L^p -weak regularity and asymptotic behavior of solutions for critical equations with singular potentials on Carnot groups

Annunziata Loiudice

Abstract. In this paper, we consider a large class of critical problems with singular potentials on Carnot groups. In particular, we prove sharp regularity results in Marcinkiewicz spaces and the exact rate of decay of the solutions. Our general results apply, for instance, to the equation satisfied by the extremals of some relevant Hardy–Sobolev type inequalities on Stratified groups.

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1. Introduction and main results

Let $\mathbb{G} = (\mathbb{R}^N, \circ)$ be a homogeneous Carnot group with homogeneous dimension Q , and let $\Delta_{\mathbb{G}}$ be a fixed sublaplacian on $\mathbb{G}$. In this paper we consider the following semilinear subelliptic problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \sum_{i=1}^n a_i(\xi) |u|^{p_i-2} u, \\ u \in S_0^1(\Omega) \end{cases} \quad (1.1)$$

where $\Omega \subseteq \mathbb{R}^N$ is an open unbounded domain, n is a positive integer, $2 < p_i \leq 2^*$ for $1 \leq i \leq n$, where 2^* denotes the critical Sobolev exponent in this context, i.e. $2^* = 2Q/(Q-2)$, and $S_0^1(\Omega)$ is the appropriate Sobolev space, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = \|\nabla_{\mathbb{G}} u\|_2$.

As regard the coefficients a_i , they are only required to satisfy the following summability assumptions

$$a_i \in M^{q_i}(\Omega), \quad \frac{1}{q_i} + \frac{p_i}{2^*} = 1, \quad \forall i = 1, \dots, n \quad (1.2)$$

where M^{q_i} denotes the classical Marcinkiewicz space.

Let us remark that a singular coefficient of the type $\frac{1}{d(\xi)^\alpha}$, where d is any homogeneous norm on $\mathbb{G}$, is only weakly summable in the sense of Marcinkiewicz. Indeed, it belongs to the weak L^p -space, with $p = \frac{Q}{\alpha}$. Therefore, a good setting for the study of semilinear subelliptic problems with singular coefficients seems to be that of weak L^p -spaces or, more appropriately, that of Lorentz spaces, which include weak and classical L^p -spaces as particular cases.

Let us observe that the summability assumptions (1.2) are quite natural if we want the right hand side of (1.1) to belong to the space $S^{-1}(\Omega)$, i.e. the dual space of $S_0^1(\Omega)$. Indeed, if $u \in S_0^1(\Omega)$, then $u \in L^{2^*,2}(\Omega)$ (see the improved version of the classical Folland–Stein embedding we prove in Sect. 3) and so $|u|^{p_i-1} \in L^{\frac{2^*}{p_i-1}, \frac{2}{p_i-1}}(\Omega) \subset L^{\frac{2^*}{p_i-1}, 2}(\Omega)$, being $p_i > 2$. Hence, by the appropriate Hölder inequality for functions in conjugate Lorentz spaces (see Proposition 3.2 below), and taking into account the assumptions (1.2), we get that $a_i|u|^{p_i-1} \in L^{r,2}$ where

$$\frac{1}{r} = \frac{1}{q_i} + \frac{p_i-1}{2^*} = 1 - \frac{1}{2^*},$$

and therefore

$$\sum_{i=1}^n a_i|u|^{p_i-2}u\varphi \in L^{1,1}(\Omega) = L^1(\Omega), \quad \forall \varphi \in S_0^1(\Omega),$$

which gives sense to the variational formulation of problem (1.1).

Let us observe that problem (1.1) may be suitably rescaled. Given any $\lambda > 0$, and denoted by δ_λ the natural anisotropic dilations on the group $\mathbb{G}$, let us consider the following two family of rescaled functions and coefficients

$$\begin{aligned} u_\lambda(\xi) &= \lambda^{\frac{Q-2}{2}} u(\delta_\lambda(\xi)) \\ a_i^\lambda(\xi) &= \lambda^{\frac{Q}{q_i}} a_i(\delta_\lambda(\xi)). \end{aligned}$$

Observe that the norms $\|\nabla_{\mathbb{G}} u_\lambda\|_{L^2}$, $\|u_\lambda\|_{L^{2^*}}$, and $\|a_i^\lambda\|_{M^{q_i}}$ keep constant when λ changes, and moreover u_λ solves Eq. (1.1) with coefficients a_i^λ on the rescaled domain $\delta_\lambda^{-1}(\Omega)$. Therefore, in particular, if the coefficients a_i are homogeneous of degree $-Q/q_i$ with respect to δ_λ (this is the case, for instance, of singular coefficients of the form $a_i = d^{-Q/q_i}$, d being a homogeneous norm on $\mathbb{G}$) and the domain is δ_λ -invariant, then u_λ exactly solves the same equation as u , for any $\lambda > 0$. Due to this scale invariance, problem (1.1) may exhibit a lack of compactness and for this reason, it can be classified as a problem with “critical growth”.

We point out that in the last years, a great deal of interest has been paid to semilinear problems with critical growth on Stratified groups (see, e.g. [3, 11, 12, 18–20] and the references therein), but at the present time, the greatest part of the literature concerns non-singular problems.

Our aim in this paper is to study regularity, in the setting of Lorentz spaces, of weak solutions to problem (1.1) and to deduce their asymptotic

behavior at infinity. We observe that this a priori analysis is a first fundamental step in order to investigate existence and non-existence results.

The technique we use here to get asymptotic results from L^p regularity is the one developed by Uguzzoni [24], and subsequently employed by Lanconelli and Uguzzoni [18, 19] to discuss the case of semilinear subelliptic problems with autonomous critical nonlinearities.

