Nonlinear elliptic boundary value problem with equivalued surface in a domain with thin layer

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Abstract. This paper deals with certain kinds of boundary value problems with equivalued surface of nonlinear elliptic equations on a domain with thin layer. We introduce the concept of renormalized solution to this problem. Existence and uniqueness of renormalized solutions are given, and the limit behaviour of solutions is studied in this paper.

Mathematics Subject Classification (2000). 35B37, 35B40, 49J20.

Keywords. Renormalized solution, Existence, Uniqueness, Equivalued surface, Equivalued interface, Nonlinear elliptic equations, Limit behaviour of solutions.

1. Introduction

In many practical applications, especially in resistivity well-logging in petroleum exploitation, boundary value problems with equivalued surface are formulated in 1970s. It is a kind of nonlocal boundary value problems, which can also be used to give mathematical descriptions for some other problems in physics and mechanics. For this kind of boundary value problem, at least in linear case, a complete theory has been established (see [8, 9, 11]). In recent years some progress on the study of nonlinear boundary value problems with equivalued surface has been made (see [5–7, 11, 12]).

In resistivity well-logging, one may encounter a formation with crack domain, the resistivity of which is often difficult to be obtained. However, this crack domain is a thin layer compared with the whole formation (see [11]). In practical calculation, the variation of solutions near the thin layer should be quite large, and then in finite element procedure, it is necessary to have a refined partition of elements near the thin layer. This causes a complexity in

FIGURE 1. The division of Ω with thin layer $\tilde{\Omega}$

computation. In order to overcome this difficulty, when the thin layer is rather thin, the thin layer can be approximately regarded as an interface and correspondingly the boundary value problem with equivalued surface on the thin layer can be approximately replaced by the boundary value problem with equivalued interface. To prove the above conclusion, we need to study existence, uniqueness and limit behaviour of solutions for boundary value problems with equivalued surface in a domain with thin layer. The case of linear problems has been studied in [10]. In this paper, our main goal is to extend these results of [10] to nonlinear problems. For the sake of clarity in the exposition, we only consider the case that the thin layer $\tilde{\Omega}$ (see Fig. 1) is a good conductor (that is the potential u should be a constant to be determined on $\tilde{\Omega}$), which is an important case in practical applications. Nevertheless, the methods involved in this paper can be applied to other cases in [10].

Here we consider the following nonlinear elliptic boundary value problems with equivalued surface on a domain with thin layer:

$$(P_1) \begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} (\tilde{a}_{ij}(x, u) \frac{\partial u}{\partial x_j}) = F(x) & \text{in } \tilde{\Omega}_1 \cup \tilde{\Omega}_2, \\ u = 0 & \text{on } \Gamma, \\ u = \tilde{C} \text{ (a constant to be determined)} & \text{on } \tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2, \\ \int_{\tilde{\Gamma}_1} \frac{\partial u}{\partial n_L} ds = \int_{\tilde{\Gamma}_2} \frac{\partial u}{\partial n_L} ds + \tilde{A} \text{ (a given constant)}, \end{cases}$$

Let Ω be a bounded domain in R^N ($N \geq 2$) with smooth outer boundary Γ (see Fig. 1). Suppose that Ω is composed of three non-overlapping subdomains $\tilde{\Omega}_1, \tilde{\Omega}_2$ and $\tilde{\Omega}$, and $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are the interfaces of thin layer $\tilde{\Omega}$ with $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ respectively. The unit outward normal vector $\vec{n} = (n_1, \dots, n_N)$ takes the inward and outward directions (or vice versa) for the domain $\tilde{\Omega}$ on $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$.

$$\frac{\partial u}{\partial n_L} = \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) \frac{\partial u}{\partial x_j} n_i$$

denotes the conormal derivative.

In this paper we will discuss the existence, uniqueness and limit behaviour of renormalized solutions to problem (P_1) .

This paper is organized as follows: In Sect. 2, we will prove the existence and uniqueness of renormalized solutions to problems (P_1) . In Sect. 3 we will discuss nonlinear elliptic boundary value problem (P) with equivalued interface. In Sect. 4 the limit behaviour of solutions to problems (P_1) will be studied.

2. Existence and uniqueness of renormalized solution to problem (P_1)

In this section, we will discuss the existence and uniqueness of renormalized solution to problem (P_1) . We first make the following assumptions:

$(\tilde{H}_1)$ $\tilde{a}_{ij}(x, s) = \tilde{a}_{ji}(x, s)$, a.e. $x \in \Omega$, $\forall s \in R$. Moreover there exists a positive constant α such that for every $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in R^N$,

$$\sum_{i,j=1}^N \tilde{a}_{ij}(x, s) \xi_i \xi_j \geq \alpha |\xi|^2, \quad \forall s \in R, \text{ a.e. } x \in \Omega. \quad (2.1)$$

$(\tilde{H}_2)$ Let $\tilde{a}_{ij} : \Omega \times R \longrightarrow R$ be Carathéodory functions with $1 \leq i, j \leq N$. Assume that $\tilde{a}_{ij}(\cdot, 0) \in L^\infty(\Omega)$ and there exists a positive function $d \in L^\infty(\Omega)$ such that for every $s_1, s_2 \in R$ and a.e. $x \in \Omega$,

$$|\tilde{a}_{ij}(x, s_1) - \tilde{a}_{ij}(x, s_2)| \leq d(x) |s_1 - s_2|.$$

We also assume that $F \in L^2(\Omega)$.

Set

$$V_1 = \{v \in H_0^1(\Omega), v|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} = \text{constant}\}.$$

Definition 2.1. A measurable function $u \in V_1$ is called a weak solution to problem (P_1) , if $\tilde{a}_{ij}(x, u) D_j u \in L^2(\Omega)$ and

$$\int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_j u D_i v \, dx = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F v \, dx + \tilde{A} v|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}, \quad \forall v \in V_1. \quad (2.2)$$

Remark 2.2. Under the above assumptions, it is impossible to obtain a L^∞ estimate on weak solutions to problem (P_1) , moreover since $\tilde{a}_{ij}(x, u)$ is unbounded with respect to u , $\tilde{a}_{ij}(x, u) D_j u$ can not belong to $L^2(\Omega)$ and the term on the left hand side of (2.2) may not be well defined. Hence we are not able to prove the existence and uniqueness of weak solutions in the sense of Definition 2.1 to problem (P_1) . In order to overcome this difficulty, we will use the concept of renormalized solution introduced by Diperna and Lions in [3] for Boltzmann equations (see also [1, 2, 7]). The renormalized solution is different from the weak solution or the distribution solution because it is defined by truncation functions.

Definition 2.3. A measurable function $u \in V_1$ is a renormalized solution to problem (P_1) , if for all $h(\cdot) \in W^{1,\infty}(R)$ with compact support and $\xi \in V_1 \cap L^\infty(\Omega)$, u satisfies

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_j u D_i (h(u) \xi) \, dx \\ &= \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F h(u) \xi \, dx + \tilde{A} h(u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) \xi|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} \end{aligned}$$

and

$$\lim_{m \rightarrow \infty} \int_{\{x \in \Omega: m \leq |u(x)| \leq m+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_j u D_i u \, dx = 0.$$

For any given $k > 0$, we defined the truncation function T_k as follows:

$$T_k(v) = \begin{cases} k & v > k, \\ v & |v| \leq k, \\ -k & v < -k. \end{cases}$$

Now we can state the existence and uniqueness result to problem (P_1) as follows.

Theorem 2.4. Under hypotheses $(\tilde{H}_1)$, $(\tilde{H}_2)$ and $F \in L^2(\Omega)$, problem (P_1) admits a unique renormalized solution $u \in V_1$ in the sense of Definition 2.3.

Proof. (1) Proof of existence

Here we will use the method in [7]. We first consider the following problem:

$$(P_{1n}) \begin{cases} - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} (\tilde{a}_{ij}(x, T_n(u_n)) \frac{\partial u_n}{\partial x_j}) = F(x) & \text{in } \tilde{\Omega}_1 \cup \tilde{\Omega}_2, \\ u_n = 0 & \text{on } \Gamma, \\ u_n = \tilde{C}_n \text{ (a constant to be determined)} & \text{on } \tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2, \\ \int_{\tilde{\Gamma}_1} \frac{\partial u_n}{\partial n_L} \, ds = \int_{\tilde{\Gamma}_2} \frac{\partial u_n}{\partial n_L} \, ds + \tilde{A} \text{ (a given constant)}. \end{cases}$$

Similarly to the first step in the proof of Theorem 3.1 in [12], using the pseudo-monotone operator theory (cf. [13]), for any fixed $n > 0$, problem (P_{1n}) admits a weak solution $u_n \in V_1$ such that

$$\int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_n(u_n)) D_j u_n D_i v \, dx = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F v \, dx + \tilde{A} v|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}, \quad \forall v \in V_1. \quad (2.3)$$

Taking $v = u_n$ in (2.3), by $(\tilde{H}_1)$ and trace theorem, we get

$$\alpha \int_{\Omega} |Du_n|^2 \, dx \leq \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F u_n \, dx + C_1 |\tilde{A}| \|u_n\|_{V_1}, \quad (2.4)$$

where C_1 is a positive constant independent of n .

