Multilinear commutators of fractional integrals over Morrey spaces with non-doubling measures

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Abstract. In this paper, the authors study the boundedness of multilinear fractional integrals on the product Morrey space with non-doubling measure, and investigate the Morrey boundedness properties of the multilinear commutators generated by multilinear fractional integral operators with a tuple of *RBMO* functions.

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1. Introduction and main results

The classical result of commutator is due to Coifman et al. [1]: if $b \in BMO$ and T is a Calderón–Zygmund operator, then the commutator $[b, T]$ is bounded on L^p spaces for $1 < p < \infty$. In 2001, Tolsa [2] developed the theory of Calderón–Zygmund operators and their commutators with *RBMO* functions in the setting of non-doubling measures. Later on, Chen and Sawyer [3] modified the definition of *RBMO* to investigate the commutators of the potential operators and *RBMO* functions. Recently, Sawano and Tanaka [4, 5] defined the Morrey spaces for non-doubling measures, and obtained the boundedness of the commutators generated by fractional integrals with *RBMO* functions in Morrey spaces. Here we point out that the commutator estimates for non-doubling measures play an important role in solving the long-standing open questions, like the Painlevé problem.

In recent years, the theory on multilinear integral operators and multilinear commutators have attracted much attentions as a rapid developing field in

harmonic analysis. Hu et al. [6] studied the L^p -boundedness and certain weak type endpoint estimates for multilinear commutators generated by multilinear Calderón-Zygmund singular integral with $RBMO$ functions. In this paper, we will extend the works of Sawano and Tanaka [4, 5, 7] and Tao et al. [8] to the context for multilinear commutators generated by multilinear fractional integrals and $RBMO$ functions over the Morrey spaces with non-doubling measures.

Throughout this paper μ will be a positive Radon measures on $\mathbb{R}^d$ satisfying the growth condition: there exist a constant $C > 0$ and $n \in (0, d]$ such that

$$\mu(Q) \leq Cl(Q)^n \quad (1.1)$$

for any cube $Q \in \mathbb{R}^d$ with sides parallel to the coordinate axes. $Q(x, l(Q))$ will be the cube centered at x with side length $l(Q)$. For $r > 0$, rQ will denote the cube with the same center as Q and with $l(rQ) = rl(Q)$. The set of all cubes $Q \subset \mathbb{R}^d$ satisfying $\mu(Q) > 0$ is denoted by $\mathcal{Q}(\mu)$.

Before stating our main results, we fix some notations and define some terminologies. Given $\beta_d \geq 2^{d+1}$ large enough but depending only on the dimension d , we say that a cube $Q \subset \mathbb{R}^d$ is doubling if $\mu(2Q) \leq \beta_d \mu(Q)$. For any fixed cube $Q \subset \mathbb{R}^d$, let $N \geq 0$ be the smallest integer such that $2^N Q$ is doubling. We denote this cube by $\tilde{Q}$.

Let $0 \leq \alpha < n$, for two cubes $Q \subset R$ in $\mathbb{R}^d$, we always set

$$K_{Q,R}^{(\alpha)} = 1 + \sum_{k=1}^{N_{Q,R}} \left[\frac{\mu(2^k Q)}{l(2^k Q)^n} \right]^{1 - \frac{\alpha}{mn}},$$

where $N_{Q,R}$ is the first positive integer k such that $l(2^k Q) \geq l(R)$. If $\alpha = 0$, then $K_{Q,R}^{(\alpha)} = K_{Q,R}$. The latter was introduced by Tolsa in [2].

Denote by $m_Q f$ the mean value of f on Q , namely, $m_Q f = \frac{1}{\mu(Q)} \int_Q f(x) d\mu$. Let $\eta > 1$ be a fixed constant, we say that $f \in L_{loc}^1(\mu)$ is in $RBMO(\mu)$ if there exist a constant $\mathcal{A}$ such that

$$\frac{1}{\mu(\eta Q)} \int_Q |f(y) - m_{\tilde{Q}} f| d\mu(y) \leq \mathcal{A} \quad (1.2)$$

for any cube Q , and

$$|m_Q f - m_R f| \leq \mathcal{A} K_{Q,R} \quad (1.3)$$

for any two doubling cubes $Q \subset R$. The minimal constant $\mathcal{A}$ is the $RBMO(\mu)$ norm of f , and it will be denoted by $\|f\|_*$.

We recall the definition of the Morrey space with non-doubling measure.

Definition 1.1. Let $k > 1$, and $1 \leq q \leq p < \infty$, the Morrey space $M_q^p(k, \mu)$ is defined as

$$M_q^p(k, \mu) := \{f \in L_{loc}^q(\mu); \|f\|_{M_q^p(k, \mu)} < \infty\}$$

with the norm

$$\|f\|_{M_q^p(k,\mu)} := \sup_{Q \in \mathcal{Q}(\mu)} \mu(kQ)^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q |f|^q d\mu \right)^{\frac{1}{q}}.$$

As is easily seen, the space $M_q^p(k, \mu)$ is a Banach space with its norm. The Morrey space norm reflects local regularity of f more precisely than the Lebesgue space norm. It is easy to see from the Hölder inequality that $L^p(\mu) = M_p^p(k, \mu) \subset M_{q_1}^p(k, \mu) \subset M_{q_2}^p(k, \mu)$ whenever $1 \leq q_2 \leq q_1 \leq p < \infty$. Moreover, the definition of the spaces is independent of the constant $k > 1$, and the norms for different choice of $k > 1$ are equivalent, see [4, 5, 7] for details. We will denote $M_q^p(2, \mu)$ by $M_q^p(\mu)$.

Denoting by $\vec{f} = (f_1, f_2, \dots, f_m)$, we consider in this article the multilinear fractional integral operator $I_{\alpha, m}$ as follows,

$$I_{\alpha, m}(\vec{f})(x) = \int_{(\mathbb{R}^d)^m} \frac{f_1(y_1) \cdots f_m(y_m) d\mu(y_1) \cdots d\mu(y_m)}{|(x - y_1, x - y_2, \dots, x - y_m)|^{mn - \alpha}} \quad \text{for } x \in \mathbb{R}^d. \quad (1.4)$$

Let $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, m$, and let $\vec{b} = (b_1, b_2, \dots, b_m)$, then the multilinear commutator $[\vec{b}, I_{\alpha, m}]$ is formally defined as

$$[\vec{b}, I_{\alpha, m}](\vec{f})(x) = \int_{(\mathbb{R}^d)^m} \prod_{j=1}^m (b_j(x) - b_j(y_j)) \frac{f_1(y_1) \cdots f_m(y_m) d\mu(y_1) \cdots d\mu(y_m)}{|(x - y_1, x - y_2, \dots, x - y_m)|^{mn - \alpha}}.$$

More generally, denote by C_i^m the family of all subsets $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_i\}$ of i different elements of $\{1, 2, \dots, m\}$, and let $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$ and $\vec{b}_\sigma = \{b_{\sigma_1}, b_{\sigma_2}, \dots, b_{\sigma_i}\}$. For any $\sigma \in C_i^m$, we define

$$\begin{aligned} [\vec{b}_\sigma, I_{\alpha, m}](\vec{f})(x) &= \int_{(\mathbb{R}^d)^m} \prod_{\sigma_j \in \sigma} (b_{\sigma_j}(x) - b_{\sigma_j}(y_j)) \\ &\quad \times \frac{f_1(y_1) \cdots f_m(y_m) d\mu(y_1) \cdots d\mu(y_m)}{|(x - y_1, x - y_2, \dots, x - y_m)|^{mn - \alpha}}. \end{aligned}$$

In case $\sigma = \{1, 2, \dots, m\}$, one sees that $[\vec{b}_\sigma, I_{\alpha, m}]$ is just the commutator $[\vec{b}, I_{\alpha, m}]$.

Here we remark that the multilinear commutator $[\vec{b}, I_{\alpha, m}]$ can be equivalently rewritten as

$$[\vec{b}, I_{\alpha, m}](\vec{f})(x) = [b_m, [b_{m-1}, \dots, [b_1, I_{\alpha, m}]](\vec{f})(x). \quad (1.5)$$

The main results in this paper can be stated as follows.

Theorem 1.2. Suppose that $0 < \alpha < mn$ and $1 < q_i \leq p_i < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} - \frac{\alpha}{n} > 0$, then

$$\|I_{\alpha, m}(\vec{f})\|_{M_q^p(\mu)} \leq C \prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)}$$

with the constant C independent of $\vec{f}$.

Remark 1.3. In the case $p_i = q_i$, Theorem 1.2 implies the L^p -boundedness for the operator $I_{\alpha,m}$, precisely,

$$\|I_{\alpha,m}(\vec{f})\|_{L^p(\mu)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$ for any $1 < p_i < \infty$.

Theorem 1.4. Let $0 < \alpha < mn$ and $1 < q_i \leq p_i < \infty$, and let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} - \frac{\alpha}{n} > 0$. Suppose $\|\mu\| = \infty$, and that $b_i \in RBMO(\mu)$ for $i = 1, 2, \dots, m$. Then, for all $\sigma \subseteq \{1, 2, \dots, m\}$, the commutators $[\vec{b}_\sigma, I_{\alpha,m}]$ are bounded from $M_{q_1}^{p_1}(\mu) \times M_{q_2}^{p_2}(\mu) \times \cdots \times M_{q_1}^{p_1}(\mu)$ to $M_q^p(\mu)$, moreover,

$$\left\| [\vec{b}_\sigma, I_{\alpha,m}](\vec{f}) \right\|_{M_q^p(\mu)} \leq C \prod_{j \in \sigma} \|b_j\|_* \prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)},$$

and particularly,

$$\left\| [\vec{b}, I_{\alpha,m}](\vec{f}) \right\|_{M_q^p(\mu)} \leq C \prod_{i=1}^m \|b_i\|_* \|f_i\|_{M_{q_i}^{p_i}(\mu)},$$

where $C > 0$ is the constant independent of $\vec{f}$, $\vec{b}$ and $\vec{b}_\sigma$.

