On the global wellposedness for the nonlinear Schrödinger equations with L^p -large initial data

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Abstract. We consider the Cauchy problem for the nonlinear Schrödinger equations

$$\begin{aligned} iu_t + \Delta u \pm |u|^{p-1}u &= 0, & x \in \mathbb{R}^d, \quad t \in \mathbb{R} \\ u(x, 0) &= u_0(x), & x \in \mathbb{R}^d \end{aligned}$$

for $1 < p < 1 + 4/d$ and prove that there is a $\rho(p, d) \in (1, 2)$ such that the initial value problem is globally well posed for $u_0 \in L^\rho$ with $\rho(p, d) < \rho < 2$.

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1. Introduction

We consider the Cauchy problem for nonlinear Schrödinger equations in $\mathbb{R}^{1+d}$

$$iu_t + \Delta u \pm |u|^{p-1}u = 0, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R} \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^d \quad (1.2)$$

where $u = u(t, x) : \mathbb{R}^{1+d} \rightarrow \mathbb{C}$ is the unknown function, $d \in \mathbb{N}$ and $1 < p < 1 + 4/d$. Let $u_0 \in L^\rho(\mathbb{R}^d)$. We would like to solve the following problem:

Problem Under what condition for p, d and ρ the Cauchy problem (1.1), (1.2) in $\mathbb{R}^{1+d}$ has a unique (global) solution for any $u_0 \in L^\rho(\mathbb{R}^d)$?

First of all, it is well known (see [12]) that the Cauchy problem has a unique global solution for large L^2 initial data. When $\rho \neq 2$, there are not many authors who reported on this subject (see Kato[8]). However, the authors think the problem is very interesting. For instance, LIA model for the vortex filament can be reduced to (1.1), (1.2) with $p = 3, d = 1$ via the Hasimoto transform, and through this process, applying the inverse scattering method,

a lot of physically interesting dynamics arise where the L^2 -theory no longer works (See Vargas and Vega [13] and references there). So, it is natural to seek a solution of (1.1), (1.2) with infinite L^2 -data. In this paper, motivated by such interests, we construct a global solution especially for large L^p -initial data. And we expect the existence when $\rho < 2$ and ρ is near 2 for two reasons. Firstly, there are recent works (Christ, Colliander and Tao [5], Koch and Tataru [9]) that obtain some local results for large data u_0 belonging to negative order Sobolev spaces, which implies the existence of a solution at least locally, and even a possibility of the well-posedness in L^p spaces with $\rho < 2$. Moreover, when ρ is close to 2, Bourgain's method of constructing a global solution would be applicable and the local solution can be extended globally. Finally, we also note that, related to this problem, the Cauchy problem (1.1), (1.2) can be solved globally for small initial data in weak L^p spaces when p is large (see Cazenave, Vega, and Vilela [4]).

As we state above, this paper is devoted to the proof of the existence of the global solution of (1.1), (1.2) for the initial data belonging to $L^\rho(\mathbb{R}^d)$ with $\rho_0 < \rho \leq 2$ for some $\rho_0 > 1$ depending on space dimension d and exponent of nonlinearity p . More precisely, we seek a solution of (1.1), (1.2) when $\rho := \rho_\theta$ is written as

$$\frac{1}{\rho_\theta} = \frac{1-\theta}{r'_0} + \frac{\theta}{2}, \quad 0 < \theta \leq 1$$

where r'_0 is defined by

$$r'_0(p, d) = \max \left\{ \frac{2p}{2p-1}, \frac{2pd}{d+4} \right\} = \begin{cases} \frac{2p}{2p-1} & \text{if } 1 < p \leq 1 + \frac{2}{d} \\ \frac{2pd}{d+4} & \text{if } 1 + \frac{2}{d} < p < 1 + \frac{3}{d}. \end{cases} \quad (1.3)$$

In our previous work [7] we proved the global existence of (1.1), (1.2) with $u_0 \in L^{\rho_\theta}(\mathbb{R}^d)$ if $\theta \in (\theta_0, 1]$ where

$$\theta_0(p, d) = \begin{cases} 0 & \text{if } 1 < p \leq p_0 \\ \frac{4p^2 + (d-4)p - (d+4)}{(-d+4)p^2 + 2dp - (d+4)} & \text{if } p_0 < p < 1 + \frac{2}{d} \end{cases} \quad (1.4)$$

and

$$p_0 = \frac{1}{2} \left\{ 1 - \frac{d}{4} + \sqrt{\left(1 - \frac{d}{4}\right)^2 + d + 4} \right\}$$

when $d \leq 4$ and $p \in (1, 1 + \frac{2}{d}]$ (in such case, $r'_0(p, d) = 2p/2p-1$). Our aim in this paper is to construct a global solution in general space dimension d and p belonging to wider range $(1, 1 + \frac{3}{d})$ when $d \geq 2$ and $(1, \frac{7}{2})$ when $d = 1$. Moreover, we will refine our previous results even in the case of $d \leq 4$ and $p \in (1, 1 + \frac{2}{d}]$. Indeed, our main theorem states that if we choose θ_1 by

$$\theta_1(p, d) = \begin{cases} 0 & \text{if } 1 < p \leq p_1 \\ \frac{(d+8)p^2 - (d+14)p + 2}{2(p-1)(4p-1)} & \text{if } p_1 < p < 1 + \frac{2}{d} \end{cases} \quad (1.5)$$

where

$$p_1 = \frac{3d + 8 + \sqrt{d^2 + 32d + 64}}{4(d + 2)} < p \leq 1 + \frac{2}{d},$$

then for any initial data $u_0 \in L^\rho$ with $\rho \in (\rho_{\theta_0}, 2]$ there exists a unique global solution of (1.1), (1.2). It is not difficult to see that $p_0 < p_1$ and

$$\begin{aligned} \rho_{\theta_0} - \rho_{\theta_1} &= K_{p,d} [p - (1 + 2/d)] [(p - (1 + 4/d))] \geq 0 \\ K_{p,d} &: \text{some positive constant} \end{aligned}$$

when $p_1 < p \leq 1 + \frac{2}{d}$ and $d \leq 4$. This means that the main result (Theorem 3.4) of this paper can partly refine our previous work [7].

To prove the main result we first prove a local and global existence theorem (Theorem 3.2) for (1.1), (1.2) with the initial data u_0 which can be decomposed as

$$u_0 = \varphi_N + \psi_N \quad (1.6)$$

with

$$\|\varphi_N\|_{L^2} \sim N^\alpha, \quad \sup_{I: |I|=1} \|U(t)\psi_N\|_{L_t^{\frac{q}{2}-\varepsilon} L^r} \lesssim N^{-1} \quad \text{uniformly in } N > 1, \quad (1.7)$$

see the next section for precise definitions. Then, we will prove that any $u_0 \in L^{\rho_\theta}(\mathbb{R}^d)$ can be decomposed as (1.6), (1.7) for some $q, r, \alpha, \varepsilon$ under an appropriate choice of θ, p, d , which means the existence of a solution with the initial data lying in $L^{\rho_\theta}(\mathbb{R}^d)$. Theorem 3.2 is motivated by the work by Vargas and Vega [13] for the one space dimension and $p = 3$ and it can be interpreted as one of the generalizations of their work. Their strategy is originally developed by Bourgain [2] for proving the global existence of large solutions with H^s , $s > 0$ for critical nonlinearity. In [7] we extend their results to the case $d \leq 4$ and $1 < p < 1 + \frac{4}{d}$ by using weak L^p type homogeneous Strichartz estimates. Here, in this paper we employ the L^p -space version of the Strichartz estimates for inhomogeneous equations with different indices of admissible pairs from those in [7] to remove the condition $d \leq 4$ and as well to adapt to the initial data $u_0 \in L^\rho$ under additional assumptions on p and indices of Strichartz norms.

