Singularly perturbed 1D Cahn–Hilliard equation revisited

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Abstract. We consider a singular perturbation of the one-dimensional Cahn–Hilliard equation subject to periodic boundary conditions. We construct a family of exponential attractors $\{\mathcal{M}_\epsilon\}$, $\epsilon \geq 0$ being the perturbation parameter, such that the map $\epsilon \mapsto \mathcal{M}_\epsilon$ is Hölder continuous. Besides, the continuity at $\epsilon = 0$ is obtained with respect to a metric independent of ϵ . Continuity properties of global attractors and inertial manifolds are also examined.

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1. Introduction

The long time behavior of solutions to many nonlinear evolution partial differential equations is described by certain compact invariant subsets of the phase space that attract uniformly the trajectories starting from bounded sets when time goes to infinity. These sets are called global attractors and, in many cases, they have finite fractal dimension (see, e.g., [30, 35, 44]). However, we do not know in general how trajectories approach the global attractor. In particular, the rate of attraction of the trajectories may be small and very sensitive to perturbations and approximations, including numerical simulations. Furthermore, the known results on the continuity of global attractors with respect to perturbations make use of restrictive assumptions on the stationary solutions and their unstable sets (see, for instance, [35] and references therein). These drawbacks led Foias et al. to introduce a new object called exponential attractor. This is a compact and positively invariant set with finite fractal dimension which attracts all the trajectories starting from bounded sets at a uniform exponential rate (see [10]). Clearly, an exponential attractor contains the global attractor. In addition, exponential attractors are more robust than global attractors with respect to perturbations and approximations (see [35] and its references).

The classical construction of exponential attractors makes use of orthogonal projectors with finite rank and are valid in Hilbert spaces only (see, e.g., [1, 10]). Rather recently, this construction has been extended to Banach spaces (see [11], cf. also [9]). In particular, the new strategy devised in [11] allows to construct a robust (under perturbations) family of exponential attractors (see also [12–14, 24, 34] and references therein). This family is characterized by an explicit estimate on the symmetric distance between exponential attractors of the unperturbed and perturbed problems, and the continuity with respect to the perturbations does not involve time shifts as in the previous results.

Here we want to use this machinery for refining some results obtained in the literature for the singularly perturbed Cahn–Hilliard equation in one spatial dimension. We recall that the Cahn–Hilliard equation is central to Materials Science [4, 39]. It is a conservation law (i.e., the spatial average of the order parameter is conserved) and describes important qualitative features of phase separation processes, namely, the transport of atoms between unit cells. In the one-dimensional case, this equation reads

$$\rho_t - (-\alpha\rho_{xx} + f'(\rho))_{xx} = 0,$$

on some interval $(0, L)$, $L > 0$, where $\alpha > 0$ is given. Here ρ is the order parameter (corresponding to a rescaled density of atoms) and f represents a coarse-grain free energy (e.g., a double-well potential) which accounts for the presence of two different phases.

We want to compare the large time behavior of the above equation with the one of its singular perturbation

$$\epsilon\rho_{tt} + \rho_t - (-\alpha\rho_{xx} + f'(\rho))_{xx} = 0, \quad (1.1)$$

for $\epsilon \in (0, \epsilon_0]$, $0 < \epsilon_0 \leq 1$ being fixed, which has been proposed to model the early stages of spinodal decomposition in certain glasses (see [16–21, 32]).

The first mathematical analysis of the longtime behavior of equation (1.1) with no-flux boundary conditions can be found in [7]. There the author proved the existence of the (weak) global attractor and its upper semicontinuity with respect to ϵ . These results were improved in [46], with different boundary conditions, by establishing the existence of the (strong) global attractor and its smoothness, provided that ϵ is small enough. The same authors proved in [45] the existence of an exponential attractor and an inertial manifold, always assuming ϵ small enough. We recall that inertial manifolds and their dependence on ϵ were also analyzed in [2, 3] in the viscous case. The existence a family of robust exponential attractors with no restriction on ϵ was proven in [22] by using the new construction mentioned above which is no longer based on the squeezing property (see also [23] for a version with memory). However, in [22], the convergence of the exponential attractors, as ϵ goes to 0, was obtained with respect to a norm that depends on ϵ .

Inspired by the recent contributions [13] and [34], the main goal of this paper is to construct a family of exponential attractors which is continuous with respect to ϵ by using a metric which does not depend on ϵ as ϵ goes to 0. More precisely, in Sects. 2 and 3, we recall and demonstrate some basic results on the perturbed problem and the unperturbed one. Then, in Sect. 4, using

a decomposition proposed in [41], we provide estimates of the differences of trajectories associated with perturbation parameters ϵ_1 and $\epsilon_2 < \epsilon_1$ (possibly $\epsilon_2 = 0$) with respect to a norm which does not depend on the parameters themselves. Section 5 contains the main results. We first prove the existence of a family of exponential attractors which is continuous with respect to ϵ . More precisely, we first give explicit estimates of the Hausdorff symmetric distance between exponential attractors of two given perturbed problems on the one hand, and a proper exponential attractor of the perturbed problem and the natural lifting of the corresponding exponential attractor in the unperturbed one on the other hand. These estimates are given with respect to a metric that depends on the perturbation parameter. Then, we show the upper and lower semicontinuity of the exponential attractors at $\epsilon = 0$ with respect to a metric independent of ϵ . Also, we establish the upper semicontinuity of the global attractor at any ϵ , including the limit case $\epsilon = 0$. Here we report an alternative proof along the lines of [17] (see also [2]) which differs from all the previous ones based on a contradiction argument. Section 6 is dedicated to analyze further properties of the global attractors under suitable assumptions on the nature of the stationary points. Finally, in Sect. 7, we discuss the existence and stability of inertial manifolds. The results of this section complete the ones proven in [45] for Dirichlet-type boundary conditions; in particular, our construction allows us to establish some continuity properties of inertial manifolds with respect to ϵ .

Equation (1.1) has recently been studied in more than one spatial dimension (see [6, 26–28, 42]). In that case, the mathematical analysis is much more involved. We conclude by mentioning that equation (1.1) has also been examined from the numerical viewpoint (cf. [25]).

2. Functional setting and preliminary results

If W is a Sobolev-type space, then we set

$$\dot{W} = \{q \in W, m(q) = 0\},$$

where

$$m(q) = \frac{1}{L} \int_{\Omega} q(x) dx, \quad \Omega = (0, L).$$

Moreover, we set

$$\bar{q} = q - m(q).$$

Let us define the linear unbounded operator

$$N = -\frac{d^2}{dx^2} : H_{\text{per}}^2(\Omega) \rightarrow \dot{L}^2(\Omega)$$

which is self-adjoint and nonnegative. If N is restricted to $\dot{H}_{\text{per}}^2(\Omega)$, then it turns out to be positive with compact inverse N^{-1} . We consider the initial

and boundary value problem

$$\begin{cases} \rho_t + N(\alpha N \rho + f'(\rho)) = 0, \\ \rho|_{t=0} = \rho_0, \\ \rho \text{ is } \Omega\text{-periodic}, \end{cases} \quad (2.1)$$

and its singular perturbation

$$\begin{cases} \epsilon \rho_{tt} + \rho_t + N(\alpha N \rho + f'(\rho)) = 0, \\ \rho|_{t=0} = \rho_0, \quad \rho_t|_{t=0} = \rho_1, \\ \rho \text{ is } \Omega\text{-periodic}. \end{cases} \quad (2.2)$$

We assume $f \in C^6(\mathbb{R})$ to be such that

$$\liminf_{|r| \rightarrow +\infty} f''(r) > -\lambda_1, \quad (2.3)$$

where $\lambda_1 > 0$ is the first eigenvalue of N . Observe that condition (2.3) implies

$$f''(r) \geq -C_1, \quad \forall r \in \mathbb{R}, \quad (2.4)$$

for some $C_1 \geq 0$. For instance, polynomials of degree $2p + 2$ with strictly positive leading coefficients (e.g., $f(r) = (r^2 - 1)^{p+1}$) satisfy (2.3).

From now on, the same letter c (and sometimes c_r, c'_r , and $c_i, i \in \mathbb{N}$) denotes positive constants that may change from line to line, but are always independent of ϵ . We denote by $\|\cdot\|$ and $(\cdot, \cdot)$ the usual norm and scalar product in $L^2(\Omega)$.

We denote by W' the dual space of W . The space $\overline{X}^Y$ denotes the closure of a metric space $X \subset Y$ in the topology of the complete metric space Y . For every $r > 0$, we endow $(H_{\text{per}}^r(\Omega))'$ with the norm $\|q\|_{-r} = (\|N^{-r/2}\bar{q}\|^2 + |m(q)|^2)^{1/2}$. We also note that $\|q\|_r = (\|N^{r/2}\bar{q}\|^2 + |m(q)|^2)^{1/2}$ is a norm on $H_{\text{per}}^r(\Omega)$ which is equivalent to the usual $H^r(\Omega)$ -norm (see, e.g., [43]). Furthermore, there exist two positive constants C_2, C_3 such that

$$\|\bar{q}\|_{-1} \leq C_2 \|\bar{q}\| \leq C_3 \|\nabla q\|, \quad \forall q \in H_{\text{per}}^1(\Omega).$$

We endow the Banach spaces $\mathcal{H}_\epsilon^0 = H_{\text{per}}^1(\Omega) \times (H_{\text{per}}^1(\Omega))'$, $\mathcal{H}_\epsilon^1 = H_{\text{per}}^2(\Omega) \times L^2(\Omega)$ and $\mathcal{H}_\epsilon^j = H_{\text{per}}^{j+1}(\Omega) \times H_{\text{per}}^{j-1}(\Omega)$, $j = 2, 3, \dots$, with the norms

$$\begin{aligned} \|(p, q)\|_{\mathcal{H}_\epsilon^0} &= (\|p\|_1^2 + \epsilon \|q\|_{-1}^2)^{1/2}, \\ \|(p, q)\|_{\mathcal{H}_\epsilon^1} &= (\|p\|_2^2 + \epsilon \|q\|^2)^{1/2}, \\ \|(p, q)\|_{\mathcal{H}_\epsilon^j} &= (\|p\|_{j+1}^2 + \epsilon \|q\|_{j-1}^2)^{1/2}, \end{aligned}$$

respectively. For $\epsilon = 0$, we agree that $\mathcal{H}_0^j = H_{\text{per}}^{j+1}(\Omega)$ for any $j \in \mathbb{N}$. Then we set

$$\begin{aligned} K_\delta &= \{u \in H_{\text{per}}^1(\Omega), |m(u)| \leq \delta\}, \\ \tilde{K}_\delta &= \{(u, v) \in \mathcal{H}_\epsilon^0, |m(u)| + \epsilon_0 |m(v)| \leq \delta\}, \\ K_\delta^j &= K_\delta \cap H_{\text{per}}^{j+1}(\Omega), \\ \tilde{K}_\delta^j &= \tilde{K}_\delta \cap \mathcal{H}_\epsilon^j, \end{aligned}$$

for some $\delta \geq 0$, where $j \in \mathbb{N}$.

We start by recalling the following result (see [43]).

Theorem 2.1. *We assume that (2.3) hold and that $\rho_0 \in H_{\text{per}}^1(\Omega)$. Then problem (2.1) possesses a unique solution ρ such that*

$$\rho \in \mathcal{C}([0, T]; H_{\text{per}}^1(\Omega)) \cap L^2(0, T; H_{\text{per}}^3(\Omega)), \quad m(\rho(t)) = m(\rho_0),$$

for any $T > 0$. Furthermore, if $\rho_0 \in H_{\text{per}}^2(\Omega)$, then

$$\rho \in \mathcal{C}([0, T]; H_{\text{per}}^2(\Omega)) \cap L^2(0, T; H_{\text{per}}^4(\Omega)).$$

Thanks to this result, we can define the semigroup

$$S(t) : H_{\text{per}}^1(\Omega) \rightarrow H_{\text{per}}^1(\Omega), \quad \rho_0 \mapsto \rho(t), \quad t \geq 0,$$

where $\rho(t)$ is the solution to (2.1) at time t . This semigroup possesses the global attractor $\mathcal{A}_\delta$ on K_δ which is bounded in K_δ^4 (cf., e.g., [38]).

Let us now introduce the following weak formulation of problem (2.2).

Problem $\mathbf{P}_\epsilon$. For any given $T > 0$, find $\rho : [0, T] \rightarrow H_{\text{per}}^2(\Omega)$ such that

$$\rho(0) = \rho_0, \quad \rho_t(0) = \rho_1,$$

and, for almost every $t \in [0, T]$ and all $q \in H_{\text{per}}^3(\Omega)$,

$$\frac{d}{dt} [\epsilon(\rho_t, q) + (\rho, q)] + \alpha(\rho_x, (Nq)_x) + ((f'(\rho))_x, q_x) = 0. \quad (2.5)$$

Observe that, taking $q = 1$ in (2.5), we obtain

$$\epsilon \frac{d^2}{dt^2} m(\rho) + \frac{d}{dt} m(\rho) = 0,$$

so that

$$m(\rho(t)) = m(\rho_0) + \epsilon m(\rho_1)(1 - e^{-t/\epsilon}), \quad \forall t \geq 0, \quad (2.6)$$

and

$$m(\rho_t(t)) = m(\rho_1)e^{-t/\epsilon}, \quad \forall t \geq 0. \quad (2.7)$$

Thus, when t goes to infinity, $m(\rho_t(t)) \rightarrow 0$ and $m(\rho(t)) \rightarrow m(\rho_0) + \epsilon m(\rho_1)$.

We now report a well-posedness result for $\mathbf{P}_\epsilon$ which is proven in [7] (see also [22] and [46]).

Theorem 2.2. *We assume that (2.3) holds and that $(\rho_0, \rho_1) \in \mathcal{H}_\epsilon^0$. Then $\mathbf{P}_\epsilon$ possesses a unique solution ρ such that*

$$(\rho, \rho_t) \in \mathcal{C}([0, T]; \mathcal{H}_\epsilon^0)$$

and

$$\rho_{tt} \in L^2(0, T; (H_{\text{per}}^3(\Omega))'),$$

for any $T > 0$. Moreover, if $(\rho_0, \rho_1) \in \mathcal{H}_\epsilon^1$, then

$$(\rho, \rho_t) \in \mathcal{C}([0, T]; \mathcal{H}_\epsilon^1), \quad \rho_{tt} \in L^2(0, T; (H_{\text{per}}^2(\Omega))').$$

Thanks to Theorem 2.2, we can define the semigroup

$$S_\epsilon(t) : \mathcal{H}_\epsilon^0 \rightarrow \mathcal{H}_\epsilon^0, \quad (\rho_0, \rho_1) \mapsto (\rho(t), \rho_t(t)), \quad t \geq 0,$$

where $\rho(t)$ is the solution to $\mathbf{P}_\epsilon$ at time t . Moreover, arguing as in [22], for every $\epsilon \in (0, \epsilon_0]$, we can prove the existence of the global attractor $\mathcal{A}_{\epsilon, \delta}$ for $S_\epsilon(t)$ on $\tilde{K}_\delta$ that is bounded in $\tilde{K}_\delta^4$.