Hölder's inequality, Young's inequality and (2.4) imply that

$$\int_{\Omega} |Du_n|^2 dx \leq C_2(\|F\|_{L^2(\Omega)}^2 + |\tilde{A}|^2), \quad (2.5)$$

where C_2 is also a positive constant independent of n .

Inequality (2.5) implies that

$$\|u_n\|_{V_1} \leq C_3(\|F\|_{L^2(\Omega)} + |\tilde{A}|), \quad (2.6)$$

where C_3 is a positive constant independent of n .

By (2.6), there is a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$u_n \rightharpoonup u \text{ weakly in } V_1. \quad (2.7)$$

Sobolev imbedding theorem and trace theorem yield

$$u_n \rightarrow u \text{ strongly in } L^q(\Omega), \quad (2.8)$$

where

$$\begin{cases} 1 \leq q < \frac{2N}{N-2}, & \text{if } N > 2, \\ 1 \leq q < +\infty, & \text{if } N = 2 \end{cases}$$

and

$$u_n|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} \rightarrow u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}, \quad (2.9)$$

$$u_n \rightarrow u \text{ a.e. in } \Omega. \quad (2.10)$$

In the following, the proof will be divided into two steps.

Step 1. We will prove that for every $k > 0$, $T_k(u_n)$ strongly converges to $T_k(u)$ in V_1 .

For any fixed $k > 0$, let $h > k$ and take $v = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ (introduced by Leone and Porretta in [4]) as a test function in (2.3). Setting $M = 4k + h$, it is easy to check that $Dv = 0$ where $|u_n| > M$, therefore for $n > M$, we can write (2.3) as

$$\int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_M(u_n)) D_j u_n D_i v dx = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} Fv dx + \tilde{A}v|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}, \quad \forall v \in V_1. \quad (2.11)$$

Splitting the integral on the left side of (2.11) on the sets $\{x \in \Omega : |u_n(x)| \leq k\}$ and $\{x \in \Omega : |u_n(x)| > k\}$, by (2.1) we get

$$\begin{aligned}
& \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_M(u_n)) D_j u_n D_i T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\
&= \int_{\{x \in \Omega: |u_n(x)| \leq k\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_M(u_n)) D_j u_n D_i T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\
&\quad + \int_{\{x \in \Omega: |u_n(x)| > k\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_M(u_n)) D_j u_n D_i T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\
&\geq \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_k(u_n)) D_j T_k(u_n) D_i (T_k(u_n) - T_k(u)) \, dx \\
&\quad - \int_{\{x \in \Omega: |u_n(x)| > k\}} \sum_{i,j=1}^N |\tilde{a}_{ij}(x, T_M(u_n))| |D_j u_n| |D_i T_k(u)| \, dx. \tag{2.12}
\end{aligned}$$

Relations (2.11) and (2.12) imply that

$$\begin{aligned}
& \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_k(u_n)) D_j (T_k(u_n) - T_k(u)) D_i (T_k(u_n) - T_k(u)) \, dx \\
&\leq \int_{\{x \in \Omega: |u_n(x)| > k\}} \sum_{i,j=1}^N |\tilde{a}_{ij}(x, T_M(u_n))| |D_j u_n| |D_i T_k(u)| \, dx \\
&\quad + \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \, dx \\
&\quad + \tilde{A} T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} \\
&\quad - \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_k(u_n)) D_j T_k(u) D_i (T_k(u_n) - T_k(u)) \, dx \\
&= I_1 + I_2 + I_3 - I_4. \tag{2.13}
\end{aligned}$$

For any fixed $h > 0$, $(\tilde{H}_2)$ and (2.5) imply that $|\tilde{a}_{ij}(\cdot, T_M(u_n))| |D_j(u_n)|$ is bounded in $L^2(\Omega)$, while $\chi_{\{|u_n| > k\}} |D_i T_k(u)|$ converges strongly to zero in $L^2(\Omega)$ as n tends to the infinity, hence we have

$$\lim_{n \rightarrow \infty} I_1 = \lim_{n \rightarrow \infty} \int_{\Omega} \sum_{i,j=1}^N |\tilde{a}_{ij}(x, T_M(u_n))| |D_j(u_n)| |D_i T_k(u)| \chi_{\{|u_n| > k\}} \, dx = 0. \tag{2.14}$$

Relation (2.10) and Lebesgue's dominated convergence theorem imply that

$$\lim_{n \rightarrow \infty} I_2 = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F T_{2k}(u - T_h(u)) \, dx. \tag{2.15}$$

We derive from (2.9) that

$$\lim_{n \rightarrow \infty} I_3 = \tilde{A} T_{2k}(u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} - T_h(u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2})). \tag{2.16}$$

By $(\tilde{H}_2)$, (2.10) and Lebesgue's dominated convergence theorem, we have

$$\tilde{a}_{ij}(\cdot, T_k(u_n)) D_j T_k(u) \rightarrow \tilde{a}_{ij}(\cdot, T_k(u)) D_j T_k(u) \text{ strongly in } L^2(\Omega). \tag{2.17}$$

Thus, from (2.7) and (2.17) it follows that

$$\lim_{n \rightarrow \infty} I_4 = 0. \quad (2.18)$$

Let n tend to the infinity in (2.13). We derive from (2.1), (2.13)–(2.16) and (2.18) that

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \alpha \int_{\Omega} |DT_k(u_n) - DT_k(u)|^2 \\ & \leq \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} FT_{2k}(u - T_h(u)) \, dx \\ & \quad + \tilde{A}T_{2k}(u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} - T_h(u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2})). \end{aligned} \quad (2.19)$$

Let $h \rightarrow \infty$ in (2.19), we get

$$\overline{\lim}_{n \rightarrow \infty} \alpha \int_{\Omega} |DT_k(u_n) - DT_k(u)|^2 \leq 0,$$

which implies that

$$DT_k(u_n) \rightarrow DT_k(u) \text{ in } L^2(\Omega), \quad \text{as } n \rightarrow \infty. \quad (2.20)$$

Step 2. We will prove that u is a renormalized solution to problem (P_1) .

For any given $\xi \in V_1 \cap L^\infty(\Omega)$ and $h(\cdot) \in W^{1,\infty}(R)$ with compact support belonging to $[-k, k]$, taking $v = h(u_n)\xi$ in (2.3), we get

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_n(u_n)) D_j u_n D_i(h(u_n)\xi) \, dx \\ & = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} Fh(u_n)\xi \, dx + \tilde{A}h(u_n|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2})\xi|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}. \end{aligned} \quad (2.21)$$

For any given $n > k$, we have

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_n(u_n)) D_j u_n D_i(h(u_n)\xi) \, dx \\ & = \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_k(u_n)) D_j T_k(u_n) D_i(h(u_n)\xi) \, dx \\ & = \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_k(u_n)) D_j T_k(u_n) D_i T_k(u_n) h'(u_n) \xi \, dx \\ & \quad + \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_k(u_n)) D_j T_k(u_n) D_i \xi h(u_n) \, dx \\ & = J_1 + J_2. \end{aligned} \quad (2.22)$$

By $(\tilde{H}_2)$, (2.10) and (2.20) we get

$$\lim_{n \rightarrow \infty} J_1 = \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_k(u)) D_j T_k(u) D_i T_k(u) h'(u) \xi \, dx \quad (2.23)$$

and

$$\lim_{n \rightarrow \infty} J_2 = \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_k(u)) D_j T_k(u) D_i \xi h(u) \, dx. \quad (2.24)$$

Let $n \rightarrow \infty$ in (2.21), it follows from (2.8)–(2.10), (2.22)–(2.24) that

$$\int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_j u D_i (h(u) \xi) \, dx = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} Fh(u) \xi \, dx + \tilde{A}h(u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) \xi|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}. \quad (2.25)$$

For any given $m > 0$, taking $v = T_{m+1}(u_n) - T_m(u_n)$ in (2.3), we get

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_n(u_n)) D_j u_n D_i [T_{m+1}(u_n) - T_m(u_n)] \, dx \\ &= \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F[T_{m+1}(u_n) - T_m(u_n)] \, dx \\ & \quad + \tilde{A}[T_{m+1}(u_n|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) - T_m(u_n|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2})]. \end{aligned}$$

For $n > m + 1$, we get

$$\begin{aligned} & \int_{\{x \in \Omega: m \leq |u_n(x)| \leq m+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_n) D_j u_n D_i u_n \, dx \\ &= \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_n(u_n)) D_j u_n D_i [T_{m+1}(u_n) - T_m(u_n)] \, dx \\ &= \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F[T_{m+1}(u_n) - T_m(u_n)] \, dx \\ & \quad + \tilde{A}[T_{m+1}(u_n|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) - T_m(u_n|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2})]. \end{aligned} \quad (2.26)$$