Finally, we also remark that in the local case we can put $\theta_1 = 0$ for any $p \in (1, 1 + \frac{3}{d})$ and $d \in \mathbb{N}$, that is, we can prove a local existence of (1.1), (1.2) with $u_0 \in L^\rho(\mathbb{R}^d)$ for any $r'_0 < \rho \leq 2$ (Theorem 3.3). In global case, we will see that it is possible to construct a unique solution in such ρ when d is large ($d \geq 15$) or p is close to 1. However, for general p, d whether or not a unique global solution exists for any $\rho \in (r'_0, 2]$ remains unsolved.

2. Notations and preliminary results

2.1. Strichartz estimate and $L_I^q L^r$ -inequalities

Let $I \subset \mathbb{R}$ be an interval and $0 < q, r \leq \infty$. The space $L_I^q L^r$ is defined by all measurable functions f on $I \times \mathbb{R}^d$ satisfying

$$\|f\|_{L_I^q L^r} = \begin{cases} \left(\int_I \|f(t, \cdot)\|_{L^r(\mathbb{R}^d)}^q dt \right)^{1/q} < \infty, & 0 < q < \infty \\ \text{ess sup}_{t \in I} \|f(t, \cdot)\|_{L^r(\mathbb{R}^d)} < \infty. & q = \infty \end{cases}$$

Recall that a pair of two positive numbers (q, r) is admissible (with respect to a space dimension d) if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \quad (2.1)$$

The well-known Strichartz estimates for the Schrödinger group $U(t) = e^{it\Delta}$ is as follows.

Theorem 2.1. *Let $I \subset \mathbb{R}$. Let (q, r) be admissible with $q > 2$. Then, there exists $C_1 > 0$ independent of I such that*

$$\|U(t)\varphi\|_{L_I^q L^r} \leq C_1 \|\varphi\|_{L^2} \quad (2.2)$$

for any $\varphi \in L^2(\mathbb{R}^d)$.

Let $(q, r), (\gamma, \rho)$ be admissible with $q, \gamma > 2$. Then, there exists $C_2 > 0$ independent of I such that

$$\left\| \int_0^t U(\cdot - \tau) f(\tau) d\tau \right\|_{L_I^q L^r \cap L_I^\infty L^2} \leq C_2 \|f\|_{L_I^{\gamma'} L^{\sigma'}}. \quad (2.3)$$

for any $f \in L_I^{\gamma'} L^{\sigma'}$ where γ', σ' is the conjugate of γ and σ , respectively.

As a corollary to the Strichartz estimates we will prove $L_I^p L^q$ -type estimates which will be needed later. Let us prepare some notations.

Definition 2.2. (i) We define the open interval J_d by

$$J_d = \begin{cases} (1, \frac{7}{2}) & \text{if } d = 1 \\ (1, 1 + \frac{3}{d}) & \text{if } d \geq 2. \end{cases}$$

(ii) Let $d \in \mathbb{N}$ and $p \in J_d$. We define

$$q_{\min}(p, d) = \max \left\{ \frac{4p}{d(p-1)}, \frac{4p}{4-d(p-1)} \right\}$$

and

$$q_{\max}(p, 1) = \begin{cases} \infty & \text{if } p \in (1, 2] \\ \frac{4p}{p-2} & \text{if } p \in [2, \frac{7}{2}) \end{cases}$$

when $d = 1$, and

$$q_{\max}(p, d) = \begin{cases} \infty & \text{if } p \in (1, 1 + \frac{2}{d}] \\ \frac{4p}{pd-d-2} & \text{if } p \in [1 + \frac{2}{d}, 1 + \frac{3}{d}) \end{cases}$$

when $d \geq 2$.

Remark 2.3. (i) Note that $q_{\max}(p, d) > q_{\min}(p, d)$ for any $d \in \mathbb{N}$ and $p \in J_d$.
(ii) It is easy to see that the pair $(q_{\min}, r_0)$ is admissible, where r_0 is the conjugate of r'_0 which appeared in the introduction.

We define two complex-valued functions $F, \tilde{F}$ by

$$F(u, v) = |u + v|^{p-1}(u + v) - |u|^{p-1}u, \quad u, v \in \mathbb{C}, \quad (2.4)$$

$$\tilde{F}(u, v_1, v_2) = F(u, v_1) - F(u, v_2), \quad u, v_1, v_2 \in \mathbb{C}. \quad (2.5)$$

It is easy to see that $\tilde{F}(u, v, 0) = F(u, v)$ and there is a constant $C_3 > 0$ such that

$$|\tilde{F}(u, v_1, v_2)| \leq C_3(|u|^{p-1} + |v_1|^{p-1} + |v_2|^{p-1})|v_1 - v_2| \quad (2.6)$$

for any $u, v_1, v_2 \in \mathbb{C}$.

The next corollary is an easy consequence of the Strichartz estimates, Hölder's inequality, and (2.6). Let $I_T = [0, T]$ for $T > 0$.

Corollary 2.4. Let $d \in \mathbb{N}, p \in J_d$ and let (q, r) be an admissible pair satisfying $q_{\min}(p, d) < q < q_{\max}(p, d)$. Then, following statements (i), (ii) hold.

(i) There exists a positive constant C_2 such that for any $T > 0$ we have

$$\left\| \int_0^t U(\cdot - \tau)(|u|^{p-1}u)(\tau) d\tau \right\|_{L^q_{I_T} L^r} \leq C_2 T^{\frac{4-d(p-1)}{4}} \|u\|_{L^q_{I_T} L^r}^p \quad (2.7)$$

for all $u \in L^q_{I_T} L^r$.

(ii) Assume moreover that

$$\frac{q}{2} - \varepsilon > 1, \quad \frac{4 - d(p-1)}{4p} - \frac{1}{q} - c_q(\varepsilon) \geq 0 \quad (2.8)$$

for some $\varepsilon \geq 0$ where

$$c_q(\varepsilon) := \left(\frac{q}{2} - \varepsilon\right)^{-1} - \frac{2}{q}. \quad (2.9)$$

Then, there is a positive constant C_3 depending only on q, p, d such that following estimates are true for any $T > 0$:

$$\begin{aligned} \left\| \int_0^t U(\cdot - \tau)(\tilde{F}(u, v_1, v_2))(\tau) d\tau \right\|_{L^{\frac{q}{2}-\varepsilon}_{I_T} L^r} &\leq C_3 \left[T^{\frac{4-d(p-1)}{4}} \|u\|_{L^q_{I_T} L^r}^{p-1} \|v_1 - v_2\|_{L^{\frac{q}{2}-\varepsilon}_{I_T} L^r} \right. \\ &\quad \left. + T^{\frac{4-d(p-1)}{4} - \frac{p-1}{q} - (p-1)c_q(\varepsilon)} \|v_1 - v_2\|_{L^{\frac{q}{2}-\varepsilon}_{I_T} L^r} \left(\|v_1\|_{L^{\frac{q}{2}-\varepsilon}_{I_T} L^r}^{p-1} + \|v_2\|_{L^{\frac{q}{2}-\varepsilon}_{I_T} L^r}^{p-1} \right) \right] \end{aligned} \quad (2.10)$$

for all $u \in L^q_{I_T} L^r$ and $v_1, v_2 \in L^{\frac{q}{2}-\varepsilon}_{I_T} L^r$.

$$\left\| \int_0^t U(t-\tau)(\tilde{F}(u, v, 0))(\tau) d\tau \right\|_{L^2} \leq C_3 \left[T^{\frac{4-d(p-1)}{4} - \frac{1}{q} - c_q(\varepsilon)} \|u\|_{L_{I_T}^q L^r}^{p-1} \|v\|_{L_{I_T}^{\frac{q}{2}-\varepsilon} L^r} \right. \\ \left. + T^{\frac{4-d(p-1)}{4} - \frac{p}{q} - pc_q(\varepsilon)} \|v\|_{L_{I_T}^{\frac{q}{2}-\varepsilon} L^r}^p \right] \quad (2.11)$$

for all $u \in L_{I_T}^q L^r, v \in L_{I_T}^{\frac{q}{2}-\varepsilon} L^r$ and $t \in [0, T]$.

