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Multiple solutions for resonant nonlinear periodic equations

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Abstract. We consider a nonlinear periodic problem driven by the scalar p-Laplacian, with an asymptotically $(p-1)$-linear nonlinearity. We permit resonance with respect to the second positive eigenvalue of the negative periodic scalar p-Laplacian and we assume nonuniform nonresonance with respect to the first positive eigenvalue. Using a combination of variational methods, with truncation techniques and Morse theory, we show that the problem has at least three nontrivial solutions.

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1. Introduction

In this paper we study the following nonlinear periodic problem driven by the scalar p-Laplacian, $1 < p < \infty$,

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = f(t, u(t)) & \text{a.e. on } T = [0, b], \\ u(0) = u(b), \ u'(0) = u'(b), \end{cases} \tag{1.1}$$

with $b > 0$. Our goal is to prove a multiplicity theorem for problem (1.1), when resonance occurs at infinity with respect to a higher eigenvalue of the negative periodic scalar p-Laplacian. More precisely, we assume that asymptotically at $+\infty$ the quotients $\frac{f(t,x)}{|x|^{p-2}x}$ are within the spectral interval $[\lambda_1, \lambda_2]$, $0 < \lambda_1 < \lambda_2$ being the first two nonzero eigenvalues of the negative periodic scalar p-Laplacian, with resonance possible with respect to λ_2. With respect to λ_1, we allow nonuniform nonresonance. It is an interesting open problem whether our work here can be extended to allow for resonance also at λ_1. This is the so-called "double resonance" situation (see Fabry–Fonda [13] for semilinear (i.e., $p = 2$) periodic problems and Kyritsi–Papageorgiou [18] for nonlinear periodic problems). Multiplicity results for periodic p-Laplacian equations can be found in the works of Aizicovici–Papageorgiou–Staicu [2], Del Pino–Manasevich–Murua [10], Gasinski–Papageorgiou [14] and Yang [22]. However, none of the aforementioned works allows for resonance to occur. In addition, our approach here is different from those papers.

In this paper, using minimax methods based on the critical point theory together with suitable truncation techniques and Morse theory (critical groups), we prove a "three nontrivial solutions theorem" for problem (1.1). In dealing with problem (1.1) we face serious difficulties in the handling of the resonance situation. First, now we work outside the convenient Hilbert space framework of the semilinear problem and we no longer have the orthogonal direct sum decomposition of the ambient space in terms of eigenspaces. Second, a related difficulty is that we do not have convenient variational characterizations of the higher eigenvalues $\{\lambda_k\}_{k\geq 2}$. These difficulties force us to consider the case where resonance occurs with respect to λ_2. Whether our multiplicity result can be extended to other eigenvalues different from λ_2 is an open problem.

In the next section, we recall the main mathematical tools that we shall use in the sequel.

2. Mathematical preliminaries

First, we recall some basic facts from the critical point theory. Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the "Palais-Smale condition" (PS-condition, for short), if the following holds: every sequence $\{x_n\}_{n\geq 1} \subset X$ such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded and $\varphi'(x_n) \to 0$ in X^* as $n \to \infty$ admits a strongly convergent subsequence.

Using this notion, we have the following theorem for the critical values of a C^1-functional, known in the literature as the "mountain pass theorem".

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ and satisfies the PS-condition, $x_0, x_1 \in X$, $r > 0$, $\|x_1 - x_0\| > r$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} =: \eta_r,$$

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t)), \quad \textit{where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geq \eta_r$ and c is a critical value of φ.

Let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets: $\varphi^c = \{x \in X : \varphi(x) \leq c\}$ and $K_\varphi = \{x \in X : \varphi'(x) = 0\}$. For a topological pair (Y_1, Y_2) such that $Y_2 \subset Y_1 \subset X$ and for every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_1, Y_2) with coefficients in $\mathbb{Z}$. The critical groups of φ at an isolated critical point x_0 with $c = \varphi(x_0)$, are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}) \quad \text{for all integers } k \geq 0,$$

where U is a neighborhood of x_0 such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$ (see [6,19]). The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all integers } k \geq 0$$

(see Bartsch and Li [4]). The deformation theorem implies that the above definition of critical groups at infinity is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

We assume that K_φ is finite. We define

$$M(t, x) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, x) t^k \quad \text{for every } x \in K_\varphi$$

and

$$P(t, \infty) = \sum_{k \geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k.$$

The “Morse relation” holds, namely

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t)Q(t), \tag{2.1}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a polynomial with nonnegative integer coefficients (see [6, p. 36] and [19, p. 184]).

In the study of problem (1.1), we shall use the Sobolev space

$$W^{1,p}_{\text{per}}(0, b) = \{u \in W^{1,p}(0, b) : u(0) = u(b)\}$$

endowed with the norm $\|u\| = (\|u\|_p^p + \|u'\|_p^p)^{1/p}$, where $\|\cdot\|_p$ stands for the norm in $L^p(T)$ (recall that $W^{1,p}(0, b)$ is embedded compactly in $C(T)$ and so the evaluations of u at $t = 0$ and $t = b$ make sense). We shall also use the Banach space

$$\hat{C}(T) = C^1(T) \cap W^{1,p}_{\text{per}}(0, b) = \{u \in C^1(T) : u(0) = u(b)\}.$$

This is an ordered Banach space with the order cone

$$\hat{C}_+ = \{u \in \hat{C}(T) : u(t) \geq 0 \quad \text{for all } t \in T\}.$$

The above cone has nonempty interior given by

$$\operatorname{int} \hat{C}_+ = \{u \in \hat{C}_+ : u(t) > 0 \quad \text{for all } t \in T\}.$$

Finally let us recall some basic facts about the spectrum of the negative periodic scalar p-Laplacian. We consider the following nonlinear eigenvalue problem:

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = \lambda |u(t)|^{p-2}u(t) & \text{a.e. on } T, \\ u(0) = u(b), \ u'(0) = u'(b). \end{cases} \tag{2.2}$$

Evidently, a necessary condition for $\lambda \in \mathbb{R}$ to be an eigenvalue is that $\lambda \geq 0$. In fact, $\lambda_0 = 0$ is an eigenvalue and the corresponding eigenspace is $\mathbb{R}$. Moreover, $\lambda_0 = 0$ is the only eigenvalue whose eigenfunctions have constant sign. Every eigenfunction corresponding to an eigenvalue $\lambda > 0$ is nodal (i.e., sign changing). Let $\pi_p = \frac{2\pi(p-1)^{1/p}}{p \sin \frac{\pi}{p}}$. The sequence $\{\lambda_n = (\frac{2n\pi_p}{b})^p\}_{n\geq 0}$ is the set of all eigenvalues of problem (2.2). If $p = 2$ (linear eigenvalue problem), then $\pi_2 = \pi$ and so we recover the well-known sequence of eigenvalues of the negative periodic Laplacian, given by $\{\lambda_n = (\frac{2n\pi}{b})^2\}_{n\geq 0}$. Every eigenfunction $u \in C^1(T)$ of (2.2) satisfies $u(t) \neq 0$ a.e. on T and, more precisely, it has a finite number of zeros. For details, we refer to Binding–Rynne [5], Drabek–Manasevich [11] and Gasinski–Papageorgiou [15].