However

$$\begin{aligned} & \int_{\{x \in \Omega: m \leq |u_n(x)| \leq m+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_n) D_j u_n D_i u_n \, dx \\ &= \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_{m+1}(u_n)) D_j T_{m+1}(u_n) \\ & \quad \times D_i T_{m+1}(u_n) \chi_{\{x \in \Omega: m \leq |u_n(x)| \leq m+1\}} \, dx. \end{aligned} \quad (2.27)$$

By $(\tilde{H}_2)$, (2.10) and replacing k by $m+1$ in (2.20), we get

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_{m+1}(u_n)) D_j T_{m+1}(u_n) D_i T_{m+1}(u_n) \chi_{\{x \in \Omega: m \leq |u_n(x)| \leq m+1\}} dx \\ & \longrightarrow \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, T_{m+1}(u)) D_j T_{m+1}(u) D_i T_{m+1}(u) \chi_{\{x \in \Omega: m \leq |u(x)| \leq m+1\}} dx \\ & = \int_{\{x \in \Omega: m \leq |u(x)| \leq m+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_j u D_i u dx. \end{aligned} \quad (2.28)$$

Let n tend to the infinity in (2.26). We conclude from (2.8)–(2.10) and (2.27), (2.28) that

$$\begin{aligned} & \int_{\{x \in \Omega: m \leq |u(x)| \leq m+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_j u D_i u dx \\ & = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F[T_{m+1}(u) - T_m(u)] dx \\ & \quad + \tilde{A}[T_{m+1}(u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) - T_m(u|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2})]. \end{aligned} \quad (2.29)$$

We conclude from (2.29) and Lebesgue's dominated convergence theorem that

$$\lim_{m \rightarrow \infty} \int_{\{x \in \Omega: m \leq |u(x)| \leq m+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_j u D_i u dx = 0. \quad (2.30)$$

From (2.25) and (2.30), we can deduce that u is a renormalized solution to problem (P_1) .

(2) Proof of uniqueness

Here the idea of this proof comes from [1]. Let $u_1, u_2 \in V_1$ be two renormalized solutions to problem (P_1) in the sense of Definition 2.3. Then for all $h(\cdot) \in W^{1,\infty}(R)$ with compact support and $\xi \in V_1 \cap L^\infty(\Omega)$, we have

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_s) D_j u_s D_i (h(u_s) \xi) dx \\ & = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} Fh(u_s) \xi dx + \tilde{A}h(u_s|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) \xi|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} \end{aligned} \quad (2.31)$$

and

$$\lim_{m \rightarrow \infty} \int_{\{x \in \Omega: m \leq |u_s(x)| \leq m+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_s) D_j u_s D_i u_s dx = 0, \quad s = 1, 2. \quad (2.32)$$

Let $h_n(\cdot) \in W^{1,\infty}(R)$ with compact support be defined as follows

$$h_n(t) = 1 - |T_{n+1}(t) - T_n(t)|, \quad \forall t \in R. \quad (2.33)$$

For any given $0 \leq k < 1$ and $n > 0$, taking $h(\cdot) = h_n(\cdot)$, $\xi = T_k(u_1 - u_2)$ in (2.31), we get

$$\begin{aligned}
 & \int_{\Omega} \sum_{i,j=1}^N [\tilde{a}_{ij}(x, u_1) D_j u_1 h_n(u_1) - \tilde{a}_{ij}(x, u_2) D_j u_2 h_n(u_2)] D_i T_k(u_1 - u_2) \, dx \\
 &= \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2 h'_n(u_2) T_k(u_1 - u_2) \, dx \\
 &\quad - \int_{\Omega} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 h'_n(u_1) T_k(u_1 - u_2) \, dx \\
 &\quad + \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F[h_n(u_1) - h_n(u_2)] T_k(u_1 - u_2) \, dx \\
 &\quad + \tilde{A}[h_n(u_1|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) - h_n(u_2|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2})] T_k((u_1 - u_2)|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) \\
 &= K_1 - K_2 + K_3 + K_4.
 \end{aligned} \tag{2.34}$$

The left term in (2.34) can be written as

$$\begin{aligned}
 & \int_{\Omega} \sum_{i,j=1}^N [\tilde{a}_{ij}(x, u_1) D_j u_1 h_n(u_1) - \tilde{a}_{ij}(x, u_2) D_j u_2 h_n(u_2)] D_i T_k(u_1 - u_2) \, dx \\
 &= \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N [h_n(u_1) + h_n(u_2)] [\tilde{a}_{ij}(x, u_1) D_j u_1 - \tilde{a}_{ij}(x, u_2) D_j u_2] \\
 &\quad \times D_i T_k(u_1 - u_2) \, dx \\
 &\quad + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N [h_n(u_1) - h_n(u_2)] [\tilde{a}_{ij}(x, u_1) D_j u_1 + \tilde{a}_{ij}(x, u_2) D_j u_2] \\
 &\quad \times D_i T_k(u_1 - u_2) \, dx \\
 &= A(n, k) + B(n, k).
 \end{aligned} \tag{2.35}$$

Writing $A(n, k)$ as

$$\begin{aligned}
 A(n, k) &= \frac{1}{4} \int_{\Omega} \sum_{i,j=1}^N [h_n(u_1) + h_n(u_2)] [\tilde{a}_{ij}(x, u_1) - \tilde{a}_{ij}(x, u_2)] \\
 &\quad \times D_j(u_1 + u_2) D_i T_k(u_1 - u_2) \, dx \\
 &\quad + \frac{1}{4} \int_{\Omega} \sum_{i,j=1}^N [h_n(u_1) + h_n(u_2)] [\tilde{a}_{ij}(x, u_1) + \tilde{a}_{ij}(x, u_2)] \\
 &\quad \times D_j(u_1 - u_2) D_i T_k(u_1 - u_2) \, dx \\
 &= C(n, k) + D(n, k).
 \end{aligned} \tag{2.36}$$

Noting (2.33), we have

$$\begin{aligned} |K_1| &\leq k \int_{\Omega} \sum_{i,j=1}^N |h'_n(u_2)| \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2 \, dx \\ &= k \int_{\{x \in \Omega: n \leq |u_2(x)| \leq n+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2 \, dx \end{aligned} \quad (2.37)$$

and

$$\begin{aligned} |K_2| &\leq k \int_{\Omega} \sum_{i,j=1}^N |h'_n(u_1)| \tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 \, dx \\ &= k \int_{\{x \in \Omega: n \leq |u_1(x)| \leq n+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 \, dx. \end{aligned} \quad (2.38)$$

From (2.34)–(2.38), we have

$$\begin{aligned} D(n, k) &\leq k \int_{\{x \in \Omega: n \leq |u_1(x)| \leq n+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 \, dx \\ &\quad + k \int_{\{x \in \Omega: n \leq |u_2(x)| \leq n+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2 \, dx \\ &\quad + K_3 + K_4 + |B(n, k)| + |C(n, k)|. \end{aligned}$$

For any given $y, z \in R^N$, by $(\tilde{H}_1)$, it is easy to get the following inequality

$$\left| \sum_{i,j=1}^N \tilde{a}_{ij}(x, s) y_i z_j \right| \leq \left(\sum_{i,j=1}^N \tilde{a}_{ij}(x, s) y_i y_j \right)^{1/2} \left(\sum_{i,j=1}^N \tilde{a}_{ij}(x, s) z_i z_j \right)^{1/2}. \quad (2.39)$$

For any given $\varepsilon > 0$, (2.39) and Young's inequality yield

$$\begin{aligned} |B(n, k)| &\leq \frac{\varepsilon}{2} \int_{\Omega} \sum_{i,j=1}^N [h_n(u_1) + h_n(u_2)] [\tilde{a}_{ij}(x, u_1) + \tilde{a}_{ij}(x, u_2)] \\ &\quad \times D_j T_k(u_1 - u_2) D_i T_k(u_1 - u_2) \, dx \\ &\quad + \frac{1}{2\varepsilon} \int_{\{|u_1 - u_2| < k\}} \sum_{i,j=1}^N |h_n(u_1) - h_n(u_2)| \\ &\quad \times [\tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 + \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2] \, dx. \end{aligned} \quad (2.40)$$

