Multiple solutions for nonhomogeneous Schrödinger–Maxwell and Klein–Gordon–Maxwell equations on $\mathbf{R}^3$

Shang-Jie Chen and Chun-Lei Tang

Abstract. In this paper we study the following nonhomogeneous Schrödinger–Maxwell equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u) + h(x), & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbf{R}^3, \end{cases}$$

where f satisfies the Ambrosetti–Rabinowitz type condition. Under appropriate assumptions on V , f and h , the existence of multiple solutions is proved by using the Ekeland’s variational principle and the Mountain Pass Theorem in critical point theory. Similar results for the nonhomogeneous Klein–Gordon–Maxwell equations

$$\begin{cases} -\Delta u + [m^2 - (\omega + \phi)^2]u = |u|^{q-2}u + h(x), & \text{in } \mathbf{R}^3, \\ -\Delta \phi + \phi u^2 = -\omega u^2, & \text{in } \mathbf{R}^3, \end{cases}$$

are also obtained when $2 < q < 6$.

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1. Introduction and main results

In the first part of this paper, we study the following system of Schrödinger–Maxwell equations:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u) + h(x), & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbf{R}^3. \end{cases} \quad (1)$$

Such a system, also called Schrödinger–Poisson equations, arises in an interesting physical context. In fact, according to a classical model, the interaction of a charge particle with an electro-magnetic field can be described by coupling the nonlinear Schrödinger’s and Maxwell’s equations (we refer the reader to [8] and the references therein for more details on the physical aspects). In particular, if we are looking for electrostatic-type solutions, we just have to solve problem (1).

Let us recall some recent results in the literature of nonlinear Schrödinger–Maxwell equations (1). The case of $h \equiv 0$, that is the homogeneous case, has been studied widely in [4, 14–17, 20, 22, 24, 25]) when V is a constant or radially symmetric, and in [27, 29] when V is not radially symmetric. Very recently, Azzollni and Pomponio in [1] proved the existence of a ground state solution (namely for solution which minimizes the action functional among all the solutions) for system (1) with $f(x, u) = |u|^{s-2}u$ ($4 < s < 6$) and non-constant potential V which may be unbounded from below; Zhao and Zhao [28] established the existence of a positive solution for problem (1) with a critical Sobolev exponent and constant potential V ; Chen and Tang [13] obtained the existence of infinitely many large energy solutions for system (1) with $f(x, u)$ satisfying Ambrosetti–Rabinowitz type condition [see (f2)] and V being non-radially symmetric.

The nonhomogeneous case, that is $h \neq 0$, has been also studied in [25], and the existence of multiple radially symmetric solutions for problem (1) was obtained.

The object of the first part of this paper is to consider the case that $h \neq 0$ and V is not radially symmetric for the Schrödinger–Maxwell equations (1). Precisely, we make the following assumptions.

- (v1) $V \in C(\mathbf{R}^3, \mathbf{R})$ satisfies $\inf_{x \in \mathbf{R}^3} V(x) \geq a_1 > 0$, where $a_1 > 0$ is a constant. Moreover, for every $M > 0$, $meas(\{x \in \mathbf{R}^3 : V(x) \leq M\}) < \infty$, where $meas$ denote the Lebesgue measure in $\mathbf{R}^3$.
- (f1) $f \in C(\mathbf{R}^3 \times \mathbf{R}, \mathbf{R})$ and, for some $2 < p < 2^* = 6$, $a_2 > 0$,

$$|f(x, z)| \leq a_2(1 + |z|^{p-1}),$$

for a.e. $x \in \mathbf{R}^3$ and all $z \in \mathbf{R}$.

- (f2) There exists $\mu > 4$ such that

$$\mu F(x, z) := \mu \int_0^z f(x, y) dy \leq z f(x, z),$$

for every $x \in \mathbf{R}^3$ and all $z \in \mathbf{R}$.

- (f3) $f(x, z)/z \rightarrow 0$, as $z \rightarrow 0$, uniformly for $x \in \mathbf{R}^3$.
- (f4)

$$\inf_{x \in \mathbf{R}^3, |z|=1} F(x, z) > 0.$$

Before stating our main results, we give several notations. Define the function space

$$H^1(\mathbf{R}^3) := \{u \in L^2(\mathbf{R}^3) : |\nabla u| \in L^2(\mathbf{R}^3)\}$$

with the usual norm

$$\|u\|_{H^1} := \left(\int_{\mathbf{R}^3} (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}},$$

and define the function space

$$D^{1,2}(\mathbf{R}^3) := \left\{ u \in L^{2^*}(\mathbf{R}^3) : |\nabla u| \in L^2(\mathbf{R}^3) \right\}$$

with the norm

$$\|u\|_{D^{1,2}} := \left(\int_{\mathbf{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Let

$$E := \left\{ u \in H^1(\mathbf{R}^3) : \int_{\mathbf{R}^3} (|\nabla u|^2 + V(x)u^2) dx < \infty \right\}.$$

Then E is a Hilbert space with the inner product

$$(u, v)_E = \int_{\mathbf{R}^3} (\nabla u \cdot \nabla v + V(x)uv) dx$$

and the norm $\|u\|_E = (u, u)_E^{1/2}$. Obviously, the embedding $E \hookrightarrow L^s(\mathbf{R}^3)$ is continuous, for any $s \in [2, 2^*]$.

We can now state the first result:

Theorem 1.1. *Suppose that $h \in L^2(\mathbf{R}^3)$ and $h \not\equiv 0$. Let (v1), (f1–f4) hold, then there exists a constant $m_0 > 0$ such that problem (1) admits at least two different solutions in $E \times D^{1,2}(\mathbf{R}^3)$ when $\|h\|_{L^2} < m_0$.*

Remark 1.2. (a) The condition in (v1), which implies the compactness of embedding of the working space E (see Lemma 2.1) and contains the coercivity condition: $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, was first introduced by Bartsch and Wang in [5] to overcome the lack of compactness.