Remark 2.5. When proving the inequalities in Corollary 2.4, we apply (2.3) with

$$\gamma = \left(\frac{pd}{4} - \frac{p}{q} - \frac{d}{4} \right)^{-1}.$$

We need

$$0 < \frac{pd}{4} - \frac{p}{q} - \frac{d}{4} < \begin{cases} \frac{1}{4} & \text{if } d = 1 \\ \frac{1}{2} & \text{if } d \geq 2 \end{cases}$$

in order that (γ', σ') is admissible for an suitable choice of σ . That is why we need the upper bound $q_{\max}$ for q .

Moreover, we need an $L_I^q L^r$ -estimate of a L^2 -solution of (1.1), (1.2).

Corollary 2.6. Let u be the solution of (1.1), (1.2) with the initial data $u_0 \in L^2(\mathbb{R}^d)$. Let C_1, C_2 be constants in (2.2) and (2.7), respectively. Assume that p , the exponent of the nonlinearity in (1.2) belongs to J_d and an admissible pair (q, r) and δ satisfy

$$q_{\min}(p, d) < q < q_{\max}(p, d), \quad 0 \leq \delta \leq \left(2^{\frac{p+1}{p-1}} C_1 C_2^{\frac{1}{p-1}} \|u_0\|_{L^2} \right)^{-\frac{4(p-1)}{4-d(p-1)}}. \quad (2.12)$$

Then, one has

$$\|u\|_{L_{I_\delta}^q L^r} \leq 2C_1 \|u_0\|_{L^2}. \quad (2.13)$$

Proof. Put

$$\delta^* = \left(2^{\frac{p+1}{p-1}} C_1 C_2^{\frac{1}{p-1}} \|u_0\|_{L^2} \right)^{-\frac{4(p-1)}{4-d(p-1)}}.$$

Since $\delta = 0$ satisfies (2.13), it suffices to prove that the set

$$I = \{\delta \in [0, \delta^*] : \|u\|_{L_{I_\delta}^q L^r} \leq 2C_1 \|u_0\|_{L^2}\}$$

is both open and closed. I is closed since $\delta \mapsto \|\cdot\|_{L_{I_\delta}^q L^r}$ is continuous. We will show that I is open. For a fixed $\delta \in I$ with $\delta < \delta^*$ take $\varepsilon > 0$ satisfying $\delta + \varepsilon \leq \delta^*$. Then, by the Duhamel's principle, we have

$$u(t) = U(t)u_0 \pm i \int_0^t U(t-\tau)(|u|^{p-1}u)(\tau) d\tau.$$

To estimate the right hand side of the above inequality, we use (2.2) and (2.7), which gives

$$\begin{aligned} \|u\|_{L^q_{I_{\delta+\varepsilon}} L^r} &\leq \|U(t)u_0\|_{L^q_{I_{\delta+\varepsilon}} L^r} + \left\| i \int_0^t U(t-\tau)(|u|^{p-1}u)(\tau)d\tau \right\|_{L^q_{I_{\delta+\varepsilon}} L^r} \\ &= C_1 \|u_0\|_{L^2} + \left\| \int_0^t U(t-\tau)(|u|^{p-1}u)(\tau)d\tau \right\|_{L^q_{I_\delta} L^r} \\ &\quad + \left\| \int_0^t U(t-\tau)(|u|^{p-1}u)(\tau)d\tau \right\|_{L^q_{[\delta, \delta+\varepsilon]} L^r} \\ &\leq C_1 \|u_0\|_{L^2} + C_2 \left(\delta^{\frac{4-d(p-1)}{4}} \|u\|_{L^q_{I_\delta} L^r}^p + \varepsilon^{\frac{4-d(p-1)}{4}} \|u\|_{L^q_{[\delta, \delta+\varepsilon]} L^r}^p \right). \end{aligned}$$

We have

$$C_2 \varepsilon^{\frac{4-d(p-1)}{4}} \|u\|_{L^q_{[\delta, \delta+\varepsilon]} L^r}^p \leq \frac{C_1}{2} \|u_0\|_{L^2}^p$$

if $\varepsilon > 0$ is taken sufficiently small. Since $\delta \in I$ and $\delta \leq \delta^*$, we have

$$\begin{aligned} \|u\|_{L^q_{I_{\delta+\varepsilon}} L^r} &\leq \frac{3C_1}{2} \|u_0\|_{L^2} + C_2 \delta^{\frac{4-d(p-1)}{4}} \|u\|_{L^q_{I_\delta} L^r}^p \\ &\leq \frac{3C_1}{2} \|u_0\|_{L^2} + C_2 (\delta^*)^{\frac{4-d(p-1)}{4}} \times (2C_1 \|u_0\|_{L^2})^p \\ &= \left(\frac{3C_1}{2} + \frac{C_1}{2} \right) \|u_0\|_{L^2} = 2C_1 \|u_0\|_{L^2}. \end{aligned}$$

Hence, we get $\delta + \varepsilon \in I$, which completes the proof. $\square$

2.2. New spaces of functions

Definition 2.7. (i) Let (q, r) be admissible, and let $\varepsilon \geq 0$ be a nonnegative constant satisfying $\frac{q}{2} - \varepsilon > 1$. Define the space $Z^{\frac{q}{2}-\varepsilon, r}(\mathbb{R}^d)$ by

$$Z^{\frac{q}{2}-\varepsilon, r}(\mathbb{R}^d) = \{\psi : U(t)\psi \in L^{\frac{q}{2}-\varepsilon}_I L^r \text{ for all intervals of unit length } I\}$$

endowed with the norm

$$\|\psi\|_{Z^{\frac{q}{2}-\varepsilon, r}(\mathbb{R}^d)} = \sup_{I \subset \mathbb{R}: |I|=1} \|U(t)\psi\|_{L^{\frac{q}{2}-\varepsilon}_I L^r}.$$

(ii) Let (q, r) be admissible, and let $\alpha \geq 0$ and $\varepsilon \geq 0$ satisfy $\frac{q}{2} - \varepsilon > 1$. One denotes by $\mathcal{Y}^{q, r, \varepsilon}_\alpha(\mathbb{R}^d)$ the set of all functions u_0 such that there exist a constant $C > 1$ and sequences

$$(\varphi^N)_{N>1} \subseteq L^2(\mathbb{R}^d), \quad (\psi^N)_{N>1} \subseteq Z^{\frac{q}{2}-\varepsilon, r}(\mathbb{R}^d)$$

indexed by $N \in (1, \infty)$ such that

$$u_0 = \varphi^N + \psi^N$$

and

$$C^{-1}N^\alpha \leq \|\varphi^N\|_{L^2} \leq CN^\alpha, \quad \|\psi^N\|_{Z^{\frac{q}{2}-\varepsilon, r}(\mathbb{R}^d)} \leq C \frac{1}{N}$$

for all $N > 1$.