Remark 1.5. In the case $m = 1$, Theorem 1.2 and 1.4 follow that the fractional integral operator I_α and its commutator $[b, I_\alpha]$ are all bounded from $M_{q_1}^{p_1}(\mu)$ to $M_q^p(\mu)$, which are just Theorem 3.3 of [4] and Theorem 4.6 of [5], so our theorems contain their conclusions. In [8] the authors obtained the same result as Theorem 1.2 if μ is the Lebesgue measure.

To show Theorem 1.2 in Sect. 2, we will use a maximal function $M_k f(x)$ and give a pointwise estimate for the multilinear fractional integral $I_{\alpha,m}(\vec{f})(x)$ in the setting of non-doubling measures. In order to derive Theorem 1.4, we will exploit the decomposition technique for the product of functions and overcome the difficult brought from the multilinear operators and non-doubling measures. We will establish a pointwise estimate for the sharp maximal function of the commutator $[\vec{b}, I_{\alpha,m}](\vec{f})$ by using the non-centered maximal operator $M_{\tau,\eta}^{(\alpha)}$, see Lemma 3.4 below. We will prove Theorem 1.4 in Sect. 3.

2. The proof of Theorem 1.2

Before proving the theorem, we recall the following maximal operator,

$$M_k f(x) = \sup_{x \in Q \subset \mathcal{Q}(\mu)} \frac{1}{\mu(kQ)} \int_Q |f(y)| d\mu(y),$$

and some lemmas which will be used in this paper.

Lemma 2.1. [4] *If $1 < q \leq p < \infty$, then there exist a constant C independent of f such that*

$$\|M_k f\|_{M_q^p(\mu)} \leq C \|f\|_{M_q^p(\mu)}.$$

Lemma 2.2. *For $0 < \alpha < mn$, $1 < q_i \leq p_i < \infty$, $\frac{1}{h} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, $\frac{1}{p} = \frac{1}{h} - \frac{\alpha}{n} > 0$, we have*

$$|I_{\alpha,m} \vec{f}(x)| \leq C \left(\prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)} \right)^{1-\frac{h}{p}} \left(\prod_{i=1}^m M_k f_i(x) \right)^{\frac{h}{p}}$$

with the absolute constant C independent of $\vec{f}$.

Proof. For $x, y_i \in \mathbb{R}^d$ and $x \neq y_i$, $i = 1, 2, \dots, m$, we can see from the Fubini theorem that

$$\begin{aligned} I_{\alpha,m}(\vec{f})(x) &\leq \int_{(\mathbb{R}^d)^m} \frac{|f_1(y_1)| |f_2(y_2)| \cdots |f_m(y_m)| d\mu(y_1) d\mu(y_2) \cdots d\mu(y_m)}{|(x - y_1, x - y_2, \dots, x - y_m)|^{mn-\alpha}} \\ &\leq C \int_{(\mathbb{R}^d)^m} \int_{|(x-y_1, x-y_2, \dots, x-y_m)|}^{\infty} l^{\alpha-mn-1} dl \left(\prod_{i=1}^m |f_i(y_i)| d\mu(y_i) \right) \\ &\leq C \int_0^{\infty} \left(\prod_{i=1}^m \frac{1}{l^n} \int_{|(x-y_i)| < l} |f_i(y_i)| d\mu(y_i) \right) l^{\alpha-1} dl. \end{aligned} \quad (2.1)$$

Using the Hölder inequality and the growth condition (1.1) for the non-doubling measure μ , and letting cube $Q = Q(x, 2l)$, then we have

$$\begin{aligned} &\int_0^{\lambda} \left(\prod_{i=1}^m \frac{1}{l^n} \int_{|(x-y_i)| < l} |f_i(y_i)| d\mu(y_i) \right) l^{\alpha-1} dl \\ &\leq C \int_0^{\lambda} \left(\prod_{i=1}^m \frac{1}{\mu(kQ)} \int_Q |f_i(y_i)| d\mu(y_i) \right) l^{\alpha-1} dl \\ &\leq C \int_0^{\lambda} \left(\prod_{i=1}^m M_k f_i(x) \right) l^{\alpha-1} dl \leq C \lambda^{\alpha} \prod_{i=1}^m M_k f_i(x). \end{aligned}$$

On the other hand, with the similar ways as above we have

$$\begin{aligned} \frac{1}{l^n} \int_{|(x-y_i)| < l} |f_i(y_i)| d\mu(y_i) &\leq \frac{C}{l^n} \int_Q |f_i(y_i)| d\mu(y_i) \\ &\leq \frac{C}{l^n} \left(\int_Q |f_i(y_i)|^{q_i} d\mu(y_i) \right)^{\frac{1}{q_i}} \mu(Q)^{1-\frac{1}{q_i}} \\ &\leq C l^{-\frac{n}{p_i}} \|f_i\|_{M_{q_i}^{p_i}(\mu)} \end{aligned}$$

which means that

$$\begin{aligned} & \int_{\lambda}^{\infty} \left(\prod_{i=1}^m \frac{1}{l^n} \int_{|(x-y_i)| < l} |f_i(y_i)| d\mu(y_i) \right) l^{\alpha-1} dl \\ & \leq C \int_{\lambda}^{\infty} \left(\prod_{i=1}^m l^{-\frac{n}{p_i}} \|f_i\|_{M_{q_i}^{p_i}(\mu)} \right) l^{\alpha-1} dl \leq C \lambda^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)}. \end{aligned}$$

Thus, by (2.1), we obtain that

$$|I_{\alpha,m} \vec{f}(x)| \leq C \left[\lambda^{\alpha} \prod_{i=1}^m M_k f_i(x) + \lambda^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)} \right].$$

This yields the lemma by choosing the suitable positive real number λ . $\square$

Now we give **the proof of theorem 1.2**. Note $0 < \alpha < mn$, $1 < q_i \leq p_i < \infty$, $\frac{1}{h} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, and let $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m}$. By the lemma 2.2, one can get that

$$\begin{aligned} \|I_{\alpha,m} \vec{f}\|_{M_q^p(\mu)} & \leq C \left(\prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)} \right)^{1-\frac{h}{p}} \\ & \quad \times \sup_{Q \in \mathcal{Q}(\mu)} |\mu(kQ)|^{\frac{1}{p}-\frac{1}{q}} \left\{ \int_Q \left(\prod_{i=1}^m M_k f_i(y) \right)^{\frac{qh}{p}} d\mu(y) \right\}^{\frac{1}{q}}. \end{aligned}$$

Since $\frac{pr}{qh} = \frac{n-r\alpha}{n-h\alpha} > 1$, hence by the Hölder inequality and Lemma 2.1 it follows

$$\begin{aligned} & |\mu(kQ)|^{\frac{1}{p}-\frac{1}{q}} \left\{ \int_Q \left(\prod_{i=1}^m M_k f_i(y) \right)^{\frac{qh}{p}} d\mu(y) \right\}^{\frac{1}{q}} \\ & \leq C |\mu(kQ)|^{\frac{1}{p}-\frac{h}{pr}} \left\{ \int_Q \left(\prod_{i=1}^m M_k f_i(y) \right)^r d\mu(y) \right\}^{\frac{h}{pr}} \\ & \leq C \left\{ |\mu(kQ)|^{\frac{1}{h}-\frac{1}{r}} \left\{ \int_Q \left(\prod_{i=1}^m M_k f_i(y) \right)^r d\mu(y) \right\}^{\frac{1}{r}} \right\}^{\frac{h}{p}} \\ & \leq C \left\{ \prod_{i=1}^m |\mu(kQ)|^{\frac{1}{p_i}-\frac{1}{q_i}} \left(\int_Q (M_k f_i(y))^{q_i} d\mu(y) \right)^{\frac{1}{q_i}} \right\}^{\frac{h}{p}} \\ & \leq C \left(\prod_{i=1}^m \|M_k f_i\|_{M_{q_i}^{p_i}(\mu)} \right)^{\frac{h}{p}} \leq C \left(\prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)} \right)^{\frac{h}{p}}. \end{aligned}$$

Thus the theorem is obtained.

3. The proof of Theorem 1.4

In this section we will use the sharp maximal estimates. Let f be a function in $L^1_{loc}(\mu)$, the sharp maximal function of f is defined by

$$M^{\sharp,(\alpha)}f(x) = \sup_{Q \ni x} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{\tilde{Q}}f| d\mu(y) + \sup_{R \supset Q \ni x, R \text{ doubling}} \frac{|m_Q f - m_R f|}{K_{Q,R}^{(\alpha)}}. \quad (3.1)$$

The non-centered doubling maximal operator is defined by

$$Nf(x) = \sup_{\substack{Q \ni x \\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y). \quad (3.2)$$

By the Lebesgue differential theorem, it is easy to see that $|f(x)| \leq Nf(x)$ for any $f \in L^1_{loc}(\mu)$ and μ -a.e. $x \in \mathbb{R}^d$. Define the non-centered maximal operator,

$$M_{\tau,\eta}^{(\alpha)}f(x) = \sup_{Q \ni x} \left(\frac{1}{\mu(\eta Q)^{1-\frac{\tau\alpha}{n}}} \int_Q |f(y)|^\tau d\mu(y) \right)^{\frac{1}{\tau}} \quad (3.3)$$

for $\eta > 1$ and $\tau > 1$, where the supremum is taking over all the cubes Q containing the point x . Usually we denote $M^{\sharp,(0)}(f)$ simply by $M^{\sharp}(f)$ and $M_{\tau,\eta}^{(0)}(f)$ by $M_{\tau,\eta}(f)$.