In our case, however, the presence of coefficients which are only weakly summable suggest to refine the analysis in the scale of Lorentz spaces.

For what concern regularity, by adapting to the present context the Euclidean elliptic proof in [16], we prove that a weak solution $u \in S_0^1(\Omega)$ to problem (1.1) belongs not only to $L^p(\Omega)$ for any $\frac{2^*}{2} < p \leq \infty$, but also to the weak space $M^{\frac{2^*}{2}}(\Omega) = L^{\frac{2^*}{2}, \infty}(\Omega)$.

This sharp regularity result, which completes in the setting of Lorentz space the analysis performed in the L^p framework in [18, 19], turns out to be a crucial ingredient in order to study the behavior of solutions at infinity, since it allows to optimize the method by Uguzzoni and directly get the exact asymptotic decay, without passing through any partial decay result.

Denoted by d the homogeneous norm such that $\Gamma = Cd^{2-Q}$ is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0 for a suitable positive constant C , our result is the following.

Theorem 1.1. *Let Ω be an arbitrary unbounded open subset of $\mathbb{G}$. If $u \in S_0^1(\Omega)$ is a weak solution of the Dirichlet problem (1.1), then there exist $C, R > 0$ such that*

$$|u(\xi)| \leq Cd(\xi)^{2-Q}, \quad d(\xi) \geq R. \quad (1.3)$$

So, we obtain that the solutions of our semilinear problem with multiple critical nonlinearities are bounded from above, for d large, by a constant multiple of the fundamental solution of the involved operator.

We observe that the Euclidean analog of the above result was obtained in [16] with different techniques and assuming decaying assumptions on the coefficients a_i , which are here removed.

We remark at this point that if, in particular, Ω is an *exterior domain* (i.e. Ω^C is bounded), and u is a non-negative solution of (1.1) with non-negative coefficients a_i , then u is also bounded from below by a constant multiple of the fundamental solution. Indeed, if u is a non-negative non-trivial solution of (1.1) with $a_i \geq 0$, then $-\Delta_{\mathbb{G}}u \geq 0$ and therefore, by Bony's strong maximum principle [4], u is strictly positive. So, there exists $R > 0$ such that $u \geq Cd^{2-Q}$ on $\partial B_d(0, R)$, u is superharmonic in $B_d(0, R)^C$ and vanishes at ∞ , and therefore $u \geq Cd^{2-Q}$ in $B_d(0, R)^C$ by the weak maximum principle for $\Delta_{\mathbb{G}}$.

Hence, in particular, entire non-negative solutions (i.e. solutions in the whole space $\mathbb{G}$) of problem (1.1) with $a_i \geq 0$ exactly behaves like the fundamental solution Γ at infinity.

It is worthwhile to note that our general regularity and asymptotic results apply, for instance, to the study of qualitative properties of the extremal functions for some relevant Hardy–Sobolev type inequalities on Stratified groups.

Due to the fundamental role of such inequalities and the growing interest they have received in connection with subelliptic pde's, this application can be considered the main motivation of our abstract analysis. We discuss it in the last part of the paper.

The subsequent sections are organized as follows. In Sect. 2 we recall the definition and the main properties of the Marcinkiewicz and Lorentz spaces, which will occur in our proofs. We also prove an improved version of the Folland–Stein embedding, namely the embedding $S_0^1(\Omega) \hookrightarrow L^{2^*,2}(\Omega)$, which is stronger than the usual one into $L^{2^*,2^*}(\Omega) = L^{2^*}(\Omega)$. In Sect. 3, we prove sharp regularity results in the scale of Lorentz spaces for solutions to general Schrödinger-type inequalities under suitable L^p -weak assumptions on the potential. Consequently, we derive the regularity of solutions to the nonlinear problem (1.1). In Sect. 4, we prove our main result, namely the asymptotic decay of solutions. Finally, in Sect. 5, as an application of our general results, we obtain the asymptotic behavior of Hardy–Sobolev type extremals on Carnot groups.

2. Notations and definitions

We briefly recall the relevant definitions about Carnot groups and fix the notation we shall use in the sequel. For a complete treatment, we refer the reader to the wide monograph [2] and the classical papers [9, 10].

A *Carnot group* (or *Stratified group*) $(\mathbb{G}, \circ)$ is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely a decomposition $\mathfrak{g} = \bigoplus_{j=1}^r \mathfrak{g}_j$, such that $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$ for $1 \leq j < r$, and $[\mathfrak{g}_1, \mathfrak{g}_r] = \{0\}$.

The integer $Q = \sum_{i=1}^r i \dim(\mathfrak{g}_i)$ is called the *homogeneous dimension* of G . We shall assume throughout that $Q \geq 3$ (if $Q = 2$, then $\mathbb{G}$ is necessarily the ordinary Euclidean space $\mathbb{G} = (\mathbb{R}^2, +)$). The number r is called the *step* of the group $\mathbb{G}$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map (which we shall assume throughout), it is not restrictive to suppose that $\mathbb{G}$ be a homogeneous Lie group on $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \dots \times \mathbb{R}^{N_r}$, with $N_i = \dim(\mathfrak{g}_i)$, equipped with a family of group-automorphisms (called *dilations*) δ_λ of the form

$$\delta_\lambda(\xi) = (\lambda \xi^{(1)}, \lambda^2 \xi^{(2)}, \dots, \lambda^r \xi^{(r)}),$$

where $\xi^{(j)} \in \mathbb{R}^{N_j}$ for $j = 1, \dots, r$.