Remark 2.8. We will see that $L^\rho \subseteq \mathcal{Y}_\alpha^{q,r,\varepsilon}(\mathbb{R}^d)$ for an appropriate choice of indexes. Indeed, we can prove that $L^{\rho_\theta}(\mathbb{R}^d) \subseteq \mathcal{Y}_{\frac{1-\theta}{\theta}}^{q,r,\varepsilon}(\mathbb{R}^d)$ when $q - q_{\min}$ and ε are sufficiently small positive number, where ρ_θ is defined in the introduction. For details, see Proposition 5.2 and 5.3.

3. Main results

3.1. Local and global existence

Firstly, we define the function space $\mathcal{X}^{q,r,\varepsilon}$ on $\mathbb{R}^{1+d}$ to which the global solutions of (1.1), (1.2) belong.

Definition 3.1. For any admissible pair (q, r) and $\varepsilon \geq 0$ define

$$\begin{aligned} \mathcal{X}^{q,r,\varepsilon}(\mathbb{R}^{1+d}) &= L_{loc}^\infty L^2 \cap L_{loc}^q L^r + L_{loc}^{\frac{q}{2}-\varepsilon} L^r \\ &= \{u(t, x) = U(t, x) + V(t, x) : U \in L_{loc}^\infty L^2 \cap L_{loc}^q L^r, V \in L_{loc}^{\frac{q}{2}-\varepsilon} L^r\}. \end{aligned}$$

Our first result is the local and global existence theorem for (1.1), (1.2) with the initial data $u_0 \in \mathcal{Y}_\alpha^{q,r,\varepsilon}(\mathbb{R}^d)$ when q is larger than $q_{\min}$ but close enough to $q_{\min}$ and $\varepsilon \geq 0$ is sufficiently small.

In what follows $d \in \mathbb{N}$ is a space dimension and $p \in J_d$ is the exponent of nonlinearity.

Theorem 3.2. Let $\alpha \geq 0$, $\varepsilon \geq 0$, and let (q, r) be an admissible pair satisfying

$$q_{\min}(p, d) < q < q_{\max}(p, d), \quad \frac{q}{2} - \varepsilon > 1 \quad (3.1)$$

and

$$\frac{4 - d(p-1)}{4p} - \frac{1}{q} - c_q(\varepsilon) \geq 0 \quad (3.2)$$

where $c_q(\varepsilon)$ is defined as (2.9).

(i) For any $u_0 \in \mathcal{Y}_\alpha^{q,r,\varepsilon}(\mathbb{R}^d)$, there exists a sequence of positive real numbers $(T_N)_{N>1}$ such that for any $N > 1$, the initial value problem (1.1), (1.2) has a unique solution u on the interval $[-T_N, T_N]$ in the form of

$$u = U + V, \quad U \in L_{I_{T_N}}^\infty L^2 \cap L_{I_{T_N}}^q L^r, \quad V \in L_{I_{T_N}}^{\frac{q}{2}-\varepsilon} L^r. \quad (3.3)$$

Moreover, T_N can be written as

$$T_N = cN^{1-\alpha(\frac{4(p-1)}{4-d(p-1)}(1+\frac{1}{q}+c_q(\varepsilon))-1)} \quad (3.4)$$

where $c > 0$ is a constant independent of N .

(ii) Assume moreover that $p, q, d, \alpha, \varepsilon$ satisfy

$$-\infty < \alpha \left(\frac{4(p-1)}{4-d(p-1)} \left(1 + \frac{1}{q} + c_q(\varepsilon) \right) - 1 \right) < 1. \quad (3.5)$$

Then, for any $u_0 \in \mathcal{Y}_\alpha^{q,r,\varepsilon}(\mathbb{R}^d)$, there exists a unique global solution $u \in \mathcal{X}^{q,r,\varepsilon}(\mathbb{R}^{1+d})$ of (1.1), (1.2). Moreover, the mapping $u_0 \mapsto u(t)$ is locally Lipschitz from $L^2 + Z^{\frac{q}{2}-\varepsilon,r}$ into $L_T^q L^r$ for any $T > 0$.

3.2. Global existence for initial data belonging to L^ρ

Applying Theorem 3.2, we can construct a global solution of (1.1), (1.2) for L^ρ -initial data when $\rho < 2$ and ρ is close to 2.

Theorem 3.3. (Local existence)

(a) Assume that

$$1 < p \leq 1 + \frac{2}{d}.$$

Then, for any $u_0 \in L^\rho(\mathbb{R}^d)$ with

$$\frac{2p}{2p-1} < \rho \leq 2$$

there exist an admissible pair (q, r) and $\varepsilon > 0$ satisfying (3.1), (3.2) such that the initial value problem (1.1), (1.2) has a unique solution u in the form of

$$u = U + V, \quad U \in L_{I_T}^\infty L^2 \cap L_{I_T}^q L^r, \quad V \in L_{I_T}^{\frac{q}{2}-\varepsilon} L^r \quad (3.6)$$

for some $T > 0$.

(b) Assume that

$$1 + \frac{2}{d} < p < \begin{cases} \frac{7}{2} & \text{when } d = 1 \\ 1 + \frac{3}{d} & \text{when } d \geq 2. \end{cases}$$

Then, for any $u_0 \in L^\rho(\mathbb{R}^d)$ with

$$\frac{2pd}{d+4} < \rho \leq 2$$

there exist an admissible pair (q, r) and $\varepsilon > 0$ satisfying (3.1), (3.2) such that the initial value problem (1.1), (1.2) has a unique solution u in the form of

$$u = U + V, \quad U \in L_{I_T}^\infty L^2 \cap L_{I_T}^q L^r, \quad V \in L_{I_T}^{\frac{q}{2}-\varepsilon} L^r \quad (3.7)$$

for some $T > 0$.

Similarly, the global existence theorem can be obtained as follows.

Theorem 3.4. For any $u_0 \in L^\rho(\mathbb{R}^d)$ the Cauchy problem (1.1), (1.2) has a unique global solution $u \in \mathcal{X}^{q,r,\varepsilon}(\mathbb{R}^{1+d})$ for some admissible pair (q, r) , $\varepsilon > 0$ and $\alpha \geq 0$ satisfying (3.1), (3.2), (3.5) which can be specified below in the proof, provided that any of the following conditions holds:

Case 1. Let $1 \leq d \leq 5$.

(1-a) If $1 < p < \frac{3d+8+\sqrt{d^2+32d+64}}{4(d+2)}$, then $\frac{2p}{2p-1} < \rho \leq 2$.

(1-b) If $\frac{3d+8+\sqrt{d^2+32d+64}}{4(d+2)} \leq p \leq 1 + \frac{2}{d}$, then $\frac{4p+d(p-1)}{2(p+1)} < \rho \leq 2$.

(1-c) If $1 + \frac{2}{d} < p < \begin{cases} \frac{7}{2} & \text{if } d = 1 \\ 1 + \frac{3}{d} & \text{if } d \geq 2 \end{cases}$, then $\frac{d(p-1)(4p+4-d(p-1))}{8+2d(p-1)^2} < \rho \leq 2$.

Case 2. Let $6 \leq d \leq 14$.

(2-a) If $1 < p \leq 1 + \frac{2}{d}$, then $\frac{2p}{2p-1} < \rho \leq 2$.