From the work in [2], we know that $M_{\tau,\eta}$ is bounded on $L^q(\mu)$ for $q > \tau > 1$ and $\eta > 1$. Moreover, we have the following lemma.

Lemma 3.1. *Let $q > \tau > 1$, $\eta > 1$, and $1 < q \leq p < \infty$, then the operator $M_{\tau,\eta}$ is bounded on $M_q^p(\mu)$ and*

$$\|M_{\tau,\eta}(f)\|_{M_q^p(\mu)} \leq C \|f\|_{M_q^p(\mu)}$$

with the constant C independent of f .

Proof. The lemma could be proved along the similar lines as that of Theorem 2.3 in [4], i.e. Lemma 2.1 of this paper. Fix a cube $Q_0 \in \mathcal{Q}(\mu)$ and put $L = \frac{l(Q_0)}{2}$, and let $f_1 = \chi_{\frac{\eta+7}{\eta-1}Q_0}f$ and $f_2 = f - f_1$, then we have for all $y \in Q_0$ that

$$M_{\tau,\eta}f(y) \leq M_{\tau,\eta}f_1(y) + M_{\tau,\eta}f_2(y). \quad (3.4)$$

Noting for $y \in Q_0$,

$$\begin{aligned} M_{\tau,\eta}f_2(y) &\leq \sup_{\substack{y \in Q \in \mathcal{Q}(\mu) \\ l(Q) \geq \frac{8L}{\eta-1}}} \left\{ \frac{1}{\mu(\eta Q)} \int_Q |f|^\tau d\mu \right\}^{\frac{1}{\tau}} \\ &\leq \sup_{Q_0 \subset Q \in \mathcal{Q}(\mu)} \left\{ \frac{1}{\mu\left(\frac{2\eta}{1+\eta}Q\right)} \int_Q |f|^\tau d\mu \right\}^{\frac{1}{\tau}} \end{aligned} \quad (3.5)$$

because of the fact that $y \in Q_0 \cap Q$ and $l(Q) \geq \frac{8L}{(\eta-1)}$ yields $Q_0 \subset \frac{1+\eta}{2}Q$. Thus, from the inequalities (3.4) and (3.5), the $L^q(\mu)$ -boundedness for $M_{\tau,\eta}$, and the Hölder inequality we get that

$$\begin{aligned}
& \mu \left(\frac{2\eta(\eta+7)}{\eta^2-1} Q_0 \right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_0} (M_{\tau,\eta} f)^q d\mu \right)^{\frac{1}{q}} \\
& \leq \mu \left(\frac{2\eta(\eta+7)}{\eta^2-1} Q_0 \right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_0} (M_{\tau,\eta} f_1)^q d\mu \right)^{\frac{1}{q}} \\
& \quad + \mu (Q_0)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_0} (M_{\tau,\eta} f_2)^q d\mu \right)^{\frac{1}{q}} \\
& \leq \mu \left(\frac{2\eta(\eta+7)}{\eta^2-1} Q_0 \right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\frac{\eta+7}{\eta-1} Q_0} |f|^q d\mu \right)^{\frac{1}{q}} \\
& \quad + \sup_{Q_0 \subset Q \in \mathcal{Q}(\mu)} \frac{\mu(Q_0)^{\frac{1}{p}}}{\mu \left(\frac{2\eta}{1+\eta} Q \right)^{\frac{1}{\tau}}} \left(\int_Q |f|^\tau d\mu \right)^{\frac{1}{\tau}} \\
& \leq C \sup_{Q_0 \subset Q \in \mathcal{Q}(\mu)} \mu \left(\frac{2\eta}{1+\eta} Q \right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q d\mu \right)^{\frac{1}{q}}
\end{aligned}$$

which implies the lemma. $\square$

Lemma 3.2. *Let $1 < \tau < \frac{n}{\alpha}$, $1 < \tau < q \leq p < \infty$, $1 < t \leq s < \infty$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{q}{p} = \frac{t}{s}$, then*

$$\|M_{\tau,\eta}^{(\alpha)} f\|_{M_t^s(\mu)} \leq C \|f\|_{M_q^p(\mu)}$$

with the constant C independent of f .

Proof. Noting the condition $\frac{q}{p} = \frac{t}{s}$, it is enough to show the following pointwise estimates

$$M_{\tau,\eta}^{(\alpha)}(f)(x) \leq C (\|f\|_{M_q^p(\mu)})^{1-\frac{p}{s}} (M_{\tau,\eta} f(x))^{\frac{p}{s}}. \quad (3.6)$$

To do this, we fixed the point x such that $M_{\tau,\eta}^{(\alpha)}(f)(x) > 0$. We choose $\beta_x = \frac{\|f\|_{M_q^p(\mu)}}{M_{\tau,\eta} f(x)^p}$ and then write

$$\begin{aligned}
M_{\tau,\eta}^{(\alpha)} f(x) & \leq \sup_{\substack{x \in Q \in \mathcal{Q}(\mu) \\ \mu(\eta Q) \leq \beta_x}} \left(\frac{1}{\mu(\eta Q)^{1-\frac{\tau\alpha}{n}}} \int_Q |f(y)|^\tau d\mu(y) \right)^{\frac{1}{\tau}} \\
& \quad + \sup_{\substack{x \in Q \in \mathcal{Q}(\mu) \\ \mu(\eta Q) > \beta_x}} \left(\frac{1}{\mu(\eta Q)^{1-\frac{\tau\alpha}{n}}} \int_Q |f(y)|^\tau d\mu(y) \right)^{\frac{1}{\tau}} \\
& \leq (\beta_x)^{\frac{\alpha}{n}} \sup_{\substack{x \in Q \in \mathcal{Q}(\mu) \\ \mu(\eta Q) \leq \beta_x}} \left(\frac{1}{\mu(\eta Q)} \int_Q |f(y)|^\tau d\mu(y) \right)^{\frac{1}{\tau}}
\end{aligned}$$

$$+ \sup_{\substack{x \in Q \in \mathcal{Q}(\mu) \\ \mu(\eta Q) > \beta_x}} \mu(\eta Q)^{-(\frac{1}{p} - \frac{\alpha}{n})} \|f\|_{M_q^p(\mu)}$$

by using the Hölder inequality. Hence we have

$$M_{\tau, \eta}^{(\alpha)} f(x) \leq (\beta_x)^{\frac{\alpha}{n}} M_{\tau, \eta} f(x) + (\beta_x)^{-\frac{1}{s}} \|f\|_{M_q^p(\mu)},$$

which yields the desired inequality (3.6). The lemma is proved. $\square$

Lemma 3.3. [5] *Suppose that $1 < q \leq p < \infty$, and there exists an increasing sequence of concentric doubling cubes, $I_0 \subset I_1 \subset \cdots \subset I_k \subset \cdots$, such that*

$$\lim_{k \rightarrow \infty} m_{I_k}(f) = 0, \quad \text{and} \quad \bigcup_{k=0}^{\infty} I_k = \mathbb{R}^d.$$

Then there exist a constant $C > 0$ independent on f such that

$$\|Nf\|_{M_q^p(\mu)} \leq C \|M^{\sharp, (\alpha)} f\|_{M_q^p(\mu)}.$$

In order to investigate the multilinear commutators $[\vec{b}, I_{\alpha, m}]$ and $[\vec{b}_{\sigma}, I_{\alpha, m}]$, we introduce the following decomposition technique. For any $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$, writing $\vec{b}_{\sigma}(x) - \vec{b}_{\sigma}(y) = (\vec{b}_{\sigma}(x) - \vec{\lambda}_{\sigma}) - (\vec{b}_{\sigma}(y) - \vec{\lambda}_{\sigma})$, thus it is clear that $[\vec{b}_{\sigma}, I_{\alpha, m}](\vec{f}) = [(\vec{b}_{\sigma} - \vec{\lambda}_{\sigma}), I_{\alpha, m}](\vec{f})$. Moreover,

$$\begin{aligned} [\vec{b}, I_{\alpha, m}](\vec{f})(x) &= \sum_{i=0}^m \sum_{\sigma \in C_i^m} (-1)^{m-i} \prod_{\sigma_j \in \sigma} (b_{\sigma_j}(x) - \lambda_{\sigma_j}) \\ &\quad \times \int_{(\mathbb{R}^d)^m} \prod_{\sigma_k \in \sigma'} (b_{\sigma_k}(y) - \lambda_{\sigma_k}) \\ &\quad \times \frac{f_1(y_1) \cdots f_m(y_m) d\mu(y_1) \cdots d\mu(y_m)}{|(x - y_1, x - y_2, \dots, x - y_m)|^{mn-\alpha}}. \end{aligned} \quad (3.7)$$

By expanding $b_{\sigma_k}(y) - \lambda_{\sigma_k} = (b_{\sigma_k}(y) - b_{\sigma_k}(x)) + (b_{\sigma_k}(x) - \lambda_{\sigma_k})$ and

$$\prod_{\sigma_k \in \sigma'} (b_{\sigma_k}(y) - \lambda_{\sigma_k}) = \sum_{\substack{\sigma^{(1)} \cup \sigma^{(2)} = \sigma' \\ \sigma^{(1)} \cap \sigma^{(2)} = \emptyset}} \prod_{\sigma_u \in \sigma^{(1)}} (b_{\sigma_u}(y) - b_{\sigma_u}(x)) \prod_{\sigma_v \in \sigma^{(2)}} (b_{\sigma_v}(x) - \lambda_{\sigma_v}),$$

hence we can obtain from the equality (3.7) that

$$\begin{aligned} [\vec{b}, I_{\alpha, m}](\vec{f})(x) &= (-1)^{m+1} \prod_{j=1}^m (b_j(x) - \lambda_j) I_{\alpha, m}(\vec{f})(x) \\ &\quad + (-1)^m I_{\alpha, m} \left((\vec{b} - \vec{\lambda}) \vec{f} \right)(x) \\ &\quad + \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} C_{m, i} \prod_{\sigma_j \in \sigma} (b_{\sigma_j}(x) - \lambda_{\sigma_j}) \\ &\quad \times [\vec{b}_{\sigma'}, I_{\alpha, m}](\vec{f})(x), \end{aligned} \quad (3.8)$$

where $C_{m, i}$ are constants depending only on m and i .