The Lebesgue measure is invariant w.r.t. left and right translations on $\mathbb{G}$ and the volumes scale as λ^Q , i.e. denoted by $|E|$ the Lebesgue measure of a set E , we have that $|\delta_\lambda(E)| = \lambda^Q |E|$.

Let $X_1, \dots, X_{N_1}$ be any basis of $\mathfrak{g}_1$. The second order differential operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{N_1} X_i^2$$

is called a *sublaplacian* on $\mathbb{G}$. We shall denote by $\nabla_{\mathbb{G}} = (X_1, \dots, X_{N_1})$ the associated gradient. Note that $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$ are left-translation invariant w.r.t. the group action and δ_λ -homogeneous, respectively, of degree one and two. In other words, $\nabla_{\mathbb{G}}(u \circ \tau_\xi) = \nabla_{\mathbb{G}}u \circ \tau_\xi$, $\nabla_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda \nabla_{\mathbb{G}}u \circ \delta_\lambda$, $\Delta_{\mathbb{G}}(u \circ \tau_\xi) = \Delta_{\mathbb{G}}u \circ \tau_\xi$, $\Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda^2 \Delta_{\mathbb{G}}u \circ \delta_\lambda$. Moreover, due the stratification condition, the Lie algebra generated by $X_1, \dots, X_{N_1}$ is the whole $\mathfrak{g}$, and therefore it is everywhere of rank N ; therefore, the sublaplacian operator $\Delta_{\mathbb{G}}$ satisfies the well-known Hörmander's hypoellipticity condition.

By definition, a *homogeneous norm* on $\mathbb{G}$ is a continuous function $d : \mathbb{G} \rightarrow [0, +\infty)$, smooth away from the origin, such that $d(\delta_\lambda(\xi)) = \lambda d(\xi)$, $d(\xi^{-1}) = d(\xi)$ and $d(\xi) = 0$ iff $\xi = 0$. Moreover, any homogeneous norm satisfies the following pseudo-triangular inequality $d(\xi \circ \eta) \leq \beta(d(\xi) + d(\eta))$ for a suitable constant $\beta > 0$. It can be easily seen that the homogeneous norms on $\mathbb{G}$ are all equivalent.

A remarkable fact is that there exists a suitable homogeneous norm d on $\mathbb{G}$ such that

$$\Gamma(\xi) = \frac{C}{d(\xi)^{Q-2}} \quad (2.1)$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$ (see, e.g. [9]). Throughout the paper, we shall indicate by d this particular homogeneous norm. We recall that, if we define $d(\xi, \eta) = d(\eta^{-1} \circ \xi)$, then d is a pseudo-distance on $\mathbb{G}$. We shall denote by $B_d(\xi, r)$ the d -ball with center at ξ and radius r .

3. The functional setting of Lorentz spaces

We begin with a brief recall about classical Lorentz spaces. For a more complete description, we refer the reader to the book [1], but also to the papers [21–23]. We also quote [16, 17, 26], where the role of the Lorentz spaces in studying variational problems with singular potentials in the Euclidean setting is emphasized.

First of all, we remark that in this paper, in order to denote the Lorentz spaces, we shall adopt the notation $L^{p,q}$, usually reserved to Morrey spaces, instead of the usual one $L(p, q)$.

In order to define the Lorentz spaces, let us introduce some preliminary definitions. If u is a measurable function on a measurable set $\Omega \subset \mathbb{G}$, its distributional function $\lambda_u : (0, \infty) \rightarrow [0, \infty]$ is defined by

$$\lambda_u(s) = |\{\xi \in \Omega : |u(\xi)| > s\}|$$

and its decreasing rearrangement $u^* : [0, \infty) \rightarrow [0, \infty]$ is defined by

$$u^*(t) = \inf\{s : \lambda_u(s) \leq t\}.$$

Observe that, as it is well know,

$$\int_{\mathbb{G}} |u(\xi)|^p d\xi = p \int_0^\infty s^{p-1} \lambda_u(s) ds = \int_0^\infty u^*(t)^p dt.$$

We can now define the Lorentz spaces.

Definition 3.1. For a given pair of indexes (p, q) , $1 \leq p, q \leq \infty$, we define the Lorentz space $L^{p,q}(\Omega)$ as the space of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that

$$\|u\|_{L^{p,q}(\Omega)} := \begin{cases} \left(\int_0^\infty [u^*(t)]^q t^{q/p-1} dt \right)^{1/q} < \infty, & \text{if } q < \infty \\ \sup_{t>0} u^*(t) t^{1/p} = (\sup_{t>0} t^p \lambda_u(t))^{1/p} < \infty & \text{if } q = \infty. \end{cases}$$

Unless $q = p$, the above quantity $\|\cdot\|_{p,q}$ does not define a norm, but a quasi-norm.

A monotonicity property holds with respect to the secondary index, namely if $q_1 \leq q_2$, then we have $L^{p,q_1} \subset L^{p,q_2}$. So the strongest case of the Lorentz space with principal index p is the space $L^{p,1}$, while the weakest one is $L^{p,\infty}$. The most familiar one is the intermediate case when $p = q$, since in this case $L^{p,p}$ is equivalent to the classical L^p space.

The above mentioned weakest space $L^{p,\infty}$ is, on the other hand, equivalent to the classical weak- L^p space, or Marcinkiewicz space M^p , consisting, for $p > 1$, of all measurable functions u such that

$$\|u\|_{M^p(\Omega)} = \sup_A \frac{\int_A |u(\xi)| d\xi}{|A|^{1/p'}} < \infty \quad (3.1)$$

where the supremum is taken over all measurable sets $A \subset \Omega$, $|A| < +\infty$ and p' is the conjugate exponent of p .