By Hölder's inequality, Young's inequality and $(\tilde{H}_1) - (\tilde{H}_2)$, we find that

$$\begin{aligned}
|C(n, k)| &\leq \frac{1}{4} \int_{\{0 < |u_1 - u_2| < k\}} \sum_{i,j=1}^N [h_n(u_1) + h_n(u_2)] d(x) |u_1 - u_2| \\
&\quad \times |D_j(u_1 + u_2)| |D_i T_k(u_1 - u_2)| \, dx \\
&\leq \frac{k}{4} \int_{\{0 < |u_1 - u_2| < k\}} \sum_{i,j=1}^N [h_n(u_1) + h_n(u_2)] \|d\|_{L^\infty(\Omega)} \\
&\quad \times |D_j(u_1 + u_2)| |D_i T_k(u_1 - u_2)| \, dx \\
&\leq \frac{k^2}{4\varepsilon} \|d\|_{L^\infty(\Omega)}^2 \int_{\{0 < |u_1 - u_2| < k\}} [h_n(u_1) + h_n(u_2)] (|Du_1|^2 + |Du_2|^2) \, dx \\
&\quad + \frac{\varepsilon}{4} \int_{\{0 < |u_1 - u_2| < k\}} [h_n(u_1) + h_n(u_2)] |DT_k(u_1 - u_2)|^2 \, dx \\
&\leq \frac{k^2}{4\varepsilon} \|d\|_{L^\infty(\Omega)}^2 \int_{\{0 < |u_1 - u_2| < k\}} [h_n(u_1) + h_n(u_2)] (|Du_1|^2 + |Du_2|^2) \, dx \\
&\quad + \frac{\varepsilon}{8\alpha} \int_{\Omega} \sum_{i,j=1}^N [h_n(u_1) + h_n(u_2)] [\tilde{a}_{ij}(x, u_1) + \tilde{a}_{ij}(x, u_2)] \\
&\quad \times D_j T_k(u_1 - u_2) D_i T_k(u_1 - u_2) \, dx. \tag{2.41}
\end{aligned}$$

Taking $\varepsilon = \frac{\alpha}{4\alpha+1}$ in (2.40) and (2.41), we get

$$\begin{aligned}
&\int_{\Omega} \sum_{i,j=1}^N [h_n(u_1) + h_n(u_2)] [\tilde{a}_{ij}(x, u_1) + \tilde{a}_{ij}(x, u_2)] \\
&\quad \times D_j T_k(u_1 - u_2) D_i T_k(u_1 - u_2) \, dx \\
&\leq C_4 \{ k \int_{\{x \in \Omega: n \leq |u_1(x)| \leq n+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 \, dx \\
&\quad + k \int_{\{x \in \Omega: n \leq |u_2(x)| \leq n+1\}} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2 \, dx \\
&\quad + \int_{\{0 < |u_1 - u_2| < k\}} \sum_{i,j=1}^N |h_n(u_1) - h_n(u_2)| \\
&\quad \times [\tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 + \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2] \, dx \\
&\quad + K_3 + K_4 + k^2 \|d\|_{L^\infty(\Omega)}^2 \\
&\quad \times \int_{0 < |u_1 - u_2| < k} [h_n(u_1) + h_n(u_2)] (|Du_1|^2 + |Du_2|^2) \, dx \}, \tag{2.42}
\end{aligned}$$

where C_4 is a positive constant independent of n .

From (2.33), it follows that $|h'_n(t)| = 1$ for $n \leq |t| \leq n+1$ and $h'_n(t) = 0$ for $|t| \geq n+1$ or $|t| \leq n$. Furthermore we have $h_n(u_1) - h_n(u_2) = h'_n(\theta)(u_1 - u_2)$ where θ is between u_1 and u_2 . Let $0 \leq k < 1$, then

$$\begin{aligned} & \int_{\{0 < |u_1 - u_2| < k\}} \sum_{i,j=1}^N |h_n(u_1) - h_n(u_2)| [\tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 + \tilde{a}_{ij}(x, u_2) \\ & \quad \times D_j u_2 D_i u_2] dx \\ & \leq \int_{\{0 < |u_1 - u_2| < k\} \cap \{n-1 \leq |u_1| \leq n+2\} \cap \{n-1 \leq |u_2| \leq n+2\}} \\ & \quad \times k \sum_{i,j=1}^N [\tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 + \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2] dx. \end{aligned} \quad (2.43)$$

By (2.32) and (2.43), we find that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\{0 < |u_1 - u_2| < k\}} \sum_{i,j=1}^N |h_n(u_1) - h_n(u_2)| \\ & \quad \times [\tilde{a}_{ij}(x, u_1) D_j u_1 D_i u_1 + \tilde{a}_{ij}(x, u_2) D_j u_2 D_i u_2] dx = 0. \end{aligned} \quad (2.44)$$

Since

$$h_n(u_s) \longrightarrow 1 \text{ a.e. } x \in \Omega, \quad s = 1, 2 \quad (2.45)$$

and

$$h_n(u_s) \xrightarrow{\text{weak}^*} 1 \text{ in } L^\infty(\Omega), \quad (2.46)$$

we have

$$\lim_{n \rightarrow \infty} K_3 = \lim_{n \rightarrow \infty} \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} F[h_n(u_1) - h_n(u_2)] T_k(u_1 - u_2) dx = 0, \quad (2.47)$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} K_4 = \lim_{n \rightarrow \infty} \tilde{A}[h_n(u_1|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) - h_n(u_2|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2})] \\ & \quad \times T_k((u_1 - u_2)|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2}) dx = 0. \end{aligned} \quad (2.48)$$

Let $n \rightarrow \infty$ in (2.42). Assumption $(\tilde{H}_1)$, (2.32), (2.42)–(2.44) and (2.47)–(2.48) imply that

$$\int_{\Omega} |DT_k(u_1 - u_2)|^2 dx \leq C_5 k^2 \int_{\{0 < |u_1 - u_2| < k\}} |Du_1|^2 + |Du_2|^2 dx, \quad (2.49)$$

where C_5 is a positive constant independent of n and k .

Inequality (2.49) and the absolute continuity of integral imply that

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{\Omega} |DT_k(u_1 - u_2)|^2 dx = 0. \quad (2.50)$$

Setting $E_k = \{x \in \Omega : |u_1 - u_2| > k\}$, $\Omega_1 = \{x \in \Omega : u_1(x) \neq u_2(x)\}$, (2.50) and Poincaré's inequality yield

$$\begin{aligned} \text{meas } E_k &= \frac{1}{k^2} \int_{E_k} k^2 \, dx \leq \frac{1}{k^2} \int_{\Omega} |T_k(u_1 - u_2)|^2 \, dx \\ &\leq C_6 \frac{1}{k^2} \int_{\Omega} |DT_k(u_1 - u_2)|^2 \, dx \longrightarrow 0, \quad \text{as } k \rightarrow 0, \end{aligned} \quad (2.51)$$

where C_6 is a positive constant independent of k .

From (2.51) it follows $\text{meas } \Omega_1 = 0$. Thus $u_1(x) = u_2(x)$, a.e. $x \in \Omega$. $\square$

Remark 2.5. In [7] the author has discussed the existence and uniqueness of renormalized solution nonlinear elliptic boundary value problem with equivalued surface but without Dirichlet boundary conditions and thin layer. Furthermore, we remark that the zero-order term in the equation is crucial to the proof of uniqueness of renormalized solution (see Theorem 3.1 in [7]). Here the equation in problem (P_1) has not zero-order term. Thus we do not use the same approach as that of [7] to the proof of uniqueness.

Similarly to Theorem 4.1 in [7], it is easy to prove the following theorem.

Theorem 2.6. (i) Assume that $u \in V_1$ and $a_{ij}(\cdot, u)D_j u \in L^2(\Omega)$, $i, j = 1, 2, \dots, N$. Then u is a weak solution to problem (P_1) if and only if u is a renormalized solution to problem (P_1) .
(ii) If $u \in V_1 \cap L^\infty(\Omega)$, then u is a weak solution to problem (P_1) if and only if u is a renormalized solution to problem (P_1) .

3. Nonlinear elliptic boundary value problem with equivalued interface

In order to study the limit behaviour of solutions to problem (P_1) , we need to study the following equivalued interface problem (P) . Here we give another division of Ω as shown in Fig. 2. Ω is composed of two non-overlapping sub-domains Ω_1 and Ω_2 , and $\tilde{\Gamma}$ is the interface of Ω_1 and Ω_2 .

FIGURE 2. The division of Ω with an interface $\tilde{\Gamma}$

In this section we will consider the following nonlinear elliptic boundary value problem with equivalued interface:

$$(P) \begin{cases} - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} (a_{ij}(x, u) \frac{\partial u}{\partial x_j}) = F(x) & \text{in } \Omega_1 \cup \Omega_2, \\ u = 0 & \text{on } \Gamma, \\ u_+ = u_- = C \text{ (a constant to be determined)} & \text{on } \tilde{\Gamma}, \\ \int_{\tilde{\Gamma}} \left(\frac{\partial u}{\partial n_L} \right)_+ ds = \int_{\tilde{\Gamma}} \left(\frac{\partial u}{\partial n_L} \right)_- ds + A \text{ (a given constant),} \end{cases}$$

where the subscripts $+$ and $-$ denote the values on both sides of $\tilde{\Gamma}$.