In order to compare the dynamics of problems (2.1) and (2.2) (i.e., $\mathbf{P}_\epsilon$), we introduce the canonical lifting of the exponential attractor in the unperturbed case (see [30]; cf. also [14, 24]). Here the second component is reconstructed by means of the unperturbed equation, namely, we define a mapping $L : H_{\text{per}}^5(\Omega) \rightarrow \dot{H}_{\text{per}}^1(\Omega)$ by setting

$$L\rho = -N(\alpha N\rho + f'(\rho)). \quad (2.8)$$

Then we introduce the lifting of an exponential attractor $\mathcal{M}_\delta$ for $S(t)$ on K_δ^4 ,

$$(\mathcal{M}_\delta)_0 = \{(\rho, L\rho) \in \mathcal{H}_\epsilon^2, \rho \in \mathcal{M}_\delta\}.$$

More generally, if B is a bounded set in $H_{\text{per}}^5(\Omega)$, we indicate its lifting by $(B)_0$.

We now recall that, following the proof of [22, Theorem 4.1], it is possible to construct a bounded absorbing set for the semigroup $S_\epsilon(t)$ on $\tilde{K}_\delta^4$ of the form

$$B_{5, \delta} = \{(u, v) \in \tilde{K}_\delta^4, \|(u, v)\|_{\mathcal{H}_\epsilon^4} \leq r\},$$

with r independent of ϵ . Sometimes, $B_{5, \delta}$ will be denoted by $B_{5, \delta}^\epsilon$. We assume that

$$B_\delta = \{u \in K_\delta^4, \|u\|_5 \leq r\}$$

is a bounded absorbing set for $\{S(t)\}_{t \geq 0}$ in K_δ^4 (see [38]). That is, there exists $t^* > 0$ such that $S_\epsilon(t)B_{5, \delta} \subset B_{5, \delta}$ and $S(t)B_\delta \subset B_\delta$ for all $t \geq t^*$. From now on, we set

$$\tilde{B}_{5, \delta} = S_\epsilon(t^*)B_{5, \delta}, \quad \tilde{B}_\delta = S(t^*)B_\delta$$

and we will always assume that $t^* \geq 1$. Then we have

$$\tilde{B}_{5, \delta} \subset \{(u, v) \in \tilde{K}_\delta^4, \|(u, v)\|_{\mathcal{H}_\epsilon^4} \leq r\}$$

and

$$\tilde{B}_\delta \subset \{u \in K_\delta^4, \|u\|_5 \leq r\}.$$

Note that $\tilde{B}_{5, \delta}$ and $\tilde{B}_\delta$ are bounded attracting and absorbing sets for $S_\epsilon(t)$ and $S(t)$, respectively, as well. The Hausdorff semi-distance with respect to the metric of E is defined as:

$$\text{dist}_E(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_E,$$

whereas the symmetric Hausdorff distance between A and B reads

$$\text{dist}_E^{\text{sym}}(A, B) = \max\{\text{dist}_E(A, B), \text{dist}_E(B, A)\}.$$

A problem similar to $\mathbf{P}_\epsilon$, but characterized by the following boundary conditions

$$\rho(0, t) = \rho(L, t) = \rho_{xx}(0, t) = \rho_{xx}(L, t) = 0, \quad \forall t \geq 0,$$

was considered in [22]. Setting $A = -\frac{d^2}{dx^2}$, with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, it was defined, for every $\epsilon \geq 0$ and $s \in \mathbb{R}$, the family of Hilbert spaces $\mathfrak{H}_\epsilon^s = \mathcal{D}(A^{(s+1)/2}) \times \mathcal{D}(A^{(s-1)/2})$, with the norm

$$\|(p, q)\|_{\mathfrak{H}_\epsilon^s} = (\|A^{(s+1)/2}p\|^2 + \epsilon\|A^{(s-1)/2}q\|^2)^{1/2}.$$

As we already mentioned, it was shown the existence of a closed bounded attracting set B_5 for $S_\epsilon(t)$ on $\mathfrak{H}_\epsilon^4$. This set is exponentially attracting in $\mathfrak{H}_\epsilon^0$ as well (that is, any bounded set $B \subset \mathfrak{H}_\epsilon^0$ is attracted exponentially to B_5 with respect to metric induced by the $\mathfrak{H}_\epsilon^0$ -norm), due to the transitivity of exponential attraction property (see [22, Lemma 4.4]), first devised in [14, Theorem 5.1]. Then, it was constructed a robust family of exponential attractors $\mathcal{M}_\epsilon$ on B_5 with respect to the metric induced by the $\mathfrak{H}_\epsilon^0$ -norm, whose common basin of attraction is the whole phase-space $\mathfrak{H}_\epsilon^0$, with a uniform upper bound on the fractal dimension and also a uniform rate of attraction of trajectories. Moreover, it was obtained an explicit estimate of the symmetric distance between $\mathcal{M}_\epsilon$ and the canonical lifting $(\mathcal{M})_0$ of the corresponding unperturbed exponential attractor $\mathcal{M}$, that is,

$$\text{dist}_{\mathfrak{H}_\epsilon^0}^{\text{sym}}(\mathcal{M}_\epsilon, (\mathcal{M})_0) \leq C\epsilon^\sigma, \quad (2.9)$$

where C and $0 < \sigma \leq \frac{1}{8}$ are independent of ϵ . More specifically, let a sequence $(u_\epsilon, v_\epsilon) \in \mathcal{M}_\epsilon$ converge in the $\mathfrak{H}_\epsilon^0$ -norm to an element $(u, Lu) \in (\mathcal{M})_0$ when $\epsilon \rightarrow 0$. Then, $\|u_\epsilon - u\|_{H^1(\Omega)} \rightarrow 0$ and $\sqrt{\epsilon}\|v_\epsilon - Lu\|_{-1} \rightarrow 0$, when $\epsilon \rightarrow 0$, and therefore, $u_\epsilon \rightarrow u$ in $H_0^1(\Omega)$ and $\sqrt{\epsilon}v_\epsilon \rightarrow 0$ in $H^{-1}(\Omega)$, though we cannot say whether v_ϵ converges to Lu or not. In the present paper, as in [24], we construct for $S_\epsilon(t)$ a robust family of exponential attractors $\mathcal{M}_{\epsilon, \delta}$ on $\widetilde{B_{5, \delta}^0}^{\mathcal{H}_\epsilon^0}$ with respect to the metric induced by the $\mathcal{H}_\epsilon^0$ -norm, whose common basin of attraction is the whole phase-space $\widetilde{K}_\delta$, with a uniform upper bound on the fractal dimension as well as a uniform rate of attraction of trajectories. Here, inspired by [13], we show that the map $\epsilon \mapsto \mathcal{M}_{\epsilon, \delta}$ is Hölder continuous in the following sense

$$\text{dist}_{\mathcal{H}_{\epsilon_1}^0}^{\text{sym}}(\mathcal{M}_{\epsilon_1, \delta}, \mathcal{M}_{\epsilon_2, \delta}) \leq C'|\epsilon_1 - \epsilon_2|^\sigma, \quad (2.10)$$

for all $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$, where C' and $0 < \sigma \leq \frac{1}{4}$ are independent of ϵ_1 and ϵ_2 . In addition, we also prove that

$$\text{dist}_{\mathcal{H}_1^0}(\mathcal{M}_{\epsilon, \delta}, (\mathcal{M}_\delta)_0) \leq C''\epsilon^\nu, \quad (2.11)$$

where C'' and $0 < \nu \leq \frac{1}{2}$ are independent of ϵ , and

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{H}_1^0}((\mathcal{M}_\delta)_0, \mathcal{M}_{\epsilon, \delta}) = 0. \quad (2.12)$$

In other words, if $(u_\epsilon, v_\epsilon) \in \mathcal{M}_{\epsilon, \delta}$ converges in the $\mathcal{H}_1^0$ -norm to $(u, Lu) \in (\mathcal{M}_\delta)_0$, then $u_\epsilon \rightarrow u$ in $H_{\text{per}}^1(\Omega)$ and $v_\epsilon \rightarrow Lu$ in $(H_{\text{per}}^1(\Omega))'$ as $\epsilon \rightarrow 0$.

In the next Sects. 3 and 4, we derive useful estimates which will serve in establishing the main results of the paper (cf. Theorems 5.2, 5.3, 6.1 and 7.3) in Sects. 5 and 7.

3. Uniform estimates

In this section we prove some bounds which will be useful in Sect. 4.

Proposition 3.1. *For any solution ρ given by Theorem 2.2 such that the trajectory $(\rho(t), \rho_t(t))_{t \geq 0}$ lies in $B_{5,\delta}$, and any $\epsilon \in (0, \epsilon_0]$, we have*

$$\int_t^{t+1} \|\rho_t(s)\|_3^2 ds \leq c_r, \quad \forall t \geq 0. \quad (3.1)$$

Proof. Let us take $q = N^3 \bar{\rho}_t(t)$ in (2.5). This gives

$$\frac{d}{dt} [\epsilon \|\bar{\rho}_t\|_3^2] + \alpha((N^{3/2} \rho)_x, N^3 \bar{\rho}_t) + 2\|\bar{\rho}_t\|_3^2 - 2((N^2 f'(\rho))_x, N^{3/2} \bar{\rho}_t) = 0. \quad (3.2)$$

We have

$$\|(N^2 f'(\rho))_x\| \leq c_r,$$

since $\|\rho(t)\|_{H^5(\Omega)} \leq c_r, \forall t \geq 0$. We then deduce, recalling (2.7), that

$$\frac{d}{dt} [\epsilon \|\bar{\rho}_t\|_3^2 + \alpha \|\bar{\rho}\|_5^2] + c_1 \|\rho_t\|_3^2 \leq c_r, \quad (3.3)$$

whence (3.1). $\square$

Proposition 3.2. *For any solution ρ given by Theorem 2.2 such that the trajectory $(\rho(t), \rho_t(t))_{t \geq 0}$ lies in $B_{5,\delta}$, we have*

$$\|(\rho(t), \rho_t(t))\|_{\mathcal{H}_1^1}^2 \leq c_r, \quad \forall t \geq 1, \quad (3.4)$$

for any $\epsilon \in (0, \epsilon_0]$.

Proof. Taking $q = N \bar{\rho}_t(t)$ in (2.5) we obtain

$$\frac{d}{dt} [\epsilon \|\bar{\rho}_t\|_1^2] + 2\|\bar{\rho}_t\|_1^2 + \alpha((N^2 \rho)_x, N^{1/2} \bar{\rho}_t) + 2((N f'(\rho))_x, N^{1/2} \bar{\rho}_t) = 0. \quad (3.5)$$

Therefore, we have

$$\frac{d}{dt} [\epsilon \|\bar{\rho}_t\|_1^2] + \|\bar{\rho}_t\|_1^2 \leq c_r, \quad (3.6)$$

so that

$$\frac{d}{dt} [\epsilon \|\bar{\rho}_t\|_1^2 e^{t/\epsilon}] \leq c_r e^{t/\epsilon}. \quad (3.7)$$

We integrate (3.7) between s and $t + 1$, for any $s \leq t + 1$. This yields

$$\epsilon \|\bar{\rho}_t(t + 1)\|_1^2 e^{(t+1)/\epsilon} \leq c \epsilon \|\bar{\rho}_t(s)\|_1^2 e^{s/\epsilon} + c_r \epsilon (e^{(t+1)/\epsilon} - e^{s/\epsilon}). \quad (3.8)$$

Integrating now (3.8) between t and $t + 1$ with respect to s , we find

$$\begin{aligned} & \epsilon \|\bar{\rho}_t(t+1)\|_1^2 e^{(t+1)/\epsilon} \\ & \leq c\epsilon \int_t^{t+1} \|\bar{\rho}_t(s)\|_1^2 e^{s/\epsilon} ds + c_r \epsilon \int_t^{t+1} \left(e^{(t+1)/\epsilon} - e^{s/\epsilon} \right) ds. \end{aligned} \quad (3.9)$$

Therefore, we have

$$\epsilon \|\bar{\rho}_t(t+1)\|_1^2 \leq \epsilon \int_t^{t+1} \|\bar{\rho}_t(s)\|_1^2 ds + c_r \epsilon. \quad (3.10)$$

Thus, due to (3.1), we deduce

$$\|\bar{\rho}_t(t)\|_1^2 \leq c_r, \quad \forall t \geq 1, \quad (3.11)$$

which gives

$$\|\rho_t(t)\|_1^2 \leq c_r, \quad \forall t \geq 1, \quad (3.12)$$

on account of (2.7). In other words, trajectories lying in $\tilde{B}_{5,\delta}$ are not only bounded in the $\mathcal{H}_\epsilon^4$ -norm but also in the $\mathcal{H}_1^1$ -norm uniformly with respect to ϵ . $\square$

Proposition 3.3. *For any solution ρ given by Theorem 2.2 such that the trajectory $(\rho(t), \rho_t(t))_{t \geq 0}$ lies in $\tilde{B}_{5,\delta}$, and any $\epsilon \in (0, \epsilon_0]$, we have*

$$\int_0^t \epsilon \|\rho_{tt}(s)\|_1^2 ds \leq c_r(1+t), \quad \forall t \geq 0. \quad (3.13)$$

Proof. We take $q = N\bar{\rho}_{tt}(t)$ in (2.5) and we find

$$\begin{aligned} & \frac{d}{dt} [\|\bar{\rho}_t\|_1^2 + 2((Nf'(\rho))_x, N^{1/2}\bar{\rho}_t) + 2\alpha((N^2\rho)_x, N^{1/2}\bar{\rho}_t)] + 2\epsilon \|\bar{\rho}_{tt}\|_1^2 \\ & = 2\alpha((N\rho_t)_x, N^{3/2}\bar{\rho}_t) + 2((f''(\rho)\rho_t)_x, N^{3/2}\bar{\rho}_t). \end{aligned} \quad (3.14)$$

Therefore, we get

$$\begin{aligned} & \frac{d}{dt} [\|\bar{\rho}_t\|_1^2 + 2((Nf'(\rho))_x, N^{1/2}\bar{\rho}_t) + 2\alpha((N^2\rho)_x, N^{1/2}\bar{\rho}_t)] + 2\epsilon \|\bar{\rho}_{tt}\|_1^2 \\ & \leq c_r \|\rho_t\|_3^2. \end{aligned} \quad (3.15)$$

Integrating (3.15) between 0 and t , we find, on account of (3.4),

$$\int_0^t \epsilon \|\bar{\rho}_{tt}(s)\|_1^2 ds \leq c_r(1+t), \quad \forall t \geq 0, \quad (3.16)$$

and (3.13) follows, noting that $\int_0^t m(\rho_{tt}(s))^2 ds \leq c$, $\forall t \geq 0$. $\square$

4. Estimates on the difference of two solutions

We now introduce the (nonlinear) manifold $\mathbb{M} = \{(u, v) \in \mathcal{H}_\epsilon^0 : v = Lu\}$ and we define the continuous semigroup $S_0(t) : \mathbb{M} \rightarrow \mathbb{M}$ by setting

$$S_0(t)(\rho_0, L\rho_0) = (S(t)\rho_0, LS(t)\rho_0), \quad \forall t \geq 0.$$

Note that $S_0(t)$ has the global attractor $(\mathcal{A}_\delta)_0$ on $\tilde{K}_\delta \cap \mathbb{M}$.

Let us first prove a useful bound on the solution to (2.1).