3. Two solutions of constant sign

The hypotheses on the nonlinearity $f(t, x)$ are the following:

(H) $f : T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $t \mapsto f(t, x)$ is measurable;

(ii) for a.a. $t \in T$, $x \mapsto f(t, x)$ is continuous and $f(t, 0) = 0$;

(iii) for every $r > 0$, there exists $a_r \in L^{p'}(T)_+$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ such that $|f(t, x)| \leq a_r(t)$ for a.a. $t \in T$, all $|x| \leq r$;

(iv) there exists $\beta \in L^\infty(T)$ such that $\lambda_1 \leq \beta(t)$ a.e. on T, with strict inequality on a set of positive measure, and

$$\beta(t) \leq \liminf_{|x|\to\infty} \frac{f(t, x)}{|x|^{p-2}x} \leq \limsup_{|x|\to\infty} \frac{f(t, x)}{|x|^{p-2}x} \leq \lambda_2 \quad \text{uniformly for a.a. } t \in T,$$

and if $F(t, x) = \int_0^x f(t, s)\, ds$, then there exists a function $\eta \in L^\infty(T)_+$ such that $\eta(t) \leq \lambda_2$ a.e. on T, with strict inequality on a set of positive measure, and

$$\limsup_{|x|\to\infty} \frac{pF(t, x)}{|x|^p} \leq \eta(t) \quad \text{uniformly for a.a. } t \in T;$$

(v) there exists $\delta > 0$ such that $F(t, x) \leq 0$ for a.a. $t \in T$, all $|x| \leq \delta$;

(vi) there exists $c_0 > 0$ such that $f(t, x)x \geq -c_0|x|^p$ for a.a. $t \in T$, all $x \in \mathbb{R}$.

Remark 3.1. (a) Hypothesis (H)(iv) implies that at infinity we can have resonance with respect to the second nonzero eigenvalue λ_2, that is

$$\lim_{|x|\to\infty} \frac{f(t, x)}{|x|^{p-2}x} = \lambda_2 \quad \text{uniformly for a.a. } t \in T.$$

With respect to λ_1, we have nonuniform nonresonance. To deal with the possibility of resonance, we need an extra condition, which is provided by the second part of hypothesis H(iv), involving the potential $F(t, x)$. A similar condition for semilinear Neumann partial differential equations can be found in Gossez–Omari [16].

(b) Hypothesis (H)(iv) implies that $k_1|x|^p \leq F(t, x) \leq k_2|x|^p$ for a.a. $t \in T$ and all $|x| \geq d$, with constants $d > 0$ and $0 < k_1 < k_2$. Therefore, there exist constants $c_- < 0 < c_+$ such that $\int_0^b F(t, c_\pm)\, dt \geq 0$.

Example. Consider the following potential function $F(x)$ (for the sake of simplicity we drop the t-dependence):

$$F(x) = \begin{cases} \frac{c_0}{r}|x|^r \ln|x| & \text{if } |x| \le 1 \\ \frac{\eta}{p}|x|^p + \frac{\lambda_2-\eta}{p}\sin|x|^p + c_1 & \text{if } |x| > 1, \end{cases}$$

with $c_0 = r(\eta + (\lambda_2 - \eta)\cos 1)$, $c_1 = -\frac{\eta}{p} - \frac{\lambda_2-\eta}{p}\sin 1$, $1 < p < r < \infty$, $\eta \in (\frac{\lambda_1+\lambda_2}{2}, \lambda_2)$. Then $f(x) = F'(x)$ satisfies hypothesis (H).

As already mentioned in the Sect. 1, our approach involves also truncation techniques. We introduce the positive and negative truncations of the nonlinearity $f(t,\cdot)$ defined by

$$f_-(t,x) = \begin{cases} f(t,x) & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}, \quad f_+(t,x) = \begin{cases} 0 & \text{if } x \le 0 \\ f(t,x) & \text{if } x > 0 \end{cases}.$$

Both are Carathéodory functions. We set $F_\pm(t,x) = \int_0^x f_\pm(t,s)\,ds$ and then, for $\varepsilon \in (0,1)$, we introduce the functionals $\varphi_\pm^\varepsilon : W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ defined by

$$\varphi_\pm^\varepsilon(u) = \frac{1}{p}\|u'\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \int_0^b F_\pm(t,u(t))\,dt - \frac{\varepsilon}{p}\|u^\pm\|_p^p \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b).$$

Here $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$. For $u \in W^{1,p}_{\mathrm{per}}(0,b)$, we know that $u^+, u^- \in W^{1,p}_{\mathrm{per}}(0,b)$. We also consider the Euler functional $\varphi : W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|u'\|_p^p - \int_0^b F(t,u(t))\,dt \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b).$$

Evidently, $\varphi_\pm^\varepsilon, \varphi \in C^1(W^{1,p}_{\mathrm{per}}(0,b))$.

Proposition 3.2. *If hypotheses* (H) *hold, then the functionals* $\varphi_\pm^\varepsilon$ *and* φ *satisfy the* PS-*condition.*

Proof. First we do the proof for the functional φ_+^ε. Consider a sequence $\{u_n\}_{n\ge 1} \subset W^{1,p}_{\mathrm{per}}(0,b)$ such that

$$|\varphi_+^\varepsilon(u_n)| \le M_1 \quad \text{for all } n \ge 1, \tag{3.1}$$

for some $M_1 > 0$, and

$$(\varphi_+^\varepsilon)'(u_n) \to 0 \text{ in } W^{1,p}_{\mathrm{per}}(0,b)^* \text{ as } n \to \infty. \tag{3.2}$$

Note that

$$(\varphi_+^\varepsilon)'(u) = A(u) + \varepsilon|u|^{p-2}u - N_+(u) - \varepsilon(u^+)^{p-1} \quad \text{for all } u \in W^{1,p}_{\mathrm{per}}(0,b),$$

where $A : W^{1,p}_{\mathrm{per}}(0,b) \to W^{1,p}_{\mathrm{per}}(0,b)^*$ is the nonlinear map defined by

$$\langle A(u), y\rangle = \int_0^b |u'(t)|^{p-2}u'(t)y'(t)\,dt \quad \text{for all } u, y \in W^{1,p}_{\mathrm{per}}(0,b)$$

and $N_+(u)(\cdot) = f_+(\cdot, u(\cdot))$ for all $u \in W^{1,p}_{\mathrm{per}}(0,b)$. By virtue of (3.2) we have

$$\left| \langle A(u_n), v\rangle + \varepsilon \int_0^b |u_n|^{p-2} u_n v\,dt - \int_0^b f_+(t,u_n) v\,dt - \varepsilon \int_0^b (u_n^+)^{p-1} v\,dt \right| \leq \varepsilon_n \|v\| \tag{3.3}$$

for all $v \in W^{1,p}_{\mathrm{per}}(0,b)$, with $\varepsilon_n \downarrow 0$.