(b) It is not difficult to find out functions f satisfying (f1–f4), for example,

$$f(x, z) = |z|^{s-2}z, \quad 4 < s < 6.$$

(c) Compared with the results in [25] where the existence of multiple radially symmetric solutions on $\mathbf{R}^3$ for both the homogeneous and the nonhomogeneous system (1) is proved, in Theorem 1.1 multiple non-radially symmetrical solutions for system (1) can be obtained because the potential V may be not radially symmetric in our results. By the way, we would like to point out that in [11] the authors study the existence of standing waves which are solutions of a nonlinear Schrödinger equation stated in a ball and coupled with Maxwell's equations when a nonhomogeneous term breaks the symmetry of the associated functional.

In the second part of this paper, we deal with the following system of Klein–Gordon–Maxwell equations

$$\begin{cases} -\Delta u + [m^2 - (\omega + e\phi)^2]u = f(x, u) + h(x), & \text{in } \mathbf{R}^3, \\ -\Delta \phi + e^2 \phi u^2 = -e\omega u^2, & \text{in } \mathbf{R}^3, \end{cases} \quad (2)$$

where m , ω and e are real constants. Such a system has been first introduced in [6] as a model describing solitary waves for the nonlinear stationary Klein–Gordon equation in the three-dimensional space interacting with the electrostatic field. Here m and e are the mass and the charge of the particle respectively, while ω denote the phase. The unknowns of the system are the field u associated to the particle and the electric potential ϕ . The presence of the nonlinear term simulates the interaction between many particles or external nonlinear perturbations.

In recent years, some existence and nonexistence results for the Klein–Gordon–Maxwell equations (2) have been proved in the case $f(x, u) = |u|^{q-2}u$ and $h(x) \equiv 0$. In [6, 7] the authors found the existence of infinitely many radially symmetric solutions (u, ϕ) in $H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ for system (2) when $4 < q < 6$; in [17] the range $q \in (2, 4]$ was also covered; in [2] the existence of a ground state solution $(u, \phi) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ for problem (2) was get when $4 \leq q < 6$ and $m > \omega$ or $2 < q \leq 4$ and $m\sqrt{q-2} > \omega\sqrt{6-q}$; in [16], the nonexistence results for system (2) were obtained for $q \leq 2$ or $q \geq 6$, respectively. For the case $f(x, u) = \lambda|u|^{q-2}u + |u|^{2^*-2}u$ and $h \equiv 0$, Cassani in [12] get that problem (2) has at least a radially symmetric (nontrivial) solution $(u, \phi) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ when $4 < q < 6$ or $q = 4$ provided that λ is sufficiently large.

The object of the second part of this paper is to consider the case of $h \not\equiv 0$ in the Klein–Gordon–Maxwell equations (2), and we have the following result:

Theorem 1.3. *Suppose that $h \in C^1(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$ is a radial function and $h \not\equiv 0$. If $m > \omega > 0$, $e = 1$ and $f(x, u) = |u|^{q-2}u$ ($2 < q < 6$), there exists a constant $m_1 > 0$ such that the problem in (2) has at least two different solutions in $H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ when $\|h\|_{L^2} < m_0$.*

Remark 1.4. To the best of our knowledge, it seems that Theorem 1.3 is the first result about the existence of multiple solutions for the nonhomogeneous Klein–Gordon–Maxwell equations on $\mathbf{R}^3$.

2. Nonlinear Schrödinger equations coupled with Maxwell equations

In this section, we consider the following nonhomogeneous Schrödinger–Maxwell system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u) + h(x), & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbf{R}^3, \end{cases} \quad (1)$$

where $h \in L^2(\mathbf{R}^3)$, $h \not\equiv 0$, V satisfies (v1), and f satisfies (f1–f4).

In order to apply critical point theory, we first state a key property of E to our proof, then reduce the problem of finding solutions of system (1) to the one of seeking the critical points of a corresponding variational functional.

In Sect. 1, we know that the embedding $E \hookrightarrow L^s(\mathbf{R}^3)$ is continuous, for any $s \in [2, 2^*]$. Furthermore, we have the following result:

Lemma 2.1. ([30], Lemma 3.4) *Under assumption (v1), the embedding $E \hookrightarrow L^s(\mathbf{R}^3)$ is compact for any $s \in [2, 2^*)$.*

System (1) has a variational structure. Indeed we consider the functional

$$J : E \times D^{1,2}(\mathbf{R}^3) \rightarrow \mathbf{R}$$

defined by

$$\begin{aligned} J(u, \phi) = & \frac{1}{2} \|u\|_E^2 - \frac{1}{4} \int_{\mathbf{R}^3} |\nabla \phi|^2 dx + \frac{1}{2} \int_{\mathbf{R}^3} \phi u^2 dx \\ & - \int_{\mathbf{R}^3} F(x, u) dx - \int_{\mathbf{R}^3} h(x) u dx. \end{aligned}$$

Evidently, the action functional J belongs to $C^1(E \times D^{1,2}(\mathbf{R}^3), \mathbf{R})$ and the partial derivatives in (u, ϕ) are given, for $\zeta \in E$ and $\eta \in D^{1,2}(\mathbf{R}^3)$, by

$$\begin{aligned} \left\langle \frac{\partial J}{\partial u}(u, \phi), \zeta \right\rangle &= \int_{\mathbf{R}^3} (\nabla u \cdot \nabla \zeta + V(x) u \zeta + \phi u \zeta - f(x, u) \zeta - h(x) \zeta) dx, \\ \left\langle \frac{\partial J}{\partial \phi}(u, \phi), \eta \right\rangle &= -\frac{1}{2} \int_{\mathbf{R}^3} \nabla \phi \cdot \nabla \eta dx + \frac{1}{2} \int_{\mathbf{R}^3} u^2 \eta dx. \end{aligned}$$

Thus, we have the following result:

Proposition 2.2. *The pair (u, ϕ) is a weak solution of problem (1) if and only if it is a critical point of J in $E \times D^{1,2}(\mathbf{R}^3)$.*

The functional J exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [9], by which we are led to study a one variable functional that does not present such a strongly indefinite nature.