We resume below the formulas for estimating the Lorentz norms of pointwise products and convolutions. (Recall that the convolution of two functions on a homogeneous group $\mathbb{G}$ with group law $\circ$ is defined as $(u * v)(\xi) = \int u(\xi^{-1} \circ \eta) v(\eta) d\eta$, for a.e. $\xi \in \mathbb{G}$, if the integral converges.)

Proposition 3.2. (Hölder-type inequality) *Let $1 \leq p_1, p_2, q_1, q_2 \leq +\infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. If $u \in L^{p_1,q_1}(\Omega)$ and $v \in L^{p_2,q_2}(\Omega)$, then $uv \in L^{p,q}(\Omega)$ and*

$$\|uv\|_{L^{p,q}(\Omega)} \leq C \|u\|_{L^{p_1,q_1}(\Omega)} \|v\|_{L^{p_2,q_2}(\Omega)} \quad (3.2)$$

(Young-type inequality) *Let $1 < p_1, p_2 < +\infty$ and $1 \leq q_1, q_2 \leq +\infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. If $u \in L^{p_1,q_1}(\Omega)$ and $v \in L^{p_2,q_2}(\Omega)$, then $u * v \in L^{r,q}(\Omega)$ and*

$$\|u * v\|_{L^{r,q}(\Omega)} \leq C \|u\|_{L^{p_1,q_1}(\Omega)} \|v\|_{L^{p_2,q_2}(\Omega)}. \quad (3.3)$$

For the proof of the above formulas see, e.g. the classical paper by O'Neil [21]. Some particular cases only concerning convolutions between L^p and L^p -weak functions can be found in [9, 10].

Note that when $p_1 = q_1$ and $p_2 = q_2$, formula (3.3) is stronger than the analogous one for the L^p spaces, since in this case $u * v \in L^{r,q}$ with $q < r$, which is stronger than $u * v \in L^{r,r} = L^r$.

To conclude the description of the functional setting, we present an improved version of the Sobolev-type embedding inequality for the subelliptic

gradient, obtained by means of the Lorentz norms. We recall that the classical one involving the L^p -norm is due to Folland and Stein and it can be found in [9].

Proposition 3.3. *Let Ω be an open subset of $\mathbb{G}$. Then, for any $q \geq 1$, there exists a constant $C = C(\Omega, q) > 0$ such that the following inequality holds*

$$\|u\|_{L^{2^*,q}(\Omega)} \leq C \|\nabla_{\mathbb{G}} u\|_{L^{2,q}(\Omega)}, \quad \forall u \in C_0^\infty(\Omega). \quad (3.4)$$

Proof. Take into account the following surface mean value formula, which holds for any $u \in C^2(\Omega)$ (see for details [2]): for a.e. $\xi \in \Omega$ and every $r > 0$ such that $\overline{B_d(\xi, r)} \subseteq \Omega$, then

$$u(\xi) = \mathcal{M}_r(u)(\xi) + \beta_d \int_{B_d(\xi, r)} \langle \nabla_{\mathbb{G}} \Gamma_\xi, \nabla_{\mathbb{G}} u \rangle,$$

where $\Gamma_\xi = \Gamma(\xi, \cdot)$, $\beta_d > 0$ and $\mathcal{M}_r$ is defined as

$$\mathcal{M}_r(u)(\xi) = \frac{(Q-2)\beta_d}{r^{Q-1}} \int_{\partial B_d(\xi, r)} \mathcal{K}(\xi, \eta) u(\eta) dH^{N-1}(\eta),$$

with $\mathcal{K}(\xi, \eta)$ a suitable non-negative kernel (see [2]).

Now, if u has compact support, then $\mathcal{M}_r(u) = 0$ for r sufficiently large, and so

$$|u| \leq C |\nabla_{\mathbb{G}} \Gamma| * |\nabla_{\mathbb{G}} u|.$$

Therefore, since $\nabla_{\mathbb{G}} \Gamma \in M^{Q/(Q-1)}$, by (3.3) we get

$$\|u\|_{L^{2^*,q}(\Omega)} \leq C \|\nabla_{\mathbb{G}} \Gamma\|_{M^{Q/(Q-1)}(\Omega)} \|\nabla_{\mathbb{G}} u\|_{L^{2,q}(\Omega)},$$

and we are done. $\square$

We explicitly observe that, in the case $q = 2$, the inequality (3.4) reads like

$$\|u\|_{L^{2^*,2}(\Omega)} \leq C \|\nabla_{\mathbb{G}} u\|_{L^2(\Omega)}, \quad \forall u \in C_0^\infty(\Omega),$$

and so the following continuous embedding holds

$$S_0^1(\Omega) \hookrightarrow L^{2^*,2}(\Omega),$$

which is sharper than the usual Folland–Stein embedding into $L^{2^*,2^*}(\Omega) = L^{2^*}(\Omega)$.

4. L^p -weak regularity results

In what follows, we shall regard our initial problem (1.1) as an equation of the form

$$-\Delta_{\mathbb{G}} u(\xi) = V(\xi)u(\xi), \quad u \in S_0^1(\Omega) \quad (4.1)$$

where $V(\xi) = \sum_{i=1}^n a_i(\xi)|u|^{p_i-2}$, with the natural summability assumption $V \in M^{\frac{Q}{2}}$, which ensures the right hand side of (4.1) to belong to $S^{-1}(\Omega)$, that is the dual space of $S_0^1(\Omega)$.