We claim that $\{u_n\}_{n\geq 1}$ is bounded in $W^{1,p}_{\mathrm{per}}(0,b)$. To this end, in (3.3) we choose $v = -u_n^- \in W^{1,p}_{\mathrm{per}}(0,b)$ and we have

$$\|(u_n^-)'\|_p^p + \varepsilon \|u_n^-\|_p^p \leq \varepsilon_n \|u_n^-\|,$$

which implies

$$u_n^- \to 0 \ \text{ in } W^{1,p}_{\mathrm{per}}(0,b). \tag{3.4}$$

Reasoning by contradiction, suppose that $\|u_n^+\| \to \infty$. Set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \geq 1$. Then $\|y_n\| = 1$ and so we may assume that $y_n \xrightarrow{\mathrm{w}} y$ in $W^{1,p}_{\mathrm{per}}(0,b)$ and $y_n \to y$ in $C(T)$, with $y \geq 0$. Note that hypotheses (H)(iii), (iv) imply that

$$\begin{gathered} |f(t,x)| \leq a(t) + c|x|^{p-1} \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R}, \\ \text{with } a \in L^{p'}(T)_+,\ c > 0. \end{gathered} \tag{3.5}$$

Therefore we see that

$$\frac{|f_+(t,u_n(t))|}{\|u_n^+\|^{p-1}} \leq \frac{a(t)}{\|u_n^+\|^{p-1}} + c|y_n(t)|^{p-1} \quad \text{for a.a. } t \in T,$$

which yields that $\{\frac{N_+(u_n)}{\|u_n^+\|^{p-1}}\}_{n\geq 1}$ is bounded in $L^{p'}(T)$. So, along a subsequence,

$$\frac{N_+(u_n)}{\|u_n^+\|^{p-1}} \xrightarrow{\mathrm{w}} h \ \text{ in } L^{p'}(T) \ \text{ as } n \to \infty.$$

Using hypothesis H(iv), as in [20] (see the proof of Proposition 5 therein), we can show that

$$h(t) = g(t) y(t)^{p-1} \quad \text{a.e. on } T, \ \text{ with } \beta(t) \leq g(t) \leq \lambda_2 \ \text{ a.e. on } T. \tag{3.6}$$

Now in (3.3) we choose $v = y_n - y \in W^{1,p}_{\mathrm{per}}(0,b)$, multiply by $\frac{1}{\|u_n^+\|^{p-1}}$ and finally pass to the limit as $n \to \infty$. Then, by (3.4), we obtain $\lim\limits_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0$. Since A is a map of type $(S)_+$ (see, for example, Kyritsi–Papageorgiou [18]), it follows that

$$y_n \to y \ \text{ in } W^{1,p}_{\mathrm{per}}(0,b), \ \text{ hence } \|y\| = 1. \tag{3.7}$$

Once again we return to (3.3), multiply with $\frac{1}{\|u_n^+\|^{p-1}}$ and pass to the limit as $n \to \infty$. Then, by virtue of (3.4) and (3.6), we obtain

$$\langle A(y), v\rangle = \int_0^b g y^{p-1} v\,dt \quad \text{for all } v \in W^{1,p}_{\mathrm{per}}(0,b).$$

This reads as

$$\begin{cases} -(|y'(t)|^{p-2} y'(t))' = g(t) y(t)^{p-1} & \text{a.e. on } T, \\ y(0) = y(b),\ y'(0) = y'(b). \end{cases} \tag{3.8}$$

We consider two distinct cases depending on the position of the weight function $g \in L^{\infty}(T)$ within the spectral interval $[\lambda_1, \lambda_2]$.

Case 1: $g(t) = \lambda_2$ a.e. on T.

By (3.7) and (3.8) it follows that in this case y is an eigenfunction corresponding to the eigenvalue λ_2. Hence y must be nodal, a contradiction to the fact that $y \geq 0$.

Case 2: $\beta(t) \leq g(t) \leq \lambda_2$ a.e. on T and the last inequality is strict on a set of positive measure.

By (3.8) and invoking Proposition 2 of Aizicovici–Papageorgiou–Staicu [1] we infer that $y = 0$, a contradiction to (3.7).

Therefore $\{u_n^+\}_{n\geq 1}$ is bounded in $W^{1,p}_{\text{per}}(0,b)$, thus $\{u_n\}_{n\geq 1}$ is bounded in $W^{1,p}_{\text{per}}(0,b)$. So, we may assume that

$$u_n \xrightarrow{\text{w}} u \ \text{ in } W^{1,p}_{\text{per}}(0,b) \quad \text{and} \quad u_n \to u \ \text{ in } C(T) \ \text{ as } n \to \infty. \tag{3.9}$$

In (3.3) we choose $v = u_n - u \in W^{1,p}_{\text{per}}(0,b)$ and pass to the limit as $n \to \infty$. Then, using (3.9), we find that $\lim\limits_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0$, which implies that $u_n \to u$ in $W^{1,p}_{\text{per}}(0,b)$.

This proves that φ_+^{ε} satisfies the PS-condition. The proof for the functional φ_-^{ε} is similar.

Finally, for the functional φ we argue as follows. In this case, (3.4) is no longer true. We suppose that $\|u_n\| \to \infty$ and let $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ and we may assume that $y_n \xrightarrow{\text{w}} y$ in $W^{1,p}_{\text{per}}(0,b)$ and $y_n \to y$ in $C(T)$ as $n \to \infty$. As before exploiting the fact that $\{u_n\}_{n\geq 1} \subset W^{1,p}_{\text{per}}(0,b)$ is a PS-sequence and that A is a map of type $(S)_+$, we derive that $y_n \to y$ in $W^{1,p}_{\text{per}}(0,b)$ (hence $\|y\| = 1$) and in this case y satisfies

$$\begin{cases} -(|y'(t)|^{p-2}y'(t))' = g(t)|y(t)|^{p-2}y(t) \quad \text{a.e. on } T, \\ y(0) = y(b), \ \ y'(0) = y'(b). \end{cases} \tag{3.10}$$

with $\beta(t) \leq g(t) \leq \lambda_2$ a.e. on T.

Again we consider two cases depending on the position of $g(\cdot)$ in $[\lambda_1, \lambda_2]$.

Case $1'$*:* $g(t) = \lambda_2$ a.e. on T.