Remember these notations and the equality (3.8) in mind, we turn to prove the following lemma, which is crucial to us.

Lemma 3.4. *Let $0 < \alpha < mn$, $\tau > 1$, $s_i > 1$, $b_i \in RBMO(\mu)$ and $f_i \in L^{q_i}(\mu)$, $i = 1, 2, \dots, m$, then there exist a constant $C > 0$ independent of b_i and f_i such that*

$$\begin{aligned} M^{\sharp,(\alpha)}\left(\left[\vec{b}, I_{\alpha,m}\right](\vec{f})\right)(x) &\leq C\left[\left\|b_1\right\|_* \cdots\left\|b_m\right\|_* M_{\tau, \frac{3}{2}}\left(I_{\alpha,m}(\vec{f})\right)(x)\right. \\ &\quad +\sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \prod_{j \in \sigma}\left\|b_j\right\|_* M_{\tau, \frac{3}{2}} \\ &\quad \times\left(\left[\vec{b}_{\sigma'}, I_{\alpha,m}\right](\vec{f})\right)(x) \\ &\quad \left.+\prod_{i=1}^m\left\|b_i\right\|_* M_{s_i, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_i(x)\right] . \end{aligned} \quad (3.9)$$

Proof. For simplicity, we denote by $\mathcal{R}(\vec{b}, \vec{f})(x)$ the quantities on the right hand side of the inequality (3.9). Recall the definition of the sharp maximal operator $M^{\sharp,(\alpha)}$, and use the standard technique, see [3] for example, we only need to prove that, for any $x \in \mathbb{R}^d$ and a cube $Q \ni x$, there is a quantity h_Q such that

$$\frac{1}{\mu\left(\frac{3}{2} Q\right)} \int_Q\left|\left[\vec{b}, I_{\alpha,m}\right](\vec{f})(z)-h_Q\right| d \mu(z) \leq C \mathcal{R}(\vec{b}, \vec{f})(x) \quad (3.10)$$

and

$$\left|h_Q-h_R\right| \leq C\left(K_{Q, R} K_{Q, R}^{(\alpha)}\right)^m \mathcal{R}(\vec{b}, \vec{f})(x) \quad (3.11)$$

with the absolute constant C independent of $\vec{b}, \vec{f}, Q$ and R , where R is any doubling cube with $Q \subset R$. In fact, we take

$$\begin{aligned} h_Q &=(-1)^m m_Q\left(I_{\alpha,m}\left(\left(m_{\tilde{Q}}\left(b_1\right)-b_1\right) f_1 \chi_{\mathbb{R}^d \setminus \frac{4}{3} Q}, \ldots,\left(m_{\tilde{Q}}\left(b_m\right)\right.\right.\right. \\ &\quad \left.\left.\left.-b_m\right) f_m \chi_{\mathbb{R}^d \setminus \frac{4}{3} Q}\right)\right), \end{aligned}$$

and clearly

$$\begin{aligned} h_R &=(-1)^m m_R\left(I_{\alpha,m}\left(\left(m_R\left(b_1\right)-b_1\right) f_1 \chi_{\mathbb{R}^d \setminus \frac{4}{3} R}, \ldots,\left(m_R\left(b_m\right)\right.\right.\right. \\ &\quad \left.\left.\left.-b_m\right) f_m \chi_{\mathbb{R}^d \setminus \frac{4}{3} R}\right)\right) . \end{aligned}$$

Recall the equality (3.8), for any $z \in Q$, we have that

$$\begin{aligned}
 \left| \left[\vec{b}, I_{\alpha, m} \right] (\vec{f})(z) - h_Q \right| &\leq \left| \prod_{j=1}^m (b_j(x) - m_{\vec{Q}}(b_j)) I_{\alpha, m}(\vec{f})(z) \right| \\
 &+ \left| \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} C_{m, i} \prod_{\sigma_j \in \sigma} (b_{\sigma_j}(z) - m_{\vec{Q}}(b_{\sigma_j})) \right. \\
 &\times \left. \left[\vec{b}_{\sigma'}, I_{\alpha, m} \right] (\vec{f})(z) \right| \\
 &+ \left| (-1)^m I_{\alpha, m} \left((\vec{b} - m_{\vec{Q}}(\vec{b})) \vec{f} \right) (z) - h_Q \right| \\
 &=: I(z) + II(z) + III(z). \tag{3.12}
 \end{aligned}$$

In order to show the inequality (3.10), we will calculate the integrals for the three functions above, respectively. Firstly, for $\tau > 1$, by the Hölder inequality one sees that

$$\begin{aligned}
 &\frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |I(z)| d\mu(z) \\
 &\leq \prod_{i=1}^m \left(\frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |b_i(z) - m_{\vec{Q}}(b_i)|^{\tau_i} d\mu(z) \right)^{\frac{1}{\tau_i}} \\
 &\times \left(\frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |I_{\alpha, m}(\vec{f})|^\tau d\mu(z) \right)^{\frac{1}{\tau}} \\
 &\leq C \|b_1\|_* \|b_2\|_* \cdots \|b_m\|_* M_{\tau, \frac{3}{2}}(I_{\alpha, m}(\vec{f}))(x), \tag{3.13}
 \end{aligned}$$

where we have choose $\tau_i > 1$ such that $\frac{1}{\tau_1} + \frac{1}{\tau_2} + \cdots + \frac{1}{\tau_m} + \frac{1}{\tau} = 1$.

Similarly, for $\tau > 1$, by the Hölder inequality, we also deduce that

$$\begin{aligned}
 &\frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |II(z)| d\mu(z) \\
 &\leq C \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \prod_{\sigma_j \in \sigma} \|b_{\sigma_j}\|_* M_{\tau, \frac{3}{2}} \left(\left[\vec{b}_{\sigma'}, I_{\alpha, m} \right] (\vec{f}) \right) (x). \tag{3.14}
 \end{aligned}$$

To estimate the integral related to the function $III(z)$, we split each f_i as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f \chi_{\frac{4}{3}Q}$ and $f_i^\infty = f_i - f_i^0$, this yields

$$\begin{aligned}
 \prod_{j=1}^m f_j(y_j) &= \sum_{\alpha_1, \alpha_2, \dots, \alpha_m \in \{0, \infty\}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m) \\
 &= \prod_{j=1}^m f_j^0(y_j) + \sum_{*} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m) + \prod_{j=1}^m f_j^\infty(y_j), \tag{3.15}
 \end{aligned}$$

where each term in $\sum_*$ satisfies that $\alpha_{j_1} = \alpha_{j_2} = \cdots = \alpha_{j_\lambda} = 0$, for some $1 \leq \lambda < m$ and some $\{j_1, j_2, \dots, j_\lambda\} \subset \{1, 2, \dots, m\}$. So we can decompose

the function $III(z)$ further into three parts as follows

$$\begin{aligned}
III(z) &\leq \left| I_{\alpha,m} \left((b_1 - m_{\tilde{Q}}(b_1))f_1^0, \dots, (b_m - m_{\tilde{Q}}(b_m))f_m^0 \right) (z) \right| \\
&\quad + \sum_* \left| I_{\alpha,m} \left((b_1 - m_{\tilde{Q}}(b_1))f_1^{\alpha_1}, \dots, (b_m - m_{\tilde{Q}}(b_m))f_m^{\alpha_m} \right) (z) \right| \\
&\quad + \left| (-1)^m I_{\alpha,m} \left((b_1 - m_{\tilde{Q}}(b_1))f_1^\infty, \dots, (b_m - m_{\tilde{Q}}(b_m))f_m^\infty \right) (z) - h_Q \right| \\
&=: III_1(z) + III_2(z) + III_3(z).
\end{aligned} \tag{3.16}$$