For every $u \in E$, the Lax–Milgram theorem (see [19]) implies that there exists a unique function $\phi_u \in D^{1,2}(\mathbf{R}^3)$ such that

$$-\Delta \phi_u = u^2$$

in a weak sense. Moreover, we can write an integral expression for ϕ_u in the form:

$$\phi_u = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{u^2(y)}{|x-y|} dy, \quad (3)$$

for any $u \in E$ (for detail, see Sect. 2 of [13]). So, we can consider the functional $I : E \rightarrow \mathbf{R}$ defined by $I(u) = J(u, \phi_u)$. After multiplying $-\Delta \phi_u = u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbf{R}^3} |\nabla \phi_u|^2 dx = \int_{\mathbf{R}^3} \phi_u u^2 dx.$$

Therefore, the reduced functional takes the form

$$I(u) = \frac{1}{2} \|u\|_E^2 + \frac{1}{4} \int_{\mathbf{R}^3} \phi_u u^2 dx - \int_{\mathbf{R}^3} F(x, u) dx - \int_{\mathbf{R}^3} h(x) u dx. \quad (4)$$

Next, we show that I is well defined. Actually, using integration by parts, $-\Delta\phi_u = u^2$, the Hölder inequality and the Sobolev inequality, for any $u \in E$ we get

$$\begin{aligned}\|\phi_u\|_{D^{1,2}}^2 &= -\int_{\mathbf{R}^3} \phi_u \Delta\phi_u dx \\ &= \int_{\mathbf{R}^3} \phi_u u^2 dx \\ &\leq \|\phi_u\|_{L^6} \|u\|_{L^{12/5}}^2 \\ &\leq a_3 \|\phi_u\|_{D^{1,2}} \|u\|_{L^{12/5}}^2,\end{aligned}$$

where $a_3 > 0$ is a constant and

$$\|u\|_{L^p} := \left(\int_{\mathbf{R}^3} |u|^p dx \right)^{\frac{1}{p}} \text{ for any } p \in [1, +\infty),$$

is the norm of the usual Lebesgue space $L^p(\mathbf{R}^3)$. It follows that

$$\|\phi_u\|_{D^{1,2}} \leq a_3 \|u\|_{L^{12/5}}^2, \quad (5)$$

hence there exists a constant $a_4 > 0$ such that

$$\int_{\mathbf{R}^3} \phi_u u^2 dx \leq a_3 \|\phi_u\|_{D^{1,2}} \|u\|_{L^{12/5}}^2 \leq a_3^2 \|u\|_{L^{12/5}}^4 \leq a_4 \|u\|_E^4, \quad (6)$$

because $E \hookrightarrow L^s(\mathbf{R}^3)$, for any $s \in [2, 2^*]$. Thus, I is well defined. Furthermore, I is a C^1 functional with derivative given by

$$\langle I'(u), v \rangle = \int_{\mathbf{R}^3} \left(\nabla u \cdot \nabla v + V(x)uv + \phi_u uv - f(x, u)v - h(x)v \right) dx. \quad (7)$$

Now, we can apply Theorem 2.3 of [9] to our functional J and obtain:

Proposition 2.3. *The following statements are equivalent:*

- (1) $(u, \phi) \in E \times D^{1,2}(\mathbf{R}^3)$ is a critical point of J (i.e. (u, ϕ) is a solution of system (1));
- (2) u is a critical point of I and $\phi = \phi_u$.

Now we give a sketch of how to look for two distinct critical points of the functional I . First, we consider a minimization of I constrained in a neighborhood of zero, by using the Ekeland's variational principle (see [18], or [21]), we can find a critical point of I which achieves the local minimum of I . Moreover, this local minimum is negative, see the Step 1 in the proof of Theorem 1.1. Next, around “zero”, by the Mountain Pass Theorem (see [3], or [23]), we can also get a critical point of I and its level is positive (see Step 2 in the proof of Theorem 1.1). Because these two critical points are in different level, they must be distinct.

Before going to the proof of Theorem 1.1, we give some useful preliminary results.

Lemma 2.4. *Suppose that (v1), (f1) and (f3) hold. Let $h \in L^2(\mathbf{R}^3)$. Then there exist some constants $\rho, \alpha, m_0 > 0$ such that $I(u)|_{\|u\|_E=\rho} \geq \alpha$ for all h satisfying $\|h\|_{L^2} < m_0$.*

Proof. It follows from (f1), (f3) and the equality $F(x, z) = \int_0^1 f(x, tz)z dt$ that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$F(x, z) \leq \varepsilon |z|^2 + C_\varepsilon |z|^p$$

for any $x \in \mathbf{R}^3$ and all $z \in \mathbf{R}$. Thus, using $\phi_u \geq 0$ [see (3)], the Hölder inequality and $H^1(\mathbf{R}^3) \hookrightarrow L^s(\mathbf{R}^3)$ for any $s \in [2, 2^*]$, we have

$$\begin{aligned} I(u) &\geq \frac{1}{2} \|u\|_E^2 - \varepsilon \|u\|_{L^2}^2 - C_\varepsilon \|u\|_{L^p}^p - \|h\|_{L^2} \|u\|_{L^2} \\ &\geq \frac{1}{2} \|u\|_E^2 - \frac{\varepsilon}{a_1} \|u\|_E^2 - a_5 \|u\|_E^p - \frac{1}{\sqrt{a_1}} \|h\|_{L^2} \|u\|_E \\ &= \|u\|_E \left[\left(\frac{1}{2} - \frac{\varepsilon}{a_1} \right) \|u\|_E - a_5 \|u\|_E^{p-1} - \frac{1}{\sqrt{a_1}} \|h\|_{L^2} \right], \end{aligned}$$

where a_1 is a lower bound of the potential V from (v1) and $a_5 > 0$ is a constant. Taking $\varepsilon = a_1/4$ and setting

$$g(t) = \frac{1}{4}t - a_5 t^{p-1}$$

for $t \geq 0$, we see that there exists a constant $\rho > 0$ such that $\max_{t \geq 0} g(t) = g(\rho) > 0$. Taking $m_0 := \frac{1}{2}\sqrt{a_1}g(\rho)$, then it follows that there exists a constant $\alpha > 0$ such that $I(u)|_{\|u\|_E=\rho} \geq \alpha$ for all h satisfying $\|h\|_{L^2} < m_0$. $\square$