Proposition 4.1. *Let $\rho_0 \in \widetilde{B}_\delta$. Then the solution ρ given by Theorem 2.1 satisfies*

$$\int_0^t \|\rho_{tt}(s)\|_{-1}^2 ds \leq c_r(1+t), \quad \forall t \geq 0. \quad (4.1)$$

Proof. First, we observe that $\tilde{\rho} = \rho_t$ is the solution to the following linearized problem:

$$\tilde{\rho}_t + N(\alpha N \tilde{\rho} + f''(\rho) \tilde{\rho}) = 0, \quad (4.2)$$

$$\tilde{\rho}|_{t=0} = L\rho_0, \quad (4.3)$$

where $\rho(t) = S(t)\rho_0$. We multiply (4.2) by $N^{-1}\tilde{\rho}_t$ and find

$$\alpha \frac{d}{dt} \|\tilde{\rho}\|_1^2 + 2\|\tilde{\rho}_t\|_{-1}^2 + 2 \left((f''(\rho)\tilde{\rho})_x, N^{-1/2}\tilde{\rho}_t \right) = 0. \quad (4.4)$$

Since we have

$$\|(f''(\rho(t)))_x \tilde{\rho}(t)\| \leq c_r \|\rho_t(t)\|_1, \quad \forall t \geq 0,$$

then (4.4) yields

$$\frac{d}{dt} \|\tilde{\rho}\|_1^2 + c \|\tilde{\rho}_t\|_{-1}^2 \leq c_r \|\tilde{\rho}\|_1^2. \quad (4.5)$$

If $\rho_0 \in \widetilde{B}_\delta$, then we have $\|\rho(t)\|_5 \leq c_r$ and $\|\rho_t(t)\|_1 \leq c_r, \forall t \geq 0$. Thus, (4.1) follows from (4.5). $\square$

We now show the first basic estimate.

Proposition 4.2. *Let (2.3) hold. Then, there exists $t_\star > 0$, independent of ϵ , such that*

$$\|S_\epsilon(t)(\rho_0, \rho_1) - S_0(t)(\rho_0, L\rho_0)\|_{\mathcal{H}_1^0}^2 \leq \epsilon c(r, t), \quad \forall t \geq t_\star, \quad (4.6)$$

for any $(\rho_0, \rho_1) \in \widetilde{B}_{5,\delta}$ and any $\epsilon \in (0, \epsilon_0]$.

Proof. We follow the strategy devised in [41]. For any solution ρ^ϵ of (2.2), we set $\rho^\epsilon = \rho^1 + \rho^2$, where ρ^1 and ρ^2 solve the following problems

$$\epsilon \rho_{tt}^1 + \rho_t^1 + \alpha N^2 \rho^1 = 0, \quad (4.7)$$

$$\rho^1|_{t=0} = 0, \quad \rho_t^1|_{t=0} = \rho_1, \quad (4.8)$$

and

$$\epsilon \rho_{tt}^2 + \rho_t^2 + N(\alpha N \rho^2 + f'(\rho^\epsilon)) = 0, \quad (4.9)$$

$$\rho^2|_{t=0} = \rho_0, \quad \rho_t^2|_{t=0} = 0, \quad (4.10)$$

respectively. We then consider the difference of the solutions ρ and ρ^ϵ to problems (2.1) and (2.2) and we assume that the corresponding trajectories $(\rho^\epsilon(t), \rho_t^\epsilon(t))_{t \geq 0}$ and $(\rho(t), \rho_t(t))_{t \geq 0}$ lie in the absorbing sets $\widetilde{B}_{5,\delta}$ and $\widetilde{B}_\delta$, respectively. Thus we have

$$(\rho^\epsilon(t), \rho_t^\epsilon(t)) = S_\epsilon(t)(\rho_0, \rho_1), \quad (\rho(t), \rho_t(t)) = S_0(t)(\rho_0, L\rho_0),$$

for (ρ_0, ρ_1) in $\widetilde{B}_{5,\delta}$. Note that, due to (3.4), we have $\|\rho_1\|_{H^1(\Omega)} \leq c_r$, and recalling (2.8), we can see that

$$\|L\rho_0\|_1 \leq c_r.$$

We now set $P = \rho^\epsilon - \rho$ and write $P = \rho^1 + v$, where $v = \rho^2 - \rho$ satisfies the following problem:

$$\epsilon v_{tt} + v_t + N(\alpha N v + f'(\rho^\epsilon) - f'(\rho)) = -\epsilon \rho_{tt}, \quad (4.11)$$

$$v|_{t=0} = 0, \quad v_t|_{t=0} = -L\rho_0. \quad (4.12)$$

Then we observe that

$$m(\rho^1(t)) = \epsilon m(\rho_1)(1 - e^{-t/\epsilon}), \quad m(\rho_t^1(t)) = m(\rho_1)e^{-t/\epsilon}, \quad \forall t \geq 0, \quad (4.13)$$

which imply

$$m(\rho^1(t))^2 \leq \epsilon^2 m(\rho_1)^2, \quad \int_0^t m(\rho_s^1)^2 ds \leq \epsilon m(\rho_1)^2, \quad \forall t \geq 0. \quad (4.14)$$

Let us take the L^2 -scalar product of (4.7) with $\epsilon^{-1}N^{-1}\bar{\rho}^1$ and obtain

$$\frac{d}{dt} \left[2(N^{-1/2}\bar{\rho}_t^1, N^{-1/2}\bar{\rho}^1) + \frac{1}{\epsilon} \|\bar{\rho}^1\|_{-1}^2 \right] + \frac{2\alpha}{\epsilon} \|\bar{\rho}^1\|_1^2 = 2\|\bar{\rho}_t^1\|_{-1}^2. \quad (4.15)$$

Similarly, we take the L^2 -scalar product of (4.7) with $N^{-1}\bar{\rho}_t^1$ and obtain

$$\frac{d}{dt} [\epsilon \|\bar{\rho}_t^1\|_{-1}^2 + \alpha \|\bar{\rho}^1\|_1^2] + 2\|\bar{\rho}_t^1\|_{-1}^2 = 0. \quad (4.16)$$

Adding together $\varpi(4.15)$ and (4.16), with $\varpi > 0$ small, we get

$$\frac{dE_1}{dt} + \frac{c_1}{\epsilon} \|(\bar{\rho}^1, \bar{\rho}_t^1)\|_{\mathcal{H}_\epsilon^0}^2 \leq 0, \quad (4.17)$$

where

$$E_1(t) = \varpi \left[2(N^{-1/2}\bar{\rho}_t^1, N^{-1/2}\bar{\rho}^1) + \frac{1}{\epsilon} \|\bar{\rho}^1\|_{-1}^2 \right] + [\epsilon \|\bar{\rho}_t^1\|_{-1}^2 + \alpha \|\bar{\rho}^1\|_1^2]. \quad (4.18)$$

Observe that

$$2(N^{-1/2}\bar{\rho}_t^1, N^{-1/2}\bar{\rho}^1) \geq -2\epsilon \|\bar{\rho}_t^1\|_{-1}^2 - \frac{1}{2\epsilon} \|\bar{\rho}^1\|_{-1}^2.$$

Clearly, there exists c_1 independent of ϵ such that

$$E_1(t) \geq c_1 \left(\|(\bar{\rho}^1(t), \bar{\rho}_t^1(t))\|_{\mathcal{H}_\epsilon^0}^2 + \frac{1}{\epsilon} \|\bar{\rho}^1(t)\|_{-1}^2 \right). \quad (4.19)$$

Integrating (4.17) between 0 and t and noting that $\rho^1(0) = 0$, we deduce

$$\int_0^t \|(\bar{\rho}^1(s), \bar{\rho}_t^1(s))\|_{\mathcal{H}_\epsilon^0}^2 ds \leq c_r \epsilon^2, \quad \forall t \geq 0. \quad (4.20)$$

Therefore, on account of (4.14),

$$\int_0^t \|(\rho^1(s), \rho_t^1(s))\|_{\mathcal{H}_\epsilon^0}^2 ds \leq c_r \epsilon^2 (1 + t), \quad \forall t \geq 0. \quad (4.21)$$

On the other hand, noting that $m(v(t)) = m(v_t(t)) = 0, \forall t \geq 0$, we take the L^2 -scalar product of (4.11) with $\epsilon^{-1} N^{-1} v$ and $N^{-1} v_t$, respectively, and obtain

$$\begin{aligned} & \frac{d}{dt} \left[2(N^{-1/2} v_t, N^{-1/2} v) + \frac{1}{\epsilon} \|v\|_{-1}^2 \right] + \frac{2\alpha}{\epsilon} \|v\|_1^2 - \frac{2}{\epsilon} (f'(\rho^\epsilon) - f'(\rho), v) \\ &= 2\|v_t\|_{-1}^2 - 2(N^{-1/2} \rho_{tt}, N^{-1/2} v), \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} & \frac{d}{dt} [\epsilon \|v_t\|_{-1}^2 + \alpha \|v\|_1^2] + 2\|v_t\|_{-1}^2 + 2(f'(\rho^\epsilon) - f'(\rho), v_t) \\ &= -2\epsilon (N^{-1/2} \rho_{tt}, N^{-1/2} v_t). \end{aligned} \quad (4.23)$$

We now sum (4.22) and (4.23), for $\varpi > 0$ sufficiently small. It is easy to realize that

$$\|f'(\rho^\epsilon) - f'(\rho)\|_{H^1(\Omega)} \leq c_r \|\rho^\epsilon - \rho\|_1.$$

Moreover, we have, for any $q \in (\dot{H}_{\text{per}}^1(\Omega))'$,

$$(N^{-1/2} \rho_{tt}, N^{-1/2} q) \leq c \|\rho_{tt}\|_{-1} \|q\|_{-1}.$$

We thus infer

$$\frac{dE_2}{dt} + \frac{c_1}{\epsilon} \|(v, v_t)\|_{\mathcal{H}_\epsilon^0}^2 \leq c_2 E_2 + \frac{c_3}{\epsilon} (\|\rho^1\|_1^2 + \epsilon^2 \|\rho_{tt}\|_{-1}^2), \quad (4.24)$$

where

$$E_2(t) = \varpi \left[2(N^{-1/2} v_t, N^{-1/2} v) + \frac{1}{\epsilon} \|v\|_{-1}^2 \right] + \epsilon \|v_t\|_{-1}^2 + \alpha \|v\|_1^2, \quad (4.25)$$

and

$$E_2(t) \geq c_1 \left(\|(v(t), v_t(t))\|_{\mathcal{H}_\epsilon^0}^2 + \frac{1}{\epsilon} \|v(t)\|_{-1}^2 \right). \quad (4.26)$$

We deduce from (4.24), by applying the Gronwall lemma, that

$$E_2(t) \leq \left[E_2(0) + \frac{c_3}{\epsilon} \int_0^t (\|\rho^1(s)\|_1^2 + \epsilon^2 \|\rho_{tt}(s)\|_{-1}^2) ds \right] e^{c_2 t} \leq c_r \epsilon e^{c_r' t},$$

on account of (4.1) and (4.21). Therefore,

$$\int_0^t \|(v(s), v_t(s))\|_{\mathcal{H}_\epsilon^0}^2 ds \leq c_r \epsilon^2 e^{c_r' t}, \quad \forall t \geq 0, \quad (4.27)$$

where the constants c_r and c_r' depend on r , but are independent of ϵ .

Let us multiply (4.16) by t , for any $t > 0$. This gives

$$\frac{d}{dt} [\epsilon t \|\bar{\rho}_t^1\|_{-1}^2 + \alpha t \|\bar{\rho}^1\|_1^2] + 2t \|\bar{\rho}_t^1\|_{-1}^2 = \epsilon \|\bar{\rho}_t^1\|_{-1}^2 + \alpha \|\bar{\rho}^1\|_1^2. \quad (4.28)$$

Integrating (4.28) between 0 and t , we find, on account of (4.20),

$$t\|(\bar{\rho}^1(t), \bar{\rho}_t^1(t))\|_{\mathcal{H}_e^0}^2 \leq c_r \epsilon^2, \quad \forall t \geq 0. \quad (4.29)$$

Let $t_\star > 0$, independent of ϵ , be such that $e^{-(2t_\star)/\epsilon} \leq \epsilon$. Any $t_\star \geq \frac{1}{2\epsilon}$ realizes this condition. Then, recalling (4.13), it follows that $m(\rho_t^1(t))^2 \leq \epsilon m(\rho_1)^2$, $\forall t \geq t_\star$ (note that $m(\rho^1(t))^2 \leq \epsilon^2 m(\rho_1)^2$, $\forall t \geq 0$). Thus we have

$$t\|(\rho^1(t), \rho_t^1(t))\|_{\mathcal{H}_e^0}^2 \leq c_r \epsilon^2, \quad \forall t \geq t_\star. \quad (4.30)$$

On the other hand, multiplying (4.23) by t , we find that

$$\begin{aligned} & \frac{d}{dt} [\epsilon t \|v_t\|_{-1}^2 + \alpha t \|v\|_1^2] + t \|v_t\|_{-1}^2 \\ & \leq c(1+t) \|(v, v_t)\|_{\mathcal{H}_e^0}^2 + t \epsilon^2 \|\rho_{tt}\|_{-1}^2 + t \|\rho^1\|_1^2. \end{aligned} \quad (4.31)$$

We integrate (4.31) between 0 and t and obtain, due to (4.1), (4.21) and (4.27),

$$t\|(v(t), v_t(t))\|_{\mathcal{H}_e^0}^2 \leq c_r \epsilon^2 e^{c_r' t}, \quad \forall t \geq 0. \quad (4.32)$$

Finally, it follows from (4.30) and (4.32) that

$$t\|(P(t), P_t(t))\|_{\mathcal{H}_e^0}^2 \leq c_r \epsilon^2 e^{c_r' t}, \quad \forall t \geq t_\star, \quad (4.33)$$

where c_r and c_r' depend on r , but are independent of ϵ .

Finally, estimate (4.6) follows from (4.33). $\square$

We end this section by proving the second basic estimate.

Proposition 4.3. *Let (2.3) hold and suppose $\epsilon_1, \epsilon_2 \in (0, \epsilon_0]$ are such that $\epsilon_1 > \epsilon_2$. Then, there exists $t_\star > 0$, independent of ϵ_1 and ϵ_2 , such that*

$$\begin{aligned} & \left\| S_{\epsilon_1}(t)(\rho_0, \rho_1) - S_{\epsilon_2}(t) \left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \rho_1 \right) \right\|_{\mathcal{H}_1^0}^2 \\ & \leq |\epsilon_1 - \epsilon_2|^{1/2} c(r, t), \quad \forall t \geq t_\star, \end{aligned} \quad (4.34)$$

for any $(\rho_0, \rho_1) \in \tilde{B}_{5,\delta}^{\epsilon_1}$.