We make the following assumptions:

(H₁) $a_{ij}(x, s) = a_{ji}(x, s)$, a.e. $x \in \Omega$, $\forall s \in R$. There exists a positive constant α such that for every $\xi = (\xi_1, \xi_2, \dots, \xi_N) \in R^N$,

$$\sum_{i,j=1}^N a_{ij}(x, s) \xi_i \xi_j \geq \alpha |\xi|^2, \quad \forall s \in R, \text{ a.e. } x \in \Omega.$$

(H₂) Let $a_{ij} : \Omega \times R \rightarrow R$ be Carathéodory functions with $1 \leq i, j \leq N$. Assume that $a_{ij}(\cdot, 0) \in L^\infty(\Omega)$ and there exists a positive function $d \in L^\infty(\Omega)$ such that for every $s_1, s_2 \in R$ and a.e. $x \in \Omega$,

$$|a_{ij}(x, s_1) - a_{ij}(x, s_2)| \leq d(x) |s_1 - s_2| \quad (i, j = 1, 2, \dots, N).$$

Let

$$V = \{v | v \in H_0^1(\Omega), v|_{\tilde{\Gamma}} = \text{constant}\}. \quad (3.1)$$

Definition 3.1. A measurable function $u \in V$ is a renormalized solutions to problem (P), if for all $h(\cdot) \in W^{1,\infty}(R)$ with compact support and $\xi \in V \cap L^\infty(\Omega)$, u satisfies

$$\int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i (h(u) \xi) dx = \int_{\Omega} F h(u) \xi dx + A h(u|_{\tilde{\Gamma}}) \xi|_{\tilde{\Gamma}} \quad (3.2)$$

and

$$\lim_{m \rightarrow \infty} \int_{\{x \in \Omega : m \leq |u(x)| \leq m+1\}} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i u dx = 0. \quad (3.3)$$

Theorem 3.2. Under hypotheses (H₁), (H₂) and $F \in L^2(\Omega)$, problem (P) admits a unique renormalized solution $u \in V$ in the sense of Definition 3.1.

Proof. The proof of this theorem is similar to Theorem 2.4, thus we omit the details. $\square$

4. Limit behaviour of solutions to problem (P_1)

In this section, we will study the limit behaviour of solutions to boundary value problem (P_1) . More precisely, let $\varepsilon > 0$ be a small parameter. Replace $\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}$ by $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \tilde{\Omega}^\varepsilon$, and interfaces $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ by interfaces $\tilde{\Gamma}_1^\varepsilon$ and $\tilde{\Gamma}_2^\varepsilon$, respectively as shown in Fig. 3.

Here we will discuss the following problem

$$(P_\varepsilon) \begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}^\varepsilon(x, u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_j} \right) = F(x) & \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma, \\ u_\varepsilon = \tilde{C}_\varepsilon \text{ (a constant to be determined)} & \text{on } \tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon, \\ \int_{\tilde{\Gamma}_1^\varepsilon} \frac{\partial u_\varepsilon}{\partial n_{L^\varepsilon}} ds = \int_{\tilde{\Gamma}_2^\varepsilon} \frac{\partial u_\varepsilon}{\partial n_{L^\varepsilon}} ds + A_\varepsilon \text{ (a given constant).} \end{cases}$$

We make the following assumptions:

- (H₃) $\tilde{\Gamma} \subset \tilde{\Omega}^\varepsilon$, $\forall \varepsilon > 0$; $\tilde{\Omega}^\varepsilon$ shrinks to Γ as $\varepsilon \rightarrow 0$.
 (H₄) For any given domain $\tilde{\Omega}$ such that $\tilde{\Gamma} \subset \tilde{\Omega} \subset \Omega$, then for any $\varepsilon > 0$ small enough, we have

$$\tilde{\Omega}^\varepsilon \subset \tilde{\Omega}.$$

- (H₅) For all $\varepsilon > 0$, $a_{ij}^\varepsilon(x, s) = a_{ji}^\varepsilon(x, s)$, a.e. $x \in \Omega$, $\forall s \in R$. There exists a positive constant α independent of ε such that

$$\sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) \xi_i \xi_j \geq \alpha |\xi|^2, \quad \forall \xi = (\xi_1, \xi_2, \dots, \xi_N) \in R^N, \quad \forall x \in \Omega.$$

- (H₆) Let $a_{ij}^\varepsilon : \Omega \times R \rightarrow R$ be Carathéodory functions with $1 \leq i, j \leq N$. We also assume that there exists a positive constant β independent of ε such that $\|a_{ij}^\varepsilon(\cdot, 0)\|_{L^\infty(\Omega)} \leq \beta$. Moreover there exists a positive function $d \in L^\infty(\Omega)$ such that for every $s_1, s_2 \in R$ and a.e. $x \in \Omega$,

$$|a_{ij}^\varepsilon(x, s_1) - a_{ij}^\varepsilon(x, s_2)| \leq d(x) |s_1 - s_2|, \quad \forall s_1, s_2 \in R, \text{ a.e. } x \in \Omega$$

FIGURE 3. The division of Ω with thin layer described by parameter ε

(H_7) For any given domain $\tilde{\Omega}$ such that $\tilde{\Gamma} \subset \tilde{\Omega} \subset \Omega$, as $\varepsilon \rightarrow 0$

$$a_{ij}^\varepsilon(x, s) \rightarrow a_{ij}(x, s) \quad \text{strongly in } L^\infty(\Omega \setminus \tilde{\Omega}), \quad \forall s \in R.$$

Set

$$V_\varepsilon = \{v | v \in H_0^1(\Omega), v|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon} = \text{constant}\}.$$

Definition 4.1. A measurable function $u_\varepsilon \in V_\varepsilon$ is a renormalized solutions to problem (P_ε) , if for all $h \in W^{1,\infty}(R)$ with compact support and $\xi \in V_\varepsilon \cap L^\infty(\Omega)$, u satisfies

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i (h(u_\varepsilon) \xi) \, dx \\ &= \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} F h(u_\varepsilon) \xi \, dx + A_\varepsilon h(u_\varepsilon|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon}) \xi|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon} \end{aligned} \quad (4.1)$$

and

$$\lim_{m \rightarrow \infty} \int_{\{x \in \Omega; m \leq |u_\varepsilon(x)| \leq m+1\}} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i u_\varepsilon \, dx = 0. \quad (4.2)$$

Remark 4.2. For every fixed $\varepsilon > 0$, if $(H_5) - (H_6)$ and $F \in L^2(\Omega)$ hold, we can similarly prove that problem (P_ε) admits a unique solution $u_\varepsilon \in V_\varepsilon$ in the sense of Definition 4.1.

Now we give the limit behaviour of solutions to problem (P_ε) as follows.

Theorem 4.3. Suppose that $(H_1) - (H_7)$ and $F \in L^2(\Omega)$ hold. If $\varepsilon \rightarrow 0$, $A_\varepsilon \rightarrow A$, then for every renormalized solution u_ε to problem (P_ε) we have

$$u_\varepsilon \longrightarrow u \text{ weakly in } V,$$

where u is the renormalized solution to problem (P) and V is as in (3.1).

In order to prove Theorem 4.3, we need the following lemma.

Lemma 4.4. Under hypotheses (H_3) and (H_4) , for any given $\varphi \in V$, there exist $\varphi_\varepsilon \in V_\varepsilon$ such that as $\varepsilon \rightarrow 0$,

$$\varphi_\varepsilon \longrightarrow \varphi \quad \text{strongly in } V,$$

where V is as in (3.1).

Proof. For convenience, we may assume that the origin is an interior point of Ω_2 (see Fig. 2).

For fixed $\varepsilon > 0$ small enough, let

$$\Omega_2^\varepsilon = \{x(1-\varepsilon) | x \in \Omega_2\}, \quad \Omega_1' = \left\{ \frac{x}{1-\varepsilon} | x \in \Omega_2 \right\}, \quad \Omega_1^\varepsilon = \Omega \setminus \Omega_1', \quad \tilde{\Omega}^\varepsilon = \Omega_1' \setminus \Omega_2^\varepsilon.$$

Define $\Gamma^\varepsilon = \{x(1-\varepsilon) | x \in \Gamma\}$ and assume that $\tilde{\Gamma}_1^\varepsilon, \tilde{\Gamma}_2^\varepsilon$ are the interfaces of $\tilde{\Omega}^\varepsilon$ with Ω_1^ε and Ω_2^ε (see Fig. 4). $\square$

FIGURE 4. Scaling down or up the division of Ω in Fig. 2

We define ψ_ε as follows:

$$\psi_\varepsilon = \psi_\varepsilon^+ - \psi_\varepsilon^-,$$

where

$$\psi_\varepsilon^+ = \begin{cases} \left(\psi((1-\varepsilon)x) - \sup_{\Gamma^\varepsilon} \psi^+(x) \right)^+ & x \in \Omega_1^\varepsilon, \\ \left(\psi(x)|_{\tilde{\Gamma}} - \sup_{\Gamma^\varepsilon} \psi^+(x) \right)^+ & x \in \tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon, \\ \left(\psi\left(\frac{x}{1-\varepsilon}\right) - \sup_{\Gamma^\varepsilon} \psi^+(x) \right)^+ & x \in \Omega_2^\varepsilon, \end{cases}$$

$$\psi_\varepsilon^- = \begin{cases} \left(\psi((1-\varepsilon)x) - \inf_{\Gamma^\varepsilon} \psi^-(x) \right)^- & x \in \Omega_1^\varepsilon, \\ \left(\psi(x)|_{\tilde{\Gamma}} - \inf_{\Gamma^\varepsilon} \psi^-(x) \right)^- & x \in \tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon, \\ \left(\psi\left(\frac{x}{1-\varepsilon}\right) - \inf_{\Gamma^\varepsilon} \psi^-(x) \right)^- & x \in \Omega_2^\varepsilon. \end{cases}$$

It is easy to see that $\psi_\varepsilon^+ \in V_\varepsilon$, $\psi_\varepsilon^- \in V_\varepsilon$. Thus we have $\psi_\varepsilon \in V_\varepsilon$. We can easily prove that φ_ε^+ and φ_ε^- strongly converge to φ^+ and φ^- in V respectively. Hence φ_ε strongly converges to φ . We omit the details.