Lemma 2.5. *Suppose that (v1), (f2) and (f4) hold, then there exists a function $v \in E$ with $\|v\|_E > \rho$ such that $I(v) < 0$, where ρ is given by Lemma 2.4.*

Proof. Let

$$k(t) = t^{-\mu} F(x, tz) - F(x, z), \quad t \geq 1,$$

then, we get

$$k'(t) = t^{-\mu-1} (f(x, tz)tz - \mu F(x, tz)) \geq 0$$

for all $t \geq 1$ by (f2). Hence, it follows that $k(t) \geq k(1) = 0$ for all $t \geq 1$, that is

$$F(x, tz) \geq t^\mu F(x, z)$$

for all $x \in \mathbf{R}^3, z \in \mathbf{R}$ and $t \geq 1$. Thus, by (6), $\mu > 4$ and (f4) we have

$$\begin{aligned} I(tu) &= \frac{1}{2} t^2 \|u\|_E^2 + \frac{1}{4} \int_{\mathbf{R}^3} \phi_{tu}(tu)^2 dx - \int_{\mathbf{R}^3} F(x, tu) dx - t \int_{\mathbf{R}^3} h(x) u dx \\ &\leq \frac{1}{2} t^2 \|u\|_E^2 + \frac{t^4}{4} \|u\|_E^4 - t^\mu \int_{\mathbf{R}^3} F(x, u) dx - t \int_{\mathbf{R}^3} h(x) u dx \\ &\rightarrow -\infty, \end{aligned}$$

as $t \rightarrow +\infty$ for $u \in E, u \neq 0$. The lemma is proved by taking $v = t_0 u$ with $t_0 > 0$ large enough and $u \neq 0$. $\square$

Lemma 2.6. *Assume that (v1), (f1–f3) hold, and $\{u_n\} \subset E$ is a bounded Palais–Smale sequence of I , then $\{u_n\}$ has a strongly convergent subsequence in E .*

Proof. Consider a sequence $\{u_n\}$ in E which satisfies

$$I(u_n) \rightarrow c, \quad I'(u_n) \rightarrow 0, \quad \text{and} \quad \sup_n \|u_n\|_E < +\infty.$$

Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E . In view of Lemma 2.1, $u_n \rightarrow u$ in $L^s(\mathbf{R}^3)$ for any $s \in [2, 2^*)$. By (7), we easily get

$$\begin{aligned} \|u_n - u\|_E^2 &= \langle I'(u_n) - I'(u), u_n - u \rangle + \int_{\mathbf{R}^3} (f(x, u_n) - f(x, u))(u_n - u) dx \\ &\quad - \int_{\mathbf{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) dx. \end{aligned}$$

It is clear that

$$\langle I'(u_n) - I'(u), u_n - u \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

According to assumptions (f1) and (f3), there exists a constant $a_6 > 0$ such that

$$f(x, u) \leq \varepsilon |u| + a_6 |u|^{p-1}$$

for a.e. $x \in \mathbf{R}^3$, and all $z \in \mathbf{R}$. Using the Hölder inequality, we obtain

$$\begin{aligned} &\int_{\mathbf{R}^3} (f(x, u_n) - f(x, u))(u_n - u) dx \\ &\leq \int_{\mathbf{R}^3} [\varepsilon(|u_n| + |u|) + a_6(|u_n|^{p-1} + |u|^{p-1})] |u_n - u| dx \\ &\leq \varepsilon (\|u_n\|_{L^2} + \|u\|_{L^2}) \|u_n - u\|_{L^2} + a_6 \left(\|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1} \right) \|u_n - u\|_{L^p}. \end{aligned}$$

Since $u_n \rightarrow u$ in $L^s(\mathbf{R}^3)$ for any $s \in [2, 2^*)$, we have

$$\int_{\mathbf{R}^3} (f(x, u_n) - f(x, u))(u_n - u) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

By the Hölder inequality, the Sobolev inequality and (5), we have

$$\begin{aligned} \left| \int_{\mathbf{R}^3} \phi_{u_n} u_n (u_n - u) dx \right| &\leq \|\phi_{u_n} u_n\|_{L^2} \|u_n - u\|_{L^2} \\ &\leq \|\phi_{u_n}\|_{L^6} \|u_n\|_{L^3} \|u_n - u\|_{L^2} \\ &\leq a_7 \|\phi_{u_n}\|_{D^{1,2}} \|u_n\|_{L^3} \|u_n - u\|_{L^2} \\ &\leq a_3 a_7 \|u_n\|_{L^{12/5}}^2 \|u_n\|_{L^3} \|u_n - u\|_{L^2}, \end{aligned}$$

where $a_7 > 0$ is a constant. Again using $u_n \rightarrow u$ in $L^s(\mathbf{R}^3)$ for any $s \in [2, 2^*)$, we have

$$\int_{\mathbf{R}^3} \phi_{u_n} u_n (u_n - u) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Similarly, we also obtain

$$\int_{\mathbf{R}^3} \phi_u u (u_n - u) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Thus, we get

$$\int_{\mathbf{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

so that $\|u_n - u\|_E \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Proof of Theorem 1.1. The proof of this theorem is divided in two steps.

Step 1 There exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) < 0$.

Observe that assumptions (f1–f3) and the equality $F(x, z) = \int_0^1 f(x, tz) z dt$ imply that there exist two positive constants a_8, a_9 such that

$$F(x, z) \geq a_8 |z|^\mu - a_9 |z|^2$$

for all $x \in \mathbf{R}^3$, and all $z \in \mathbf{R}$.