Proof. We first observe that $(\rho_0, (\sqrt{\epsilon_1}/\sqrt{\epsilon_2})\rho_1) \in \tilde{B}_{5,\delta}^{\epsilon_2}$ whenever $(\rho_0, \rho_1) \in \tilde{B}_{5,\delta}^{\epsilon_1}$. Let us consider two solutions ρ^{ϵ_1} and ρ^{ϵ_2} of (2.2) with $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$, with initial conditions (ρ_0, ρ_1) and $(\rho_0, (\sqrt{\epsilon_1}/\sqrt{\epsilon_2})\rho_1)$, respectively. We assume that the trajectories $(\rho^{\epsilon_1}(t), \rho_t^{\epsilon_1}(t))_{t \geq 0}$ and $(\rho^{\epsilon_2}(t), \rho_t^{\epsilon_2}(t))_{t \geq 0}$ lie in the absorbing sets $\tilde{B}_{5,\delta}^{\epsilon_1}$ and $\tilde{B}_{5,\delta}^{\epsilon_2}$, respectively. Thus we have

$$(\rho^{\epsilon_1}(t), \rho_t^{\epsilon_1}(t)) = S_{\epsilon_1}(t)(\rho_0, \rho_1), \quad (\rho^{\epsilon_2}(t), \rho_t^{\epsilon_2}(t)) = S_{\epsilon_2}(t) \left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \rho_1 \right),$$

for $(\rho_0, \rho_1) \in \tilde{B}_{5,\delta}^{\epsilon_1}$. Let us set $\tilde{P} = \rho^{\epsilon_1} - \rho^{\epsilon_2}$ and $\epsilon = \epsilon_1 - \epsilon_2$. Note that $\tilde{P}$ solves the following problem:

$$\epsilon_2 \tilde{P}_{tt} + \tilde{P}_t + N(\alpha N \tilde{P} + f'(\rho^{\epsilon_1}) - f'(\rho^{\epsilon_2})) = -\epsilon \rho_{tt}^{\epsilon_1}, \quad (4.35)$$

$$\tilde{P}|_{t=0} = 0, \quad \tilde{P}_t|_{t=0} = \left(1 - \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \right) \rho_1, \quad (4.36)$$

and, for all $t \geq 0$, we have

$$\begin{aligned} m(\tilde{P}(t)) &= \sqrt{\epsilon_1}(\sqrt{\epsilon_1} - \sqrt{\epsilon_2})m(\rho_1) + m(\rho_1)\sqrt{\epsilon_1} \left(\sqrt{\epsilon_2}e^{-t/\epsilon_2} - \sqrt{\epsilon_1}e^{-t/\epsilon_1} \right), \\ m(\tilde{P}_t(t)) &= m(\rho_1)\sqrt{\epsilon_1} \left(\frac{1}{\sqrt{\epsilon_1}}e^{-t/\epsilon_1} - \frac{1}{\sqrt{\epsilon_2}}e^{-t/\epsilon_2} \right). \end{aligned}$$

We observe that

$$\begin{aligned} \sqrt{\epsilon_1}e^{-t/\epsilon_1} - \sqrt{\epsilon_2}e^{-t/\epsilon_2} &= \epsilon \left(\frac{1}{2\sqrt{\xi_1}} + \frac{t}{\xi_1\sqrt{\xi_1}} \right) e^{-t/\xi_1}, \\ \frac{1}{\sqrt{\epsilon_1}}e^{-t/\epsilon_1} - \frac{1}{\sqrt{\epsilon_2}}e^{-t/\epsilon_2} &= \epsilon \left(-\frac{1}{2\xi_2\sqrt{\xi_2}} + \frac{t}{\xi_2^2\sqrt{\xi_2}} \right) e^{-t/\xi_2}, \end{aligned}$$

where $\xi_1 = \varpi_1\epsilon_1 + (1 - \varpi_1)\epsilon_2$ and $\xi_2 = \varpi_2\epsilon_1 + (1 - \varpi_2)\epsilon_2$, for some $\varpi_1, \varpi_2 \in [0, 1]$. Let $t_\star > 0$, independent of ϵ_1 and ϵ_2 , be such that $t_\star \geq \frac{5}{2e}$. Then, we have $e^{-t/\xi_1} \leq \xi_1\sqrt{\xi_1}$ and $e^{-t/\xi_2} \leq \xi_2^2\sqrt{\xi_2}$, $\forall t \geq t_\star$. Thus, we deduce

$$m(\tilde{P}(t))^2 + m(\tilde{P}_t(t))^2 \leq c\epsilon(1 + t^2), \quad \forall t \geq t_\star, \quad (4.37)$$

since $(\sqrt{\epsilon_1} - \sqrt{\epsilon_2})^2 \leq \epsilon$. We take the L^2 -scalar product of (4.35) with $N^{-1}\tilde{P}_t$. This yields

$$\frac{d}{dt}[\epsilon_2\|\tilde{P}_t\|_{-1}^2 + \alpha\|\tilde{P}\|_1^2] + 2\|\tilde{P}_t\|_{-1}^2 + 2(f'(\rho^{\epsilon_1}) - f'(\rho^{\epsilon_2}), \tilde{P}_t) = -2\epsilon(\rho_{tt}^{\epsilon_1}, N^{-1}\tilde{P}_t). \quad (4.38)$$

Therefore, we get

$$\frac{d}{dt}[\epsilon_2\|\tilde{P}_t\|_{-1}^2 + \alpha\|\tilde{P}\|_1^2] + \|\tilde{P}_t\|_{-1}^2 \leq c_r\|\tilde{P}\|_{H^1(\Omega)}^2 + \epsilon^2\|\rho_{tt}^{\epsilon_1}\|_{-1}^2. \quad (4.39)$$

Applying the Gronwall lemma to (4.39), we obtain, on account of (3.13),

$$\begin{aligned} \epsilon_2\|\tilde{P}_t(t)\|_{-1}^2 + \|\tilde{P}(t)\|_1^2 &\leq c_r[(\sqrt{\epsilon_1} - \sqrt{\epsilon_2})^2 + \epsilon^2\epsilon_1^{-1}]e^{c'_rt} \\ &\leq c_r\epsilon e^{c'_rt}, \quad \forall t \geq 0, \end{aligned} \quad (4.40)$$

since $\epsilon\epsilon_1^{-1} \leq 1$ and $\|\rho_1\| \leq c_r$. Therefore, due to (4.37),

$$\epsilon_2\|\tilde{P}_t(t)\|_{-1}^2 + \|\tilde{P}(t)\|_1^2 \leq c_r\epsilon e^{c'_rt}, \quad \forall t \geq t_\star, \quad (4.41)$$

where c_r and c'_r depend on r , but not on ϵ_1 and ϵ_2 . From the equality

$$\sqrt{\epsilon_1}\tilde{P}_t = (\sqrt{\epsilon_1} - \sqrt{\epsilon_2})\tilde{P}_t + \sqrt{\epsilon_2}\tilde{P}_t,$$

we have

$$\epsilon_1\|\tilde{P}_t(t)\|_{-1}^2 - 2\epsilon_2\|\tilde{P}_t(t)\|_{-1}^2 \leq 2(\sqrt{\epsilon_1} - \sqrt{\epsilon_2})^2\|\tilde{P}_t(t)\|_{-1}^2 \leq c_r\epsilon, \quad (4.42)$$

since $\|\tilde{P}_t(t)\|_{-1}^2 \leq c_r$, $\forall t \geq 0$. Then, we infer from (4.41) and (4.42) that

$$\epsilon_1\|\tilde{P}_t(t)\|_{-1}^2 + \|\tilde{P}(t)\|_1^2 \leq c_r\epsilon e^{c'_rt}, \quad \forall t \geq t_\star. \quad (4.43)$$

We eventually deduce the estimate

$$\left\| S_{\epsilon_1}(t)(\rho_0, \rho_1) - S_{\epsilon_2}(t)\left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}\rho_1\right) \right\|_{\mathcal{H}_1^0}^2 \leq \epsilon\epsilon_1^{-1}c(r, t), \quad \forall t \geq t_\star. \quad (4.44)$$

To conclude, we note that, due to (4.6), there holds

$$\begin{aligned} & \left\| S_{\epsilon_1}(t)(\rho_0, \rho_1) - S_{\epsilon_2}(t) \left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \rho_1 \right) \right\|_{\mathcal{H}_1^0}^2 \\ & \leq \| S_{\epsilon_1}(t)(\rho_0, \rho_1) - S_0(t)(\rho_0, L\rho_0) \|_{\mathcal{H}_1^0}^2 \\ & \quad + \left\| S_{\epsilon_2}(t) \left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \rho_1 \right) - S_0(t)(\rho_0, L\rho_0) \right\|_{\mathcal{H}_1^0}^2 \\ & \leq \epsilon_1 c(r, t), \quad \forall t \geq t_*. \end{aligned} \quad (4.45)$$

Hence, it follows from (4.44) and (4.45) that

$$\left\| S_{\epsilon_1}(t)(\rho_0, \rho_1) - S_{\epsilon_2}(t) \left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \rho_1 \right) \right\|_{\mathcal{H}_1^0}^2 \leq \min\{\epsilon_1, \epsilon\epsilon_1^{-1}\} c(r, t), \quad \forall t \geq t_*. \quad (4.46)$$

Estimate (4.34) follows from (4.46) by noting that $\min\{\epsilon_1, \epsilon\epsilon_1^{-1}\} \leq \epsilon^{1/2}$. $\square$

5. Exponential and global attractors

Let E be a metric space, X be a compact subset of E and $\{S(t)\}_{t \geq 0}$ be a continuous semigroup on E .

For the reader's convenience, here below we recall the definition of an exponential attractor for the semigroup $\{S(t)\}_{t \geq 0}$.

Definition 5.1. A compact set $\mathcal{M}$ is called an exponential attractor for $S(t)$ for the topology of E if:

- (i) $\mathcal{M}$ is positively invariant under $S(t)$, that is, $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geq 0$;
- (ii) the fractal dimension of $\mathcal{M}$ is finite;
- (iii) there exists a constant $c_0 > 0$ such that, for every bounded subset $B \subset X$, there exists a constant $c_1(B) > 0$ such that

$$\text{dist}_E(S(t)B, \mathcal{M}) \leq c_1 e^{-c_0 t}, \quad \forall t \geq 0.$$

We now give sufficient conditions ensuring the existence of uniform exponential attractors that are continuous with respect to ϵ . More precisely, we have the

Theorem 5.1. Let $E^1, E^2, V^1, V^2, W^1, W^2$ be Banach spaces such that $W^i \subseteq V^i \subseteq E^i$, $i = 1, 2$. Set $E_\epsilon = E^1 \times E^2$, $V_\epsilon = V^1 \times V^2$, $W_\epsilon = W^1 \times W^2$ and endow them with the following norms

$$\begin{aligned} \|(p, q)\|_{E_\epsilon} &= (\|p\|_{E^1}^2 + \epsilon \|q\|_{E^2}^2)^{1/2}, \\ \|(p, q)\|_{V_\epsilon} &= (\|p\|_{V^1}^2 + \epsilon \|q\|_{V^2}^2)^{1/2}, \\ \|(p, q)\|_{W_\epsilon} &= (\|p\|_{W^1}^2 + \epsilon \|q\|_{W^2}^2)^{1/2}, \end{aligned}$$

respectively, where $\epsilon \in [0, 1]$, with the convention that $E_0 = E^1$, $V_0 = V^1$, and $W_0 = W^1$. Let $B_\epsilon(r)$ denote a closed ball in W_ϵ of radius $r > 0$ and centered at zero. Consider a one-parameter family of strongly continuous semigroups

$\{S_\epsilon(t)\}_\epsilon$ acting on the phase-space E_ϵ , for each $\epsilon \in [0, 1]$. Then assume that there exist $\alpha, \beta, \gamma, \vartheta \in (0, 1]$, $\kappa \in (0, \frac{1}{2})$, $\Upsilon_j \geq 0$, and $\varrho > 0$ (all independent of ϵ) such that, setting $B_\epsilon = B_\epsilon(\varrho)$, the following conditions hold:

1. There exists a map $L : B_0 \rightarrow V^2$ which is Hölder continuous of exponent α . Here B_0 is endowed with the metric topology of E^1 .
2. There exists $t^* > 0$, independent of ϵ , such that

$$S_\epsilon(t)B_\epsilon \subset B_\epsilon, \quad \forall t \geq t^*,$$

and B_ϵ is uniformly bounded (with respect to ϵ) in the E_1 -norm. Moreover, setting $S_\epsilon(t^*) = S_\epsilon$, the map S_ϵ satisfies, for every $z_1, z_2 \in B_\epsilon$,

$$S_\epsilon z_1 - S_\epsilon z_2 = L_\epsilon(z_1, z_2) + K_\epsilon(z_1, z_2),$$

where

$$\|L_\epsilon z_1 - L_\epsilon z_2\|_{E_\epsilon} \leq \kappa \|z_1 - z_2\|_{E_\epsilon}, \quad (5.1)$$

$$\|K_\epsilon z_1 - K_\epsilon z_2\|_{V_\epsilon} \leq \Upsilon_1 \|z_1 - z_2\|_{E_\epsilon}. \quad (5.2)$$

3. For any $z_1 = (\eta, \zeta) \in B_{\epsilon_1}$ and any $\epsilon_2 \in (0, \epsilon_1)$, there hold

$$\|S_{\epsilon_1}^m z_1 - S_{\epsilon_2}^m z_2\|_{E_1} \leq \Upsilon_2^m |\epsilon_1 - \epsilon_2|^\beta, \quad \forall m \in \mathbb{N}, \quad (5.3)$$

$$\|S_{\epsilon_1}(t)z_1 - S_{\epsilon_2}(t)z_2\|_{E_1} \leq \Upsilon_3 |\epsilon_1 - \epsilon_2|^\gamma, \quad \forall t \in [t^*, 2t^*], \quad (5.4)$$

where $z_2 = (\eta, (\sqrt{\epsilon_1}/\sqrt{\epsilon_2})\zeta) \in B_{\epsilon_2}$. When $\epsilon_1 > 0$ and $\epsilon_2 = 0$, $S_{\epsilon_2}^m z_2$ and $S_{\epsilon_2}(t)z_2$ are replaced in (5.3) and (5.4) by $\mathcal{L}_{\epsilon_1} S_0^m \Pi_{\epsilon_1} z_1$ and $\mathcal{L}_{\epsilon_1} S_0(t) \Pi_{\epsilon_1} z_1$, respectively. Here the “lifting” map $\mathcal{L}_\epsilon : B_0 \rightarrow E_\epsilon$ is defined by

$$\mathcal{L}_\epsilon x = \begin{cases} (x, Lx), & \text{if } \epsilon > 0, \\ x, & \text{if } \epsilon = 0, \end{cases}$$

and $\Pi_\epsilon : B_\epsilon \rightarrow B_0$ is the projection onto the first component when $\epsilon > 0$, and the identity map otherwise.

4. The map $z \mapsto S_\epsilon(t)z$ is Lipschitz continuous on B_ϵ endowed with the metric topology of E_ϵ , with a Lipschitz constant independent of ϵ and $t \in [t^*, 2t^*]$.
5. The map

$$(t, z) \mapsto S_\epsilon(t)z : [t^*, 2t^*] \times B_\epsilon \rightarrow B_\epsilon$$

is Hölder continuous of exponent ϑ , where B_ϵ is endowed with the metric topology of E_ϵ .

Then there exists a family of exponential attractors $\mathfrak{M}_\epsilon$ on $\mathcal{B}_\epsilon = \overline{B_\epsilon}^{E_\epsilon}$ with the following properties:

- (i) $\mathfrak{M}_\epsilon$ attracts $\mathcal{B}_\epsilon$ with an exponential rate which is uniform with respect to ϵ , that is,

$$\text{dist}_{E_\epsilon}(S_\epsilon(t)\mathcal{B}_\epsilon, \mathfrak{M}_\epsilon) \leq M_1 e^{-\omega t}, \quad \forall t \geq 0,$$

for some $M_1 > 0$ and some $\omega > 0$.

