad (3.4)$$

$$\sup_{0 \leq t \leq T} \left(t \int_0^\infty \dot{v}^h(x, t)^2 dx + t^2 \int_0^\infty \dot{v}_x^h(x, t)^2 dx \right) \leq C(T) \quad (3.5)$$

$$\|v^h(\cdot, t_2) - v^h(\cdot, t_1)\|_{L^2} \leq C(T)|t_2 - t_1|^{1/2}. \quad (3.6)$$

There is also a positive constant $C(T, \tau)$ depending on C_0, T , and τ , such that for any $0 < \tau \leq t_1 < t_2 \leq T$

$$|u^h(x_1, t_1) - u^h(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_2 - t_1|) \quad (3.7)$$

$$|v^h(x_1, t_1) - v^h(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_2 - t_1|) \quad (3.8)$$

$$|u_x^h(x_1, t_1) - u_x^h(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_2 - t_1|^{1/4} + h^{1/2}) \quad (3.9)$$

$$|v_x^h(x_1, t_1) - v_x^h(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_2 - t_1|^{1/4} + h^{1/2}) \quad (3.10)$$

Proof. Inequalities (3.1)–(3.5) follow immediately from (2.4), (2.5) and (2.17)–(2.21). To prove (3.6) we refer to the inequality

$$\begin{aligned} \int_0^\infty (v^h(x, t_2) - v^h(x, t_1))^2 dx &\leq \int_0^\infty \left| \int_{t_1}^{t_2} |\dot{v}^h(x, t)| dt \right|^2 dx \\ &\leq |t_2 - t_1| \int_{t_1}^{t_2} \int_0^\infty u_x^h(x, t) dx dt \leq C(T)|t_2 - t_1|. \end{aligned}$$

Next, v^h and u^h are Hölder continuous with respect to x due to (3.2) and (3.3). Also,

$$\begin{aligned} |u^h(x, t_1) - u^h(x, t_2)| &\leq \int_{t_1}^{t_2} \|\dot{u}^h(\cdot, t)\|_{L^\infty} dt \leq \int_{t_1}^{t_2} \|\dot{u}^h(\cdot, t)\|_{L^2}^{1/2} \|\dot{u}_x^h(\cdot, t)\|_{L^2}^{1/2} dt \\ &\leq C(T, \tau)|t_2 - t_1| \end{aligned}$$

by (3.4). This completes the proof of (3.7), and the proof of (3.8) is similar.

The derivation of (3.9) is the most involved, and we will treat it by three steps below:

Step one, when x_1, x_2 are two of the net pots of x_j defined as in Sect 2, we then have

$$\begin{aligned} \left| u_x^h(x_j, t) - u_x^h(x_{j'}, t) \right| &\leq \left| \sum_{j < k < j'} \delta(\delta u)_k h \right| \leq \left(\sum_{j < k < j'} \delta(\delta u)_k^2 h \right)^{1/2} \left(\sum_{j < k < j'} h \right)^{1/2} \\ &\leq C(T, \tau) |x_j - x_{j'}|^{1/2} \end{aligned} \quad (3.11)$$

follows directly from (2.19), where $\delta \dot{v}_k = \delta(\delta u)_k$.

Step two, now we derive time-regularity bound. Let J be a large positive integer. Then by (3.11)

$$\begin{aligned} |u_x^h(x_j, t_1) - u_x^h(x_j, t_2)| &\leq |u_x^h(x_j, t_1) - u_x^h(x_i, t_1)| + |u_x^h(x_i, t_1) - u_x^h(x_i, t_2)| \\ &\quad + |u_x^h(x_j, t_2) - u_x^h(x_i, t_2)| \\ &\leq |u_x^h(x_i, t_1) - u_x^h(x_i, t_2)| + C(T, \tau) |x_j - x_{j'}|^{1/2} \Rightarrow \\ |u_x^h(x_j, t_1) - u_x^h(x_j, t_2)| &\leq \frac{1}{Jh} \sum_{|i-j| < J} |u_x^h(x_i, t_1) - u_x^h(x_i, t_2)| h + C(T, \tau)(Jh)^{1/2} \\ &\leq \sqrt{\frac{t_2 - t_1}{Jh}} \left(\int_{t_1}^{t_2} \dot{u}_x^h(\cdot, t)^2 h dt \right)^{1/2} + C(T, \tau)(Jh)^{1/2} \\ &\leq C(T, \tau) \left(\sqrt{\frac{t_2 - t_1}{Jh}} + (Jh)^{1/2} \right) \\ &\leq C(T, \tau) (|t_2 - t_1|^{1/4} + (Jh)^{1/2}) \end{aligned}$$

we choose J so that $Jh - h < \sqrt{t_2 - t_1} \leq Jh$. So (3.9) holds for x_1, x_2 are two of the net pots of x_j defined as in Sect. 2, and arbitrary $t_1, t_2 > 0$.