From (3.5) we have

$$|F(t,x)| \leq a(t)|x| + c|x|^p \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R},$$

which leads to $\{\frac{pF(\cdot,u_n(\cdot))}{\|u_n\|^p}\}_{n\geq 1}$ is bounded in $L^{p'}(T)$. So, we may assume that

$$\frac{pF(\cdot,u_n(\cdot))}{\|u_n\|^p} \xrightarrow{\text{w}} \mu \ \text{ in } L^{p'}(T) \ \text{ as } n \to \infty. \tag{3.11}$$

From (3.10) we know that $y(t) \neq 0$ for a.a. $t \in T$ and so $|u_n(t)| \to \infty$ for a.a. $t \in T$ as $n \to \infty$. Hence, using hypothesis (H)(iv) as in [20], we show that

$$\mu(t) = \xi(t)|y(t)|^p \quad \text{a.e. on } T, \ \text{ with } \xi(t) \leq \eta(t) \text{ a.e. on } T. \tag{3.12}$$

Since $\{u_n\}_{n\geq 1} \subset W^{1,p}_{\text{per}}(0,b)$ is a PS-sequence, we have $|\varphi(u_n)| \leq M_1$ for all $n \geq 1$, or equivalently,

$$\left| \|y_n'\|_p^p - \int_0^b \frac{pF(t,u_n(t))|}{\|u_n\|^p}\, dt \right| \leq \frac{pM_1}{\|u_n\|^p}.$$

It turns out that

$$\|y'\|_p^p = \int_0^b \xi |y|^p\, dt \tag{3.13}$$

(see (3.11), (3.12) and because $y_n \to y$ in $W^{1,p}_{\text{per}}(0,b)$).

Recall that $y(t) \neq 0$ for a.a. $t \in T$. So, from (3.12), (3.13) and the hypothesis on $\eta(\cdot)$ (see (H)(iv)) we have

$$\|y'\|_p^p < \lambda_2 \|y\|_p^p,$$

which contradicts (3.10).

Case $2'$*:* $g(t) \leq \lambda_2$ a.e. on T, with strict inequality on a set of positive measure.

As before (see Case 2), from (3.10) and Proposition 2 of [1], we infer that $y = 0$, which contradicts the fact that $\|y\| = 1$.

So, we have that $\{u_n\}_{n\geq 1}$ is bounded in $W^{1,p}_{\text{per}}(0,b)$ and we may assume that

$$u_n \xrightarrow{\text{w}} u \ \text{ in } W^{1,p}_{\text{per}}(0,b) \quad \text{and} \quad u_n \to u \ \text{ in } C(T) \ \text{ as } n \to \infty.$$

Again exploiting the fact that $\{u_n\}_{n\geq 1}$ is a PS-sequence and that the map A is of type $(S)_+$, we conclude that $u_n \to u$ in $W^{1,p}_{\text{per}}(0,b)$, which proves that φ satisfies the PS-condition. □

Proposition 3.3. *If hypotheses* (H) *hold, then* 0 *is a local minimizer for the functionals* $\varphi_{\pm}^{\varepsilon}$ *and* φ*.*

Proof. Since $W^{1,p}_{\text{per}}(0,b)$ is compactly embedded into $C(T)$, we can find $c_1 > 0$ such that $\|u\|_\infty \leq c_1\|u\|$ for all $u \in W^{1,p}_{\text{per}}(0,b)$. Consequently, if $\delta > 0$ is as in hypothesis (H)(v) and $\delta_0 = \frac{\delta}{c_1}$, then for each $u \in W^{1,p}_{\text{per}}(0,b)$ with $\|u\| \leq \delta_0$ one has $|u(t)| \leq \delta$ for all $t \in T$. Hence, by virtue of hypothesis (H)(v), we have $F_+(t,u(t)) \leq 0$ a.e. on T for all $\|u\| \leq \delta_0$. It follows that

$$\varphi_+^{\varepsilon}(u) \geq \frac{1}{p}\|u'\|_p^p - \int_0^b F_+(t,u(t))\, dt \geq 0 = \varphi_+^{\varepsilon}(0) \quad \text{for all } \|u\| \leq \delta_0.$$

This ensures that 0 is a local minimizer of φ_+^{ε}. Similarly, for φ_-^{ε} and φ. □

Remark 3.4. Without any loss of generality, we may assume that 0 is an isolated critical point for the functionals $\varphi_{\pm}^{\varepsilon}$ and φ. Indeed, suppose we could find a sequence $\{u_n\}_{n\geq 1} \subset W^{1,p}_{\text{per}}(0,b) \setminus \{0\}$ such that $u_n \to 0$ in $W^{1,p}_{\text{per}}(0,b)$ and $(\varphi_+^{\varepsilon})'(u_n) = 0$ for all $n \geq 1$, which means

$$A(u_n) + \varepsilon |u_n|^{p-2} u_n = N_+(u_n) + \varepsilon \left(u_n^+\right)^{p-1}.$$

Acting with $-u_n^- \in W^{1,p}_{\text{per}}(0,b)$, we obtain $u_n \geq 0$, $u_n \neq 0$. So, we have $A(u_n) = N(u_n)$ for all $n \geq 1$, with $N(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in W^{1,p}_{\text{per}}(0,b)$. This entails that $u_n \in C^1(T)$ and solves (1.1). Therefore we can produce a whole sequence of distinct, nontrivial, constant sign solutions of (1.1) and so we are done. Similarly for the functionals φ_-^ε and φ.

By virtue of Proposition 3.3 and Remark 3.4, and reasoning as in the proof of Proposition 6 of [20], we can find $0 < \rho < \min\{c_+ b^{1/p}, |c_-| b^{1/p}\}$ (with the constants $c_- < 0 < c_+$ in Remark 3.1(b)) such that

$$0 = \varphi_\pm^\varepsilon(0) < \inf\{\varphi_\pm^\varepsilon(u) : \|u\| = \rho\} =: \eta_\rho^\pm. \tag{3.14}$$

Now we are ready to produce two nontrivial solutions of (1.1) of constant sign.

Proposition 3.5. *If hypotheses* (H) *hold, then problem* (1.1) *has at least two solutions* $u_0 \in \mathrm{int}\hat{C}_+$ *and* $v_0 \in -\mathrm{int}\hat{C}_+$.