- (ii) The fractal dimension of $\mathfrak{M}_\epsilon$ is uniformly bounded with respect to ϵ , that is,

$$\dim_{E_\epsilon}[\mathfrak{M}_\epsilon] \leq M_2.$$

- (iii) The family $\mathfrak{M}_\epsilon$ is Hölder continuous with respect to ϵ , that is, there exist a positive constant M_3 and $\tau \in (0, \frac{1}{2}]$ such that

$$\text{dist}_{E_{\epsilon_1}}^{\text{sym}}(\mathfrak{M}_{\epsilon_1}, \mathfrak{M}_{\epsilon_2}) \leq M_3 |\epsilon_1 - \epsilon_2|^\tau, \quad (5.5)$$

for all $0 \leq \epsilon_2 < \epsilon_1 \leq 1$, where $\mathfrak{M}_0$ must be replaced by its lifting $\mathcal{L}_{\epsilon_1} \mathfrak{M}_0$. In addition, there exist a positive constant M_4 and $\sigma \in (0, \frac{1}{2}]$ such that

$$\text{dist}_{E_1}(\mathfrak{M}_\epsilon, \mathcal{L}_\epsilon \mathfrak{M}_0) \leq M_4 \epsilon^\sigma, \quad (5.6)$$

for all $0 < \epsilon \leq 1$, and

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{E_1}(\mathcal{L}_\epsilon \mathfrak{M}_0, \mathfrak{M}_\epsilon) = 0. \quad (5.7)$$

Here ω, τ, σ and M_j are independent of ϵ, ϵ_1 and ϵ_2 , and they can be computed explicitly.

Proof. Observe first that the existence of a family of exponential attractors $\mathfrak{M}_\epsilon$ satisfying (i) and (ii) follows from [24]. In order to ensure that this family fulfills (5.5), we follow a technique used in [34], based on an application of [13, Theorem 2.2]. We introduce the auxiliary operators $\tilde{S}_\epsilon : E_1 \rightarrow E_1, \epsilon \in (0, 1], \tilde{S}_\epsilon(t) = C_\epsilon^{-1} \circ S_\epsilon(t) \circ C_\epsilon$ and $\tilde{S}_0(t) = (S(t), 0), \forall t \geq 0$, where $C_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\epsilon}} \end{pmatrix}$. We have $\tilde{S}_\epsilon = \tilde{S}_\epsilon(t^*) : B_1 \rightarrow B_1$, for some $t^* > 0$, and we set $\tilde{L}_\epsilon = C_\epsilon^{-1} \circ L_\epsilon \circ C_\epsilon$ and $\tilde{K}_\epsilon = C_\epsilon^{-1} \circ K_\epsilon \circ C_\epsilon$. For any $\tilde{z} = (u, v) \in B_1$, we set $z_i = (u, (1/\sqrt{\epsilon_i})v), i = 1, 2$, and we have $z_i \in B_{\epsilon_i}$ since $B_\epsilon = C_\epsilon B_1$. There holds $\|\tilde{z}\|_{E_1} = \|z_i\|_{E_{\epsilon_i}}$, and we can easily deduce from (5.1) and (5.2) that

$$\|\tilde{L}_\epsilon \tilde{z}_1 - \tilde{L}_\epsilon \tilde{z}_2\|_{E_1} \leq \kappa \|\tilde{z}_1 - \tilde{z}_2\|_{E_1}, \quad (5.8)$$

$$\|\tilde{K}_\epsilon \tilde{z}_1 - \tilde{K}_\epsilon \tilde{z}_2\|_{V_1} \leq \Upsilon_1 \|\tilde{z}_1 - \tilde{z}_2\|_{E_1}. \quad (5.9)$$

Also, we can deduce from (5.3) and (5.4) that

$$\|\tilde{S}_{\epsilon_1}^m \tilde{z} - \tilde{S}_{\epsilon_2}^m \tilde{z}\|_{E_1} \leq \Upsilon_2^m |\epsilon_1 - \epsilon_2|^\beta, \quad \forall m \in \mathbb{N}, \quad (5.10)$$

$$\|\tilde{S}_{\epsilon_1}(t) \tilde{z} - \tilde{S}_{\epsilon_2}(t) \tilde{z}\|_{E_1} \leq \Upsilon_3 |\epsilon_1 - \epsilon_2|^\gamma, \quad \forall t \in [t^*, 2t^*]. \quad (5.11)$$

It is thus clear that the maps $\tilde{S}_\epsilon, \tilde{L}_\epsilon$ and $\tilde{K}_\epsilon$ satisfy conditions 2. through 5. on B_1 with respect to the metric induced by the E_1 -norm. Applying [34, Theorem 3.1], there exists a robust family of exponential attractors $\mathfrak{N}_\epsilon$ for $\tilde{S}_\epsilon$ in $\overline{B_1}^{E_1}$ with respect to the metric induced by the E_1 -norm and such that

$$\text{dist}_{E_1}(\tilde{S}_\epsilon B_1, \mathfrak{N}_\epsilon) \leq M_1 e^{-\omega t}, \quad (5.12)$$

$$\dim_{E_1}[\mathfrak{N}_\epsilon] \leq M_2, \quad (5.13)$$

$$\text{dist}_{E_1}^{\text{sym}}(\mathfrak{N}_{\epsilon_1}, \mathfrak{N}_{\epsilon_2}) \leq M_3 |\epsilon_1 - \epsilon_2|^{\tau_2}, \quad (5.14)$$

where $\mathfrak{N}_0$ is replaced by the lifting $(\mathfrak{M}_0, 0)$ of $\mathfrak{M}_0$. Let us now set $\mathfrak{M}_\epsilon = C_\epsilon \mathfrak{N}_\epsilon$ for every $\epsilon \in (0, 1]$. We note that, for any $z_i = (\eta_i, \zeta_i) \in \mathfrak{M}_{\epsilon_i}$, and $z_0 = (\eta_0, L\eta_0) \in \mathcal{L}_\epsilon \mathfrak{M}_0$, we have $\tilde{z}_i = (\eta_i, \sqrt{\epsilon_i} \zeta_i) \in \mathfrak{N}_{\epsilon_i}$ and $\tilde{z}_0 = (\eta_0, 0) \in (\mathfrak{M}_0, 0)$ and there hold

$$\|z_1 - z_2\|_{E_{\epsilon_1}} \leq \|\tilde{z}_1 - \tilde{z}_2\|_{E_1} + M_4(\sqrt{\epsilon_1} - \sqrt{\epsilon_2}) \quad (5.15)$$

$$\|z_1 - z_0\|_{E_{\epsilon_1}} \leq \|\tilde{z}_1 - \tilde{z}_0\|_{E_1} + M_5 \sqrt{\epsilon_1}. \quad (5.16)$$

For every $\epsilon \in (0, 1]$, we can check that $\mathfrak{M}_\epsilon$ is an exponential attractor for the operator S_ϵ in $\mathcal{B}_\epsilon$ with respect to the metric induced by the E_ϵ -norm, and (5.5) holds. In the case $\epsilon_1 > 0$ and $\epsilon_2 = 0$ we can say more. To achieve this goal, we follow a different construction devised in [24]. There a construction of exponential attractors $\mathfrak{M}_\epsilon$ for S_ϵ satisfying (i), (ii) and (5.5) with $\epsilon_1 > 0$ and $\epsilon_2 = 0$ was given. On the other hand, a careful examination of the argument shows that the upper semicontinuity estimate (5.6) holds as well. Indeed, thanks to Assumption 1, referring to the proof of [24], if E_n is a $C(\frac{1}{2} + \kappa)^n$ -net of $S_0^n B_0$ in E_0 , then $\mathcal{L}_\epsilon E_n$ is not only a $C'(\frac{1}{2} + \kappa)^{\alpha n}$ -net of $\mathcal{L}_\epsilon S_0^n B_0$ in E_ϵ , but also in E_1 , for some positive constants C and C' . Thus, retracing the steps of the proof of the main theorem in [24] and taking advantage of (5.3), we can deduce (5.6). Finally, let us show the lower semicontinuity estimate (5.7). This is consequence of (5.5) and (5.6). Let us consider an element $z_0 = (u_0, Lu_0) \in \mathcal{L}_\epsilon \mathfrak{M}_0$. On account of (5.5), there exists a sequence $(z_{\epsilon_k})_{k \in \mathbb{N}}$ in $\mathfrak{M}_{\epsilon_k}$, with $\epsilon_k \rightarrow 0$, such that $z_{\epsilon_k} \rightarrow z_0$ in the E_{ϵ_k} -norm, as $k \rightarrow +\infty$. But, due to (5.6) and the fact that $\mathfrak{M}_\epsilon$ is uniformly bounded in V_1 , there also exist $\tilde{z}_0 = (\tilde{u}_0, L\tilde{u}_0) \in \mathcal{L}_\epsilon \mathfrak{M}_0$ and a subsequence, still denoted by $(z_{\epsilon_k})_{k \in \mathbb{N}}$, such that $z_{\epsilon_k} \rightarrow \tilde{z}_0$ in the E_1 -norm, as $k \rightarrow +\infty$. Then, it is clear that $z_0 = \tilde{z}_0$ and the result follows. This completes the proof of Theorem 5.1. $\square$

Remark 5.1. A lower semicontinuity estimate analogous to (5.6) seems out of reach since one has to use the exponential attraction property of $\mathfrak{M}_\epsilon$ that holds in the E_ϵ -norm only. We thus take this opportunity to correct estimate (78) in [2, Theorem 5.2] by replacing the symmetric Hausdorff distance with the Hausdorff semidistance.

We now want to use this result, together with the transitivity of exponentially attraction property, to prove the existence of a family of robust exponential attractors $\mathcal{M}_{\epsilon, \delta}$ for the semigroup $S_\epsilon(t)$ on $\tilde{K}_\delta$ such that $\epsilon \mapsto \mathcal{M}_{\epsilon, \delta}$ is continuous.

Our main result is the following

Theorem 5.2. *Let (2.3) hold. Then, for every $\epsilon \in (0, \epsilon_0]$, the semigroup $S_\epsilon(t)$ possesses an exponential attractor $\mathcal{M}_{\epsilon, \delta}$ in $\widetilde{B_{5, \delta}^0}$ with respect to the metric induced by the $\mathcal{H}_\epsilon^0$ -norm. Besides, the basin of attraction of each $\mathcal{M}_{\epsilon, \delta}$ coincides with the whole phase space $\tilde{K}_\delta$. Furthermore, the fractal dimension of $\mathcal{M}_{\epsilon, \delta}$ has an upper bound which is independent of ϵ and the rate of attraction of trajectories with respect to the metric induced by the $\mathcal{H}_\epsilon^0$ -norm to $\mathcal{M}_{\epsilon, \delta}$ is also uniform in ϵ . The exponential attractors $\mathcal{M}_{\epsilon, \delta}$ are Hölder continuous with respect to ϵ , i.e., for all $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$,*

$$\text{dist}_{\mathcal{H}_{\epsilon_1}^0}^{\text{sym}}(\mathcal{M}_{\epsilon_1, \delta}, \mathcal{M}_{\epsilon_2, \delta}) \leq M_6 |\epsilon_1 - \epsilon_2|^\nu, \quad (5.17)$$

where $M_6 > 0$ and $0 < \nu \leq \frac{1}{4}$ are independent of ϵ_1 and ϵ_2 . In addition, we have

$$\text{dist}_{\mathcal{H}_1^0}(\mathcal{M}_{\epsilon, \delta}, (\mathcal{M}_\delta)_0) \leq M_7 \epsilon^\sigma, \quad (5.18)$$

where the constant $M_7 > 0$ and $0 < \sigma \leq \frac{1}{2}$ are independent of ϵ , and

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{H}_1^0}((\mathcal{M}_\delta)_0, \mathcal{M}_{\epsilon, \delta}) = 0. \quad (5.19)$$

Proof. On account of Theorem 5.1, we let

$$E_\epsilon = \mathcal{H}_\epsilon^0, \quad V_\epsilon = \mathcal{H}_\epsilon^2, \quad W_\epsilon = \mathcal{H}_\epsilon^4,$$

and we check all the assumptions 1–5.

Observe first that there exists $t^* > 0$ such that $S_\epsilon(t)\tilde{B}_{5,\delta} \subset \tilde{B}_{5,\delta}$ for all $t \geq t^*$. Then, arguing as in [22], we can prove that $\tilde{B}_{5,\delta}$ is an exponentially attracting set in $\tilde{K}_\delta$. Thus we can apply Theorem 5.1 with $B_\epsilon = \tilde{B}_{5,\delta}$ and we can extend the basin of attraction to the whole phase-space by using the transitivity of the exponential attraction.

Recalling (2.8), we see that L satisfies Assumption 1. Indeed, there exists a constant c_r such that

$$\|L\rho_1 - L\rho_2\|_{-1} \leq c_r \|\rho_1 - \rho_2\|_3 \leq c_r \|\rho_1 - \rho_2\|_1^{1/2},$$

for any ρ_1 and ρ_2 in $\tilde{B}_\delta$. Observe that, for any $q \in \tilde{B}_\delta$, we have $\|q\|_3 \leq c\|q\|_5^{1/2}\|q\|_1^{1/2}$ and $\|q\|_5 \leq c_r$. We now prove Assumption 2. Let us consider two solutions ρ_1 and ρ_2 to (2.2) with initial conditions

$$\rho_i|_{t=0} = \rho_{0i}, \quad \rho_{it}|_{t=0} = \rho_{1i},$$

$i = 1, 2$, and set $\rho = \rho_1 - \rho_2$, $\tilde{\rho}_0 = \rho_{01} - \rho_{02}$ and $\tilde{\rho}_1 = \rho_{11} - \rho_{12}$. We assume that both trajectories $(\rho_i(t), \rho_{it}(t))_{t \geq 0}$, $i = 1, 2$, lie in $\tilde{B}_{5,\delta}$. We introduce the decomposition $\rho = \rho^1 + \rho^2$, where ρ^1 and ρ^2 are the solutions to the following problems, respectively,

$$\epsilon \rho_{tt}^1 + \rho_t^1 + N(\alpha N \rho^1 + \gamma \rho^1) + C_1^2 \rho^1 = 0, \quad (5.20)$$

$$\rho^1|_{t=0} = \tilde{\rho}_0, \quad \rho_t^1|_{t=0} = \tilde{\rho}_1, \quad (5.21)$$

and

$$\epsilon \rho_{tt}^2 + \rho_t^2 + N(\alpha N \rho^2 + \gamma \rho^2) + C_1^2 \rho^2 = C_1^2 \rho, \quad (5.22)$$

$$\rho^2|_{t=0} = 0, \quad \rho_t^2|_{t=0} = 0, \quad (5.23)$$

where C_1^2 is like in (2.4) and

$$\gamma(t) = \int_0^1 f''(s\rho_1(t) + (1-s)\rho_2(t))ds.$$

We first observe that

$$m(\rho^1(t)) = Ae^{-\frac{1+\sqrt{1-4\epsilon C_1^2}}{2\epsilon}t} + Be^{-\frac{1-\sqrt{1-4\epsilon C_1^2}}{2\epsilon}t}, \quad \forall t \geq 0,$$

$$m(\rho_t^1(t)) = Ce^{-\frac{1+\sqrt{1-4\epsilon C_1^2}}{2\epsilon}t} + De^{-\frac{1-\sqrt{1-4\epsilon C_1^2}}{2\epsilon}t}, \quad \forall t \geq 0,$$

where

$$\begin{aligned} A &= \frac{1}{2} \left(1 - (1 - 4\epsilon C_1^2)^{-1/2} \right) m(\tilde{\rho}_0) - \epsilon (1 - 4\epsilon C_1^2)^{-1/2} m(\tilde{\rho}_1), \\ B &= \frac{1}{2} \left(1 + (1 - 4\epsilon C_1^2)^{-1/2} \right) m(\tilde{\rho}_0) + \epsilon (1 - 4\epsilon C_1^2)^{-1/2} m(\tilde{\rho}_1), \\ C &= C_1^2 (1 - 4\epsilon C_1^2)^{-1/2} m(\tilde{\rho}_0) + \frac{1}{2} \left(1 + (1 - 4\epsilon C_1^2)^{-1/2} \right) m(\tilde{\rho}_1), \\ D &= -C_1^2 (1 - 4\epsilon C_1^2)^{-1/2} m(\tilde{\rho}_0) + \frac{1}{2} \left(1 - (1 - 4\epsilon C_1^2)^{-1/2} \right) m(\tilde{\rho}_1). \end{aligned}$$