In order to forget about the sign of u and get free of Ω , we shall pass to an inequality. Indeed, taking into account that $-\Delta_{\mathbb{G}}|u| \leq |\Delta_{\mathbb{G}} u|$ and letting

$v(\xi) = |u(\xi)|$ in Ω and $v(\xi), V(\xi) = 0$ outside Ω , we get that v satisfies the differential inequality

$$-\Delta_{\mathbb{G}}v(\xi) \leq |V(\xi)|v(\xi) \quad \text{in } S^{-1}(\mathbb{G}) \quad (4.2)$$

with $V \in M^{\frac{Q}{2}}(\mathbb{G})$.

From the above inequality, we easily get that

$$v(\xi) \leq (\Gamma * (|V|v))(\xi), \quad (4.3)$$

where Γ denotes the fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0. Indeed, if we let $w = \Gamma * |V|v$, then w belongs to $S^1(\mathbb{G})$ and weakly solves the equation $-\Delta_{\mathbb{G}}w = |V|v$ in $\mathbb{G}$, hence $-\Delta_{\mathbb{G}}v \leq -\Delta_{\mathbb{G}}w$ in Ω ; moreover, $v \leq w$ on $\partial\Omega \cup \{\infty\}$. Therefore, by the weak maximum principle for $\Delta_{\mathbb{G}}$, $v \leq w$ in Ω , that is (4.3).

Now, we give a Brezis–Kato type (see [5,6]) regularity result, which applies to every solution to inequality (4.3). The proof is an adaptation to the present subelliptic context of the Euclidean proof by Jannelli and Solimini [16].

Proposition 4.1. *Let $v \in S^1(\mathbb{G})$ be a solution to (4.3). For any $\frac{2^*}{2} \leq p < +\infty$, there exists $\varepsilon = \varepsilon(p) > 0$ such that, if $|V|$ admits a decomposition of the form $|V| = V_0 + V_1$, where V_0, V_1 are positive functions with $V_0 \in C_0^\infty(\Omega)$ and $\|V_1\|_{\frac{Q}{2}, \infty} < \varepsilon$, then $v \in L^p(\mathbb{G})$ if $p > \frac{2^*}{2}$, while $v \in M^{\frac{2^*}{2}}(\mathbb{G})$ if $p = \frac{2^*}{2}$.*

Moreover, the norms in $M^{\frac{2^}{2}}$ and in L^p may be estimated in terms of the L^{2^*} -norm of v and of suitable norms of V_0 .*

Proof. Let us denote by $\|\cdot\|$ indifferently one of the two norms of v concerned by our statement. For the time being, let us suppose that $\|v\| < +\infty$; later we shall remove this auxiliary hypothesis. We have

$$\|\Gamma * (V_1v)\| \leq C\|\Gamma\|_{M^{\frac{2^*}{2}}} \|V_1\|_{M^{\frac{Q}{2}}} \|v\| \leq C\varepsilon\|\Gamma\|_{M^{\frac{2^*}{2}}} \|v\| \leq \frac{1}{2}\|v\| \quad (4.4)$$

if ε is sufficiently small with respect to C , which in turn depends on the norm $\|\cdot\|$.

Since, by (4.3)

$$\|v\| \leq \|\Gamma * (V_0v)\| + \|\Gamma * (V_1v)\|,$$

by using (4.4) we get the crucial estimate

$$\|v\| \leq 2\|\Gamma * (V_0v)\|. \quad (4.5)$$

Being $v \in L^{2^*}$, we have, in particular, that $V_0v \in L^1$; hence

$$\|v\|_{M^{\frac{2^*}{2}}} \leq C\|\Gamma\|_{M^{\frac{2^*}{2}}} \|V_0v\|_{L^1}. \quad (4.6)$$

Therefore, $v \in M^{\frac{2^*}{2}}$, and so, by interpolation, $v \in L^p$ for any $p \in (\frac{2^*}{2}, 2^*]$.

Now, to prove the L^p -summability for $p > 2^*$, let us take into account that, if $v \in L^q$ (we shall start by $q = 2^*$) and $p \leq (q^*)^*$, then

$$\|v\|_p \leq C\|\Gamma\|_{M^{\frac{2^*}{2}}} \|V_0v\|_r \leq C\|v\|_q, \quad (4.7)$$

where $\frac{1}{r} = \frac{1}{p} + \frac{2}{Q} \geq \frac{1}{q}$. In this way, we gain summability for v : if $v \in L^q$, then $v \in L^p$ for any p such that $\frac{1}{p} \geq \frac{1}{q} - \frac{2}{Q}$. Iterating this argument, we can manage any $2^* < p < +\infty$.

Now, in order to remove the auxiliary hypothesis $\|v\| < +\infty$, we shall prove (4.5) only by knowing that $v \in L^{2^*}$.

Let us observe that there exists a sequence $\phi_k \in C_0^\infty(\mathbb{G})$ with $\|\phi_k\|' \leq 1$ such that $\|v\| = \lim_k \int_{\mathbb{G}} v \phi_k$ (the norm $\|\cdot\|'$ denotes the dual norm of $\|\cdot\|$, i.e. $\|\cdot\|' = \|\cdot\|_{\frac{Q}{2},1}$ if $\|\cdot\| = \|\cdot\|_{M^{\frac{2^*}{2}}}$, while $\|\cdot\|' = \|\cdot\|_{p'}$ if $\|\cdot\| = \|\cdot\|_p$).