(2-b) If $1 + \frac{2}{d} < p < \frac{-d+4+\sqrt{d^2+8d+80}}{8}$, then $\frac{2pd}{d+4} < \rho \leq 2$.

(2-c) If $\frac{-d+4+\sqrt{d^2+8d+80}}{8} \leq p < 1 + \frac{3}{d}$, then

$$\frac{d(p-1)(4p+4-d(p-1))}{8+2d(p-1)^2} < \rho < 2.$$

Case 3. Let $d \geq 15$.

(3-a) If $1 < p \leq 1 + \frac{2}{d}$, then $\frac{2p}{2p-1} < \rho \leq 2$.

(3-b) If $1 + \frac{2}{d} < p \leq 1 + \frac{3}{d}$, then $\frac{2pd}{d+4} < \rho \leq 2$.

4. Proof of theorem 3.2

It suffices to prove the existence of the solution in the case $t \geq 0$ and positive sign in (1.1). Note that since $u_0 \in \mathcal{Y}_\alpha^{q,r,\varepsilon}$ there exist $C_0 > 0$ and $\varphi_0^N \in L^2(\mathbb{R}^d), \psi_0^N \in Z^{\frac{q}{2}-\varepsilon,r}(\mathbb{R}^d)$ such that

$$u_0 = \varphi_0^N + \psi_0^N$$

and

$$C_0^{-1}N^\alpha \leq \|\varphi_0^N\|_{L^2} \leq C_0N^\alpha, \quad (4.1)$$

$$\|\psi_0^N\|_{Z^{\frac{q}{2}-\varepsilon,r}(\mathbb{R}^d)} \leq C_0 \frac{1}{N} \quad (4.2)$$

for all $N > 1$.

4.1. Existence of solutions on $[0, \delta_N]$

In this subsection we will construct a local solution on a small interval $[0, \delta_N]$ using a fixed point argument, where $\delta_N > 0$ will be defined below. For later use in the next subsection (see (4.8), (4.9) below), it is convenient to construct a solution under the weaker assumption

$$(2C_0)^{-1}N^\alpha \leq \|\varphi_0^N\|_{L^2} \leq 2C_0N^\alpha \quad (4.3)$$

instead of (4.1). For fixed $N > 1$ we consider two Cauchy problems (IVPL²) and (IVP₀^N):

$$(\text{IVPL}^2) \quad \begin{cases} i(u^{(0)})_t + \triangle u^{(0)} + |u^{(0)}|^{p-1}u^{(0)} = 0, & t > 0 \\ u^{(0)}(x, 0) = \varphi_0^N(x), & x \in \mathbb{R}^d, \end{cases} \quad (4.4)$$

$$(\text{IVP}_0^N) \quad \begin{cases} i(v^{(0)})_t + \triangle v^{(0)} + F(u^{(0)}, v^{(0)}) = 0, & t > 0 \\ v^{(0)}(x, 0) = \psi_0^N(x), & x \in \mathbb{R}^d \end{cases} \quad (4.5)$$

where F is the complex-valued function defined by (2.5).

It is obvious that $u = u^{(0)} + v^{(0)}$ is a solution of (1.1), (1.2). Since $\varphi_0^N \in L^2$, we have a unique solution $u^{(0)}$ of (IVPL²) on any interval $[0, T]$. Thus, to prove the existence of a solution of (1.1), (1.2) on $[0, \delta_N]$, it suffices to show the existence of a solution of (IVP₀^N) on $[0, \delta_N]$. Let us introduce some positive real constants. Let C_1 be the constant in (2.2). Similarly, let C_2, C_3 be the positive numbers in the inequalities in Corollary 2.4. We then fix $M > 1$ which is larger than any of the following constants:

$$2^{\frac{p+1}{p-1}} C_1 C_2^{\frac{1}{p-1}}, \quad 2 \cdot 3^{\frac{1}{p-1}} C_1 C_3^{\frac{1}{p-1}}, \quad 2^{1+\frac{1}{(p-1)K_{p,q,d,\varepsilon}}} \cdot 3^{\frac{p}{(p-1)K_{p,q,d,\varepsilon}}} \\ \times C_0^{1+\frac{1}{K_{p,q,d,\varepsilon}}} C_3^{\frac{1}{(p-1)K_{p,q,d,\varepsilon}}}$$

where

$$K_{p,q,d,\varepsilon} := 1 - \frac{4(p-1)}{4-d(p-1)} \left(\frac{1}{q} + c_q(\varepsilon) \right)$$

and note that $K_{p,q,d,\varepsilon} > 0$ by (3.2). Next, we define

$$\delta_N^* := (M \|\varphi_0^N\|_{L^2})^{-\frac{4(p-1)}{4-d(p-1)}}, \quad \delta_N := (M(2C_0 N^\alpha))^{-\frac{4(p-1)}{4-d(p-1)}}.$$

Remark that δ_N is independent of the value of the initial data $\|\varphi_0^N\|_{L^2}$ and since

$$\delta_N \leq \delta_N^* \leq (2^{\frac{p+1}{p-1}} C_1 C_2^{\frac{1}{p-1}} \|\varphi_0^N\|_{L^2})^{-\frac{4(p-1)}{4-d(p-1)}}$$

by the definition of M and (4.3), for any $\delta \in [0, \delta_N^*]$ we have

$$\|u^{(0)}\|_{L_{I_\delta}^q L^r} \leq 2C_1 \|\varphi_0^N\|_{L^2} \quad (4.6)$$

by (2.13).

We use a fixed point argument to obtain a solution of (IVP₀^N). For $\delta \in [0, \delta_N^*]$ define

$$\mathcal{V}_N^\delta = \left\{ v \in L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r : \|v\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r} \leq \frac{3C_0}{N} \right\}$$

and we put

$$(Tv)(t) = U(t)\psi_0^N + i \int_0^t U(t-\tau)F(u^{(0)}, v)(\tau)d\tau$$

for any $v \in L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r$. Then, we get

Proposition 4.1. *The operator $T : \mathcal{V}_N^\delta \rightarrow \mathcal{V}_N^\delta$ is well-defined and contraction mapping for all $\delta \in [0, \delta_N^*]$.*

Proof. Assume that $\delta \leq \delta_N^*$ and $v \in \mathcal{V}_N^\delta$. By the triangle inequality and Corollary 2.4, we have

$$\|Tv\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r} \leq \|U(\cdot)\psi_0^N\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r} + C_3 \delta^{\frac{4-d(p-1)}{4}} \|u^{(0)}\|_{L_{I_\delta}^q L^r}^{p-1} \|v\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r} \\ + C_3 \delta^{\frac{4-d(p-1)}{4} - \frac{p-1}{q} - (p-1)c_q(\varepsilon)} \|v\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r}^p \\ := I_1 + I_2 + I_3.$$

By the assumption, we have

$$I_1 \leq \|\psi_0^N\|_{Z^{\frac{q}{2}-\varepsilon, r}(\mathbb{R}^d)} \leq C_0 N^{-1}.$$

For I_2 we use (4.6) to obtain

$$\begin{aligned} I_2 &\leq C_3 \left\{ (M \|\varphi_0^N\|_{L^2})^{-\frac{4(p-1)}{4-d(p-1)}} \right\}^{\frac{4-d(p-1)}{4}} \times (2C_1 \|\varphi_0^N\|_{L^2})^{p-1} \times \frac{3C_0}{N} \\ &\leq 3 \cdot 2^{p-1} C_1^{p-1} C_3 M^{1-p} \cdot \frac{C_0}{N}. \end{aligned}$$

By the definition of M , the right side hand of the above inequality is bounded by $\frac{C_0}{N}$.