Proof of Theorem 4.3. For any given $\varepsilon > 0$, problem (P_ε) admits a unique renormalized solution $u_\varepsilon \in V_\varepsilon$ by Theorem 2.4 and there exists a sequence $\{u_{\varepsilon_n}\} \subset V_\varepsilon$ such that

$$u_{\varepsilon_n} \longrightarrow u_\varepsilon \text{ weakly in } V_\varepsilon, \quad \text{as } n \rightarrow \infty \quad (4.3)$$

and

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_n(u_{\varepsilon_n})) D_j u_{\varepsilon_n} D_i v \, dx \\ &= \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} Fv \, dx + A_\varepsilon v|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon}, \quad \forall v \in V_\varepsilon. \end{aligned} \quad (4.4)$$

By virtue of $v \in V_\varepsilon$ and (H_3) , $v|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon} = v|_{\tilde{\Gamma}}$. Thus, we can write (4.4) as follows:

$$\int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_n(u_{\varepsilon_n})) D_j u_{\varepsilon_n} D_i v \, dx = \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} F v \, dx + A_\varepsilon v|_{\tilde{\Gamma}}, \quad \forall v \in V_\varepsilon. \quad (4.5)$$

Take $v = u_{\varepsilon_n}$ as a test function in (4.5). From (H_5) , Sobolev imbedding theorem, trace theorem, Poincaré's inequality and Young's inequality, it follows

$$\|u_{\varepsilon_n}\|_{V_\varepsilon} \leq C_7[\|F\|_{L^2(\Omega)} + A_\varepsilon], \quad (4.6)$$

where C_7 is a positive constant depending on α , $meas \, \tilde{\Gamma}$ and $meas \, \Omega$, independent of ε, n .

Relations (4.3) and (4.6) imply that

$$\|u_\varepsilon\|_{V_\varepsilon} \leq C_7[\|F\|_{L^2(\Omega)} + A_\varepsilon].$$

Hence there exists a positive constant C_8 independent of ε such that

$$\|u_\varepsilon\|_V \leq C_8. \quad (4.7)$$

Thus there is a subsequence of $\{u_\varepsilon\}$ (still denote by $\{u_\varepsilon\}$) such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } V, \quad \text{as } \varepsilon \rightarrow 0. \quad (4.8)$$

Sobolev imbedding theorem, trace theorem and (H_3) yield

$$u_\varepsilon \rightarrow u \text{ strongly in } L^q(\Omega), \quad \text{as } \varepsilon \rightarrow 0,$$

where

$$\begin{cases} 1 \leq q < \frac{2N}{N-2}, & N > 2, \\ 1 \leq q < +\infty, & N = 2, \end{cases}$$

and

$$\begin{aligned} u_\varepsilon|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon} &= u_\varepsilon|_{\tilde{\Gamma}} \rightarrow u|_{\tilde{\Gamma}}, \\ u_\varepsilon &\rightarrow u \text{ a.e in } \Omega. \end{aligned} \quad (4.9)$$

In the following, the proof will be divided into three steps.

Step 1: We will prove that for any given $k > 0$,

$$T_k(u_\varepsilon) \rightarrow T_k(u) \text{ strongly in } V, \quad \text{as } \varepsilon \rightarrow 0. \quad (4.10)$$

By Lemma 4.4, there is a sequence $\omega_{\varepsilon_0} \in V_{\varepsilon_0}$ such that

$$\omega_{\varepsilon_0} \rightarrow u \text{ strongly in } V, \quad \text{as } \varepsilon_0 \rightarrow 0. \quad (4.11)$$

Hence for any given $k > 0$, we have

$$T_k(\omega_{\varepsilon_0}) \rightarrow T_k(u) \text{ strongly in } V, \quad \text{as } \varepsilon_0 \rightarrow 0. \quad (4.12)$$

Fixed $\varepsilon_0 > 0$, by (H_4) , for any ε with $0 < \varepsilon < \varepsilon_0$

$$\tilde{\Omega}^\varepsilon \subset \tilde{\Omega}^{\varepsilon_0} \quad \text{and} \quad \omega_{\varepsilon_0} \in V_\varepsilon. \quad (4.13)$$

Let $h > k > 0$, taking $\xi = \xi_\varepsilon = T_{2k}(u_\varepsilon - T_h(u_\varepsilon) + T_k(u_\varepsilon) - T_k(\omega_{\varepsilon_0}))$ and $h(t) = h_n(t)$ in (4.1), where $h_n(t)$ is defined in (2.33), we get

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i \xi_\varepsilon h_n(u_\varepsilon) dx \\ & \quad + \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i u_\varepsilon h'_n(u_\varepsilon) \xi_\varepsilon dx \\ & = \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} F h_n(u_\varepsilon) \xi_\varepsilon dx + A_\varepsilon h_n(u_\varepsilon|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon}) \xi_\varepsilon|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon}. \end{aligned} \quad (4.14)$$

Setting $M = 4k + h$, it is easy to see that $D\xi_\varepsilon = DT_{2k}(u_\varepsilon - T_h(u_\varepsilon) + T_k(u_\varepsilon) - T_k(\omega_{\varepsilon_0})) = 0$ as $|u_\varepsilon| > M$. Therefore for $n > M + 1$, we have

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i \xi_\varepsilon h_n(u_\varepsilon) dx \\ & = \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_M(u_\varepsilon)) D_j T_M(u_\varepsilon) D_i \xi_\varepsilon h_n(u_\varepsilon) dx. \end{aligned} \quad (4.15)$$

By (H_5) , (4.15) and (2.45)–(2.46), we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i \xi_\varepsilon h_n(u_\varepsilon) dx \\ & = \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_M(u_\varepsilon)) D_j T_M(u_\varepsilon) D_i \xi_\varepsilon dx \\ & \geq \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_k(u_\varepsilon)) D_j T_k(u_\varepsilon) D_i [T_k(u_\varepsilon) - T_k(\omega_{\varepsilon_0})] dx \\ & \quad - \int_{\{x \in \Omega: |u_\varepsilon(x)| > k\}} \sum_{i,j=1}^N |a_{ij}^\varepsilon(x, T_M(u_\varepsilon))| |D_j T_M(u_\varepsilon)| |D_i T_k(\omega_{\varepsilon_0})| dx. \end{aligned} \quad (4.16)$$

Sobolev imbedding theorem, trace theorem and (4.11) imply that

$$\omega_{\varepsilon_0} \rightarrow u \quad \text{a.e } x \in \Omega, \quad \text{as } \varepsilon_0 \rightarrow 0$$

and

$$\omega_{\varepsilon_0}|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon} = \omega_{\varepsilon_0}|_{\tilde{\Gamma}} \longrightarrow u|_{\tilde{\Gamma}}, \quad \text{as } \varepsilon_0 \rightarrow 0. \quad (4.17)$$

It follows from (4.11) and (4.12) that

$$\lim_{\varepsilon_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{\{|u_\varepsilon| > k\}} |D_i T_k(\omega_{\varepsilon_0})|^2 dx = 0.$$

For any fixed $h > 0$, (H_6) and (4.7) imply that $a_{ij}^\varepsilon(x, T_M(u_\varepsilon))D_j T_M(u_\varepsilon)$ is bounded in $L^2(\Omega)$. Thus we can deduce that

$$\lim_{\varepsilon_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{\{|u_\varepsilon| > k\}} |a_{ij}^\varepsilon(x, T_M(u_\varepsilon))| |D_j T_M(u_\varepsilon)| |D_i T_k(\omega_{\varepsilon_0})| dx = 0.$$

Obviously, we have

$$\begin{aligned} & \left| \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i u_\varepsilon h'_n(u_\varepsilon) \xi_\varepsilon dx \right| \\ & \leq 2k \int_{\{x \in \Omega: n \leq |u_\varepsilon(x)| \leq n+1\}} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i u_\varepsilon dx. \end{aligned} \quad (4.18)$$