Proof. By Remark 3.1(b), given $\varepsilon > 0$, we have

$$\varphi_+^\varepsilon(c_+) = -\int_0^b F_+(t, c_+)\, dt \le 0 = \varphi_+^\varepsilon(0).$$

This fact, together with Proposition 3.2 and (3.14), permits the use of Theorem 2.1 (the mountain pass theorem). So, we find $u_0 \in W^{1,p}_{\mathrm{per}}(0, b)$ such that

$$\eta_\rho^+ \le \varphi_+^\varepsilon(u_0) \tag{3.15}$$

and

$$(\varphi_+^\varepsilon)'(u_0) = 0. \tag{3.16}$$

From (3.14) and (3.15) it is clear that $u_0 \neq 0$, while from (3.16) we have

$$A(u_0) + \varepsilon |u_0|^{p-2} u_0 = N_+(u_0) + \varepsilon \left(u_0^+\right)^{p-1}.$$

Acting with $-u_0^- \in W^{1,p}_{\mathrm{per}}(0, b)$, we obtain $u_0 \ge 0$, $u_0 \neq 0$. Hence $A(u_0) = N(u_0)$, which gives

$$\begin{cases} -(|u_0'(t)|^{p-2} u_0'(t))' = f(t, u_0(t)) & \text{a.e. on } T, \\ u_0(0) = u_0(b), \ u_0'(0) = u_0'(b). \end{cases} \tag{3.17}$$

It follows that $u_0 \in C^1(T)$ solves problem (1.1).

Moreover, from (3.17) and hypothesis (H)(vi), we have

$$(|u_0'(t)|^{p-2} u_0'(t))' \le c_0 u_0(t)^{p-1} \quad \text{a.e. on } T$$

(recall $u_0 \ge 0$). Invoking the nonlinear maximum principle of Vazquez [21], we conclude that $u_0 \in \mathrm{int}\hat{C}_+$.

In a similar fashion, working this time with the functional φ_-^ε, we obtain $v_0 \in -\mathrm{int}\hat{C}_+$, a second constant sign solution of (1.1). □

In the next section, using Morse theory, we shall produce a third nontrivial solution.

4. Three nontrivial solutions

To implement Morse theoretic methods, we need to compute certain critical groups of the functionals $\varphi_{\pm}^{\varepsilon}$ and φ.

Proposition 4.1. *If hypotheses* (H) *hold and* $2 \leq p < \infty$, *then* $C_k(\varphi_{\pm}^{\varepsilon}, \infty) = 0$ *for all integers* $k \geq 0$.

Proof. As before, we do the proof for the functional $\varphi_{+}^{\varepsilon}$, the proof for the functional $\varphi_{-}^{\varepsilon}$ being similar.

Consider the functional $\psi_{+}^{\varepsilon} : W^{1,p}_{\mathrm{per}}(0,b) \to \mathbb{R}$ defined by

$$\psi_{+}^{\varepsilon}(u) = \frac{1}{p}\|u'\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p - \frac{\theta+\varepsilon}{p}\|u^+\|_p^p \quad \text{for } u \in W^{1,p}_{\mathrm{per}}(0,b), \text{ with } \theta \in (\lambda_1, \lambda_2).$$

We introduce the following affine homotopy

$$h_{+}^{1}(\tau, u) = (1-\tau)\varphi_{+}^{\varepsilon}(u) + \tau\psi_{+}^{\varepsilon}(u)$$

for all $(\tau, u) \in [0,1] \times W^{1,p}_{\mathrm{per}}(0,b)$.

We show that there exists $R > 0$ such that $\overline{B}_R = \{u \in W^{1,p}_{\mathrm{per}}(0,b) : \|u\| \leq R\}$ contains the critical sets of $h_{+}^{1}(\tau,\cdot)$ for all $\tau \in [0,1]$. We argue indirectly. Then there exist $\{\tau_n\}_{n\geq 1} \subset [0,1]$ and $\{u_n\}_{n\geq 1} \subset W^{1,p}_{\mathrm{per}}(0,b)$ such that

$$\tau_n \to \tau \in [0,1], \ \|u_n\| \to \infty \quad \text{and} \quad (h_{+}^{1})'(\tau_n, u_n) = 0 \ \text{ for all } n \geq 1. \tag{4.1}$$

From the equation in (4.1) we have

$$A(u_n) + \varepsilon|u_n|^{p-2}u_n = (1-\tau_n)N_{+}(u_n) + \varepsilon(u_n^+)^{p-1} + \tau_n\theta(u_n^+)^{p-1} \quad \text{for all } n \geq 1.$$

As before, acting with $-u_n^- \in W^{1,p}_{\mathrm{per}}(0,b)$, we obtain $u_n \geq 0$, so

$$A(u_n) = (1-\tau_n)N(u_n) + \tau_n\theta u_n^{p-1} \quad \text{for all } n \geq 1. \tag{4.2}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ and so we may assume that $y_n \xrightarrow{\mathrm{w}} y$ in $W^{1,p}_{\mathrm{per}}(0,b)$ and $y_n \to y$ in $C(T)$ as $n \to \infty$. Notice that (4.2) yields

$$A(y_n) = (1-\tau_n)\frac{N(u_n)}{\|u_n\|^{p-1}} + \tau_n\theta y_n^{p-1}. \tag{4.3}$$

From the proof of Proposition 3.2 we know that

$$\frac{N(u_n)}{\|u_n\|^{p-1}} \xrightarrow{\mathrm{w}} gy^{p-1} \ \text{ in } L^{p'}(T), \ \text{ with } \beta(t) \leq g(t) \leq \lambda_2 \ \text{ a.e. on } T. \tag{4.4}$$

Acting on (4.3) with $y_n - y \in W^{1,p}_{\mathrm{per}}(0,b)$, passing to the limit and using that A is a map of type $(S)_+$, it is seen that

$$y_n \to y \ \text{ in } W^{1,p}_{\mathrm{per}}(0,b), \ \text{ hence } \|y\| = 1, \ y \geq 0. \tag{4.5}$$

Passing to the limit as $n \to \infty$ in (4.3) leads to

$$\begin{cases} -(|y'(t)|^{p-2}y'(t))' = [(1-\tau)g(t) + \tau\theta]y(t)^{p-1} & \text{a.e. on } T, \\ y(0) = y(b), \ y'(0) = y'(b). \end{cases} \tag{4.6}$$

First assume that $0 < \tau \leq 1$ and set $\sigma_\tau(t) = (1-\tau)g(t) + \tau\theta$. It turns out that

$$\lambda_1 < \sigma_\tau(t) < \lambda_2 \quad \text{a.e. on } T.$$

Then, from (4.6) and Proposition 2 of [1], we infer that $y = 0$, which contradicts (4.5).

Next we suppose that $\tau = 0$. Then we have

$$\begin{cases} -(|y'(t)|^{p-2}y'(t))' = g(t)y(t)^{p-1} & \text{a.e. on } T, \\ y(0) = y(b), \ y'(0) = y'(b). \end{cases}$$

Again we consider the following distinct cases: $g(t) = \lambda_2$ a.e. on T and $g(t) \leq \lambda_2$ a.e. on T, strictly on a set of positive measure. Arguing as in the proof of Proposition 3.2, we see that each case leads to a contradiction. Therefore, we can find $R > 0$ such that $\overline{B}_R = \{u \in W^{1,p}_{\text{per}}(0,b) : \|u\| \leq R\}$ contains the critical sets of $h^1_+(\tau,\cdot)$ for all $\tau \in [0,1]$.