On the other hand, since $0 < \epsilon \leq \epsilon_0$, there exist $c_0, c_1 > 0$, independent of ϵ , such that

$$\epsilon m(\rho_t^1(t))^2 + m(\rho^1(t))^2 \leq c_0 e^{-c_1 t} (\epsilon m(\tilde{\rho}_1)^2 + m(\tilde{\rho}_0)^2), \quad \forall t \geq 0.$$

Now, proceeding as in the proof of [22, Lemma 5.3], we can show that

$$\|(\rho^1(t), \rho_t^1(t))\|_{\mathcal{H}_\epsilon^0}^2 \leq c_r e^{-c_r t} \|(\tilde{\rho}_0, \tilde{\rho}_1)\|_{\mathcal{H}_\epsilon^0}^2, \quad \forall t \geq 0, \quad (5.24)$$

and

$$\|(\rho^2(t), \rho_t^2(t))\|_{\mathcal{H}_\epsilon^2}^2 \leq c'_r e^{c'_r t} \|(\tilde{\rho}_0, \tilde{\rho}_1)\|_{\mathcal{H}_\epsilon^0}^2, \quad \forall t \geq 0, \quad (5.25)$$

where c_r and c'_r depend on r , but not on ϵ . Estimates (5.24) and (5.25) entail Assumption 2, provided we choose $t^* > 0$ large enough in order to ensure that $c_r e^{-c_r t^*} < \frac{1}{2}$. Note that t^* can be also chosen in such a way that (4.6) and (4.34) hold so that Assumptions 3 follows.

Thus we are left to prove Assumptions 4 and 5. Indeed, defining

$$(\rho(t), \rho_t(t)) = S_\epsilon(t)(\rho_{01}, \rho_{11}), \quad (\rho(t'), \rho_t(t')) = S_\epsilon(t')(\rho_{02}, \rho_{12}),$$

with $(\rho_{01}, \rho_{11}), (\rho_{02}, \rho_{12}) \in \tilde{B}_{5,\delta}$, and $t, t' \in [t^*, 2t^*]$, we obtain

$$\begin{aligned} &\|S_\epsilon(t)(\rho_{01}, \rho_{11}) - S_\epsilon(t')(\rho_{02}, \rho_{12})\|_{\mathcal{H}_\epsilon^0} \\ &\leq \|S_\epsilon(t)(\rho_{01}, \rho_{11}) - S_\epsilon(t')(\rho_{01}, \rho_{11})\|_{\mathcal{H}_\epsilon^0} \\ &\quad + \|S_\epsilon(t')(\rho_{01}, \rho_{11}) - S_\epsilon(t')(\rho_{02}, \rho_{12})\|_{\mathcal{H}_\epsilon^0}. \end{aligned} \quad (5.26)$$

On the one hand, there holds

$$\begin{aligned} &\|S_\epsilon(t)(\rho_{01}, \rho_{11}) - S_\epsilon(t')(\rho_{01}, \rho_{11})\|_{\mathcal{H}_\epsilon^0} \\ &\leq c (\|\rho(t) - \rho(t')\|_1 + \sqrt{\epsilon} \|\rho_t(t) - \rho_t(t')\|_{-1}) \\ &\leq c \int_t^{t'} \|\rho_t(s)\|_1 ds + c \int_t^{t'} \sqrt{\epsilon} \|\rho_{tt}(s)\|_{-1} ds \\ &\leq c(r, t^*) |t' - t|. \end{aligned} \quad (5.27)$$

On the other hand, it follows from (5.24)–(5.25) that

$$\begin{aligned} &\|S_\epsilon(t')(\rho_{01}, \rho_{11}) - S_\epsilon(t')(\rho_{02}, \rho_{12})\|_{\mathcal{H}_\epsilon^0} \\ &\leq c(r, t^*) \|(\rho_{01}, \rho_{11}) - (\rho_{02}, \rho_{12})\|_{\mathcal{H}_\epsilon^0}, \quad \forall t' > 0. \end{aligned} \quad (5.28)$$

Hence, we conclude with

$$\begin{aligned} & \|S_\epsilon(t)(\rho_{01}, \rho_{11}) - S_\epsilon(t')(\rho_{02}, \rho_{12})\|_{\mathcal{H}_\epsilon^0} \\ & \leq c(r, t^*)(|t' - t| + \|(\rho_{01}, \rho_{11}) - (\rho_{02}, \rho_{12})\|_{\mathcal{H}_\epsilon^0}). \end{aligned} \quad (5.29)$$

This finishes the proof. $\square$

We conclude this section by proving the upper semicontinuity of the global attractor $\mathcal{A}_{\epsilon, \delta}$ at any $\epsilon_2 > 0$ in the $\mathcal{H}_{\epsilon_1}^0$ -norm, for all $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$, and at $\epsilon_2 = 0$ in the $\mathcal{H}_1^0$ -norm.

Theorem 5.3. *Let the assumptions of Theorem 5.2 hold. Then the global attractors $\mathcal{A}_{\epsilon, \delta}$ for the semigroup $\{S_\epsilon(t)\}_{t \geq 0}$ are upper semicontinuous at any $\epsilon_2 > 0$ with respect to the metric induced by the $\mathcal{H}_{\epsilon_1}^0$ -norm and at $\epsilon_2 = 0$ with respect to the metric induced by the $\mathcal{H}_1^0$ -norm, that is, for all $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$,*

$$\lim_{\epsilon_1 \rightarrow \epsilon_2} \text{dist}_{\mathcal{H}_{\epsilon_1}^0}(\mathcal{A}_{\epsilon_1, \delta}, \mathcal{A}_{\epsilon_2, \delta}) = 0 \quad (5.30)$$

and

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{H}_1^0}(\mathcal{A}_{\epsilon, \delta}, (\mathcal{A}_\delta)_0) = 0. \quad (5.31)$$

Proof. In order to prove (5.30), we use the auxiliary operators $\tilde{S}_\epsilon$ introduced in the proof of Theorem 5.1 once again. We denote by $\mathfrak{A}_{\epsilon, \delta}$ the global attractor generated by this semigroup with respect to the metric induced by the $\mathcal{H}_1^0$ -norm. We have $\mathcal{A}_{\epsilon, \delta} = C_\epsilon \mathfrak{A}_{\epsilon, \delta}$, for any $0 < \epsilon \leq \epsilon_0$, and

$$\text{dist}_{\mathcal{H}_1^0}(\tilde{S}_\epsilon(t)B_1, \mathfrak{A}_{\epsilon, \delta}) \rightarrow 0, \quad \text{when } t \rightarrow +\infty. \quad (5.32)$$

Furthermore,

$$\|\tilde{S}_{\epsilon_1}(t)(u, v) - \tilde{S}_{\epsilon_2}(t)(u, v)\|_{\mathcal{H}_1^0}^2 \leq c(r, t)|\epsilon_1 - \epsilon_2|^{1/2}, \quad \forall t \geq t_*, \quad (5.33)$$

for any $(u, v) \in B_{5, \delta}^1$ (cf. (4.34) and (5.11)). We are now ready to show that

$$\lim_{\epsilon_1 \rightarrow \epsilon_2} \text{dist}_{\mathcal{H}_1^0}(\mathfrak{A}_{\epsilon_1, \delta}, \mathfrak{A}_{\epsilon_2, \delta}) = 0. \quad (5.34)$$

To do so, let $(u, v) \in \mathfrak{A}_{\epsilon_1, \delta}$. Recalling that $\mathfrak{A}_{\epsilon_1, \delta} \subset B_{5, \delta}^1$, we have (cf. (5.33))

$$\|\tilde{S}_{\epsilon_1}(t)(u, v) - \tilde{S}_{\epsilon_2}(t)(u, v)\|_{\mathcal{H}_1^0}^2 \leq c(r, t)|\epsilon_1 - \epsilon_2|^{1/2}, \quad \forall t \geq t_*. \quad (5.35)$$

The definition of the global attractor $\mathfrak{A}_{\epsilon_2, \delta}$ implies

$$\text{dist}_{\mathcal{H}_1^0}(\tilde{S}_{\epsilon_2}(t)(u, v), \mathfrak{A}_{\epsilon_2, \delta}) \rightarrow 0, \quad \text{when } t \rightarrow +\infty. \quad (5.36)$$

This shows that, if $\eta > 0$, then there exist (φ, ϕ) belonging to $\mathfrak{A}_{\epsilon_2, \delta}$ and $t_\eta \geq t_*$ depending only on η such that

$$\|\tilde{S}_{\epsilon_2}(t_\eta)(u, v) - (\varphi, \phi)\|_{\mathcal{H}_1^0}^2 \leq \frac{\eta^2}{2}. \quad (5.37)$$

We now choose ϵ_η (which only depends on η) such that $c(r, t_\eta)|\epsilon_\eta - \epsilon_2|^{1/2} \leq \eta^2/2$. For any $\epsilon_2 < \epsilon_1 \leq \epsilon_\eta$, we have, on account of (5.35),

$$\|\tilde{S}_{\epsilon_1}(t_\eta)(u, v) - \tilde{S}_{\epsilon_2}(t_\eta)(u, v)\|_{\mathcal{H}_1^0}^2 \leq \frac{\eta^2}{2}. \quad (5.38)$$

Hence, we deduce from (5.37) and (5.38) that

$$\|\tilde{S}_{\epsilon_1}(t_\eta)(u, v) - (\varphi, \phi)\|_{\mathcal{H}_1^0}^2 \leq \eta^2. \quad (5.39)$$

This result is uniform with respect to (u, v) , so that

$$\text{dist}_{\mathcal{H}_{\epsilon_2}^0}(\tilde{S}_{\epsilon_1}(t_\eta)\mathfrak{A}_{\epsilon_1, \delta}, \mathfrak{A}_{\epsilon_2, \delta}) \leq \eta, \quad (5.40)$$

whenever $\epsilon_2 < \epsilon_1 \leq \epsilon_\eta$. Finally, the invariance property of $\mathfrak{A}_{\epsilon_1, \delta}$ entails that $\tilde{S}_{\epsilon_1}(t_\eta)\mathfrak{A}_{\epsilon_1, \delta} = \mathfrak{A}_{\epsilon_1, \delta}$. This completes the proof of (5.34). Recalling now (5.15), it is clear that

$$\text{dist}_{\mathcal{H}_{\epsilon_1}^0}(\mathcal{A}_{\epsilon_1, \delta}, \mathcal{A}_{\epsilon_2, \delta}) \leq \text{dist}_{\mathcal{H}_1^0}(\mathfrak{A}_{\epsilon_1, \delta}, \mathfrak{A}_{\epsilon_2, \delta}) + M_4(\sqrt{\epsilon_1} - \sqrt{\epsilon_2}), \quad (5.41)$$

and, passing to the limit $\epsilon_1 \rightarrow \epsilon_2$, (5.30) follows. As far as the proof of (5.31) is concerned, we need not use the auxiliary operators $\tilde{S}_\epsilon$. A direct proof is available, as it was the case for (5.6). Indeed, let $(\rho_0, \rho_1) \in \mathcal{A}_{\epsilon, \delta}$. Recalling that $\mathcal{A}_{\epsilon, \delta} \subset \tilde{B}_{5, \delta}$, the following estimate holds (cf. (4.6)):

$$\|S_\epsilon(t)(\rho_0, \rho_1) - S_0(t)(\rho_0, L\rho_0)\|_{\mathcal{H}_1^0}^2 \leq c(r, t)\epsilon, \quad \forall t \geq t_\star. \quad (5.42)$$

Thus, owing to the definition of the global attractor $(\mathcal{A}_\delta)_0$,

$$\text{dist}_{\mathcal{H}_1^0}(S_0(t)(\rho_0, L\rho_0), (\mathcal{A}_\delta)_0) \rightarrow 0, \quad \text{when } t \rightarrow +\infty. \quad (5.43)$$

This shows that, if $\eta > 0$, then there exist (φ, ϕ) belonging to $(\mathcal{A}_\delta)_0$ and $t_\eta \geq t_\star$ depending only on η such that

$$\|S_0(t_\eta)(\rho_0, L\rho_0) - (\varphi, \phi)\|_{\mathcal{H}_1^0}^2 \leq \frac{\eta^2}{2}. \quad (5.44)$$

We now choose ϵ_η (which only depends on η) such that $c(r, t_\eta)\epsilon_\eta \leq \eta^2/2$. For any $0 < \epsilon \leq \epsilon_\eta$, we have, on account of (5.42),

$$\|S_\epsilon(t_\eta)(\rho_0, \rho_1) - S_0(t_\eta)(\rho_0, L\rho_0)\|_{\mathcal{H}_1^0}^2 \leq \frac{\eta^2}{2}. \quad (5.45)$$

Hence, we deduce from (5.44) and (5.45) that

$$\|S_\epsilon(t_\eta)(\rho_0, \rho_1) - (\varphi, \phi)\|_{\mathcal{H}_1^0}^2 \leq \eta^2. \quad (5.46)$$

This result is uniform with respect to (ρ_0, ρ_1) , so that

$$\text{dist}_{\mathcal{H}_1^0}(S_\epsilon(t_\eta)\mathcal{A}_{\epsilon, \delta}, (\mathcal{A}_\delta)_0) \leq \eta, \quad (5.47)$$

whenever $0 < \epsilon \leq \epsilon_\eta$. Finally, the invariance property of $\mathcal{A}_{\epsilon, \delta}$ entails that $S_\epsilon(t_\eta)\mathcal{A}_{\epsilon, \delta} = \mathcal{A}_{\epsilon, \delta}$. This completes the proof of (5.31) (cf. also [2, Theorem 6.1]). $\square$

6. Further results on the global attractors

This section is devoted to the study of the structure of the global attractors and their stability properties with respect to ϵ under suitable assumptions on the nonconstant equilibria.

We start by showing the following result.