By (4.2) and (4.18), we see that

$$\lim_{n \rightarrow \infty} \left| \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i u_\varepsilon h'_n(u_\varepsilon) \xi_\varepsilon dx \right| = 0.$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{h \rightarrow \infty} \lim_{\varepsilon_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} F h_n(u_\varepsilon) \xi_\varepsilon dx = 0.$$

According to (2.45), (2.46), (4.9) and (4.17), it is easy to get

$$\lim_{h \rightarrow \infty} \lim_{\varepsilon_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} A_\varepsilon h_n(u_\varepsilon|_{\bar{\Gamma}_1^\varepsilon \cup \bar{\Omega}^\varepsilon \cup \bar{\Gamma}_2^\varepsilon}) \xi_\varepsilon|_{\bar{\Gamma}_1^\varepsilon \cup \bar{\Omega}^\varepsilon \cup \bar{\Gamma}_2^\varepsilon} = 0.$$

Let n tend to the infinity in (4.14). By the above relations, we get

$$\overline{\lim}_{\varepsilon_0 \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_k(u_\varepsilon)) D_j T_k(u_\varepsilon) D_i [T_k(u_\varepsilon) - T_k(\omega_{\varepsilon_0})] dx = 0. \quad (4.19)$$

For any given $k > 0$, (4.8) yields

$$T_k(u_\varepsilon) \rightarrow T_k(u) \quad \text{weakly in } V. \quad (4.20)$$

For any fixed $h > 0$ and $k > 0$, (H_6) implies that $a_{ij}^\varepsilon(x, T_k(u_\varepsilon))D_j T_k(u)$ is bounded in $L^2(\Omega)$. By (H_7) , it is easy to get as $\varepsilon \rightarrow 0$

$$a_{ij}^\varepsilon(x, T_k(u_\varepsilon))D_j T_k(\omega_{\varepsilon_0}) \rightarrow a_{ij}(x, T_k(u))D_j T_k(\omega_{\varepsilon_0}) \quad \text{strongly in } L^2(\Omega). \quad (4.21)$$

By (4.12) and (4.20)–(4.21), we obtain

$$\lim_{\varepsilon_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_k(u_\varepsilon)) D_j T_k(\omega_{\varepsilon_0}) D_i [T_k(u_\varepsilon) - T_k(\omega_{\varepsilon_0})] dx = 0. \quad (4.22)$$

From (4.19) and (4.22), it follows

$$\begin{aligned} & \overline{\lim}_{\varepsilon_0 \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_k(u_\varepsilon)) D_j [T_k(u_\varepsilon) - T_k(\omega_{\varepsilon_0})] D_i [T_k(u_\varepsilon) \\ & \quad - T_k(\omega_{\varepsilon_0})] dx = 0. \end{aligned} \quad (4.23)$$

Assumption (H_5) and (4.23) yield

$$\overline{\lim_{\varepsilon_0 \rightarrow 0}} \overline{\lim_{\varepsilon \rightarrow 0}} \|DT_k(u_\varepsilon) - DT_k(u_{\varepsilon_0})\|_{L^2(\Omega)} = 0. \quad (4.24)$$

Using (4.12) and (4.24), we obtain

$$DT_k(u_\varepsilon) \rightarrow DT_k(u) \text{ strongly in } L^2(\Omega), \quad \text{as } \varepsilon \rightarrow 0. \quad (4.25)$$

Thus, it follows from (4.25) that (4.10) holds.

Step 2: We will prove that u satisfies (3.2).

By Lemma 4.4 and its proof, for any given $\eta \in V \cap L^\infty(\Omega)$, there is a sequence $\eta_{\varepsilon_0} \in V_{\varepsilon_0} \cap L^\infty(\Omega)$ such that

$$\eta_{\varepsilon_0} \rightarrow \eta \text{ strongly in } V \cap L^\infty(\Omega), \quad \text{as } \varepsilon_0 \rightarrow 0. \quad (4.26)$$

Sobolev imbedding theorem and (4.26) yield

$$\eta_{\varepsilon_0} \rightarrow \eta \text{ a.e in } \Omega, \quad \text{as } \varepsilon_0 \rightarrow 0$$

and

$$\eta_{\varepsilon_0}|_{\tilde{\Gamma}_1 \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2} = \eta_{\varepsilon_0}|_{\tilde{\Gamma}} \rightarrow u|_{\tilde{\Gamma}}, \quad \text{as } \varepsilon_0 \rightarrow 0.$$

For fixed $\varepsilon_0 > 0$ and any ε with $0 < \varepsilon < \varepsilon_0$, by (4.13) we have $\eta_{\varepsilon_0} \in V_\varepsilon \cap L^\infty(\Omega)$. For any given $h \in W^{1,\infty}(R)$ with compact $\text{supp } h \subset [-L, L]$, let

$$Q = \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i u_\varepsilon h'(u_\varepsilon) \eta_{\varepsilon_0} \, dx.$$

For any given $\tilde{\Omega}$ such that $\tilde{\Gamma} \subset \tilde{\Omega} \subset \Omega$, we have

$$\begin{aligned} Q &= \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_L(u_\varepsilon)) D_j T_L(u_\varepsilon) D_i T_L(u_\varepsilon) h'(u_\varepsilon) \eta_{\varepsilon_0} \, dx \\ &= \int_{\Omega \setminus \tilde{\Omega}} \sum_{i,j=1}^N [a_{ij}^\varepsilon(x, T_L(u_\varepsilon)) - a_{ij}(x, T_L(u_\varepsilon))] D_j T_L(u_\varepsilon) D_i T_L(u_\varepsilon) h'(u_\varepsilon) \eta_{\varepsilon_0} \, dx \\ &\quad + \int_{\tilde{\Omega}} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, T_L(u_\varepsilon)) D_j T_L(u_\varepsilon) D_i T_L(u_\varepsilon) h'(u_\varepsilon) \eta_{\varepsilon_0} \, dx \\ &\quad + \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, T_L(u_\varepsilon)) D_j T_L(u_\varepsilon) D_i T_L(u_\varepsilon) h'(u_\varepsilon) \eta_{\varepsilon_0} \, dx \end{aligned}$$

$$\begin{aligned}
& - \int_{\tilde{\Omega}} \sum_{i,j=1}^N a_{ij}(x, T_L(u_\varepsilon)) D_j T_L(u_\varepsilon) D_i T_L(u_\varepsilon) h'(u_\varepsilon) \eta_{\varepsilon_0} \, dx \\
& = Q_1 + Q_2 + Q_3 + Q_4.
\end{aligned} \tag{4.27}$$

For any $\delta > 0$, using (H_2) , (H_6) , (4.10) and the absolute continuity of integral, we can take *meas* $\tilde{\Omega}$ so small that as $0 < \varepsilon < \varepsilon_1 < \varepsilon_0$,

$$|Q_2| + |Q_4| < \frac{\delta}{2}. \tag{4.28}$$

By the above choice of $\tilde{\Omega}$, using (4.8), (4.25) and Vitali theorem, we get

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} Q_3 &= \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, T_L(u)) D_j T_L(u) D_i T_L(u) h'(u) \eta_{\varepsilon_0} \, dx \\
&= \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i u h'(u) \eta_{\varepsilon_0} \, dx.
\end{aligned} \tag{4.29}$$

By (H_7) and (4.7), there exists $0 < \varepsilon_2 \leq \varepsilon_1$ such that for any ε with $0 < \varepsilon < \varepsilon_2$

$$|Q_1| < \frac{\delta}{2}. \tag{4.30}$$

From (4.27)–(4.30), it follows

$$\lim_{\varepsilon \rightarrow 0} Q = \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i u h'(u) \eta_{\varepsilon_0} \, dx. \tag{4.31}$$

Arguing as in the proof of (4.31), we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \sum_{i,j=1}^N a_{ij}^\varepsilon(x, u_\varepsilon) D_j u_\varepsilon D_i \eta_{\varepsilon_0} h(u_\varepsilon) \, dx \\
&= \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i \eta_{\varepsilon_0} h(u) \, dx.
\end{aligned} \tag{4.32}$$

It is easy to see that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon} F h(u_\varepsilon) \eta_{\varepsilon_0} \, dx = \int_{\Omega} F(x) h(u) \eta_{\varepsilon_0} \, dx \tag{4.33}$$

and

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon h(u_\varepsilon)|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon} \eta_{\varepsilon_0}|_{\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon} = A h(u)|_{\tilde{\Gamma}} \eta_{\varepsilon_0}|_{\tilde{\Gamma}}. \tag{4.34}$$

Taking $\xi = \eta_{\varepsilon_0}$ in (4.1) and let $\varepsilon \rightarrow 0$, (4.31)–(4.34) yield

$$\begin{aligned}
& \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i \eta_{\varepsilon_0} h(u) \, dx + \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i u h'(u) \eta_{\varepsilon_0} \, dx \\
&= \int_{\Omega} F h(u) \eta_{\varepsilon_0} \, dx + A h(u)|_{\tilde{\Gamma}} \eta_{\varepsilon_0}|_{\tilde{\Gamma}}.
\end{aligned} \tag{4.35}$$

Taking $\varepsilon_0 \rightarrow 0$ in (4.35), (4.26) and (4.35) imply that

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i \eta h(u) \, dx + \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i u h'(u) \eta \, dx \\ &= \int_{\Omega} F h(u) \eta \, dx + A h(u|_{\tilde{\Gamma}}) \eta|_{\tilde{\Gamma}}. \end{aligned}$$

Step 3: We will prove that u satisfies (3.3).