In the similar manner, by (4.3), I_3 is smaller than

$$\begin{aligned} &3^p \cdot 2^{(p-1)K_{p,q,d,\varepsilon}} C_0^{(p-1)K_{p,q,d,\varepsilon}+p} C_3 M^{(1-p)K_{p,q,d,\varepsilon}} N^{-\alpha K_{p,q,d,\varepsilon}(p-1)} \\ &\quad \times N^{-p} \\ &\leq 3^p \cdot 2^{(p-1)K_{p,q,d,\varepsilon}} C_0^{(p-1)K_{p,q,d,\varepsilon}+p} C_3 M^{(1-p)K_{p,q,d,\varepsilon}} \times N^{-1}, \end{aligned}$$

which can be bounded by $\frac{C_0}{N}$ by using the definition of M again.

Consequently, $\|Tv\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r} \leq \frac{3C_0}{N}$. Thus, we have $Tv \in \mathcal{V}_N^\delta$ if $v \in \mathcal{V}_N^\delta$.

Similarly, we get

$$\begin{aligned} &\|Tv_1 - Tv_2\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r} \\ &\leq C_3 \left\{ 2^{p-1} C_1^{p-1} M^{1-p} + 2^{(p-1)K_{p,q,d,\varepsilon}} \cdot 3^{p-1} C_0^{(p-1)(K_{p,q,d,\varepsilon}+1)} \right. \\ &\quad \left. \times M^{(1-p)K_{p,q,d,\varepsilon}} N^{(1-p)(\alpha K_{p,q,d,\varepsilon}+1)} \right\} \|v_1 - v_2\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r} \\ &\leq \left(\frac{1}{3} + \frac{1}{3} \right) \|v_1 - v_2\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r} = \frac{2}{3} \|v_1 - v_2\|_{L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r}. \end{aligned}$$

□

Hence, by means of a fixed point theorem, we can construct a solution $v^{(0)}$ of (IVP₀^N) on $[0, \delta]$ for any $\delta \leq \delta_N^*$. Moreover, we have:

Lemma 4.2. *There exists $C_4 > 0$ such that we have*

$$\left\| \int_0^t U(t-\tau) F(u^{(0)}, v^{(0)}) d\tau \right\|_{L^2} \leq C_4 N^{-1+\frac{4(p-1)}{4-d(p-1)}} \left(\frac{1}{q} + c_q(\varepsilon) \right)^\alpha$$

for any $N > 1$ and $t \in [0, \delta_N^*]$.

Proof. The proof is much the same as the one of Proposition 4.1. We use (2.11) instead of (2.10) which gives

$$\begin{aligned}
& \left\| \int_0^t U(t-\tau)F(u^{(0)}, v^{(0)})d\tau \right\|_{L^2} \\
& \leq C \left(\delta^{\frac{4-d(p-1)}{4} - \frac{1}{q} - c_q(\varepsilon)} \|u^{(0)}\|_{L_{I_{\delta_N^*}}^q L^r}^{p-1} \|v^{(0)}\|_{L_{I_{\delta_N^*}}^{\frac{q}{2}-\varepsilon} L^r} \right. \\
& \quad \left. + \delta^{\frac{4-d(p-1)}{4} - \frac{p}{q} - pc_q(\varepsilon)} \|v^{(0)}\|_{L_{I_{\delta_N^*}}^{\frac{q}{2}-\varepsilon} L^r}^p \right) \\
& \leq C \left(N^{\alpha \{ -\frac{4(p-1)}{4-d(p-1)} (\frac{4-d(p-1)}{4} - \frac{1}{q} - c_q(\varepsilon)) + p-1 \} - 1} \right. \\
& \quad \left. + N^{\alpha \{ -\frac{4(p-1)}{4-d(p-1)} (\frac{4-d(p-1)}{4} - \frac{p}{q} - pc_q(\varepsilon)) \} - p} \right) \\
& \leq C \left(N^{-1 + \frac{4(p-1)}{4-d(p-1)} (\frac{1}{q} + c_q(\varepsilon))\alpha} + N^{-p + [1-p + \frac{4(p-1)}{4-d(p-1)} (\frac{p}{q} + pc_q(\varepsilon))]\alpha} \right)
\end{aligned}$$

where $C > 0$ is a constant independent of N . The assumption (3.2) implies that the second term of the right hand side can be bounded by the first one. Hence the assertion follows. $\square$

4.2. Extending solution to the time T_N

In the previous subsection, we construct a solution of (1.1), (1.2) on $[0, \delta]$ in the form of

$$u = u^{(0)} + v^{(0)} \in L_{I_\delta}^q L^r + L_{I_\delta}^{\frac{q}{2}-\varepsilon} L^r.$$

for any $\delta \in [0, \delta_N^*]$. In particular, we have a unique solution u on the interval $[0, \delta_N]$.

For any $N > 1$ we consider the Cauchy problem

$$(\text{IVP}_1^N) \quad \begin{cases} iu_t + \triangle u + |u|^{p-1}u = 0, & x \in \mathbb{R}^d, \quad t > \delta_N \\ u(x, \delta_N) = u^{(0)}(x, \delta_N) + v^{(0)}(x, \delta_N) = \varphi_1^N(x) + \psi_1^N(x) \end{cases} \quad (4.7)$$

where we define

$$\varphi_1^N(x) = u^{(0)}(x, \delta_N) + \int_0^{\delta_N} U(\delta_N - \tau)F(u^{(0)}, v^{(0)})d\tau$$

and

$$\psi_1^N(x) = U(\delta_N)\psi_0^N(x).$$

Then, by the conservation law, (4.1) and Lemma 4.2, we have

$$\begin{aligned}
\|\varphi_1^N(\cdot)\|_{L^2} & \leq \|u^{(0)}(\cdot, \delta_N)\|_{L^2} + \left\| \int_0^{\delta_N} U(\delta_N - \tau)F(u^{(0)}, v^{(0)})d\tau \right\|_{L^2} \\
& \leq C_0 N^\alpha + C_4 N^{-1 + \frac{4(p-1)}{4-d(p-1)} (\frac{1}{q} + c_q(\varepsilon))\alpha}.
\end{aligned}$$

and

$$\|\varphi_1^N(\cdot)\|_{L^2} \geq C_0^{-1} N^\alpha - C_4 N^{-1 + \frac{4(p-1)}{4-d(p-1)} (\frac{1}{q} + c_q(\varepsilon))\alpha}.$$

We also have

$$\|\psi_1^N\|_{Z^{\frac{q}{2}-\varepsilon}(\mathbb{R}^d)} \leq C_0 \frac{1}{N}.$$