Since $h \in L^2(\mathbf{R}^3)$ and $h \not\equiv 0$, we can choose a function $\psi \in E$ such that

$$\int_{\mathbf{R}^3} h(x) \psi(x) dx > 0.$$

Hence, we have

$$I(t\psi) \leq \frac{1}{2} t^2 \|\psi\|_E^2 + \frac{t^4}{4} \|\psi\|_E^4 - a_8 t^\mu \|\psi\|_{L^\mu}^\mu + a_9 t^2 \|\psi\|_{L^2}^2 - t \int_{\mathbf{R}^3} h(x) \psi dx < 0$$

for $t > 0$ small enough, thus, we obtain

$$c_0 = \inf\{I(u) : u \in \overline{B}_\rho\} < 0,$$

where $\rho > 0$ is given by Lemma 2.4, $B_\rho = \{u \in E : \|u\|_E \leq \rho\}$. By the Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \leq I(u_n) < c_0 + \frac{1}{n},$$

and

$$I(w) \geq I(u_n) - \frac{1}{n} \|w - u_n\|_E$$

for all $w \in \overline{B}_\rho$. Then by a standard procedure, we can show that $\{u_n\}$ is a bounded Palais–Smale sequence of I . Therefore Lemma 2.6 implies that there exists a function $u_0 \in E$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

Step 2 There exists a function $\tilde{u}_0 \in E$ such that $I'(\tilde{u}_0) = 0$ and $I(\tilde{u}_0) > 0$.

From Lemmas 2.4, 2.5 and the Mountain Pass Theorem, there is a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \rightarrow \tilde{c}_0 > 0, \quad \text{and} \quad I'(u_n) \rightarrow 0.$$

In view of Lemma 2.6, we only need to check that $\{u_n\}$ is bounded in E . By (7) and (f2) we have

$$\begin{aligned}
\tilde{c}_0 + 1 + \|u_n\|_E &\geq I(u_n) - \frac{1}{\mu} \langle I'(u_n), u_n \rangle \\
&= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|_E^2 + \left(\frac{1}{4} - \frac{1}{\mu} \right) \int_{\mathbf{R}^3} \phi_{u_n} u_n^2 dx \\
&\quad + \int_{\mathbf{R}^3} \left(\frac{1}{\mu} f(x, u_n) u_n - F(x, u_n) \right) dx \\
&\quad + \left(\frac{1}{\mu} - 1 \right) \int_{\mathbf{R}^3} h(x) u_n dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|_E^2 + \left(\frac{1}{4} - \frac{1}{\mu} \right) \int_{\mathbf{R}^3} \phi_{u_n} u_n^2 dx \\
&\quad + \left(\frac{1}{\mu} - 1 \right) \frac{1}{\sqrt{a_1}} \|h\|_{L^2} \|u_n\|_E
\end{aligned}$$

for n large enough. Since $\mu > 4$ and $\|h\|_{L^2} < m_0$, it follows from (3) that $\{u_n\}$ is bounded in E . $\square$

3. Nonlinear Klein–Gordon equations coupled with Maxwell equations

In this section, we consider the problem in (2) with $e = 1$ and $f(x, u) = |u|^{q-2}u$, that is,

$$\begin{cases} -\Delta u + [m^2 - (\omega + \phi)^2]u = |u|^{q-2}u + h(x), & \text{in } \mathbf{R}^3, \\ -\Delta \phi + \phi u^2 = -\omega u^2, & \text{in } \mathbf{R}^3, \end{cases} \quad (8)$$

where $h \in L^2(\mathbf{R}^3) \cap C^1(\mathbf{R}^3)$ is a radial function, $h \not\equiv 0$, $m > \omega > 0$ and $2 < q < 6$.

System (8) has a variational structure. Indeed we consider the functional

$$\mathcal{J} : H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3) \rightarrow \mathbf{R}$$

defined by

$$\begin{aligned}
\mathcal{J}(u, \phi) &= \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 - |\nabla \phi|^2 + [m^2 - (\omega + \phi)^2]u^2) dx \\
&\quad - \frac{1}{q} \int_{\mathbf{R}^3} |u|^q dx - \int_{\mathbf{R}^3} h(x) u dx.
\end{aligned}$$

Evidently, the action functional $\mathcal{J}$ belongs to $C^1(H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3), \mathbf{R})$ and the partial derivatives in (u, ϕ) are given, for $\zeta \in H^1(\mathbf{R}^3)$ and $\eta \in D^{1,2}(\mathbf{R}^3)$, by

$$\begin{aligned}
\left\langle \frac{\partial \mathcal{J}}{\partial u}(u, \phi), \zeta \right\rangle &= \int_{\mathbf{R}^3} (\nabla u \cdot \nabla \zeta + [m^2 - (\omega + \phi)^2]u\zeta - |u|^{q-2}u\zeta - h(x)\zeta) dx, \\
\left\langle \frac{\partial \mathcal{J}}{\partial \phi}(u, \phi), \eta \right\rangle &= \int_{\mathbf{R}^3} (-\nabla \phi \cdot \nabla \eta - (\phi + \omega)u^2\eta) dx.
\end{aligned}$$

Thus, we have the following result:

Proposition 3.1. *The pair (u, ϕ) is a weak solution of system (8) if and only if it is a critical point of $\mathcal{J}$ in $H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$.*

Moreover, we have the following lemma.

Lemma 3.2. ([16] and [17]) *For every $u \in H^1(\mathbf{R}^3)$ there exists a unique function $\phi_u \in D^{1,2}(\mathbf{R}^3)$ which solves the second equation of system (8). Furthermore*

- (i) $-\omega \leq \phi_u \leq 0$ in $\mathbf{R}^3$;
- (ii) *if u is radially symmetric, ϕ_u is radial too.*

So, we can consider the functional $\mathcal{I} : H^1(\mathbf{R}^3) \rightarrow \mathbf{R}$ defined by $\mathcal{I}(u) = \mathcal{J}(u, \phi_u)$. After multiplying $-\Delta\phi_u + \phi_u u^2 = -\omega u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbf{R}^3} (|\nabla\phi_u|^2 + (\phi_u)^2 u^2) dx = -\omega \int_{\mathbf{R}^3} \phi_u u^2 dx.$$

Therefore, the reduced functional takes the form

$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + [m^2 - \omega^2] u^2 - \omega \phi_u u^2) dx - \frac{1}{q} \int_{\mathbf{R}^3} |u|^q dx - \int_{\mathbf{R}^3} h(x) u dx.$$

Furthermore $\mathcal{I}$ is C^1 and we have, for any $u, v \in H^1(\mathbf{R}^3)$,

$$\langle \mathcal{I}'(u), v \rangle = \int_{\mathbf{R}^3} (\nabla u \cdot \nabla v + [m^2 - \omega^2] uv - \omega \phi_u uv - |u|^{q-2} uv - h(x)v) dx. \quad (9)$$

Now, we can apply Theorem 2.3 of [9] to our functional $\mathcal{J}$ and obtain:

Proposition 3.3. *The following statements are equivalent:*

- (1) $(u, \phi) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ is a critical point of $\mathcal{J}$ (i.e. (u, ϕ) is a solution of system (8));
- (2) u is a critical point of $\mathcal{I}$ and $\phi = \phi_u$.