Step three, when $x_1 \in (x_j, x_{j+1})$ then:

$$u_x^h(x_1, t) = \delta u_j(t), \quad \text{when } x_1 \in (x_j, x_{j+1/2})$$

or else

$$u_x^h(x_1, t) = \delta u_{j+1}(t), \quad \text{when } x_1 \in (x_{j+1/2}, x_{j+1})$$

and when $x_1 = x_{j+1/2}$, we can define supplement by $u_x^h(x_1, t) = \delta u_j(t)$. Thus else

$$|u_x^h(x_1, t) - u_x^h(x_j, t)| = 0$$

or else

$$|u_x^h(x_1, t) - u_x^h(x_j, t)| = |\delta u_j - \delta u_{j+1}| \leq C(T, \tau) h^{1/2}$$

this proved (3.9). And we have completed this Lemma. $\square$

Lemma 3.2. *There is a sequence $h = \{h_l\} \rightarrow 0$ and functions $u, (v - v') \in H^1([0, \infty))$, such that:*

$$\begin{aligned} u^h - u, u_x^h - u_x, v^h - v, v_x^h - v_x &\rightarrow 0 \text{ uniformly on compact sets in} \\ [0, \infty) \times (t > 0). \end{aligned} \quad (3.12)$$

Proof. By the Hölder continuous and bounded estimate. It's a direct consequence of Lemma 3.1. $\square$

Now we define a new variable w_*^h by $w_*^h(x_k, t) = \delta v_k$ and taking w_*^h to be linear on $[x_k, x_{k+1}]$, $k = 0, 1, \dots$. We also introduce w^h :

$$w^h(x, t) = v^h(0, t) + \int_0^x w_x^h(y, t) dy. \quad (3.13)$$

Both variables are necessary to show that the limit of v^h as $h \rightarrow 0$ satisfies high order regularity estimates. Below we list several properties of the new variables.

Lemma 3.3. *There is a positive constant $C(T, \tau)$ depending on C_0 , T , and τ , such that for any $0 < \tau \leq t_1 < t_2 \leq T$:*

$$\sup_{0 \leq t \leq T} \left(t \int_0^\infty w_{*x}^h(x, t)^2 dx + t^4 \int_0^\infty \dot{w}_{*x}^h(x, t)^2 dx + t^2 \int_0^\infty w_{xxx}^h(x, t)^2 dx \right) \leq C(T), \quad (3.14)$$

$$|w_*^h(x_1, t_1) - w_*^h(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_2 - t_1|), \quad (3.15)$$

$$|w_{*x}^h(x_1, t_1) - w_{*x}^h(x_2, t_2)| \leq C(T, \tau)(|x_1 - x_2|^{1/2} + |t_2 - t_1|^{1/4} + h^{1/2}). \quad (3.16)$$

Proof. Inequality (3.14) follows from (2.18), (2.20) and (2.21), and the proof (3.15) and (3.16) is similar to those of (3.7)–(3.10). $\square$

Lemma 3.4. *There is a sequence $h = \{h_l\} \rightarrow 0$ and functions $w_* \in H^1([0, \infty))$, $w \in H^2([0, \infty))$, such that:*

$$w_{*x}^h - w_{*x}, \quad w^h - w, \quad w_x^h - w_x, \quad w_{xx}^h - w_{xx} \rightarrow 0 \text{ uniformly on compact sets in } [0, \infty) \times (t > 0). \quad (3.17)$$

Proof. It's a direct consequence of Lemma 3.3. $\square$

Lemma 3.5. $v \equiv w$.

Proof. We observe that $v^h(x_k, t) = w^h(x_k, t)$ for any k . Obviously $w^h(0, t) = v^h(0, t)$, and

$$\begin{aligned} w^h(x_k) - w^h(x_{k-1}) &= \int_{x_{k-1}}^{x_k} w_*^h(y) dy = \frac{h}{2} (w_*^h(x_k) - w_*^h(x_{k-1})) \\ &= \frac{1}{2} (v_{k+1/2} + v_{k-3/2}) \end{aligned}$$

and

$$\begin{aligned} v^h(x_k, t) - v^h(x_{k-1}, t) &= \frac{1}{2} (v_{k+1/2} + v_{k-1/2}) - \frac{1}{2} (v_{k-1/2} - v_{k-3/2}) \\ &= \frac{1}{2} (v_{k+1/2} + v_{k-3/2}) \end{aligned}$$

for all $k \geq 1$. We now recall the space regularity of $v^h(\cdot, t)$ and $w^h(\cdot, t)$ to complete the proof. $\square$

Finally we should derive that the limiting functions v and u are in fact the weak solution of (1.1)–(1.4).