Proposition 6.1. *The semigroup $\{S_\epsilon(t)\}_{t \geq 0}$ is a (strongly) continuous semigroup of class C^1 , for any $\epsilon \in (0, \epsilon_0]$. Furthermore, $S_\epsilon = S_\epsilon(t^*)$ defines a C^1 –mapping from $\mathcal{H}_\epsilon^0$ into $\mathcal{H}_\epsilon^0$. In addition to (4.34), there hold*

$$\|S'_\epsilon(\rho_0, \rho_1)(s_0, s_1) - S'_0(\rho_0, L\rho_0)(s_0, \tilde{L}s_0)\|_{\mathcal{H}_1^0} \leq c(r, t^*)\epsilon^{1/2} \quad (6.1)$$

and

$$\begin{aligned} & \left\| S'_{\epsilon_1}(\rho_0, \rho_1)(s_0, s_1) - S'_{\epsilon_2} \left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \rho_1 \right) \left(s_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} s_1 \right) \right\|_{\mathcal{H}_1^0} \\ & \leq c(r, t^*)|\epsilon_1 - \epsilon_2|^{1/4}, \end{aligned} \quad (6.2)$$

for any $(\rho_0, \rho_1) \in \tilde{B}_5^{\epsilon_1}$ ($\tilde{B}_5^\epsilon$ for (6.1)), any $(s_0, s_1) \in \mathcal{H}_{\epsilon_1}^0$ such that $\|(s_0, s_1)\|_{\mathcal{H}_1^0} \leq c_r$, any $0 < \epsilon \leq \epsilon_0$ and any $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$.

Proof. Observe that

$$S'_\epsilon(t)(\rho_0, \rho_1)(s_0, s_1) = (s(t), s_t(t))$$

is defined by the problem

$$\epsilon s_{tt} + s_t + N(\alpha N s + s f''(\rho)) = 0, \quad (6.3)$$

$$s|_{t=0} = s_0, \quad s_t|_{t=0} = s_1, \quad (6.4)$$

where $(\rho, \rho_t) = S_\epsilon(t)(\rho_0, \rho_1)$. Similarly, $S'(t)(\rho_0)s_0 = s(t)$ is the solution to the problem

$$s_t + N(\alpha N s + s f''(\rho)) = 0, \quad (6.5)$$

$$s|_{t=0} = s_0, \quad (6.6)$$

where $\rho = S(t)\rho_0$. We now consider the difference of the solutions to (6.3) and (6.5), $Q = s_\epsilon - s$, which satisfies the following equation:

$$\epsilon Q_{tt} + Q_t + N(\alpha N Q + s^\epsilon f''(\rho^\epsilon) - s f''(\rho)) = -\epsilon s_{tt}, \quad (6.7)$$

$$Q|_{t=0} = 0, \quad Q_t|_{t=0} = s_1 - \tilde{L}s_0, \quad (6.8)$$

where $\tilde{L}s_0$ is defined like L (see (2.8)), but via the linearized equation. We note that

$$\|N^{1/2}(s^\epsilon f''(\rho^\epsilon) - s f''(\rho))\| \leq c(\|Q\|_{H^1(\Omega)} + 1),$$

for any $(\rho_0, \rho_1) \in \tilde{B}_5$. Proceeding as in Section 4, we find (6.1), where

$$S'_0(t)(\rho_0, L\rho_0)(s_0, \tilde{L}s_0) = (s(t), \tilde{L}s(t)), \quad \forall t \geq 0.$$

We now consider the difference of two solutions to (6.3)

$$\begin{aligned} (s^{\epsilon_1}(t), s_t^{\epsilon_1}(t)) &= S'_{\epsilon_1}(t)(\rho_0, \rho_1)(s_0, s_1), \\ (s^{\epsilon_2}(t), s_t^{\epsilon_2}(t)) &= S'_{\epsilon_2}(t) \left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \rho_1 \right) \left(s_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} s_1 \right). \end{aligned} \quad (6.9)$$

The function $Q = s_{\epsilon_1} - s_{\epsilon_2}$ satisfies the following equation:

$$\epsilon_2 Q_{tt} + Q_t + N(\alpha N Q + s^{\epsilon_1} f''(\rho^{\epsilon_1}) - s^{\epsilon_2} f''(\rho^{\epsilon_2})) = -\epsilon s_{tt}^{\epsilon_1}, \quad (6.10)$$

$$Q|_{t=0} = 0, \quad Q_t|_{t=0} = \left(1 - \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \right) s_1, \quad (6.11)$$

where $\epsilon = \epsilon_1 - \epsilon_2$. We can also show that

$$\begin{aligned} & \left\| S'_{\epsilon_1}(t)(\rho_0, \rho_1)(s_0, s_1) - S'_{\epsilon_2}(t) \left(\rho_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \rho_1 \right) \left(s_0, \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} s_1 \right) \right\|_{\mathcal{H}_1^0}^2 \\ & \leq \epsilon \epsilon_1^{-1} c(r, t), \quad \forall t \geq t^*, \end{aligned} \quad (6.12)$$

for any $(\rho_0, \rho_1) \in \tilde{B}_5^{\epsilon_1}$ and any $(s_0, s_1) \in \mathcal{H}_{\epsilon_1}^0$ such that $\|(s_0, s_1)\|_{\mathcal{H}_1^0} \leq c_r$. We conclude as in the proof of Proposition 4.3, on account of (6.1) and (6.12). This completes the proof. $\square$

Observe now that $\{S_0(t)\}_{t \geq 0}$ has a Lyapunov functional

$$J_0(u, v) = \int_{\Omega} \left(\frac{\alpha}{2} |\nabla u|^2 + f(u) \right) dx$$

on $(\mathcal{A}_\delta)_0$. On the other hand, for every $0 < \epsilon \leq \epsilon_0$, the semigroup $\{S_\epsilon(t)\}_{t \geq 0}$ possesses a Lyapunov functional

$$J_\epsilon(u, v) = \int_{\Omega} \left(\frac{\alpha}{2} |\nabla u|^2 + f(u) + \frac{\epsilon}{2} |N^{-1/2} v|^2 \right) dx$$

on $\mathcal{A}_{\epsilon, \delta}$. Let us remark that, if $(u, v) \in \mathcal{A}_{\epsilon, \delta}$, there exists a complete orbit $(\rho(t), \rho_t(t))_{t \in \mathbb{R}}$ in $\mathcal{A}_{\epsilon, \delta}$ such that $\rho(0) = u$ and $\rho_t = v$. We then have $m(\rho_t(t)) = m(v)e^{-t/\epsilon}$ and, therefore, $m(v) = m(\rho_t(t))e^{t/\epsilon}$, for all $t \in \mathbb{R}$. Since $m(\rho_t(t))$ is bounded, we have $m(v) = 0$ by letting $t \rightarrow -\infty$. Therefore, $m(\rho)$ is independent of t .

Setting

$$\begin{aligned} K_\delta^\mu &= \{u \in K_\delta, m(u) = \mu\}, \quad \tilde{K}_\delta^\mu = \{(u, 0) \in \tilde{K}_\delta, m(u) = \mu\}, \\ \mathcal{A}_\delta^\mu &= \mathcal{A}_\delta \cap K_\delta^\mu, \quad \mathcal{A}_{\epsilon, \delta}^\mu = \mathcal{A}_{\epsilon, \delta} \cap \tilde{K}_\delta^\mu, \end{aligned}$$

it is clear that

$$\mathcal{A}_\delta = \bigcup_{|\mu| \leq \delta} \mathcal{A}_\delta^\mu$$

and

$$\mathcal{A}_{\epsilon, \delta} = \bigcup_{|\mu| \leq \delta} \mathcal{A}_{\epsilon, \delta}^\mu.$$

Moreover, $\mathcal{A}_\delta^\mu$ and $\mathcal{A}_{\epsilon, \delta}^\mu$ are global attractors for the semigroups $S(t)$ and $S_\epsilon(t)$ restricted to K_δ^μ and $\tilde{K}_\delta^\mu$, respectively, which will be denoted by $S^\mu(t)$ and $S_\epsilon^\mu(t)$. We consider the set of stationary solutions to (2.1) (with respect to S_0) subject to the additional condition $m(\rho_0) = \mu$

$$Z(S_0^\mu) = \{(z, 0) \in \mathcal{H}_\epsilon^1, Gz = N(\alpha N z + f'(z)) = 0, m(z) = \mu\}$$

(cf. [40]) and we recall that an element $(z, 0)$ of $Z(S_0^\mu)$ is hyperbolic if the differential operator $G'(z) : \dot{H}_{\text{per}}^2(\Omega) \rightarrow L^2(\Omega)$ defined by $G'(z)v = N(\alpha N v + v f''(z))$ is invertible. We also note that, if $(z, 0) \in Z(S_0^\mu)$ is hyperbolic, then it is isolated. Also, note that $Z(S_0^\mu)$ contains the trivial solution $\rho(t) = \mu$.

A straightforward application of known results (see, e.g., [30, Sect. 3.8]) yields the following.

Theorem 6.1. Assume that $\mu \in [-\delta, \delta]$ and that all the nontrivial stationary solutions to $\mathbf{P}_0$ subject to the condition $m(\rho_0) = \mu$ are hyperbolic. Then the global attractors $\mathcal{A}_{\epsilon, \delta}^\mu$ are lower semicontinuous at any $\epsilon \in [0, \epsilon_0]$. Therefore, the global attractors $\mathcal{A}_{\epsilon, \delta}^\mu$ are continuous at any ϵ . Moreover, there hold

$$\text{dist}_{\mathcal{H}_1^0}^{\text{sym}}(\mathcal{A}_{\epsilon_1, \delta}^\mu, \mathcal{A}_{\epsilon_2, \delta}^\mu) \leq M_8 |\epsilon_1 - \epsilon_2|^\varpi, \quad (6.13)$$

for all $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$, and

$$\text{dist}_{\mathcal{H}_1^0}^{\text{sym}}(\mathcal{A}_{\epsilon, \delta}^\mu, (\mathcal{A}_\delta^\mu)_0) \leq M_9 \epsilon^\kappa, \quad (6.14)$$

for every $0 < \epsilon \leq \epsilon_0$, where the constants $M_8, M_9 > 0$ and $0 < \varpi \leq \frac{1}{4}, 0 < \kappa \leq \frac{1}{2}$ are independent of ϵ .

Proof. Since $Z(S_0^\mu) \subset (\mathcal{A}_\delta)_0$ is compact in $\mathcal{H}_\epsilon^1$, then $Z(S_0^\mu)$ is a finite set consisting of, say, m elements. Observe that, if $(z, 0)$ is a hyperbolic stationary solution to (2.1) with respect to S_0^μ , then $(z, 0)$ is a hyperbolic stationary solution to (2.2). If all the stationary solutions are hyperbolic, it is well known that the global attractors $(\mathcal{A}_\delta)_0$ and $\mathcal{A}_{\epsilon, \delta}^\mu$ are the union of the unstable manifolds with respect to S_0^μ and S_ϵ^μ of $(z_i, 0), i = 1, \dots, m$, respectively (see [29] for details). Moreover, for every $\epsilon \in (0, \epsilon_0]$, there exist local unstable manifolds $\mathcal{W}(S_\epsilon^\mu|_{\mathcal{O}_{(z_i, 0)}})$ of S_ϵ^μ at $(z_i, 0)$ and a corresponding local unstable manifold $\mathcal{W}(S_0^\mu|_{\mathcal{O}_{(z_i, 0)}})$ of S_0^μ at $(z_i, 0)$ such that the global attractors are represented, respectively, as

$$(\mathcal{A}_\delta)_0 = \bigcup_{i=1}^m \bigcup_{j=0}^{+\infty} (S_0^\mu)^j \mathcal{W}(S_0^\mu|_{\mathcal{O}_{(z_i, 0)}}), \quad \mathcal{A}_{\epsilon, \delta}^\mu = \bigcup_{i=1}^m \bigcup_{j=0}^{+\infty} (S_\epsilon^\mu)^j \mathcal{W}(S_\epsilon^\mu|_{\mathcal{O}_{(z_i, 0)}}).$$

In addition, for any $(z_i, 0), i = 1, \dots, m$, there exist a sufficiently small $\tilde{\epsilon}_1$ and a neighborhood $\mathcal{O}_{(z_i, 0)}$, chosen independently of $\epsilon, 0 < \epsilon \leq \tilde{\epsilon}_1$, such that the local unstable manifolds are graphs, that is,

$$\begin{aligned} \mathcal{W}(S_0^\mu|_{\mathcal{O}_{(z_i, 0)}}) &= \{(u, v) \in \mathcal{H}_\epsilon^1 : (u, v) \\ &= (u^+, v^+) + g_0(u^+, v^+), (u^+, v^+) \in \mathcal{O}_{(z_i, 0)} \cap E_+(z_i, 0)\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}(S_\epsilon^\mu|_{\mathcal{O}_{(z_i, 0)}}) &= \{(u, v) \in \mathcal{H}_\epsilon^1 : (u, v) \\ &= (u^+, v^+) + g_\epsilon(u^+, v^+), (u^+, v^+) \in \mathcal{O}_{(z_i, 0)} \cap E_+(z_i, 0)\}, \end{aligned}$$

where $g_0, g_\epsilon : \mathcal{O}_{(z_i, 0)} \cap E_+(z_i, 0) \rightarrow E_-(z_i, 0), g_0(z_i, 0) = 0$ and $g_\epsilon(z_i, 0) = 0$. Note that $E_+(z_i, 0)$ and $E_-(z_i, 0)$ are the invariant linear subspaces corresponding to subsets of the spectrum of $(S_\epsilon^\mu)'(z_i, 0)$ which are in the domains $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ and $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, respectively. These are independent of time and the dimension of E_+ is finite. Moreover, we have $E = E_+(z_i, 0) \oplus E_-(z_i, 0)$ (cf. [2, Definition 6.4]). It follows from [44, Theorem 1] that the mappings g_0, g_ϵ and their differentials g'_0, g'_ϵ satisfy estimates of the

form (4.6)–(4.34) and (6.1)–(6.2), respectively. This result leads to the following comparison estimates:

$$\begin{aligned} \text{dist}_{\mathcal{H}_1^1}^{\text{sym}}(\mathcal{W}(S_\epsilon^\mu |_{\mathcal{O}_{(z_i,0)}}), \mathcal{W}(S_0^\mu |_{\mathcal{O}_{(z_i,0)}})) &\leq c_0 \sqrt{\epsilon}, \\ \text{dist}_{\mathcal{H}_1^0}^{\text{sym}}(\mathcal{W}(S_{\epsilon_1} |_{\mathcal{O}_{(z_i,0)}}), \mathcal{W}(S_{\epsilon_2} |_{\mathcal{O}_{(z_i,0)}})) &\leq c_0 |\epsilon_1 - \epsilon_2|^{1/4}, \end{aligned}$$

where c_0 is independent of ϵ, ϵ_1 and ϵ_2 . Since there is a finite number of stationary solutions $(z_i, 0)$, we can compare the whole global attractors $(\mathcal{A}_\delta^\mu)_0$ and $\mathcal{A}_{\epsilon, \delta}^\mu$ (compare with [41]). $\square$

Remark 6.1. We can observe that the global attractors $\mathcal{A}_\delta$ and $\mathcal{A}_{\epsilon, \delta}$ have continua of stationary solutions, due to the trivial solutions. Thus, we cannot expect to prove the lower semicontinuity in that case. Once again, we take this opportunity to correct estimate (99) in [2, Theorem 6.2] by replacing $\mathcal{A}_\delta$ and $\mathcal{A}_{\epsilon, \delta}$ by $\mathcal{A}_\delta^\mu$ and $\mathcal{A}_{\epsilon, \delta}^\mu$, respectively.