Thus, if

$$C_0 N^\alpha + C_4 N^{-1+\frac{4(p-1)}{4-d(p-1)}(\frac{1}{q}+c_q(\varepsilon))} \leq 2C_0 N^\alpha \quad (4.8)$$

and

$$C_0^{-1} N^\alpha - C_4 N^{-1+\frac{4(p-1)}{4-d(p-1)}(\frac{1}{q}+c_q(\varepsilon))} \geq (2C_0)^{-1} N^\alpha, \quad (4.9)$$

we can apply the argument in the previous subsection (under the weak assumption (4.3)) to the initial value problem, that is, a solution of $(\mathbf{IVP}_1^N)$ exists on $[\delta_N, 2\delta_N]$ for any $N > 1$. In much the same manner, by using induction, if $k_0 \in \mathbb{N}$ satisfies

$$C_0 N^\alpha + C_4 k_0 N^{-1+\frac{4(p-1)}{4-d(p-1)}(\frac{1}{q}+c_q(\varepsilon))} \leq 2C_0 N^\alpha \quad (4.10)$$

and

$$C_0^{-1} N^\alpha - C_4 k_0 N^{-1+\frac{4(p-1)}{4-d(p-1)}(\frac{1}{q}+c_q(\varepsilon))} \geq (2C_0)^{-1} N^\alpha, \quad (4.11)$$

then, for any $k \leq k_0$ the Cauchy problem $(\mathbf{IVP}_k^N)$ defined by

$$\begin{aligned} iu_t + \triangle u + |u|^{p-1}u &= 0, \quad x \in \mathbb{R}^d, \quad t > k\delta_N \\ u(x, k\delta_N) &= u^{(0)}(x, k\delta_N) + v^{(0)}(x, k\delta_N) \\ &= \underbrace{u^{(0)}(x, k\delta_N) + \int_0^{k\delta_N} U(k\delta_N - \tau) F(u^{(0)}, v^{(0)}) d\tau}_{:= \varphi_k^N(x)} + \underbrace{U(k\delta_N) \psi_{k-1}^N(x)}_{:= \psi_k^N(x)} \end{aligned}$$

has a unique solution on $[k\delta_N, (k+1)\delta_N]$.

Solving (4.10) and (4.11), we have

$$k_0 \leq C_4^{-1} N^{1+\alpha(1-\frac{4(p-1)}{4-d(p-1)}(\frac{1}{q}+c_q(\varepsilon)))}.$$

Therefore for any $N > 1$ we have the solution u of (1.1), (1.2) to the time

$$T_N := ck_0\delta_N = cN^{1-\alpha(\frac{4(p-1)}{4-d(p-1)}(1+\frac{1}{q}+c_q(\varepsilon))-1)}.$$

where $c > 0$ is a constant independent of N . This proves Theorem 3.2 (i). The statement (ii) follows easily from (i). Indeed, (3.5) implies that the exponent of T in (3.4) is strictly positive. Thus, T_N can be taken arbitrary large.

4.3. Proof of uniqueness

It remains to prove the uniqueness of the solution. Let u_1, u_2 be two solutions of (1.1), (1.2) with the initial data u_1^0, u_2^0 respectively. Then, we have

$$u_1(t) - u_2(t) = U(t)(u_1^0 - u_2^0) + \int_0^t U(t-\tau) \left(\tilde{F}(u_1, u_2, 0) \right) d\tau.$$

By Corollary 2.4, we obtain

$$\begin{aligned} \|u_1 - u_2\|_{L_{I_T}^{\frac{q}{2}-\varepsilon} L^r} &\leq T^{\frac{4-d(p-1)}{4} - \frac{p-1}{q} - (p-1)c_q(\varepsilon)} \left(\|u_1\|_{L_{I_T}^{\frac{q}{2}-\varepsilon} L^r}^{p-1} + \|u_2\|_{L_{I_T}^{\frac{q}{2}-\varepsilon} L^r}^{p-1} \right) \\ &\quad \times \|u_1 - u_2\|_{L_{I_T}^{\frac{q}{2}-\varepsilon} L^r} + \|u_1^0 - u_2^0\|_{L^2 + Z^{\frac{q}{2}-\varepsilon, r}}. \end{aligned}$$

Therefore we get $u_1(t) = u_2(t)$, $t \in [0, T]$ if T is small from which we can easily derive the uniqueness assertion and the Lipschitz continuity of the mapping $u_0 \mapsto u(t)$ for any $T > 0$. $\square$

Remark 4.3. Note that in the statement of Theorem 3.2 we can take $\varepsilon = 0$ as far as conditions (3.1), (3.2), (plus (3.5) in the case of (ii)) are satisfied. However, as for Theorem 3.3 and 3.4, the condition $\varepsilon \neq 0$ is required to avoid a difficulty of divergence (see the proof of Proposition 5.2 and 5.3 below). If we can replace $L^{\frac{q}{2}}$ by weak $L^{\frac{q}{2}}$ -space $L^{(\frac{q}{2}, \infty)}$, we can put $\varepsilon = c_q(\varepsilon) = 0$ throughout this paper since $|t|^{-\frac{1}{\alpha}} \in L^{(\alpha, \infty)}$. In that sense, it seems more natural to consider Theorem 3.2 in the space

$$L^2(\mathbb{R}^d) + Z^{(\frac{q}{2}, r, \infty)}(\mathbb{R}^d)$$

where

$$Z^{(\frac{q}{2}, r, \infty)}(\mathbb{R}^d) = \{\psi : U(t)\psi \in L_I^{(\frac{q}{2}, \infty)} L^r \text{ for all intervals of unit length } I\}.$$

We can prove Theorem 3.2 in the above space if the weak L^p space version of inhomogeneous Strichartz estimates hold, namely if there exists a positive number $C > 0$ for any two admissible pairs (q, r) and (α, β) such that

$$\left\| \int_0^t U(t-\tau) f d\tau \right\|_{L^{(q, \infty)} L^r} \leq C \|f\|_{L^{(\alpha', \infty)} L^{\beta'}}$$

for all $f \in L^{(\alpha', \infty)} L^{\beta'}$. However it is still open whether or not the assertion holds as far as the authors know.

5. Proof of theorem 3.3 and 3.4

To prove the theorems, we need

Lemma 5.1. Suppose that $0 < p_0 < p_1 \leq \infty$, $0 < \theta < 1$ and $p > 0$ is defined by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then,

$$L^p \subset L^{p_0} + L^{p_1}.$$

Furthermore, for any $f \in L^p$ and for any $t > 0$ one can find $f_0^t \in L^{p_0}$ and $f_1^t \in L^{p_1}$ with $f = f_0^t + f_1^t$ satisfying

$$ct^{-\theta} \max\{\|f_0^t\|_{L^{p_0}}, t\|f_1^t\|_{L^{p_1}}\} \leq \|f\|_{L^p} \quad (5.1)$$

for some positive constant c depending only on p_0, p_1 .

For a proof see [11], [7].

Theorem 3.3 and 3.4 are immediate consequences of Theorem 3.2 and following Propositions.

Proposition 5.2. *Let $d \in \mathbb{N}$ and $p \in J_d$.*

(i) *If $1 < p \leq 1 + \frac{2}{d}$, then for any $\rho \in (\frac{2p}{2p-1}, 2]$ there exist an admissible pair (q, r) , $\alpha \geq 0$, and $\varepsilon > 0$ satisfying (3.1), (3.2) such that the inclusion*

$$L^\rho(\mathbb{R}^d) \subseteq \mathcal{Y}_\alpha^{q,r,\varepsilon}(\mathbb{R}^d)$$

holds.

(ii) *If $1 + \frac{2}{d} < p < \begin{cases} \frac{7}{2} & \text{if } d = 1 \\ 1 + \frac{3}{d} & \text{if } d \geq 2 \end{cases}$, then for any $\rho \in (\frac{2pd}{d+4}, 2]$ there*

exist an admissible pair (q, r) , $\alpha \geq 0$, and $\varepsilon > 0$ satisfying (3.1), (3.2) such that the inclusion

$$L^\rho(\mathbb{R}^d) \subseteq \mathcal{Y}_\alpha^{q,r,\varepsilon}(\mathbb{R}^d)$$

holds.