Now, for $h^1_+(\cdot,\cdot)$ we can find a pseudogradient vector field $\hat{v} = (v_0, v)$: $[0,1] \times (W \setminus \overline{B}_R) \to [0,1] \times W$ (here for notational simplicity, $W = W^{1,p}_{\text{per}}(0,b)$). From the construction of a pseudogradient vector field (see, e.g., Chang [6, p. 19] or Gasinski–Papageorgiou [15, p. 614]), we may assume that $v_0(\tau,u) = \partial_\tau h^1_+(\tau,u)$ and $v(\tau,u) =: v_\tau(u)$ is locally Lipschitz and for every $\tau \in [0,1]$, $v_\tau(\cdot)$ is a pseudogradient vector field of $h^1_+(\tau,\cdot)$. Therefore, the map

$$u \in W \setminus \overline{B}_R \mapsto -\frac{|\partial_\tau h^1_+(\tau,u)|}{\|(h^1_+)'_u(\tau,u)\|^2} v_\tau(u) =: w_\tau(u) \in W$$

is well defined and locally Lipschitz (note that since by hypothesis $2 \leq p < \infty$, the map is locally Lipschitz). Let

$$\mu < \inf\{h^1_+(\tau,u) : \tau \in [0,1], \ \|u\| \leq R\}. \tag{4.7}$$

Evidently, μ is not a critical value of any of $\{h^1_+(\tau,\cdot)\}_{\tau\in[0,1]}$. Since the functional φ^ε_+ is unbounded below (see (H)(iv)), we can choose $z \in (\varphi^\varepsilon_+)^\mu$ and consider the following Cauchy problem

$$\frac{d}{d\tau}x(\tau) = w_\tau(x(\tau)) \quad \text{for } \tau \in [0,1], \ \ x(0) = z. \tag{4.8}$$

By virtue of the local existence theorem, we know that (4.8) admits a local flow. Moreover, from the definition of a pseudogradient vector field, we have

$$\frac{d}{d\tau} h^1_+(\tau, x(\tau)) \leq 0$$

for all τ in the domain of the solution x, from which it follows that

$$h^1_+(\tau,x(\tau)) \leq h^1_+(0,x(0)) = \varphi^\varepsilon_+(z) \leq \mu,$$

so by (4.7) we deduce that $\|x(\tau)\| > R$. This shows that $(h^1_+)'_u(\tau,x(\tau))) \neq 0$, hence the flow is global. Furthermore, we have that

$$h^1_+(1,x(1)) = \psi^\varepsilon_+(x(1)) \leq \mu.$$

We derive that

$$(\varphi^\varepsilon_+)^\mu = h^1_+(0,\cdot)^\mu \text{ is homeomorphic to a subset of } (\psi^\varepsilon_+)^\mu = h^1_+(1,\cdot)^\mu. \tag{4.9}$$

Proceeding in a similar way with the affine homotopy defined by

$$h^2_+(\tau,u) = h^1_+(1-\tau,u) = \tau\varphi^\varepsilon_+(u) + (1-\tau)\psi^\varepsilon_+(u),$$

for all $(\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b)$, we infer that

$$(\psi_+^\varepsilon)^\mu = h_+^2(0,\cdot)^\mu \text{ is homeomorphic to a subset of } (\varphi_+^\varepsilon)^\mu = h_+^2(1,\cdot)^\mu, \tag{4.10}$$

with $\mu < \inf\{h_+^1(\tau,u), h_+^2(\tau,u) : \ \tau \in [0,1], \ \|u\| \le R\}$.

The properties given in (4.9) and (4.10) imply that

$$H_k\left(W, \left(\varphi_+^\varepsilon\right)^\mu\right) = H_k\left(W, \left(\psi_+^\varepsilon\right)^\mu\right) \quad \text{for all integers } k \ge 0$$

(see Granas–Dugundji [17, p. 387]), which means

$$C_k(\varphi_+^\varepsilon, \infty) = C_k(\psi_+^\varepsilon, \infty) \quad \text{for all integers } k \ge 0. \tag{4.11}$$

Suppose that $u \in W = W^{1,p}_{\text{per}}(0,b)$ is a critical point of the functional ψ_+^ε. Then

$$A(u) + \varepsilon|u|^{p-2}u = (\theta+\varepsilon)(u^+)^{p-1}.$$

Acting with $-u^- \in W^{1,p}_{\text{per}}(0,b)$, we see that $u \ge 0$. Hence $A(u) = \theta u^{p-1}$, that is

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = \theta u(t)^{p-1} \quad \text{a.e. on } T, \\ u(0) = u(b), \ u'(0) = u'(b). \end{cases}$$

This yields $u = 0$ since $\theta \in (\lambda_1, \lambda_2)$. Therefore 0 is the only critical point of ψ_+^ε and so

$$C_k(\psi_+^\varepsilon, \infty) = C_k(\psi_+^\varepsilon, 0) \quad \text{for all integers } k \ge 0. \tag{4.12}$$

We need to compute $C_k(\psi_+^\varepsilon, 0)$, $k \ge 0$. To this end, let $\xi \in L^\infty(T)_+$, $\xi \ne 0$, and consider the homotopy $\hat{h}(\tau,u) = \psi_+^\varepsilon(u) - \tau\xi u$. We claim that

$$(\hat{h})'_u(\tau,u) \ne 0 \quad \text{for all } \tau \in [0,1] \text{ and all } u \ne 0. \tag{4.13}$$

Suppose that (4.13) is not true. Then we can find $\tau \in [0,1]$ and $u \ne 0$ such that $(\hat{h})'_u(\tau,u) = 0$, that is

$$A(u) + \varepsilon|u|^{p-2}u = (\theta+\varepsilon)(u^+)^{p-1} + \tau\xi.$$

Acting with $-u^- \in W^{1,p}_{\text{per}}(0,b)$ and since $\xi \ge 0$, we infer that $u \ge 0$, $u \ne 0$ and so

$$A(u) = \theta u^{p-1} + \tau\xi.$$

If $\tau = 0$, then $A(u) = \theta u^{p-1}$, which implies $u = 0$ since $\theta \in (\lambda_1, \lambda_2)$, a contradiction.

If $0 < \tau \le 1$, then we get

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = \theta u(t)^{p-1} + \tau\xi(t) \quad \text{a.e. on } T, \\ u(0) = u(b), \ u'(0) = u'(b), \end{cases} \tag{4.14}$$

and then that

$$(|u'(t)|^{p-2}u'(t))' \le 0 \quad \text{a.e. on } T.$$

This ensures that $u \in \text{int}\hat{C}_+$ (see Vazquez [21]).