Since system (8) is set on $\mathbf{R}^3$, it is well known that the Sobolev embedding $H^1(\mathbf{R}^3) \hookrightarrow L^s(\mathbf{R}^3)$ ($2 \leq s \leq 6$) is not compact, and then it is usually difficult to prove that a minimizing sequence or a Palais–Smale sequence is strongly convergent if we seek solutions of system (8) by variational methods. To overcome this difficulty we restrict ourselves to radial functions $u = u(r)$, $r = |x|$. More precisely we shall consider the functional $\mathcal{I}$ on the space of the radial functions

$$H_r^1(\mathbf{R}^3) := \{u \in H^1(\mathbf{R}^3) : u = u(r), r = |x|\}.$$

$H_r^1(\mathbf{R}^3)$ is a natural constraint for $\mathcal{I}$, namely any critical point $u \in H_r^1(\mathbf{R}^3)$ of $\mathcal{I}|_{H_r^1(\mathbf{R}^3)}$ is also a critical point of $\mathcal{I}$. Then we are reduced to look for critical points of $\mathcal{I}|_{H_r^1(\mathbf{R}^3)}$. We recall [see [26] or [10]] that, for $2 < s < 6$, $H_r^1(\mathbf{R}^3)$ is compactly embedded into $L^s(\mathbf{R}^3)$. In the following, we still denote $\mathcal{I}|_{H_r^1(\mathbf{R}^3)}$ by $\mathcal{I}$.

Before going to the proof of Theorem 1.3, we give some useful preliminary results.

Lemma 3.4. *Let $m > \omega > 0$, $2 < q < 6$, and $h \in L^2(\mathbf{R}^3) \cap C^1(\mathbf{R}^3)$ is a radial function. Then there exist some constants $\rho_1, \alpha_1, m_1 > 0$ such that $\mathcal{I}(u)|_{\|u\|_{H^1}=\rho_1} \geq \alpha_1$ for all h satisfying $\|h\|_{L^2} < m_1$.*

Proof. Using $\phi_u \leq 0$ (see (i) of Lemma 3.2), the Hölder inequality and $H_r^1(\mathbf{R}^3) \hookrightarrow L^s(\mathbf{R}^3)$ for any $s \in [2, 2^*]$, we have

$$\begin{aligned} \mathcal{I}(u) &\geq \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + [m^2 - \omega^2]u^2) dx - \frac{1}{q} \|u\|_{L^q}^q - \|h\|_{L^2} \|u\|_{L^2} \\ &\geq \frac{1}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + [m^2 - \omega^2]u^2) dx - a_{10} \|u\|_{H^1}^q - \|h\|_{L^2} \|u\|_{H^1} \\ &= \|u\|_{H^1} \left[a_{11} \|u\|_{H^1} - a_{10} \|u\|_{H^1}^{q-1} - \|h\|_{L^2} \right], \end{aligned}$$

where $a_{11} = \frac{\min\{1, m^2 - \omega^2\}}{2}$ and a_{10} are positive constants. Setting

$$g_1(t) = a_{11}t - a_{10}t^{q-1}$$

for $t \geq 0$, we see that there exists a constant $\rho_1 > 0$ such that $\max_{t \geq 0} g_1(t) = g_1(\rho_1) > 0$. Taking $m_1 := \frac{1}{2}g_1(\rho_1)$, then it follows that there exists a constant $\alpha_1 > 0$ such that $\mathcal{I}(u)|_{\|u\|_{H^1}=\rho_1} \geq \alpha_1$ for all h satisfying $\|h\|_{L^2} < m_1$. $\square$

Lemma 3.5. *Let $m > \omega > 0$ and $2 < q < 6$. Then there exists a function $v_1 \in E$ with $\|v_1\|_{H^1} > \rho_1$ such that $\mathcal{I}(v_1) < 0$, where ρ_1 is given by Lemma 3.4.*

Proof. By (i) of Lemma 3.2 and $q > 2$ we have

$$\begin{aligned} \mathcal{I}(tu) &= \frac{t^2}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + [m^2 - \omega^2]u^2 - \omega\phi_{tu}u^2) dx \\ &\quad - \frac{t^q}{q} \int_{\mathbf{R}^3} |u|^q dx - t \int_{\mathbf{R}^3} h(x)u dx \\ &\leq \frac{t^2}{2} \int_{\mathbf{R}^3} (|\nabla u|^2 + m^2u^2) dx - \frac{t^q}{q} \int_{\mathbf{R}^3} |u|^q dx - t \int_{\mathbf{R}^3} h(x)u dx \\ &\rightarrow -\infty, \end{aligned}$$

as $t \rightarrow +\infty$ for $u \in H_r^1(\mathbf{R}^3)$, $u \neq 0$. The lemma is proved by taking $v = t_1u$ with $t_1 > 0$ large enough and $u \neq 0$. $\square$

Lemma 3.6. *Assume that $m > \omega > 0$, $2 < q < 6$, and $\{u_n\} \subset H_r^1(\mathbf{R}^3)$ is a bounded Palais–Smale sequence of $\mathcal{I}$, then $\{u_n\}$ has a strongly convergent subsequence in $H_r^1(\mathbf{R}^3)$.*

Proof. Consider a sequence $\{u_n\}$ in $H_r^1(\mathbf{R}^3)$ which satisfies

$$\mathcal{I}(u_n) \rightarrow c, \quad \mathcal{I}'(u_n) \rightarrow 0, \quad \text{and} \quad \sup_n \|u_n\|_{H^1} < +\infty.$$

Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H_r^1(\mathbf{R}^3)$. Since the embedding $H_r^1(\mathbf{R}^3) \hookrightarrow L^s(\mathbf{R}^3)$ is compact for any $s \in (2, 2^*)$, we have $u_n \rightarrow u$ in $L^s(\mathbf{R}^3)$ for any $s \in (2, 2^*)$. By (9), we easily get

$$\begin{aligned} &\int_{\mathbf{R}^3} (|\nabla(u_n - u)|^2 + [m^2 - \omega^2](u_n - u)^2) dx \\ &= \langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u \rangle + \int_{\mathbf{R}^3} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u) dx \\ &\quad + \omega \int_{\mathbf{R}^3} (\phi_{u_n}u_n - \phi_uu)(u_n - u) dx. \end{aligned}$$

It is clear that

$$\langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Since $u_n \rightarrow u$ in $L^s(\mathbf{R}^3)$ for any $s \in (2, 2^*)$, we have

$$\int_{\mathbf{R}^3} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

By the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned} \left| \int_{\mathbf{R}^3} (\phi_{u_n} - \phi_u)u_n(u_n - u)dx \right| &\leq \|(\phi_{u_n} - \phi_u)u_n\|_{L^2} \|u_n - u\|_{L^2} \\ &\leq \|\phi_{u_n} - \phi_u\|_{L^6} \|u_n\|_{L^3} \|u_n - u\|_{L^2} \\ &\leq a_{12} \|\phi_{u_n} - \phi_u\|_{D^{1,2}} \|u_n\|_{L^3} \|u_n - u\|_{L^2}, \end{aligned}$$

where $a_{13} > 0$ is a constant. Because $\phi_{u_n} \rightarrow \phi_u$ in $D^{1,2}(\mathbf{R}^3)$ (see Lemma 3.2 of [12]), we have

$$\int_{\mathbf{R}^3} (\phi_{u_n} - \phi_u)u_n(u_n - u)dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Using $u_n \rightarrow u$ in $L^s(\mathbf{R}^3)$ for any $s \in (2, 2^*)$, we obtain

$$\int_{\mathbf{R}^3} \phi_u(u_n - u)(u_n - u)dx \leq \|\phi_u\|_{L^6} \|u_n - u\|_{L^3} \|u_n - u\|_{L^2}, \quad \text{as } n \rightarrow \infty.$$

Thus, we get

$$\int_{\mathbf{R}^3} (\phi_{u_n}u_n - \phi_uu)(u_n - u)dx \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

so that

$$a_{13} \|u_n - u\|_{H^1} \leq \int_{\mathbf{R}^3} (|\nabla(u_n - u)|^2 + [m^2 - \omega^2](u_n - u)^2) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where $a_{13} = \min\{1, m^2 - \omega^2\}$. $\square$

Proof of Theorem 1.3. Since $h \in L^2(\mathbf{R}^3) \cap C^1(\mathbf{R}^3)$ is a radial function and $h \not\equiv 0$, we can choose a function $\psi \in H_r^1(\mathbf{R}^3)$ such that

$$\int_{\mathbf{R}^3} h(x)\psi(x)dx > 0.$$

Hence, we have

$$\mathcal{I}(tu) \leq \frac{t^2}{2} \int_{\mathbf{R}^3} (|\nabla\psi|^2 + m^2\psi^2) dx - \frac{t^q}{q} \int_{\mathbf{R}^3} |\psi|^q dx - t \int_{\mathbf{R}^3} h(x)\psi dx < 0$$

for $t > 0$ small enough, thus, we obtain

$$c_1 = \inf\{I(u) : u \in \overline{B}_{\rho_1}\} < 0,$$

where $\rho_1 > 0$ is given by Lemma 3.4, $B_{\rho_1} = \{u \in H_r^1(\mathbf{R}^3) : \|u\|_{H^1} < \rho_1\}$. By the Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_{\rho_1}$ such that

$$c_1 \leq \mathcal{I}(u_n) < c_1 + \frac{1}{n},$$

and

$$I(w) \geq \mathcal{I}(u_n) - \frac{1}{n} \|w - u_n\|_{H^1}$$

for all $w \in \overline{B}_{\rho_1}$. Then by a standard procedure, we can show that $\{u_n\}$ is bounded Palais-Smale sequence of $\mathcal{I}$. Thus Lemma 3.6 implies that there exists a function $u_1 \in H_r^1(\mathbf{R}^3)$ such that $\mathcal{I}'(u_1) = 0$ and $\mathcal{I}(u_1) = c_1 < 0$. $\square$

Next, we prove that problem (2) has a mountain pass solution $\tilde{u}_1$ satisfying $\mathcal{I}'(\tilde{u}_1) = 0$ and $\mathcal{I}(\tilde{u}_1) > 0$. From Lemmas 3.4, 3.5 and the Mountain Pass Theorem, there is a sequence $\{u_n\} \subset H_r^1(\mathbf{R}^3)$ such that

$$\mathcal{I}(u_n) \rightarrow \tilde{c}_1 > 0, \quad \text{and} \quad \mathcal{I}'(u_n) \rightarrow 0.$$

In view of Lemma 3.6, we only need to check that $\{u_n\}$ is bounded in $H_r^1(\mathbf{R}^3)$. By (9) we have

$$\begin{aligned} q(\tilde{c}_1 + 1) + \|u_n\|_{H^1} &\geq q\mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n \rangle \\ &= \left(\frac{q}{2} - 1\right) \int_{\mathbf{R}^3} (|\nabla u_n|^2 + [m^2 - \omega^2]u_n^2) dx \\ &\quad - \left(\frac{q}{2} - 1\right) \int_{\mathbf{R}^3} \omega \phi_{u_n} u_n^2 \\ &\quad - (q - 1) \int_{\mathbf{R}^3} h(x) u_n dx \\ &\geq \left(\frac{q}{2} - 1\right) \int_{\mathbf{R}^3} (|\nabla u_n|^2 + [m^2 - \omega^2]u_n^2) dx \\ &\quad - \left(\frac{q}{2} - 1\right) \int_{\mathbf{R}^3} \omega \phi_{u_n} u_n^2 \\ &\quad - (q - 1) \|h\|_{L^2} \|u_n\|_{H^1} \end{aligned}$$

for n large enough. Since $q > 2$ and $\|h\|_{L^2} < m_1$, it follows from (i) of Lemma 3.2 that $\{u_n\}$ is bounded in $H_r^1(\mathbf{R}^3)$.