Since $\|v\|_{2^*} < +\infty$, then

$$S_k = \sup_{\substack{\|\phi\|' \leq \|\phi_k\|' \leq 1 \\ \|\phi\|_{2^{*'}} \leq \|\phi_k\|_{2^{*'}}}} \int_{\mathbb{G}} v \phi < +\infty, \quad (4.8)$$

so we may always choose ϕ_k in such a way that $\int v \phi_k > \frac{2}{3} S_k$.

Now, the smallness of ε implies, just as in (4.4), that

$$\|V_1(\Gamma * \phi_k)\|' \leq \frac{1}{2} \|\phi_k\|'; \quad (4.9)$$

$$\|V_1(\Gamma * \phi_k)\|_{2^{*'}} \leq \frac{1}{2} \|\phi_k\|_{2^{*'}}. \quad (4.10)$$

Therefore

$$\int_{\mathbb{G}} v(V_1(\Gamma * \phi_k)) \leq \frac{1}{2} S_k \leq \frac{3}{4} \int_{\mathbb{G}} v \phi_k. \quad (4.11)$$

Hence, taking into account (4.3), we get

$$\begin{aligned} \int_{\mathbb{G}} v \phi_k &\leq \int_{\mathbb{G}} (\Gamma * (V_0 + V_1)v) \phi_k = \int_{\mathbb{G}} (\Gamma * V_0 v) \phi_k + \int_{\mathbb{G}} (\Gamma * \phi_k) V_1 v \\ &\leq \int_{\mathbb{G}} (\Gamma * V_0 v) \phi_k + \frac{3}{4} \int_{\mathbb{G}} v \phi_k, \end{aligned} \quad (4.12)$$

from which

$$\|v\| = \lim_k \int_{\mathbb{G}} v \phi_k \leq 4 \|\Gamma * (V_0 v)\|. \quad (4.13)$$

Now, by replacing (4.5) with (4.13), the proof can be repeated removing the auxiliary hypothesis $\|v\| < +\infty$. $\square$

Remark 4.2. Observe that the smallness of the constant ε depends on p , since the constant C in (4.4) goes to ∞ as $p \rightarrow \infty$.

Remark 4.3. We underline that the decomposition $V = V_0 + V_1$ is always possible when $V \in L^{\frac{Q}{2},q}(\mathbb{G})$ with $q < +\infty$, due to the density of $C_0^\infty(\mathbb{G})$ into $L^{\frac{Q}{2},q}(\mathbb{G})$. On the other hand, in the case $V \in M^{\frac{Q}{2}}$ this decomposition may fail and the results of Proposition 4.1 may be false. This is the case, for instance, when the potential V behaves like $\frac{1}{d(\xi)^2}$.

From the above result, the regularity of solutions for our initial problem follows.

Proposition 4.4. *Let $u \in S_0^1(\Omega)$ be a weak solution to problem (1.1). Then, $u \in M^{\frac{2^*}{2}}(\Omega)$ and $u \in L^p(\Omega)$ for any $\frac{2^*}{2} < p \leq \infty$.*

Proof. Set $V = \sum_{i=1}^n a_i |u|^{p_i-2}$. From $a_i \in M^{q_i}$, where q_i are defined by (1.2), and $u \in L^{2^*,2}$, from which $|u|^{p_i-2} \in L^{\frac{2^*}{p_i-2}, \frac{2}{p_i-2}}$, we get that $V \in L^{\frac{Q}{2}, q}$, with $q = \max_{i=1, \dots, n} \frac{2}{p_i-2}$. Hence, taking into account Remark 4.3, we can assert that the results of Proposition 4.1 apply to our case. So, $u \in M^{\frac{2^*}{2}}$ and $u \in L^p$ for any $\frac{2^*}{2} < p < \infty$.

Now, being in particular $u \in L^p$ for $p > 2^*$, it is easily seen that $V \in L^q(\Omega)$, for some $q > \frac{Q}{2}$. Therefore, by means of the standard Moser's iteration technique (see, e.g. the proof of Lemma A.1 in [19]), we get that $u \in L^\infty(\Omega)$. $\square$

5. Asymptotic behavior of solutions

In this section, we prove a general theorem about asymptotic behavior of solutions for Schrödinger-type inequalities, which is a refinement of Theorem 1.1 in [24], obtained by using the framework of Lorentz spaces.

We recall that we are concerned with differential inequalities of the type

$$-\Delta_{\mathbb{G}} u \leq V u, \quad u \in S_0^1(\Omega),$$

where Ω is an arbitrary unbounded domain of $\mathbb{G}$.

Our first improvement is that the assumption $V \in L^p(\Omega) \cap L^q(\Omega)$ for some $p < \frac{Q}{2} < q$, required on the potential V in [24] in order to get asymptotic results for the solutions, is weakened by assuming that $V \in L^{\frac{Q}{2}, 1}(\Omega)$, that is exactly the conjugate space of $M^{\frac{Q}{Q-2}}(\Omega) = L^{\frac{Q}{Q-2}, \infty}(\Omega)$.

Moreover, by the same technique as in [24], but using the sharp $M^{\frac{Q}{Q-2}}$ -regularity of solutions proved in the preceding section, we are able to directly obtain their exact asymptotic behavior at infinity, namely $u = O(\frac{1}{d^{Q-2}})$ as $d \rightarrow \infty$, rather than the partial decay result $u = O(\frac{1}{d^s})$, for any $s < Q - 2$.