For $s_i > 1$, we can take $1 < u_i < s_i$ such that $\frac{1}{v} = \frac{1}{u_1} + \frac{1}{u_2} + \dots + \frac{1}{u_m} - \frac{\alpha}{n} > 0$ and $v > 1$. Let $\frac{1}{u_i} = \frac{1}{s_i} + \frac{1}{t_i}$ for each $i = 1, 2, \dots, m$, then $1 < t_i < \infty$. Using Theorem 1.2, we know that the operator $I_{\alpha,m}$ is bounded from $L^{u_1}(\mu) \times L^{u_2}(\mu) \times \dots \times L^{u_m}(\mu)$ to $L^v(\mu)$. Hence, by this boundedness and the Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}Q)} \int_Q |III_1(z)| d\mu(z) \\
&\leq \frac{\mu(Q)^{1-\frac{1}{v}}}{\mu(\frac{3}{2}Q)} \|I_{\alpha,m} \left((b_1 - m_{\tilde{Q}}(b_1))f_1^0, \dots, (b_m - m_{\tilde{Q}}(b_m))f_m^0 \right) (z)\|_{L^v(\mu)} \\
&\leq C\mu \left(\frac{3}{2}Q \right)^{-\frac{1}{v}} \prod_{i=1}^m \| (b_i - m_{\tilde{Q}}(b_i))f_i^0 \|_{L^{u_i}(\mu)} \\
&\leq C\mu \left(\frac{3}{2}Q \right)^{-\frac{1}{v}} \prod_{i=1}^m \left(\int_{(\frac{4}{3}Q)} |f_i|^{s_i} d\mu(y_i) \right)^{\frac{1}{s_i}} \\
&\quad \times \left(\int_{(\frac{4}{3}Q)} |b_i - m_{\tilde{Q}}(b_i)|^{t_i} d\mu(y_i) \right)^{\frac{1}{t_i}} \\
&\leq C \prod_{i=1}^m \left(\frac{1}{\mu(\frac{3}{2}Q)^{1-\frac{\alpha s_i}{mn}}} \int_{(\frac{4}{3}Q)} |f_i|^{s_i} d\mu \right)^{\frac{1}{s_i}} \\
&\quad \times \left(\frac{1}{\mu(\frac{3}{2}Q)} \int_{(\frac{4}{3}Q)} |b_i - m_{\tilde{Q}}(b_i)|^{t_i} d\mu \right)^{\frac{1}{t_i}} \\
&\leq C \prod_{i=1}^m \|b_i\|_* M_{s_i, \frac{9}{8}}^{(\frac{\alpha}{m})} f_i(x).
\end{aligned} \tag{3.17}$$

In order to estimate the integral of terms $III_2(z)$ and $III_3(z)$ over Q , we will give their point-wise estimates. In fact, for $z \in Q$, since $1 \leq \lambda \leq m-1$ we observe that

$$\begin{aligned}
III_2(z) &= \sum_* \left| I_{\alpha,m} \left((b_1 - m_{\tilde{Q}}(b_1)) f_1^{\alpha_1}, \dots, (b_m - m_{\tilde{Q}}(b_m)) f_m^{\alpha_m} \right) (z) \right| \\
&\leq C \sum_* \prod_{j \in \{j_1, \dots, j_\lambda\}} \int_{\frac{4}{3}Q} |b_j(y_j) - m_{\tilde{Q}}(b_j)| |f_j(y_j)| d\mu(y_j) \\
&\quad \times \int_{(\mathbb{R}^d \setminus \frac{4}{3}Q)^{m-\lambda}} \frac{\prod_{j \notin \{j_1, \dots, j_\lambda\}} |b_j(y_j) - m_{\tilde{Q}}(b_j)| |f_j(y_j)| d\mu(y_j)}{\left(\sum_{j \notin \{j_1, \dots, j_\lambda\}} |z - y_j| \right)^{mn-\alpha}} \\
&\leq C \sum_* \mu \left(\frac{3}{2}Q \right)^{\lambda - \frac{\alpha\lambda}{mn}} \prod_{j \in \{j_1, \dots, j_\lambda\}} \\
&\quad \times \left(\frac{1}{\mu \left(\frac{3}{2}Q \right)^{1 - \frac{\alpha s_j}{mn}}} \int_{\frac{4}{3}Q} |f_j(y_j)|^{s_j} d\mu(y_j) \right)^{\frac{1}{s_j}} \\
&\quad \times \left(\frac{1}{\mu \left(\frac{3}{2}Q \right)} \int_{\frac{4}{3}Q} |b_j(y_j) - m_{\tilde{Q}}(b_j)|^{s'_j} d\mu(y_j) \right)^{\frac{1}{s'_j}} \\
&\quad \times \sum_{k=1}^{\infty} \int_{(2^k \frac{4}{3}Q)^{m-\lambda}} \frac{\prod_{j \notin \{j_1, \dots, j_\lambda\}} |b_j(y_j) - m_{\tilde{Q}}(b_j)| |f_j(y_j)| d\mu(y_j)}{l \left(2^k \frac{4}{3}Q \right)^{mn-\alpha}} \\
&\leq C \sum_* \left(\prod_{j \in \{j_1, \dots, j_\lambda\}} \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m} \right)} f_j(x) \right) \sum_{k=1}^{\infty} 2^{-(k-1)(n - \frac{\alpha}{m})\lambda} \\
&\quad \times \int_{(2^k \frac{4}{3}Q)^{m-\lambda}} \frac{\prod_{j \notin \{j_1, \dots, j_\lambda\}} |b_j(y_j) - m_{\tilde{Q}}(b_j)| |f_j(y_j)| d\mu(y_j)}{l \left(2^k \frac{4}{3}Q \right)^{(n - \frac{\alpha}{m})(m-\lambda)}} \\
&\leq C \sum_* \left(\prod_{j \in \{j_1, \dots, j_\lambda\}} \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m} \right)} f_j(x) \right) \sum_{k=1}^{\infty} 2^{-(k-1)(n - \frac{\alpha}{m})\lambda} \\
&\quad \times \prod_{j \notin \{j_1, \dots, j_\lambda\}} \frac{1}{l \left(2^k \frac{4}{3}Q \right)^{(n - \frac{\alpha}{m})}} \int_{(2^k \frac{4}{3}Q)} |b_j(y_j) - m_{\tilde{Q}}(b_j)| |f_j(y_j)| d\mu(y_j) \\
&\leq C \sum_* \left(\prod_{j \in \{j_1, \dots, j_\lambda\}} \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m} \right)} f_j(x) \right) \sum_{k=1}^{\infty} 2^{-(k-1)(n - \frac{\alpha}{m})\lambda} \\
&\quad \times \prod_{j \notin \{j_1, \dots, j_\lambda\}} \left[\left(\frac{1}{l \left(2^k \frac{4}{3}Q \right)^n} \int_{(2^k \frac{4}{3}Q)} |b_j(y_j) - m_{\tilde{Q}}(b_j)|^{s'_j} d\mu(y_j) \right)^{\frac{1}{s'_j}} \right. \\
&\quad \left. \times \left(\frac{1}{l \left(2^k \frac{4}{3}Q \right)^{(n - \frac{\alpha s_j}{m})}} \int_{(2^k \frac{4}{3}Q)} |f_j(y_j)|^{s_j} d\mu(y_j) \right)^{\frac{1}{s_j}} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{*} \prod_{j \in \{j_1, \dots, j_\lambda\}} \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x) \sum_{k=1}^{\infty} 2^{-k(n-\frac{\alpha}{m})\lambda} \\
&\quad \times \prod_{j \notin \{j_1, \dots, j_\lambda\}} \|b_j\|_* (k+1) M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x) \\
&\leq C \prod_{j=1}^m \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x),
\end{aligned}$$

where we have used the fact (see [2]) that, there is an absolute constant C such that, for any $b \in RBMO$, integer $k \geq 0$ and cubes Q ,

$$|m_{\widetilde{2^k \frac{4}{3}Q}}(b) - m_{\widetilde{Q}}(b)| \leq C \|b\|_* K_{\widetilde{Q}, 2^k \frac{4}{3}Q} \leq C \|b\|_* K_{Q, 2^k \frac{4}{3}Q} \leq Ck \|b\|_*. \quad (3.18)$$

On the other hand, for $III_3(z)$, we note for any $z, y \in Q$ that

$$\begin{aligned}
&\left| I_{\alpha, m} \left((b_1 - m_{\widetilde{Q}}(b_1)) f_1^\infty, \dots, (b_m - m_{\widetilde{Q}}(b_m)) f_m^\infty \right) (z) \right. \\
&\quad \left. - I_{\alpha, m} \left((b_1 - m_{\widetilde{Q}}(b_1)) f_1^\infty, \dots, (b_m - m_{\widetilde{Q}}(b_m)) f_m^\infty \right) (y) \right| \\
&\leq \int_{(\mathbb{R}^d \setminus \frac{4}{3}Q)^m} \left| \frac{1}{|(z - y_1, \dots, z - y_m)|^{mn-\alpha}} - \frac{1}{|(y - y_1, \dots, y - y_m)|^{mn-\alpha}} \right| \\
&\quad \times \left| \prod_{i=1}^m (b_i(y_i) - m_{\widetilde{Q}}(b_i)) f_i^\infty \right| d\mu(y_1) d\mu(y_2) \cdots d\mu(y_m) \\
&\leq C \int_{(\mathbb{R}^d \setminus \frac{4}{3}Q)^m} \frac{|z - y| \prod_{i=1}^m |(b_i(y_i) - m_{\widetilde{Q}}(b_i)) f_i^\infty|}{|(y - y_1, \dots, y - y_m)|^{mn-\alpha+1}} \\
&\quad \times d\mu(y_1) d\mu(y_2) \cdots d\mu(y_m) \\
&\leq C \prod_{i=1}^m \sum_{k=1}^{\infty} \int_{2^k \frac{4}{3}Q \setminus 2^{k-1} \frac{4}{3}Q} \frac{l(Q)^{\frac{1}{m}}}{l \left(2^k \frac{3}{2}Q \right)^{n-\frac{\alpha}{m}+\frac{1}{m}}} |(b_i(y_i) - m_{\widetilde{Q}}(b_i)) f_i| d\mu(y_i) \\
&\leq C \prod_{i=1}^m \sum_{k=1}^{\infty} 2^{-k/m} \left(\frac{1}{l \left(2^k \frac{3}{2}Q \right)^n} \int_{2^k \frac{4}{3}Q} |(b_i(y_i) - m_{\widetilde{Q}}(b_i))|^{s_i'} d\mu(y_i) \right)^{\frac{1}{s_i'}} \\
&\quad \times \left(\frac{1}{l \left(2^k \frac{3}{2}Q \right)^{n-(\alpha s_i/m)}} \int_{2^k \frac{4}{3}Q} |f_i(y_i)|^{s_i} d\mu(y_i) \right)^{\frac{1}{s_i}} \\
&\leq C \prod_{i=1}^m \sum_{k=1}^{\infty} 2^{-k/m} (k+1) \|b_i\|_* M_{s_i, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_i(x) \leq C \prod_{i=1}^m \|b_i\|_* M_{s_i, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_i(x),
\end{aligned}$$

where we have used the fact (3.18) again.