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References

- [1] Azzollini, A., Pomponio, A.: Ground state solutions for the nonlinear Schrödinger–Maxwell equations. *J. Math. Anal. Appl.* **345**(1), 90–108 (2008)
- [2] Azzollini, A., Pomponio, A.: Ground state solutions for the nonlinear Klein–Gordon–Maxwell equations. Preprint
- [3] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14**, 349–381 (1973)

- [4] Ambrosetti, A., Ruiz, D.: Multiple bound states for the Schrödinger–Poisson problem. *Commun. Contemp. Math.* **10**(3), 391–404 (2008)
- [5] Bartsch, T., Wang, Z.-Q.: Existence and multiplicity results for some superlinear elliptic problems on $\mathbf{R}^N$. *Comm. Partial Differ. Equ.* **20**(9–10), 1725–1741 (1995)
- [6] Benci, V., Fortunato, D.: Solitary waves of the nonlinear Klein–Gordon equation coupled with the Maxwell equations. *Rev. Math. Phys.* **14**(4), 409–420 (2002)
- [7] Benci, V., Fortunato, D.: The nonlinear Klein–Gordon equation coupled with the Maxwell equations. In: *Proceedings of the Third World Congress of Nonlinear Analysts, Part 9* (Catania, 2000). *Nonlinear Anal.*, vol. 47(9), 6065–6072 (2001)
- [8] Benci, V., Fortunato, D.: An eigenvalue problem for the Schrödinger–Maxwell equations. *Topol. Methods Nonlinear Anal.* **11**(2), 283–293 (1998)
- [9] Benci, V., Fortunato, D., Masiello, A., Pisani, L.: Solitons and the electromagnetic field. *Math. Z.* **232**(1), 73–102 (1999)
- [10] Berestycki, H., Lions, P.-L.: Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Ration. Mech. Anal.* **82**(4), 313–345 (1983)
- [11] Candela, A.M., Salvatore, A.: Multiple solitary waves for non-homogeneous Schrödinger–Maxwell equations. *Mediterr. J. Math.* **3**(3–4), 483–493 (2006)
- [12] Cassani, D.: Existence and non-existence of solitary waves for the critical Klein–Gordon equation coupled with Maxwell’s equations. *Nonlinear Anal.* **58**(7–8), 733–747 (2004)
- [13] Chen, S.-J., Tang, C.-L.: High energy solutions for the superlinear Schrödinger–Maxwell equations. *Nonlinear Anal.* **71**(10), 4927–4934 (2009)
- [14] Coclite, G.M.: A multiplicity result for the nonlinear Schrödinger–Maxwell equations. *Commun. Appl. Anal.* **7**(2–3), 417–423 (2003)
- [15] Coclite, G.M.: A multiplicity result for the Schrödinger–Maxwell equations with negative potential. *Ann. Polon. Math.* **79**(1), 21–30 (2002)
- [16] D’Aprile, T., Mugnai, D.: Non-existence results for the coupled Klein–Gordon–Maxwell equations. *Adv. Nonlinear Stud.* **4**(3), 307–322 (2004)
- [17] D’Aprile, T., Mugnai, D.: Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations. *Proc. R. Soc. Edinb. Sect. A* **134**(5), 893–906 (2004)
- [18] Ekeland, I.: On the variational principle. *J. Math. Anal. Appl.* **47**, 324–353 (1974)
- [19] Gilbarg, D., Trudinger, N.S.: *Elliptic partial differential equations of second order*. Reprint of the 1998 edition. Springer, Berlin (2001)
- [20] Kikuchi, H.: On the existence of a solution for elliptic system related to the Maxwell–Schrödinger equations. *Nonlinear Anal.* **67**(5), 1445–1456 (2007)

- [21] Mawhin, J., Willem, M.: *Critical Point Theory and Hamiltonian Systems*. Applied Mathematical Sciences, 74. Springer, New York (1989)
- [22] Mercuri, C.: Positive solutions of nonlinear Schrödinger–Poisson systems with radial potentials vanishing at infinity. (English summary) *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **19**(3), 211–227 (2008)
- [23] Rabinowitz, P.H.: *Minimax methods in critical point theory with applications to differential equations*. CBMS Regional Conference Series in Mathematics, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC. American Mathematical Society, Providence (1986)
- [24] Ruiz, D.: The Schrödinger–Poisson equation under the effect of a nonlinear local term. *J. Funct. Anal.* **237**(2), 655–674 (2006)
- [25] Salvatore, A.: Multiple solitary waves for a non-homogeneous Schrödinger–Maxwell system in $\mathbf{R}^3$. *Adv. Nonlinear Stud.* **6**(2), 157–169 (2006)
- [26] Strauss, W.A.: Existence of solitary waves in higher dimensions. *Comm. Math. Phys.* **55**(2), 149–162 (1977)
- [27] Wang, Z.-P., Zhou, H.-S.: Positive solution for a nonlinear stationary Schrödinger–Poisson system in $\mathbf{R}^3$. *Discrete Contin. Dyn. Syst.* **18**(4), 809–816 (2007)
- [28] Zhao, L.-G., Zhao, F.-K.: Positive solutions for Schrödinger–Poisson equations with a critical exponent. *Nonlinear Anal.* **70**(6), 2150–2164 (2009)
- [29] Zhao, L.-G., Zhao, F.-K.: On the existence of solutions for the Schrödinger–Poisson equations. *J. Math. Anal. Appl.* **346**(1), 155–169 (2008)
- [30] Zou, W.-M., Schechter, M.: *Critical point theory and its applications*. Springer, New York (2006)

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