Before stating the theorem, we recall that the method relies on the use of the following solid mean representation formula related to $\Delta_{\mathbb{G}}$: if $v \in C^2(\Omega)$ and $\overline{B_d(\xi, r)} \subseteq \Omega$, the following formula holds (see for details [2, 7])

$$v(\xi) = M_r(v)(\xi) - \frac{Q}{r^Q} \int_0^r \rho^{Q-1} \left(\int_{B_d(\xi, \rho)} \left(\Gamma(\xi, \eta) - \frac{c_Q}{\rho^{Q-2}} \right) \Delta_{\mathbb{G}} v(\eta) d\eta \right) d\rho \quad (5.1)$$

where

$$M_r(v)(\xi) = \frac{m_Q}{r^Q} \int_{B_d(\xi, r)} \psi^2(\xi^{-1} \circ \eta) v(\eta) d\eta,$$

being $\psi := |\nabla_{\mathbb{G}} d|$.

Theorem 5.1. *Let $\Omega \subseteq \mathbb{G}$ be an unbounded open set and let $V \in L^{\frac{Q}{2},1}(\Omega)$. If u is a non-negative weak solution of*

$$\begin{cases} -\Delta_{\mathbb{G}} u \leq Vu & \text{in } \Omega \\ u \in S_0^1(\Omega), \end{cases} \quad (5.2)$$

then $u(\xi) = O(\frac{1}{d(\xi)^{Q-2}})$ as $d(\xi) \rightarrow \infty$.

Remark 5.2. The above theorem holds true also removing the sign assumption on u if we replace the inequality $-\Delta_{\mathbb{G}} u \leq Vu$ with $|\Delta_{\mathbb{G}} u| \leq |Vu|$. In particular, the result holds for the equation $-\Delta_{\mathbb{G}} u = Vu$, with u that may change sign.

Proof. Let us set as before

$$w = \Gamma * |V|u$$

where V and u are set to be 0 outside Ω .

As previously observed,

$$0 \leq u \leq w \quad \text{in } \Omega.$$

For every $\xi \in \mathbb{G}$ and $r > 0$, we define

$$M_r(\xi) = (M_r w)(\xi), \quad N_r(\xi) = \int_{B_d(\xi,r)} \Gamma_{\xi} |V|, \quad I_r(\xi) = \int_{B_d(\xi,r)} \Gamma_{\xi} |V|w.$$

By the regularity result previously proved, we know that $w \in M^{\frac{Q}{Q-2}}$. Hence, by the characterization of the weak norm recalled in (3.1), we get the sharp estimate

$$M_r(w) \leq \frac{c}{r^Q} \int_{B_d(\xi,r)} w \leq \frac{c}{r^Q} |B_d(\xi,r)|^{\frac{2}{Q}} = \frac{c}{r^{Q-2}}.$$

Moreover

$$N_r(\xi) \leq \|\Gamma_{\xi}\|_{M^{\frac{Q}{Q-2}}(B_d(\xi,r))} \|V\|_{L^{\frac{Q}{2},1}(B_d(\xi,r))} \leq c \|V\|_{L^{\frac{Q}{2},1}(B_d(\xi,r))}, \quad (5.3)$$

so, in particular, being $V \in L^{\frac{Q}{2},1}(\mathbb{G})$, we get

$$\sup_{\xi \in \mathbb{G}, r > 0} N_r(\xi) \leq c.$$

Then, by writing (5.1) for w , and taking into account that $-\Delta_{\mathbb{G}} w = |V|u \leq |V|w$, we get

$$\begin{aligned} w(\xi) &\leq M_r(\xi) + \frac{Q}{r^Q} \int_0^r \rho^{Q-1} \left(\int_{B_d(\xi,\rho)} \left(\Gamma_{\xi} - \frac{cQ}{\rho^{Q-2}} \right) |V|w \right) d\rho \\ &\leq M_r(\xi) + \frac{Q}{r^Q} \int_0^r \rho^{Q-1} \left(\int_{B_d(\xi,\rho)} \Gamma_{\xi} |V|w \right) d\rho \\ &\leq M_r(\xi) + I_r(\xi) \leq \frac{c}{r^{Q-2}} + I_r(\xi). \end{aligned} \quad (5.4)$$

Hence

$$\begin{aligned} I_r(\xi) &\leq \int_{B_d(\xi, r)} \Gamma_\xi(\eta) |V|(\eta) \left(\frac{c}{r^{Q-2}} + I_r(\eta) \right) d\eta \\ &\leq \frac{c}{r^{Q-2}} N_r(\xi) + \int_{B_d(\xi, r)} \Gamma_\xi |V| I_r \leq \frac{c}{r^{Q-2}} + \int_{B_d(\xi, r)} \Gamma_\xi |V| I_r. \end{aligned} \quad (5.5)$$

Moreover, reasoning as in [24], we get

$$\int_{B_d(\xi, r)} \Gamma_\xi |V| I_r \leq \sup_{\zeta \in B_d(\xi, 2r)} N_r(\zeta) \left(\int_{B_d(\xi, 2r)} \Gamma_\xi |V| w \right). \quad (5.6)$$

Now, take $r = r(\xi) = d(\xi)/4$. From (5.3) we obtain

$$\sup_{\zeta \in B_d(\xi, 2r(\xi))} N_{r(\xi)}(\zeta) \leq c \|V\|_{L^{\frac{Q}{2}, 1}(\mathbb{G} \setminus B_d(0, r(\xi)))} \rightarrow 0 \quad \text{as } d(\xi) \rightarrow \infty. \quad (5.7)$$