Taking the mean over $y \in Q$, we can obtain that

$$\frac{1}{\mu \left(\frac{3}{2}Q \right)} \int_Q (|III_2(z)| + |III_3(z)|) d\mu(z) \leq C \prod_{i=1}^m \|b_i\|_* M_{s_i, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_i(x). \quad (3.19)$$

Combing the inequalities (3.12), (3.13), (3.14), (3.16), (3.17) and (3.19), we see from the estimates of I , II , III_1 , III_2 and III_3 that the desired inequality (3.10) holds.

Next we turn to estimate the inequality (3.11). For any cubes $Q \subset R$ with $x \in Q$, where R is doubling. We denote $N_{Q,R} + 2$ by N , then $2^N Q \supset 2Q$ and $2^N Q \supset 2R$. We recall the equality (3.15) and let $f_i^0 = f_i \chi_{2^N Q \setminus \frac{4}{3}Q}$ and $f_i^R = f_i \chi_{2^N Q \setminus \frac{4}{3}R}$, and let $f_i^\infty = f_i \chi_{\mathbb{R}^d \setminus 2^N Q}$. Then we can write

$$\begin{aligned}
 & |h_Q - h_R| \\
 &= \left| m_Q \left[I_{\alpha,m} \left((m_{\tilde{Q}}(b_1) - b_1) f_1 \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q}, \dots, (m_{\tilde{Q}}(b_m) - b_m) f_m \chi_{\mathbb{R}^d \setminus \frac{4}{3}Q} \right) \right] \right. \\
 &\quad \left. - m_R \left[I_{\alpha,m} \left((m_R(b_1) - b_1) f_1 \chi_{\mathbb{R}^d \setminus \frac{4}{3}R}, \dots, (m_R(b_m) - b_m) f_m \chi_{\mathbb{R}^d \setminus \frac{4}{3}R} \right) \right] \right| \\
 &\leq \left| m_Q \left[I_{\alpha,m} \left((m_{\tilde{Q}}(b_1) - b_1) f_1^\infty, \dots, (m_{\tilde{Q}}(b_m) - b_m) f_m^\infty \right) \right] \right. \\
 &\quad \left. - m_R \left[I_{\alpha,m} \left((m_{\tilde{Q}}(b_1) - b_1) f_1^\infty, \dots, (m_{\tilde{Q}}(b_m) - b_m) f_m^\infty \right) \right] \right| \\
 &\quad + \left| m_R \left[I_{\alpha,m} \left((m_{\tilde{Q}}(b_1) - b_1) f_1^\infty, \dots, (m_{\tilde{Q}}(b_m) - b_m) f_m^\infty \right) \right] \right. \\
 &\quad \left. - m_R \left[I_{\alpha,m} \left((m_R(b_1) - b_1) f_1^\infty, \dots, (m_R(b_m) - b_m) f_m^\infty \right) \right] \right| \\
 &\quad + \sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_m \in \{0, \infty\} \\ \text{at least one } \alpha_i \neq \infty}} \left| m_Q \left[I_{\alpha,m} \left((m_{\tilde{Q}}(b_1) - b_1) f_1^{\alpha_1}, \dots, \right. \right. \right. \\
 &\quad \left. \left. \left. \times (m_{\tilde{Q}}(b_m) - b_m) f_m^{\alpha_m} \right) \right] \right| \\
 &\quad + \sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_m \in \{R, \infty\} \\ \text{at least one } \alpha_i \neq \infty}} \left| m_R \left[I_{\alpha,m} \left((m_R(b_1) - b_1) f_1^{\alpha_1}, \dots, \right. \right. \right. \\
 &\quad \left. \left. \left. \times (m_R(b_m) - b_m) f_m^{\alpha_m} \right) \right] \right| \\
 &= A_1 + A_2 + A_3 + A_4.
 \end{aligned}$$

For the term A_1 , noting

$$|m_R(b_i) - m_{\tilde{Q}}(b_i)| \leq C \|b_i\|_* K_{Q,R}$$

and the similar argument as that for the estimate of III_3 , we can obtain that

$$A_1 \leq C \prod_{i=1}^m K_{Q,R} \|b_i\|_* M_{s_i, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_i(x) \leq C (K_{Q,R})^m \prod_{i=1}^m \|b_i\|_* M_{s_i, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_i(x). \quad (3.20)$$

To estimate A_2 , we recall the notations and note that, for any sequences ξ_j and ζ_j ,

$$\prod_{j=1}^m (\xi_j + \zeta_j) = \sum_{i=0}^m \sum_{\sigma \in C_i^m} \prod_{j \in \sigma} \xi_j \prod_{j' \in \sigma'} \zeta_{j'}.$$

Using this equality and expanding $m_{\tilde{Q}}(b_j) - b_j(y) = (m_{\tilde{Q}}(b_j) - b_j(z)) + (b_j(z) - b_j(y))$, we observe that

$$\begin{aligned} I_{\alpha,m} \left((m_{\tilde{Q}}(b_1) - b_1)f_1^\infty, \dots, (m_{\tilde{Q}}(b_m) - b_m)f_m^\infty \right) (z) \\ = \sum_{i=0}^m \sum_{\sigma \in C_i^m} \prod_{j \in \sigma} (m_{\tilde{Q}}(b_j) - b_j(z)) \left[\vec{b}_{\sigma'}, I_{\alpha,m} \right] (\vec{f} \chi_{\mathbb{R}^d \setminus 2^N Q})(z). \end{aligned}$$

Similarly,

$$\begin{aligned} I_{\alpha,m} \left((m_R(b_1) - b_1)f_1^\infty, \dots, (m_R(b_m) - b_m)f_m^\infty \right) (z) \\ = \sum_{i=0}^m \sum_{\sigma \in C_i^m} \prod_{j \in \sigma} (m_R(b_j) - b_j(z)) \left[\vec{b}_{\sigma'}, I_{\alpha,m} \right] (\vec{f} \chi_{\mathbb{R}^d \setminus 2^N Q})(z). \end{aligned}$$

Thus

$$\begin{aligned} & \left| I_{\alpha,m} \left((m_{\tilde{Q}}(b_1) - b_1)f_1^\infty, \dots, (m_{\tilde{Q}}(b_m) - b_m)f_m^\infty \right) (z) \right. \\ & \quad \left. - I_{\alpha,m} \left((m_R(b_1) - b_1)f_1^\infty, \dots, (m_R(b_m) - b_m)f_m^\infty \right) (z) \right| \\ & \leq \left| \sum_{i=1}^m \sum_{\sigma \in C_i^m} \prod_{j \in \sigma} (m_{\tilde{Q}}(b_j) - b_j(z)) \left[\vec{b}_{\sigma'}, I_{\alpha,m} \right] (\vec{f} \chi_{\mathbb{R}^d \setminus 2^N Q})(z) \right| \\ & \quad + \left| \sum_{i=1}^m \sum_{\sigma \in C_i^m} \prod_{j \in \sigma} (m_R(b_j) - b_j(z)) \left[\vec{b}_{\sigma'}, I_{\alpha,m} \right] (\vec{f} \chi_{\mathbb{R}^d \setminus 2^N Q})(z) \right| \\ & =: B_1(z) + B_2(z). \end{aligned} \tag{3.21}$$

To estimate the integrals above, we recall that $f_j^\infty = f_j \chi_{\mathbb{R}^d \setminus 2^N Q}$ and let $f_j^Q = f_j \chi_{2^N Q}$, then we can write that $f_j^\infty = f_j - f_j^Q$ and $f_j = f_j^\infty + f_j^Q$, and thus we have

$$\begin{aligned} \prod_{j=1}^m f_j^\infty(y_j) &= \prod_{j=1}^m f_j(y_j) + \sum_{i=1}^m (-1)^i \sum_{\varrho \in C_i^m} \prod_{j \in \varrho} f_j^Q(y_j) \prod_{j' \in \varrho'} f_{j'}(y_{j'}) \\ &= \prod_{j=1}^m f_j(y_j) + \sum_{\lambda=1}^m \sum_{\{j_1, j_2, \dots, j_\lambda\} \subset \{1, 2, \dots, m\}} \\ & \quad \times C_{j_1, j_2, \dots, j_m} f_{j_1}^Q(y_{j_1}) \dots f_{j_\lambda}^Q(y_{j_\lambda}) f_{j_{\lambda+1}}^\infty(y_{j_{\lambda+1}}) \dots f_{j_m}^\infty(y_{j_m}) \end{aligned} \tag{3.22}$$

where $C_{j_1, j_2, \dots, j_m}$ are constants independent of $\vec{f}$ and Q . From the equality (3.22), we can deduce that

$$\begin{aligned} & \left[\vec{b}_{\sigma'}, I_{\alpha,m} \right] (\vec{f} \chi_{\mathbb{R}^d \setminus 2^N Q})(x) = \left[\vec{b}_{\sigma'}, I_{\alpha,m} \right] (\vec{f})(z) \\ & + \sum_{\lambda=1}^m \sum_{\{j_1, j_2, \dots, j_\lambda\} \subset \{1, 2, \dots, m\}} C_{j_1, j_2, \dots, j_\lambda} I_{\alpha,m}(g_1 F_1, g_2 F_2, \dots, g_m F_m)(z) \end{aligned} \tag{3.23}$$

where

$$g_j(y) = \begin{cases} b_j(z) - b_j(y), & \text{if } j \in \sigma' \\ 1, & \text{if } j \in \sigma \end{cases}$$

and

$$F_j(y) = \begin{cases} f_j^Q(y), & \text{if } j \in \{j_1, j_2, \dots, j_\lambda\} \\ f_j^\infty(y), & \text{if } j \notin \{j_1, j_2, \dots, j_\lambda\} \end{cases}.$$