Proof of Theorem 1. (1.18) is a direct consequence of (3.1) and (3.12), and we can obtain (1.19) from (3.2) and (3.3), since weak convergence is norm-decreasing.

Let ϕ be a smooth test function and set $\phi_j(t) = \phi(x_j, t)$. We multiply (2.1) by $\phi_j h$, sum, and integrate to obtain

$$\sum v_j \phi_j h|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum v_j \dot{\phi}_j h dt + \int_{t_1}^{t_2} \sum u_k \delta \phi_k h dt = 0. \quad (3.18)$$

We need to show that each term in (3.18) converges to the corresponding term in (1.14). Indeed, the equalities

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^\infty u^h \phi_x dx dt &= \int_{t_1}^{t_2} \sum_j \int_{x_j}^{x_{j+1}} u^h \phi_x dx dt \\ &= \int_{t_1}^{t_2} \left(\sum (u_k + O(h^{1/2})) (\delta \phi(x_k) + O(h)) h \right) dt \\ &= \int_{t_1}^{t_2} \sum u_k \delta \phi_k h dt + O(h^{1/2}) \end{aligned}$$

by (3.7), this proved for the last term in (3.18). The other terms in (3.18) can be treated in the same way. Take the test function φ as in Sect. 1, and abbreviate the same with above, multiplying now (2.2) by $\varphi_k h$, summing over k , and integrating with respect to t , arrive at:

$$\begin{aligned} \sum u_k \varphi_k h|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum u_k \dot{\varphi}_k h dt - \int_{t_1}^{t_2} \sum p_j \delta \varphi_j h dt \\ = - \int_{t_1}^{t_2} \sum \frac{\delta u_j}{v_j} \delta \varphi_j h dt + - \int_{t_1}^{t_2} \sum v_j \delta (\delta(\varphi))_j h dt. \end{aligned} \quad (3.19)$$

and give the argument only for the most difficult term, which is the first term on the right.

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^\infty \frac{u_x^h}{v^h} \varphi_x dx dt &= \int_{t_1}^{t_2} \sum_k \int_{x_k}^{x_{k+1}} \frac{u_x^h}{v^h} \varphi_x dx dt \\ &= \int_{t_1}^{t_2} \left(\sum (\delta u_j + O(h^{1/2})) (v_j^{-1} + O(h^{1/2})) (\delta \varphi(x_j) + O(h)) h \right) dt \\ &= \int_{t_1}^{t_2} \sum \frac{\delta u_j}{v_j} \delta \varphi_j h dt + O(h^{1/2}) \end{aligned}$$

by (3.8)–(3.9), thus showing that the first term on the right in (3.19) approaches the corresponding term in (1.15) as $h \rightarrow 0$. And the other terms can be handled in a same way.

And only leave to prove the regularity conditions (1.10) and (1.11). First, $u \in C((0, \infty); H^1([0, \infty)))$, $v \in C((0, \infty); H^2([0, \infty)))$ by (3.7)–(3.10), (3.12) and (3.15)–(3.17). Then the argument based on the uniform estimate (1.19) shows that:

$$v_x(\cdot, t) \rightharpoonup v_{ox} \quad \text{weakly in } L^2 \text{ as } t \rightarrow 0. \quad (3.20)$$

To prove strong convergence in (3.20), let $\zeta \in C_0^2((0, \infty))$ and take ζ_x as the test function in (1.14), we get

$$\begin{aligned} \int_0^\infty v \zeta_x dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^\infty (v \zeta_{xt} - u \zeta_{xx}) dx dt &= 0 \\ \int_0^\infty v_x \zeta dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_0^\infty u_x \zeta_x dx dt &= 0 \end{aligned}$$

set $t_1 = 0$,

$$\int_0^\infty v_x(x, t_2) \zeta(x) dx = \int_0^\infty v_x(x, 0) \zeta(x) dx - \int_0^{t_2} \int_0^\infty u_x \zeta_x dx dt$$

let ζ approach $v_x(x, t_2)$ to obtain

$$\int_0^\infty v_x(x, t_2)^2 dx = \int_0^\infty v_{0x}(x) v_x(x, t_2) dx - \int_0^{t_2} \int_0^\infty u_x v_{xx}(x, t_2) dx dt$$

last let $t_2 \rightarrow 0$, we obtain

$$v_x(\cdot, t) \rightarrow v_{0x} \quad \text{strongly in } L^2 \text{ as } t \rightarrow 0.$$

and then $v(\cdot, t) \in C([0, \infty); H^1([0, \infty))) \cap C((0, \infty); H^2([0, \infty)))$. A similar argument yields $u(\cdot, t) \in C([0, \infty); L^1([0, \infty))) \cap C((0, \infty); H^1([0, \infty)))$. $\square$

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Received: 9 June 2010.

Accepted: 1 February 2011.