Let $\xi = T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0}))$ and $h = h_n$ in (4.1), where $\{\omega_{\varepsilon_0}\}$ is the sequence taken in step 1. Then

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}^{\varepsilon}(x, u_{\varepsilon}) D_j u_{\varepsilon} D_i T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) h_n(u_{\varepsilon}) \, dx \\ &+ \int_{\Omega} \sum_{i,j=1}^N a_{ij}^{\varepsilon}(x, u_{\varepsilon}) D_j u_{\varepsilon} D_i u_{\varepsilon} h'_n(u_{\varepsilon}) T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) \, dx \\ &= \int_{\Omega_{\tilde{\Gamma}}^{\varepsilon} \cup \Omega_{\tilde{\Sigma}}^{\varepsilon}} F h_n(u_{\varepsilon}) T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) \, dx \\ &+ A_{\varepsilon} h_n(u_{\varepsilon}|_{\tilde{\Gamma}_{\tilde{\Gamma}}^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_{\tilde{\Sigma}}^{\varepsilon}}) T_1(u_{\varepsilon}|_{\tilde{\Gamma}_{\tilde{\Gamma}}^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_{\tilde{\Sigma}}^{\varepsilon}} - T_k(\omega_{\varepsilon_0}|_{\tilde{\Gamma}_{\tilde{\Gamma}}^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_{\tilde{\Sigma}}^{\varepsilon}})). \quad (4.36) \end{aligned}$$

From (4.36), it follows

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}^{\varepsilon}(x, u_{\varepsilon}) D_j u_{\varepsilon} D_i T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) h_n(u_{\varepsilon}) \, dx \\ &\leq \int_{\Omega_{\tilde{\Gamma}}^{\varepsilon} \cup \Omega_{\tilde{\Sigma}}^{\varepsilon}} F h_n(u_{\varepsilon}) T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) \, dx \\ &+ A_{\varepsilon} h_n(u_{\varepsilon}|_{\tilde{\Gamma}_{\tilde{\Gamma}}^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_{\tilde{\Sigma}}^{\varepsilon}}) T_1(u_{\varepsilon}|_{\tilde{\Gamma}_{\tilde{\Gamma}}^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_{\tilde{\Sigma}}^{\varepsilon}} - T_k(\omega_{\varepsilon_0}|_{\tilde{\Gamma}_{\tilde{\Gamma}}^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_{\tilde{\Sigma}}^{\varepsilon}})) \\ &+ \int_{\{x \in \Omega: n \leq |u_{\varepsilon}| \leq n+1\}} \sum_{i,j=1}^N a_{ij}^{\varepsilon}(x, u_{\varepsilon}) D_j u_{\varepsilon} D_i u_{\varepsilon} \, dx. \quad (4.37) \end{aligned}$$

Let n tend to the infinity in (4.37). We get

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}^{\varepsilon}(x, u_{\varepsilon}) D_j u_{\varepsilon} D_i T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) \, dx \\ &\leq \int_{\Omega_{\tilde{\Gamma}}^{\varepsilon} \cup \Omega_{\tilde{\Sigma}}^{\varepsilon}} F T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) \, dx + A_{\varepsilon} T_1(u_{\varepsilon}|_{\tilde{\Gamma}_{\tilde{\Gamma}}^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_{\tilde{\Sigma}}^{\varepsilon}} - T_k(\omega_{\varepsilon_0}|_{\tilde{\Gamma}_{\tilde{\Gamma}}^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_{\tilde{\Sigma}}^{\varepsilon}})). \quad (4.38) \end{aligned}$$

We can rewrite (4.38) as follows:

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}^{\varepsilon}(x, T_{k+1}(u_{\varepsilon})) D_j T_{k+1}(u_{\varepsilon}) \\ & \quad \times [D_i T_{k+1}(u_{\varepsilon}) - D_i T_k(\omega_{\varepsilon_0})] \chi_{\{|u_{\varepsilon} - T_k(\omega_{\varepsilon_0})| \leq 1\}} dx \\ & \leq \int_{\Omega_1^{\varepsilon} \cup \Omega_2^{\varepsilon}} F T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) dx + A_{\varepsilon} T_1(u_{\varepsilon}|_{\tilde{\Gamma}_1^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_2^{\varepsilon}} - T_k(\omega_{\varepsilon_0}|_{\tilde{\Gamma}_1^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_2^{\varepsilon}})). \end{aligned}$$

Arguing as in the proof of (4.31), we can prove

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \sum_{i,j=1}^N a_{ij}^{\varepsilon}(x, T_{k+1}(u_{\varepsilon})) D_j T_{k+1}(u_{\varepsilon}) \\ & \quad \times [D_i T_{k+1}(u_{\varepsilon}) - D_i T_k(\omega_{\varepsilon_0})] \chi_{\{|u_{\varepsilon} - T_k(\omega_{\varepsilon_0})| \leq 1\}} dx \\ & = \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, T_{k+1}(u)) D_j T_{k+1}(u) \\ & \quad \times [D_i T_{k+1}(u) - D_i T_k(\omega_{\varepsilon_0})] \chi_{\{|u - T_k(\omega_{\varepsilon_0})| \leq 1\}} dx. \end{aligned}$$

We can easily prove

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_1^{\varepsilon} \cup \Omega_2^{\varepsilon}} F T_1(u_{\varepsilon} - T_k(\omega_{\varepsilon_0})) dx = \int_{\Omega} F T_1(u - T_k(\omega_{\varepsilon_0})) dx$$

and

$$\lim_{\varepsilon \rightarrow 0} A_{\varepsilon} T_1(u_{\varepsilon}|_{\tilde{\Gamma}_1^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_2^{\varepsilon}} - T_k(\omega_{\varepsilon_0}|_{\tilde{\Gamma}_1^{\varepsilon} \cup \tilde{\Omega}^{\varepsilon} \cup \tilde{\Gamma}_2^{\varepsilon}})) = A T_1(u|_{\tilde{\Gamma}} - T_k(\omega_{\varepsilon_0}|_{\tilde{\Gamma}})).$$

Thus, we obtain

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, T_{k+1}(u)) D_j T_{k+1}(u) [D_i T_{k+1}(u) - D_i T_k(\omega_{\varepsilon_0})] \chi_{\{|u - T_k(\omega_{\varepsilon_0})| \leq 1\}} dx \\ & \leq \int_{\Omega} F T_1(u - T_k(\omega_{\varepsilon_0})) dx + A T_1(u|_{\tilde{\Gamma}} - T_k(\omega_{\varepsilon_0}|_{\tilde{\Gamma}})). \end{aligned} \quad (4.39)$$

Let $\varepsilon_0 \rightarrow 0$ in (4.39) and noting (4.11)–(4.13), we get

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N a_{ij}(x, T_{k+1}(u)) D_j T_{k+1}(u) [D_i T_{k+1}(u) - D_i T_k(u)] \chi_{\{|u - T_k(u)| \leq 1\}} dx \\ & \leq \int_{\Omega} F T_1(u - T_k(u)) dx + A T_1(u|_{\tilde{\Gamma}} - T_k(u|_{\tilde{\Gamma}})). \end{aligned} \quad (4.40)$$

We can rewrite (4.40) as

$$\begin{aligned} & \int_{\{x \in \Omega: k \leq |u(x)| \leq k+1\}} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i u dx \\ & \leq \int_{\{x \in \Omega: |u(x)| \geq k\}} |F| dx + A T_1(u|_{\tilde{\Gamma}} - T_k(u|_{\tilde{\Gamma}})). \end{aligned} \quad (4.41)$$

Hence, taking $k \rightarrow \infty$ in (4.41) we get

$$\lim_{k \rightarrow \infty} \int_{\{x \in \Omega: k \leq |u(x)| \leq k+1\}} \sum_{i,j=1}^N a_{ij}(x, u) D_j u D_i u \, dx = 0.$$

Thus we deduce that u is a renormalized solution to problem (P). Furthermore, by virtue of the uniqueness of renormalized solution to problem (P), (4.8) holds for the whole sequence $\{u_\varepsilon\}$. This completes the proof of Theorem 4.3. $\square$

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Received: 29 September 2007.

Revised: 12 May 2009.

Accepted: 21 May 2010.