Proposition 5.3. *Let $\rho \leq 2$. Then, we have the inclusion*

$$L^\rho(\mathbb{R}^d) \subseteq \mathcal{Y}_\alpha^{q,r,\varepsilon}(\mathbb{R}^d)$$

for some admissible pair (q, r) , $\varepsilon > 0$ and $\alpha \geq 0$ satisfying (3.1), (3.2), (3.5), provided that any of the following conditions holds:

Case 1. *Let $1 \leq d \leq 5$.*

(1-a) *If $1 < p < \frac{3d+8+\sqrt{d^2+32d+64}}{4(d+2)}$, then $\frac{2p}{2p-1} < \rho \leq 2$.*

(1-b) *If $\frac{3d+8+\sqrt{d^2+32d+64}}{4(d+2)} \leq p \leq 1 + \frac{2}{d}$, then $\frac{4p+d(p-1)}{2(p+1)} < \rho \leq 2$.*

(1-c) *If $1 + \frac{2}{d} < p < \begin{cases} \frac{7}{2} & \text{if } d = 1 \\ 1 + \frac{3}{d} & \text{if } d \geq 2 \end{cases}$, then*

$$\frac{d(p-1)(4p+4-d(p-1))}{8+2d(p-1)^2} < \rho \leq 2.$$

Case 2. *Let $6 \leq d \leq 14$.*

(2-a) *If $1 < p \leq 1 + \frac{2}{d}$, then $\frac{2p}{2p-1} < \rho \leq 2$.*

(2-b) *If $1 + \frac{2}{d} < p < \frac{-d+4+\sqrt{d^2+8d+80}}{8}$, then $\frac{2pd}{d+4} < \rho \leq 2$.*

(2-c) *If $\frac{-d+4+\sqrt{d^2+8d+80}}{8} \leq p < 1 + \frac{3}{d}$, then*

$$\frac{d(p-1)(4p+4-d(p-1))}{8+2d(p-1)^2} < \rho < 2.$$

Case 3. *Let $d \geq 15$.*

(3-a) *If $1 < p \leq 1 + \frac{2}{d}$, then $\frac{2p}{2p-1} < \rho \leq 2$.*

(3-b) If $1 + \frac{2}{d} < p \leq 1 + \frac{3}{d}$, then $\frac{2pd}{d+4} < \rho \leq 2$.

Proof of Proposition 5.2 and 5.3. We take an admissible pair (q, r) and $\varepsilon > 0$ where $q > q_{\min}$ is sufficiently close to $q_{\min}$ and ε is sufficiently small so that q, p, d, ε satisfy (3.1), (3.2). Let $0 < \theta \leq 1$. Define $\rho = \rho_\theta(q, r) > 1$ by

$$\frac{1}{\rho} = (1 - \theta) \frac{1}{r'} + \theta \frac{1}{2}.$$

Note that r' is arbitrarily close to

$$r'_0(p, d) := \begin{cases} \frac{2p}{2p-1} & \text{if } 1 < p \leq 1 + \frac{2}{d} \\ \frac{2pd}{d+4} & \text{if } 1 + \frac{2}{d} < p < 1 + \frac{3}{d}. \end{cases}$$

We will find $\theta, (q, r)$, and ε such that the inclusion

$$L^{\rho_\theta(q, r)} \subset \mathcal{Y}_\alpha^{q, r, \varepsilon}$$

holds true for some $\alpha \geq 0$. To prove this, it is sufficient to show that for any $u_0 \in L^\rho$ there exist $C > 0, \alpha \geq 0$ and $(\varphi_0^N)_{N>1} \subseteq L^2(\mathbb{R}^d), (\psi_0^N)_{N>1} \subseteq L^{r'}(\mathbb{R}^d)$ such that u_0 can be decomposed as

$$u_0 = \varphi_0^N + \psi_0^N, \quad N > 1 \quad (5.2)$$

with

$$\|\varphi_0^N\|_{L^2} = N^\alpha, \quad \|\psi_0^N\|_{L^{r'}} \leq \frac{C}{N}. \quad (5.3)$$

For a given $u_0 \in L^\rho$ we investigate whether or not u_0 allow decomposition (5.2), (5.3). If $u_0 \in L^2(\mathbb{R}^d) \setminus \{0\}$, the assertion is obvious since $L^2(\mathbb{R}^d) \setminus \{0\} \subseteq \mathcal{Y}_0^{q, r, \varepsilon}(\mathbb{R}^d)$. So, we assume $u_0 \notin L^2(\mathbb{R}^d) \setminus \{0\}$. By Lemma 5.1, for any $s > 0$ there exist $\varphi_0^s \in L^2(\mathbb{R}^d)$ and $\psi_0^s \in L^{r'}(\mathbb{R}^d)$ such that

$$u_0 = \varphi_0^s + \psi_0^s$$

and

$$c_0 s^{-\theta} \max\{\|\psi_0^s\|_{L^{r'}}, s\|\varphi_0^s\|_{L^2}\} \leq \|u_0\|_{L^\rho}.$$

For any $N > 1$ and $\alpha > 0$ we take $s_N > 0$ such that

$$\|\varphi_0^{s_N}\|_{L^2} = N^\alpha$$

and put $\varphi_0^N = \varphi_0^{s_N}$ and $\psi_0^N = \psi_0^{s_N}$. Then, we have

$$s_N^{1-\theta} \leq \frac{\|u_0\|_{L^\rho}}{c_0 N^\alpha}$$

and

$$\|\psi_0^N\|_{L^{r'}} \leq s_N^\theta \|u_0\|_{L^\rho} \leq C N^{-\frac{\theta}{1-\theta}\alpha}.$$

We take $\alpha = \frac{1-\theta}{\theta}$. Then, for any $\theta \in (0, 1)$ we get

$$L^{\rho_\theta(q, r)} \subseteq \mathcal{Y}_{\frac{1-\theta}{\theta}}^{q, r, \varepsilon}.$$

by taking $q - q_{\min}$ and ε sufficiently small according to ρ_θ . This proves Proposition 5.2. To prove Proposition 5.3 we seek θ such that $\alpha = \frac{1-\theta}{\theta}$ and q, p, d, ε satisfy (3.5). Since $q - q_{\min}$ and $\varepsilon > 0$ can be taken sufficiently small, it suffices to investigate whether or not θ satisfy

$$\left(\frac{1-\theta}{\theta}\right) \left(\frac{4(p-1)}{4-d(p-1)} \left(1 + \frac{1}{q_{\min}}\right) - 1\right) < 1, \quad (5.4)$$

solving which we get the conditions (1-a)-(3-b) in the statement of Proposition 5.3. $\square$

Remark 5.4. *In the last part of the above proof, any $\alpha = \frac{1-\theta}{\theta} \geq 0$ satisfies (5.4) when*

$$\left(\frac{4(p-1)}{4-d(p-1)} \left(1 + \frac{1}{q_{\min}}\right) - 1\right) < 0. \quad (5.5)$$

Such case corresponds to the cases (1-a), (2-a), (2-b), (3-a) and (3-c) of Proposition 5.3.

Remark 5.5. *In the above proof we note that if $s \rightarrow 0$, then*

$$\|\varphi_0^s\|_{L^2} = N^\alpha \rightarrow \infty$$

when $\rho_\theta \neq 2$. Indeed, if it does not hold, then

$$\limsup_{t \rightarrow 0} \left(\int_{t^\alpha}^{\infty} (u_0)^*(s)^2 ds \right)^{1/2} < +\infty$$

where $(u_0)^$ is the nonincreasing rearrangement of u_0 . This implies $u_0 \in L^2(\mathbb{R}^d)$.*

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