Along the same lines as that of the pointwise estimates of $III_2(z)$, we can obtain that, for $x, z \in R \subset 2^{N-1}Q$ and if $1 \leq \lambda \leq m$,

$$\begin{aligned} & |I_{\alpha, m}(g_1 F_1, g_2 F_2, \dots, g_m F_m)(z)| \\ & \leq C \prod_{j \in \{j_1, j_2, \dots, j_\lambda\}} \frac{1}{l(2^N Q)^{n - \frac{\alpha}{m}}} \int_{2^N Q} |g_j(y_j) f_j(y_j)| d\mu(y_j) \\ & \quad \times \sum_{k=1}^{\infty} 2^{-k(n - \frac{\alpha}{m})\lambda} \prod_{\substack{j \notin \{j_1, j_2, \dots, j_\lambda\} \\ 1 \leq j \leq m}} \frac{1}{l(2^k 2^N Q)^{n - \frac{\alpha}{m}}} \\ & \quad \times \int_{2^k 2^N Q} |g_j(y_j) f_j(y_j)| d\mu(y_j). \end{aligned} \quad (3.24)$$

Let $(g_j)_{k, s'_j}^* = 1$ if $j \in \sigma$; and

$$(g_j)_{k, s'_j}^* = \begin{cases} \left(\frac{1}{l(2^N Q)^n} \int_{2^N Q} |b_j(z) - b_j(y)|^{s'_j} d\mu(y) \right)^{\frac{1}{s'_j}}, & \text{if } j \in \sigma' \cap \{j_1, j_2, \dots, j_\lambda\} \\ \left(\frac{1}{l(2^{k+N} Q)^n} \int_{2^{k+N} Q} |b_j(z) - b_j(y)|^{s'_j} d\mu(y) \right)^{\frac{1}{s'_j}}, & \text{if } j \in \sigma' \setminus \{j_1, j_2, \dots, j_\lambda\} \end{cases}$$

thus for $j \in \sigma'$,

$$(g_j)_{k, s'_j}^* \leq C \|b_j\|_* + C |m_{2^N Q}(b_j) - b_j(z)| + C |m_{2^{k+N} Q}(b_j) - b_j(z)|.$$

Hence we get from (3.24) that, for $\tau > 1$,

$$\begin{aligned} & \left(\frac{1}{\mu(R)} \int_R |I_{\alpha, m}(g_1 F_1, g_2 F_2, \dots, g_m F_m)(z)|^\tau d\mu(z) \right)^{\frac{1}{\tau}} \\ & \leq C \left(\prod_{j=1}^m M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x) \right) \sum_{k=1}^{\infty} 2^{-k(n - \frac{\alpha}{m})\lambda} \\ & \quad \times \left(\frac{1}{\mu(R)} \int_R \left| \prod_{j \in \sigma'} (g_j)_{k, s'_j}^* \right|^\tau d\mu(z) \right)^{\frac{1}{\tau}} \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\prod_{j=1}^m M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x) \right) \sum_{k=1}^{\infty} 2^{-k(n-\frac{\alpha}{m})\lambda} \prod_{j \in \sigma'} \|b_j\|_* \\
&\quad \times (1 + K_{R, 2^N Q} + K_{R, 2^{k+N} Q}) \\
&\leq C \left(\prod_{j=1}^m M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x) \right) \prod_{j \in \sigma'} \|b_j\|_* \tag{3.25}
\end{aligned}$$

where we have used the fact that the cubes R and $2^N Q$ are comparable, which implies $K_{R, 2^N Q} \leq C$ and $K_{R, 2^{k+N} Q} \leq C(1+k)$. Using the inequality (3.25) above and the identity (3.23), we obtain that, for $\tau > 1$,

$$\begin{aligned}
A_2 &\leq \frac{1}{\mu(R)} \int_R (|B_1(z)| + |B_2(z)|) d\mu(z) \\
&\leq C \sum_{i=1}^m \sum_{\sigma \in C_i^m} \left(\prod_{j \in \sigma} K_{Q, R} \|b_j\|_* \right) \\
&\quad \times \left(\frac{1}{\mu(R)} \int_R \left| [\vec{b}_{\sigma'}, I_{\alpha, m}] (\vec{f} \chi_{\mathbb{R}^d \setminus 2^N Q})(z) \right|^\tau d\mu(z) \right)^{\frac{1}{\tau}} \\
&\leq C \sum_{i=1}^m \sum_{\sigma \in C_i^m} \left(\prod_{j \in \sigma} K_{Q, R} \|b_j\|_* \right) M_{\tau, \frac{3}{2}} \left([\vec{b}_{\sigma'}, I_{\alpha, m}] (\vec{f}) \right) (x) \\
&\quad + C \prod_{j=1}^m K_{Q, R} \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x).
\end{aligned}$$

The estimates of A_3 and A_4 is very similar to the one used in the estimate of A_2 . In fact, repeating the similar procedures used in (3.24) and (3.25) for $\tau = 1$, and noting that $K_{Q, 2^{k+N} Q} \leq K_{Q, R} + K_{R, 2^{k+N} Q} \leq C(1+k) + K_{Q, R}$ since $2^{N-3} Q \subset 2R \subset 2^N Q$ by the definition of N , we can deduce that

$$\begin{aligned}
&\sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_m \in \{0, \infty\} \\ \text{at least one } \alpha_i \neq \infty \\ \text{and one } \alpha_j \neq 0}} \left| m_Q \left[I_{\alpha, m} \left((m_{\tilde{Q}}(b_1) - b_1) f_1^{\alpha_1}, \dots, (m_{\tilde{Q}}(b_m) - b_m) f_m^{\alpha_m} \right) \right] \right| \\
&\quad + \sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_m \in \{R, \infty\} \\ \text{at least one } \alpha_i \neq \infty}} |m_R [I_{\alpha, m} ((m_R(b_1) - b_1) f_1^{\alpha_1}, \dots, \\
&\quad \times (m_R(b_m) - b_m) f_m^{\alpha_m})]| \\
&\leq C \prod_{j=1}^m K_{Q, R} \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x). \tag{3.26}
\end{aligned}$$

It is left to estimate the term in A_3 of the case $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$. A small modification is needed to estimate this term. For $z \in Q$ and $x \in Q$, one sees

$$I_{\alpha, m} \left((m_{\tilde{Q}}(b_1) - b_1) f_1^0, \dots, (m_{\tilde{Q}}(b_j) - b_j) f_j^0, \dots, (m_{\tilde{Q}}(b_m) - b_m) f_m^0 \right)$$

$$\begin{aligned}
&\leq C \sum_{k=1}^N \prod_{j=1}^m \frac{1}{l \left(2^k \frac{4}{3} Q\right)^{n-\frac{\alpha}{m}}} \int_{2^k \frac{4}{3} Q} |(m_{\tilde{Q}}(b_j) - b_j(y_j)) f_j(y_j)| d\mu(y_j) \\
&\leq C \sum_{k=1}^N \prod_{j=1}^m \frac{1}{l \left(2^k \frac{4}{3} Q\right)^{n-\frac{\alpha}{m}}} \left(\int_{2^k \frac{4}{3} Q} |(m_{\tilde{Q}}(b_j) - b_j)|^{s'_j} d\mu(y_j) \right)^{\frac{1}{s'_j}} \\
&\quad \times \left(\int_{2^k \frac{4}{3} Q} |f_j|^{s_j} d\mu(y_j) \right)^{\frac{1}{s_j}} \\
&\leq C \sum_{k=1}^N \prod_{j=1}^m \frac{\mu \left(2^k \frac{3}{2} Q\right)^{1-\frac{\alpha}{mn}}}{l \left(2^k \frac{4}{3} Q\right)^{n-\frac{\alpha}{m}}} \left(1 + K_{Q, \tilde{Q}} + K_{Q, 2^k \frac{4}{3} Q}\right) \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x) \\
&\leq C (K_{Q,R} K_{Q,R}^\alpha)^m \prod_{j=1}^m \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x).
\end{aligned}$$

This and the inequality (3.26) follows

$$A_3 + A_4 \leq C (K_{Q,R} K_{Q,R}^{(\alpha)})^m \prod_{j=1}^m \|b_j\|_* M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j(x).$$

Moreover, combining the estimates of A_1 , A_2 , A_3 and A_4 , we obtain the desired inequality (3.11).