So, for every $\xi \in \mathbb{G}$, such that $d(\xi) > R$, R sufficiently large, from (5.6) and (5.7) we have

$$\begin{aligned} \int_{B_d(\xi, r(\xi))} \Gamma_\xi |V| I_r &\leq \frac{1}{2} \int_{B_d(\xi, 2r(\xi))} \Gamma_\xi |V| w \\ &= \frac{1}{2} I_{r(\xi)}(\xi) + \frac{1}{2} \int_{B_d(\xi, 2r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V| w \\ &\leq \frac{1}{2} I_{r(\xi)}(\xi) + \frac{c}{r(\xi)^{Q-2}} \|V\|_{L^{\frac{Q}{2}, 1}} \|w\|_{M^{\frac{Q}{Q-2}}}. \end{aligned} \quad (5.8)$$

Finally, by (5.5) and (5.8), for $d(\xi) > R$ we get

$$\frac{1}{2} I_{r(\xi)}(\xi) \leq \frac{c}{r(\xi)^{Q-2}},$$

and this estimate, together with (5.4), allows us to conclude. $\square$

Proof of Theorem 1.1. We know that any solution u to problem (1.1) belongs to L^p , for any $p > \frac{2^*}{2}$. Hence, in particular, $u \in L^{2^*, q}$, for any $q \geq 1$. Therefore, $V = \sum_{i=1}^n a_i |u|^{p_i-2} \in L^{\frac{Q}{2}, 1}$, since $a_i |u|^{p_i-2} \in L^{\frac{Q}{2}, \frac{q}{p_i-2}} = L^{\frac{Q}{2}, 1}$ choosing $q = p_i - 2$. The conclusion, then, follows from Theorem 5.1. $\square$

6. Asymptotic results for Hardy–Sobolev type extremals

We conclude with an interesting application of our general results to the study of regularity and asymptotic behavior of Hardy–Sobolev type extremals on Carnot groups.

We point out that the problem of determining qualitative properties of extremal functions for Hardy–Sobolev inequalities is widely studied in the recent literature and it is still object of investigation even in the Euclidean elliptic setting (see, e.g. the recent study performed in [25] for the quasilinear elliptic case).

We recall that, in the context of Carnot groups, the following Hardy–Sobolev type inequality holds.

Theorem 6.1. (Hardy–Sobolev type inequality) *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and $0 \leq s \leq 2$. Then, there exists a constant $C = C(s, Q) > 0$ such that for any $u \in S^1(\mathbb{G})$ the following inequality holds*

$$\int_{\mathbb{G}} \psi(\xi)^s \frac{|u|^{p^*(s)}}{d(\xi)^s} \leq C \left(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \right)^{\frac{Q-s}{Q-2}}, \quad (6.1)$$

where $p^*(s) = \frac{2(Q-s)}{Q-2}$ and $\psi := |\nabla_{\mathbb{G}} d|$.

Note that, if one is not interested in the value of the best constant, the preceding inequality can be immediately obtained by combining the Hardy–type inequality on groups (see, e.g. [8]) with the Folland–Stein Sobolev embedding. A more direct proof, provided for the subclass of the H-type groups, can be found in [13].

In the pure Hardy case (namely when $s = 2$) the best constant in (6.1) is never attained (see, e.g. [8]); in the case $0 \leq s < 2$, instead, extremal functions for the above inequality exist and their existence can be proved by means of a suitable version of Lions’ Concentration–Compactness method applied to the stratified context (see, e.g. [12] for the case $s = 0$, and [13] for the case $0 < s < 2$ when $\mathbb{G}$ is an $\mathbb{H}$ -type group). Precisely, for a general Carnot group, the following theorem holds.

Theorem 6.2. *Let $\mathbb{G}$ be a Carnot group. Then, the infimum in the extremal problem*

$$I = \inf \left\{ \int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \mid u \in S^1(\mathbb{G}), \int_{\mathbb{G}} \psi(\xi)^s \frac{|u|^{p^*(s)}}{d(\xi)^s} = 1, 0 \leq s < 2 \right\} \quad (6.2)$$

is attained at some function $u \in S^1(\mathbb{G})$.

Observe that the explicit form of these extremal functions is not known, except for the case when $s = 0$ and $\mathbb{G}$ is the Heisenberg group $\mathbb{H}^n$, in which case they have been determined by Jerison and Lee [15]. However, we know that these functions obviously satisfy, up to a suitable stretching constant, the singular problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \psi(\xi)^s \frac{u^{p^*(s)-1}}{d(\xi)^s}, \\ u > 0 \\ u \in S^1(\mathbb{G}). \end{cases} \quad (6.3)$$

Now, since the function ψ which appears in the right hand side is δ_{λ} -homogeneous of degree 0 and therefore it is a bounded function, the coefficient of the nonlinearity $a(\xi) = \frac{\psi(\xi)^s}{d(\xi)^s}$ belongs to the weak space $M^{\frac{Q}{s}}$.

Moreover, being $\frac{\alpha}{Q} + \frac{p^*(s)}{2^*} = 1$, the coefficient $a(\xi)$ satisfies assumption (1.2) and so the above problem is of the form (1.1).

Hence, by applying our results, we get that any solution u to pb. (6.3), and in particular any extremal function for (6.2), has the sharp summability properties ensured by Proposition 4.4 and behaves like the fundamental solution $\frac{1}{d^{Q-2}}$ at infinity. We conclude by resuming below this final result.

Proposition 6.3. *Let $u \in S^1(\mathbb{G})$ be an extremal function for problem (6.2). Then, $u(\xi) \simeq d(\xi)^{2-Q}$ as $d(\xi) \rightarrow \infty$.*

This result generalizes and completes some partial results contained in [14].

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A. Loiudice

Dipartimento di Matematica, Università degli Studi di Bari,
Via Orabona, 4, 70125 Bari, Italy
e-mail: loiudice@dm.uniba.it

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