7. Inertial manifolds

We first give the definition of an inertial manifold (see, e.g., [15, 33, 43]).

Definition 7.1. Let E be a metric space and $\{S(t)\}_{t \geq 0}$ be a continuous semigroup on E . A set $\mathfrak{M}$ is called an inertial manifold for $S(t)$ if:

- (i) $\mathfrak{M}$ is a finite dimensional Lipschitz manifold in E ;
- (ii) $\mathfrak{M}$ is smooth, that is, $\mathfrak{M}$ is of class $\mathcal{C}^1$;
- (iii) $\mathfrak{M}$ is positively invariant under the flow, that is, $S(t)\mathfrak{M} \subset \mathfrak{M}$, $\forall t \geq 0$;
- (iv) $\mathfrak{M}$ is exponentially attracting, that is, there exists a constant c_0 such that, for every $u_0 \in E$, there exists a constant $c_1(u_0) > 0$ such that

$$\text{dist}_E(S(t)u_0, \mathfrak{M}) \leq c_1 e^{-c_0 t}, \quad \forall t \geq 0.$$

We know that problem (2.1) possesses a bounded absorbing set $\tilde{B}_\delta$ in $H_{\text{per}}^5(\Omega)$. Thus we can truncate the nonlinear term for large $\|\rho\|_5$. This allows to construct an inertial manifold which is globally realized as a graph for the semigroup generated by the modified equation and also by the original equation (see [45, p. 875]). Then, since the resulting modified equation is identical to the original one within the absorbing set, the intersection of this graph with the absorbing set also defines an inertial manifold (note that it is also an exponential attractor) for the original equation (cf., e.g., [15, 31]).

We now define a truncated function by setting

$$g(\rho) = \theta \left(\frac{\|\rho\|_5}{r} \right) f'(\rho), \quad (7.1)$$

where r is related with the absorbing set B_δ and is defined in Sect. 2 and $\theta : \mathbb{R}^+ \rightarrow [0, 1]$ is a $\mathcal{C}^\infty$ function such that

$$\theta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq 1, \\ 0, & \text{if } s > 2, \end{cases}$$

and

$$|\theta'(s)| \leq 2, \quad \forall s \geq 0.$$

We now consider an equation which is “equivalent” to the original equation for t large, namely,

$$\rho_t + N(N\rho + g(\rho)) = 0. \quad (7.2)$$

Then the following result holds.

Theorem 7.1. *Let the assumptions of Theorem 5.2 hold. Then there exists an inertial manifold $\mathfrak{M}_\delta$ for the semigroup $S(t)$ generated by equation (2.1) on K_δ .*

Proof. It is well known that there exists an inertial manifold $\mathfrak{M}_\delta$ for the semigroup $\tilde{S}(t)$ generated by the modified equation (7.2) on K_δ (see [38, 43]). On the other hand, arguing as in [45], one can show that $\mathfrak{M}_\delta$ is also an inertial manifold for the semigroup $S(t)$ generated by equation (2.1) on K_δ . $\square$

We now consider the following system which is “equivalent” to the original equation for t large:

$$\epsilon\rho_{tt} + \rho_t + N(N\rho + \mathfrak{g}(\rho)) = 0, \quad (7.3)$$

where $\mathfrak{g}$ is defined like g (see (7.1)) and where r is related with the absorbing set $B_{5,\delta}$. In this case, we can prove the

Theorem 7.2. *Let the assumptions of Theorem 5.2 hold. Then, for any $\epsilon \in (0, \epsilon_0]$, there exists an inertial manifold $\mathfrak{M}_{\epsilon,\delta}$ for the semigroup $S_\epsilon(t)$ generated by equation (2.2) on $\tilde{K}_\delta$ of the same dimension as $\mathfrak{M}_\delta$ (cf. Theorem 7.1).*

Proof. This result was proven in detail in [3, Theorem 5.1] for a singularly perturbed viscous Cahn–Hilliard–Gurtin equation in one and two space dimension. We can thus follow the same steps with minor modifications.

First, we introduce the following change of variables: $\rho_t = -(2\epsilon)^{-1}\rho + \epsilon^{-1/2}v$ and the auxiliary problem:

$$U_t + \mathcal{A}U + \mathcal{G}(U) = 0, \quad (7.4)$$

where

$$U = \begin{pmatrix} \rho \\ v \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \frac{1}{\sqrt{\epsilon}}N^2 - \frac{1}{4\epsilon\sqrt{\epsilon}}I & -\frac{1}{\sqrt{\epsilon}} \\ \frac{1}{\sqrt{\epsilon}}N\mathfrak{g}(\rho) & 0 \end{pmatrix}, \quad \mathcal{G}(U) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{\epsilon}}N\mathfrak{g}(\rho) \end{pmatrix}.$$

Then, we show that, for any $\epsilon \in (0, \epsilon_0]$, there exists an inertial manifold $\mathfrak{M}_{\epsilon,\delta}$ for the semigroup generated by (7.4) on $\tilde{K}_\delta$ of the same dimension as $\mathfrak{M}_\delta$ (cf. Theorem 7.1).

Now, for all $\epsilon \in (0, \epsilon_0]$, we introduce the auxiliary semigroups $T_\epsilon(t) : \mathcal{H}_\epsilon^0 \rightarrow \mathcal{H}_\epsilon^0$ defined by

$$T_\epsilon(t)(\rho_0, \rho_1) = \left(\rho(t), \sqrt{\epsilon}\rho_t(t) + \frac{1}{2\sqrt{\epsilon}}\rho(t) \right), \quad \forall t \geq 0,$$

where ρ is the solution to problem (7.3) with the initial conditions

$$\rho|_{t=0} = \rho_0, \quad \rho_t|_{t=0} = -\frac{1}{2\epsilon}\rho_0 + \frac{1}{\sqrt{\epsilon}}\rho_1.$$

Introducing the matrix

$$O_\epsilon = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2\epsilon} & \frac{1}{\sqrt{\epsilon}} \end{pmatrix},$$

we can also write

$$\tilde{S}_\epsilon(t) = O_\epsilon \circ T_\epsilon(t) \circ O_\epsilon^{-1}, \quad \forall \epsilon > 0.$$

Clearly, $\mathfrak{N}_{\epsilon,\delta}$ is an inertial manifold for $T_\epsilon(t)$, for every $0 < \epsilon \leq \epsilon_0$, with respect to the metric induced by a norm $\|\cdot\|_{\mathcal{H}_\epsilon^0}$ equivalent to $\|\cdot\|_{\mathcal{H}_\epsilon^0}$. We set

$$\mathfrak{M}_{\epsilon,\delta} = O_\epsilon \mathfrak{N}_{\epsilon,\delta}, \quad \forall \epsilon \in (0, \epsilon_0].$$

For every $\epsilon \in (0, \epsilon_0]$, $\mathfrak{M}_{\epsilon,\delta}$ is an inertial manifold for the semigroup $\tilde{S}_\epsilon(t)$ generated by (7.3) on $\tilde{K}_\delta$ with respect to the metric induced by the norm $\|\cdot\|_{\mathcal{H}_\epsilon^0} = \|C_\epsilon^{-1} \cdot\|_{\mathcal{H}_\epsilon^0}$. Indeed, we have, by definition,

$$\text{dist}_{\mathcal{H}_\epsilon^0}(\tilde{S}_\epsilon(t)B_\epsilon, \mathfrak{M}_\epsilon) = \text{dist}_{\mathcal{H}_\epsilon^0}(T_\epsilon B_1, \mathfrak{N}_\epsilon), \quad (7.5)$$

$$\dim_{\mathcal{H}_\epsilon^0} \mathfrak{M}_\epsilon = \dim_{\mathcal{H}_\epsilon^0} \mathfrak{N}_\epsilon, \quad (7.6)$$

and

$$\tilde{S}_\epsilon(t)\mathfrak{M}_\epsilon = O_\epsilon T_\epsilon(t)O_\epsilon^{-1}\mathfrak{M}_\epsilon = O_\epsilon T_\epsilon(t)\mathfrak{N}_\epsilon \subset O_\epsilon \mathfrak{N}_\epsilon = \mathfrak{M}_\epsilon, \quad \forall t \geq 0. \quad (7.7)$$

Finally, arguing as in [45], we deduce that $\mathfrak{M}_{\epsilon,\delta}$ is also an inertial manifold for the semigroup $S_\epsilon(t)$ generated by the original equation (2.2) on $\tilde{K}_\delta$. This completes the proof of the theorem. $\square$

We observe that the inertial manifolds $\mathfrak{M}_\delta$ and $\mathfrak{M}_{\epsilon,\delta}$, given by Theorems 7.1 and 7.2, respectively, are not only positively invariant, but also satisfy $\tilde{S}(t)\mathfrak{M}_\delta = \mathfrak{M}_\delta$ and $\tilde{S}_\epsilon(t)\mathfrak{M}_{\epsilon,\delta} = \mathfrak{M}_{\epsilon,\delta}, \forall t \in \mathbb{R}$ (see, e.g., [8, 31]). We also note that the lifting $(\mathfrak{M}_\delta)_0$ is an inertial manifold for the semigroups $\{\tilde{S}_0(t)\}_{t \geq 0}$ and $\{S_0(t)\}_{t \geq 0}$ on $\mathbb{M}$. We now set

$$\mathfrak{M}_{\epsilon,\delta}^r = \mathfrak{M}_{\epsilon,\delta} \cap \tilde{B}_{5,\delta}, \quad \mathfrak{M}_\delta^r = \mathfrak{M}_\delta \cap \tilde{B}_\delta.$$

As noted above, $(\mathfrak{M}_\delta^r)_0$ and $\mathfrak{M}_{\epsilon,\delta}^r$ are inertial manifolds for the semigroups generated by the original equations and are only positively invariant under the flow, that is, $S_0(t)(\mathfrak{M}_\delta^r)_0 \subset (\mathfrak{M}_\delta^r)_0$ and $S_\epsilon(t)\mathfrak{M}_{\epsilon,\delta}^r \subset \mathfrak{M}_{\epsilon,\delta}^r, \forall t \geq 0$.

Observe now that, if $(u, s) \in \mathfrak{M}_{\epsilon,\delta}^r$, then there exists a complete orbit $(\rho(t), w(t))_{t \in \mathbb{R}}$ in $\mathfrak{M}_{\epsilon,\delta}$ such that $\rho(0) = u$ and $w(0) = s$. Hence we have

$$\begin{aligned} m(\rho(t)) &= m(u) + \epsilon m(s)(1 - e^{-t/\epsilon}), \quad \forall t \geq 0, \\ m(w(t)) &= m(s)e^{-t/\epsilon}, \quad \forall t \geq 0, \end{aligned}$$

so that

$$m(s) = m(\rho(t))e^{t/\epsilon}, \quad \forall t \in \mathbb{R}.$$

Since $m(\rho(t))$ and $m(w(t))$ are bounded for all $t \leq 0$, it follows that $m(s) = 0$ by letting $t \rightarrow -\infty$, and also $m(\rho(t))$ is independent of t .

We set

$$\mathfrak{M}_\delta^{r,\mu} = \mathfrak{M}_\delta^r \cap K_\delta^\mu, \quad \mathfrak{M}_{\epsilon,\delta}^{r,\mu} = \mathfrak{M}_{\epsilon,\delta}^r \cap \tilde{K}_\delta^\mu.$$

It is clear that

$$\mathfrak{M}_\delta^r = \bigcup_{|\mu| \leq \delta} \mathfrak{M}_\delta^{r,\mu}$$

and

$$\mathfrak{M}_{\epsilon,\delta}^r = \bigcup_{|\mu| \leq \delta} \mathfrak{M}_{\epsilon,\delta}^{r,\mu}.$$

Moreover, $\mathfrak{M}_\delta^{r,\mu}$ and $\mathfrak{M}_{\epsilon,\delta}^{r,\mu}$ are inertial manifolds for the semigroups $S(t)$ and $S_\epsilon(t)$ restricted to K_δ^μ and $\tilde{K}_\delta^\mu$, respectively.

We are now able to state the following result on the upper and lower semicontinuity of the above inertial manifolds at $\epsilon = 0$. These results were also proven in detail in the viscous case (see [3, Propositions 6.2 and 6.3]).

Theorem 7.3. *Let the assumptions of Theorem 5.2 hold and let $\mu \in [-\delta, \delta]$. Then the inertial manifolds $\mathfrak{M}_{\epsilon,\delta}^r \cup (\mathfrak{M}_\delta)_0$ are upper semicontinuous at $\epsilon = 0$ and $\mathfrak{M}_{\epsilon,\delta} \cup (\mathfrak{M}_\delta^{r,\mu})_0$ are lower semicontinuous at $\epsilon = 0$, with respect to the metric induced by the $\mathcal{H}_1^0$ -norm, that is, for any $\eta > 0$, there exist $t_\eta, \bar{t}_\eta, \epsilon_\eta$ and $\bar{\epsilon}_\eta$ such that*

$$\forall \epsilon \leq \epsilon_\eta, \quad \text{dist}_{\mathcal{H}_1^0}(S_\epsilon(t_\eta)\mathfrak{M}_{\epsilon,\delta}^r, (\mathfrak{M}_\delta)_0) \leq \eta \quad (7.8)$$

and

$$\forall \epsilon \leq \bar{\epsilon}_\eta, \quad \text{dist}_{\mathcal{H}_1^0}((S(\bar{t}_\eta)\mathfrak{M}_\delta^{r,\mu})_0, \mathfrak{M}_{\epsilon,\delta}) \leq \eta. \quad (7.9)$$

Remark 7.1. Using the same arguments, we can also prove that the family of inertial manifolds $\mathfrak{M}_{\epsilon,\delta}^{r,\mu}$ are upper and lower semicontinuous at any $\epsilon_2 > 0$ with respect to the metric induced by the $\mathcal{H}_{\epsilon_1}^0$ -norm, that is, for all $0 < \epsilon_2 \leq \epsilon_1 \leq \epsilon_0$,

$$\lim_{\epsilon_1 \rightarrow \epsilon_2} \text{dist}_{\mathcal{H}_{\epsilon_1}^0}^{\text{sym}}(\mathfrak{M}_{\epsilon_1,\delta}^{r,\mu}, \mathfrak{M}_{\epsilon_2,\delta}^{r,\mu}) = 0. \quad (7.10)$$

Remark 7.2. The existence and convergence, as ϵ goes to 0, of inertial manifolds for a singularly perturbed damped wave equation in one space dimension and under stronger assumptions on the nonlinear term were obtained in [5, 36, 37]. The method used in [36, 37] works essentially for equations involving linear self-adjoint operators only. In [2], we successfully applied the latter method to a singular perturbation of the standard viscous Cahn–Hilliard equation in one and two space dimensions. This method may also be applied to problem (2.2). However, our approach works for a larger class of nonlinear evolution dynamical systems, even for equations that contain a non-self-adjoint operator (see [3]). Moreover, our results do not need stronger assumptions on g as in [5, 36, 37]. On the other hand, the results of [5] give the convergence of the first component of the inertial manifolds only. We somehow complete those results by showing the strong convergence of the inertial manifolds for both components of the semigroup $S_\epsilon(t)$.

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