Finally, let us show how to acquire the inequality (3.9) from the two inequalities (3.10) and (3.11). Fix the point x and let Q be any cube such that $x \in Q$, notice $K_{Q, \tilde{Q}} \leq C$ and $K_{Q, \tilde{Q}}^{(\alpha)} \leq C$, hence we see from the inequalities (3.10) and (3.11) that

$$\begin{aligned}
&\frac{1}{\mu \left(\frac{3}{2} Q\right)} \int_Q \left| \left[\vec{b}, I_{\alpha, m} \right] (\vec{f})(z) - m_{\tilde{Q}} \left(\left[\vec{b}, I_{\alpha, m} \right] (\vec{f}) \right) \right| d\mu(z) \\
&\leq \frac{1}{\mu \left(\frac{3}{2} Q\right)} \int_Q \left| \left[\vec{b}, I_{\alpha, m} \right] (\vec{f})(z) - h_Q \right| d\mu(z) + \frac{1}{\mu \left(\frac{3}{2} Q\right)} \int_Q \left| h_Q - h_{\tilde{Q}} \right| d\mu(z) \\
&\quad + \frac{1}{\mu \left(\frac{3}{2} Q\right)} \int_Q \left| m_{\tilde{Q}} \left[\vec{b}, I_{\alpha, m} \right] (\vec{f}) - h_{\tilde{Q}} \right| d\mu(z) \\
&\leq C \mathcal{R}(\vec{b}, \vec{f})(x).
\end{aligned} \tag{3.27}$$

On the other hand, for all doubling cubes $Q \subset R$ with $x \in Q$ such that $K_{Q,R} \leq K_{Q,R}^{(\alpha)} \leq P_0$, where P_0 is the constant in Lemma 6 in [3], using (3.11), we have

$$|h_Q - h_R| \leq C K_{Q,R}^{(\alpha)} P_0^{2m-1} \mathcal{R}(\vec{b}, \vec{f})(x), \tag{3.28}$$

and moreover the inequality (3.28) holds for any doubling cubes Q, R with $Q \subset R$. Therefore,

$$\begin{aligned}
& \left| m_Q \left(\left[\vec{b}, I_{\alpha, m} \right] (\vec{f}) \right) - m_R \left(\left[\vec{b}, I_{\alpha, m} \right] (\vec{f}) \right) \right| \\
& \leq \left| m_Q \left(\left[\vec{b}, I_{\alpha, m} \right] (\vec{f}) \right) - h_Q \right| + \left| h_R - m_R \left(\left[\vec{b}, I_{\alpha, m} \right] (\vec{f}) \right) \right| \\
& \quad + |h_Q - h_R| \\
& \leq CK_{Q, R}^{(\alpha)} \mathcal{R}(\vec{b}, \vec{f})(x).
\end{aligned} \tag{3.29}$$

According to the estimates (3.27), (3.29) and the definition of the sharp maximal function, we deduce the inequality (3.9) and so finish the proof of the lemma. $\square$

Corollary 3.5. *Using the same notations in Lemma 3.4, and letting $\sigma \subset \{1, 2, \dots, m\}$, we have*

$$\begin{aligned}
M^{\sharp, (\alpha)} \left(\left[\vec{b}_\sigma, I_{\alpha, m} \right] (\vec{f}) \right) (x) & \leq C \left[\prod_{j \in \sigma} \|b_j\|_* M_{\tau, \frac{3}{2}}(I_{\alpha, m}(\vec{f}))(x) \right. \\
& + \sum_{\substack{\sigma_1 \cup \sigma_2 = \sigma \\ \sigma_1 \neq \emptyset, \sigma_2 \neq \emptyset}} \prod_{j \in \sigma_1} \|b_j\|_* M_{\tau, \frac{3}{2}} \left(\left[\vec{b}_{\sigma_2}, I_{\alpha, m} \right] (\vec{f}) \right) (x) \\
& \left. + \prod_{j \in \sigma} \|b_j\|_* \prod_{j=1}^m M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m} \right)} f_j(x) \right]
\end{aligned} \tag{3.30}$$

and particular,

$$M^{\sharp, (\alpha)}([b_i, I_{\alpha, m}](\vec{f}))(x) \leq C \|b_i\|_* \left[M_{\tau, \frac{3}{2}}(I_{\alpha, m}(\vec{f}))(x) + \prod_{j=1}^m M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m} \right)} f_j(x) \right]. \tag{3.31}$$

Proof. The proof of (3.30) and (3.31) is very similar to (3.9), so we omit the details. $\square$

Now we are ready to give **the proof of theorem 1.4**. Using Lemma 3.2, Lemma 3.3, Corollary 3.5 and Theorem 1.2, we get that

$$\begin{aligned}
\left\| [b_i, I_{\alpha, m}](\vec{f}) \right\|_{M_q^p(\mu)} & \leq C \left\| N([b_i, I_{\alpha, m}](\vec{f})) \right\|_{M_q^p(\mu)} \\
& \leq C \left\| M^{\sharp, (\alpha)}([b_i, I_{\alpha, m}](\vec{f})) \right\|_{M_q^p(\mu)} \\
& \leq C \|b_i\|_* \left\| M_{\tau, \frac{3}{2}}(I_{\alpha, m}(\vec{f}))(x) + \prod_{i=1}^m M_{s_i, \frac{9}{8}}^{\left(\frac{\alpha}{m} \right)} f_i \right\|_{M_q^p(\mu)} \\
& \leq C \|b_i\|_* \left\| M_{\tau, \frac{3}{2}}(I_{\alpha, m}(\vec{f})) \right\|_{M_q^p(\mu)} + C \|b_i\|_*
\end{aligned}$$

$$\begin{aligned}
& \times \left\| \prod_{i=1}^m M_{s_i, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_i \right\|_{M_q^p(\mu)} \\
& \leq C \|b_i\|_* \left\| I_{\alpha, m}(\vec{f}) \right\|_{M_q^p(\mu)} + C \|b_i\|_* \prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)} \\
& \leq C \|b_i\|_* \prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}(\mu)}.
\end{aligned}$$

Similarly, applying the inequality (3.30) in Corollary 3.5, for $\sigma \subseteq \{1, 2, \dots, m\}$, we have

$$\begin{aligned}
\left\| [\vec{b}_\sigma, I_{\alpha, m}](\vec{f}) \right\|_{M_q^p(\mu)} & \leq C \|N([\vec{b}_\sigma, I_{\alpha, m}](\vec{f}))\|_{M_q^p(\mu)} \\
& \leq C \|M^{\sharp, (\alpha)}[\vec{b}_\sigma, I_{\alpha, m}](\vec{f})\|_{M_q^p(\mu)} \\
& \leq C \left\| \prod_{j \in \sigma} \|b_j\|_* M_{\tau, \frac{3}{2}}(I_{\alpha, m}(\vec{f})) \right. \\
& \quad + \sum_{\substack{\sigma_1 \cup \sigma_2 = \sigma \\ \sigma_1 \neq \emptyset, \sigma_2 \neq \emptyset}} \prod_{j \in \sigma_1} \|b_j\|_* M_{\tau, \frac{3}{2}} \left([\vec{b}_{\sigma_2}, I_{\alpha, m}](\vec{f}) \right) \\
& \quad \left. + \prod_{j \in \sigma} \|b_j\|_* \prod_{j=1}^m M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j \right\|_{M_q^p(\mu)} \\
& \leq C \prod_{j \in \sigma} \|b_j\|_* \left\| M_{\tau, \frac{3}{2}}(I_{\alpha, m}(\vec{f})) \right\|_{M_q^p(\mu)} \\
& \quad + C \sum_{\substack{\sigma_1 \cup \sigma_2 = \sigma \\ \sigma_1 \neq \emptyset, \sigma_2 \neq \emptyset}} \prod_{j \in \sigma_1} \|b_j\|_* \left\| M_{\tau, \frac{3}{2}} \left([\vec{b}_{\sigma_2}, I_{\alpha, m}](\vec{f}) \right) \right\|_{M_q^p(\mu)} \\
& \quad + C \prod_{j \in \sigma} \|b_j\|_* \left\| \prod_{j=1}^m M_{s_j, \frac{9}{8}}^{\left(\frac{\alpha}{m}\right)} f_j \right\|_{M_q^p(\mu)} \\
& \leq C \prod_{j \in \sigma} \|b_j\|_* \prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}(\mu)} \\
& \quad + C \sum_{\substack{\sigma_1 \cup \sigma_2 = \sigma \\ \sigma_1 \neq \emptyset, \sigma_2 \neq \emptyset}} \prod_{j \in \sigma_1} \|b_j\|_* \left\| M_{\tau, \frac{3}{2}} \left([\vec{b}_{\sigma_2}, I_{\alpha, m}](\vec{f}) \right) \right\|_{M_q^p(\mu)} \\
& \leq C \prod_{j \in \sigma} \|b_j\|_* \prod_{j=1}^m \|f_j\|_{M_{q_j}^{p_j}(\mu)} \\
& \quad + C \sum_{\substack{\sigma_1 \cup \sigma_2 = \sigma \\ \sigma_1 \neq \emptyset, \sigma_2 \neq \emptyset}} \prod_{j \in \sigma_1} \|b_j\|_* \left\| [\vec{b}_{\sigma_2}, I_{\alpha, m}](\vec{f}) \right\|_{M_q^p(\mu)}
\end{aligned}$$

where σ_1 and σ_2 are two nonempty subsets of σ and $\sigma_1 \cap \sigma_2 = \emptyset$. Hence, we can make use of induction on $\sigma \subseteq \{1, 2, \dots, m\}$ to get that

$$\left\| \left[\vec{b}_\sigma, I_{\alpha, m} \right] (\vec{f}) \right\|_{M_q^p(\mu)} \leq C \prod_{j \in \sigma} \|b_j\|_* \prod_{i=1}^m \|f_i\|_{M_{q_i}^{p_i}(\mu)}.$$

This completes the proof of Theorem 1.4.

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