Let $v \in \mathrm{int}\hat{C}_+$ and consider

$$R(v,u)(t) = |v'(t)|^p - |u'(t)|^{p-2}u'(t)\left(\frac{v^p}{u^{p-1}}\right)'(t) \quad \text{for all } t \in T.$$

By virtue of the generalized Picone identity (see Allegretto–Huang [3]), we have

$$\begin{aligned} 0 \le \int_0^b R(v,u)(t)\,dt &= \|v'\|_p^p - \int_0^b |u'(t)|^{p-2}u'(t)\left(\frac{v^p}{u^{p-1}}\right)'(t)\,dt \\ &= \|v'\|_p^p - \int_0^b -(|u'(t)|^{p-2}u'(t))'\frac{v^p}{u^{p-1}}(t)\,dt \\ &= \|v'\|_p^p - \theta\|v\|_p^p - \tau\int_0^b \xi(t)\frac{v^p}{u^{p-1}}(t)\,dt, \end{aligned} \tag{4.15}$$

where the integration by parts and (4.14) were utilized. In (4.15) we choose $v = 1 \in \mathrm{int}\hat{C}_+$. Then

$$0 \le \int_0^b R(1,u)(t)\,dt < 0,$$

a contradiction. This proves that (4.13) is true.

Let $\hat{\psi}_+^\varepsilon(u) = \psi_+^\varepsilon(u) - \xi u$. We have just seen that $\hat{\psi}_+^\varepsilon$ has no critical points, so

$$C_k(\hat{\psi}_+^\varepsilon, 0) = 0 \quad \text{for all integers } k \ge 0. \tag{4.16}$$

Also, by the homotopy invariance of singular homology, we have

$$C_k(\psi_+^\varepsilon, 0) = C_k(\hat{\psi}_+^\varepsilon, 0) \quad \text{for all integers } k \ge 0. \tag{4.17}$$

Combining (4.16) and (4.17), we conclude that

$$C_k(\psi_+^\varepsilon, 0) = 0 \quad \text{for all integers } k \ge 0,$$

thereby

$$C_k(\varphi_+^\varepsilon, \infty) = 0 \quad \text{for all integers } k \ge 0$$

(see (4.11) and (4.12)). In a similar fashion, we show that

$$C_k(\varphi_-^\varepsilon, \infty) = 0 \quad \text{for all integers } k \ge 0.$$

□

We want to conduct a similar computation for the functional φ. To do this, we will need the following alternative minimax characterization of $\lambda_1 > 0$.

Let $\hat{u}_0(t) = \frac{1}{b^{1/p}}$, $t \in T$, be the L^p-normalized principal eigenfunction of the negative periodic scalar p-Laplacian (i.e., the L^p-normalized eigenfunction corresponding to the eigenvalue $\lambda_0 = 0$). Let

$$\partial B_1^{L^p} = \{u \in L^p(T) : \|u\|_p = 1\}$$

and

$$C_1(p) = \left\{u \in W_{\mathrm{per}}^{1,p}(0,b) : \|u\|_p = 1,\ \int_0^b |u|^{p-2}u\,dt = 0\right\}.$$

We know (see [11], [15]) that

$$\lambda_1 = \min\{\|u'\|_p^p : u \in C_1(p)\}. \tag{4.18}$$

For our purposes, the following alternative minimax characterization of λ_1 is more convenient.

Proposition 4.2. *If $D = W^{1,p}_{\mathrm{per}}(0,b) \cap \partial B_1^{L^p}$ and $\Gamma_0 = \{\gamma_0 \in C([-1,1], D) : \gamma_0(-1) = -\hat{u}_0,\ \gamma_0(1) = \hat{u}_0\}$, then $\lambda_1 = \inf_{\gamma_0 \in \Gamma_0} \max_{-1 \le s \le 1} \|\frac{d}{dt}\gamma_0(s)\|_p^p$.*

Proof. Let $\gamma_0 \in \Gamma_0$ and consider the function $\theta_{\gamma_0} : [-1,1] \to \mathbb{R}$ defined by

$$\theta_{\gamma_0}(s) = \int_0^b |\gamma_0(s)(t)|^{p-2}\gamma_0(s)(t)\, dt.$$

Clearly, $\theta_{\gamma_0}(\cdot)$ is continuous and $\theta_{\gamma_0}(-1) < 0 < \theta_{\gamma_0}(1)$. Therefore, there is $s_0 \in [-1,1]$ such that $\theta_{\gamma_0}(s_0) = 0$. This means that $\gamma_0(s_0) \in C_1(p)$. Hence, by virtue of (4.18), we have $\|\frac{d}{dt}\gamma_0(s_0)\|_p^p \ge \lambda_1$, thus

$$\inf_{\gamma_0 \in \Gamma_0} \max_{-1 \le s \le 1} \left\| \frac{d}{dt}\gamma_0(s) \right\|_p^p \ge \lambda_1. \tag{4.19}$$

Due to (4.19), to establish the stated result we need to produce $\gamma_0^* \in \Gamma_0$ satisfying $\max_{-1 \le s \le 1} \|\frac{d}{dt}\gamma_0^*(s)\|_p^p = \lambda_1$.

Let $\hat{u}_1 \in C_1(p) \cap C^1_{\mathrm{per}}(T)$ be an eigenfunction corresponding to the eigenvalue $\lambda_1 > 0$. We consider the map $\sigma : \mathbb{R} \to D$ defined by

$$\sigma(r) = \frac{\hat{u}_1 + r}{\|\hat{u}_1 + r\|_p} \quad \text{for all } r \in \mathbb{R}.$$

We have

$$\left\| \frac{d}{dt}\sigma(r) \right\|_p^p = \frac{\|\hat{u}_1'\|_p^p}{\|\hat{u}_1 + r\|_p^p}.$$

This implies

$$\frac{d}{dr}\left\| \frac{d}{dt}\sigma(r) \right\|_p^p = -p\|\hat{u}_1'\|_p^p\ \|\hat{u}_1 + r\|_p^{p-1}\frac{\langle \mathcal{F}_p(\hat{u}_1 + r), 1\rangle_{pp'}}{\|\hat{u}_1 + r\|_p}\frac{1}{\|\hat{u}_1 + r\|_p^{2p}},$$

where $\mathcal{F}_p : L^p(T) \to L^{p'}(T)$ is the duality map for the pair $(L^p(T), L^{p'}(T))$, $\langle\cdot,\cdot\rangle_{pp'}$ denote the duality brackets for this pair and 1 is the constant function equal to one. We recall that $\mathcal{F}_p(u)(\cdot) = \frac{1}{\|u\|_p^{p-2}}|u(\cdot)|^{p-2}u(\cdot)$ for all $u \in L^p(T)$. Then

$$\frac{d}{dr}\left\| \frac{d}{dt}\sigma(r) \right\|_p^p = -p\|\hat{u}_1'\|_p^p \int_0^b |\hat{u}_1(t) + r|^{p-2}(\hat{u}_1(t) + r)\, dt \frac{1}{\|\hat{u}_1 + r\|_p^{2p}}. \tag{4.20}$$

From (4.20) it follows that the function $r \mapsto \left\|\frac{d}{dt}\sigma(r)\right\|_p^p$ has a unique maximum at $r = 0$ and this maximum value is $\|\hat{u}_1'\|_p^p = \lambda_1$ (recall $\hat{u}_1 \in C_1(p)$ and $\|\hat{u}_1\|_p = 1$). Also, for $r \neq 0$, we have

$$\sigma(r) = \frac{\hat{u}_1}{\|\hat{u}_1 + r\|_p} + \frac{r}{\|\hat{u}_1 + r\|_p} = \frac{\hat{u}_1}{\|\hat{u}_1 + r\|_p} + \frac{\mathrm{sgn}(r)}{[\int_0^b |\frac{1}{r}\hat{u}_1(t) + 1|^p\, dt]^{1/p}}.$$

It turns out that

$$\sigma(r) \to \pm \frac{1}{b^{\frac{1}{p}}} = \pm \hat{u}_0 \quad \text{as } r \to \pm\infty. \tag{4.21}$$

Let $\gamma_0^*(s) = \sigma(\frac{s}{1-s^2})$ for $s \in (-1,1)$. Evidently γ_0^* is continuous and by virtue of (4.21), it can be extended to $[-1,1]$ showing that $\gamma_0^* \in \Gamma_0$ and

$$\max_{s\in[-1,1]} \left\| \frac{d}{dt}\gamma_0^*(s) \right\|_p^p = \lambda_1,$$

which proves the proposition. □

Remark 4.3. A similar minimax characterization of the second eigenvalue of the negative Dirichlet p-Laplacian was obtained by Cuesta–de Figueiredo–Gossez [8].

Proposition 4.4. *If hypotheses* (H) *hold and* $2 \le p < \infty$*, then*

$$C_0(\varphi,\infty) = C_1(\varphi,\infty) = 0.$$

Proof. Let $\mu \in (\lambda_1, \lambda_2)$ and consider the set

$$\hat{L} = \{u \in W^{1,p}_{\text{per}}(0,b) : \|u'\|_p^p < \mu\|u\|_p^p\}.$$

Claim 1. The set $\hat{L}$ is path connected.

We use ideas from Cuesta–de Figueiredo–Gossez [8] and Dancer–Perera [9].

From Dugundji [12, p. 115] we know that it suffices to connect any $u \in \hat{L}$ and $\hat{u}_0 \in \hat{L}$ with a path lying entirely in $\hat{L}$. Let V_u be the connected component of $\hat{L}$ containing u. We note that V_u is open in $\hat{L}$ and path-connected. Letting $m_u = \inf\{\frac{\|v'\|_p^p}{\|v\|_p^p} : v \in V_u,\ v \neq 0\}$, consider $\{v_n\}_{n\ge1} \subset V_u \setminus \{0\}$ a minimizing sequence. We may assume that $\|v_n\|_p = 1$ for all $n \ge 1$. Then $\|v_n'\|_p^p \to m_u$ as $n \to \infty$. So $\{v_n\}_{n\ge1} \subset W^{1,p}_{\text{per}}(0,b)$ is bounded and we may suppose that

$$v_n \xrightarrow{\text{w}} v_0 \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } v_n \to v_0 \text{ in } C(T) \text{ as } n \to \infty.$$

Then $\|v_0\|_p = 1$ and from the lower semicontinuity of the norm in a Banach space, we have

$$\|v_0'\|_p^p \le \liminf_{n\to\infty} \|v_n'\|_p^p = m_u,\ \ \|v_0\|_p = 1,$$

which forces $\|v_0'\|_p^p = m_u$ (see also the proof of [8, Lemma 3.6]). Consequently, we derive that $v_n \to v_0$ in $W^{1,p}_{\text{per}}(0,b)$, hence $v_0 \in \overline{V_u \cap \partial B^{L^p}}$.

Let $\omega(v) = \|v'\|_p^p$, $v \in W^{1,p}_{\text{per}}(0,b)$, and $\omega_0 = \omega|_D$, with $D = W^{1,p}_{\text{per}}(0,b) \cap \partial B^{L^p}$. Notice that v_0 is a global minimum point of ω_0 on $\overline{V_u \cap \partial B^{L^p}}$. Assume that v_0 is a boundary point of $V_u \cap \partial B^{L^p}$. Using that $V_u \cap \partial B^{L^p}$ is a union of connected components of $\hat{L} \cap \partial B^{L^p}$ and the latter is open in D, Lemma 3.5 (iii) of [8] ensures that $v_0 \notin \hat{L} \cap \partial B^{L^p}$. On the other hand, by the definition of m_u one has that $m_u < \mu$, so $v_0 \in \hat{L} \cap \partial B^{L^p}$. This contradiction shows that $v_0 \in V_u \cap \partial B^{L^p}$. Hence v_0 is a critical point of ω_0 on the smooth submanifold

D of $W^{1,p}_{\text{per}}(0,b)$. Now, to prove Claim 1, it suffices to connect v_0 and $\hat{u}_0$ with a path in $\hat{L}$.

Suppose $v_0 \leq 0$. We infer that $v_0 = -\hat{u}_0$ and from Proposition 4.2, we know that $-\hat{u}_0$ and $\hat{u}_0$ can be connected with a path in $\hat{L}$. Therefore, we may assume that $v_0^+ \neq 0$. We define

$$v_\tau = \frac{v_0^+ - (1-\tau)v_0^-}{\|v_0^+ - (1-\tau)v_0^-\|_p}, \quad \tau \in [0,1].$$

Recall that

$$\langle A(v_0), h\rangle = \int_0^b m_u |v_0|^{p-2} v_0 h\, dt \quad \text{for all } h \in W^{1,p}_{\text{per}}(0,b). \tag{4.22}$$

In (4.22) first we choose $h = v_0^+ \in W^{1,p}_{\text{per}}(0,b)$ which gives

$$\|(v_0^+)'\|_p^p = m_u \|v_0^+\|_p^p. \tag{4.23}$$

Next, in (4.22), we choose $h = -v_0^- \in W^{1,p}_{\text{per}}(0,b)$ obtaining

$$\|(v_0^-)'\|_p^p = m_u \|v_0^-\|_p^p. \tag{4.24}$$

Therefore exploiting the disjointness of the supports of v_0^+ and v_0^- and using (4.23) and (4.24), we see that $\omega_0(v_\tau) = m_u$ for all $\tau \in [0,1]$. Hence $v_1 := \frac{v_0^+}{\|v_0^+\|_p} \in D$ is a local minimizer of ω_0, so a critical point of ω_0. Noting also that $v_1 \geq 0$ we find that $v_1 = \hat{u}_0$, thus $\tau \mapsto v_\tau$ is a continuous path connecting v_0 and $\hat{u}_0$. This proves Claim 1.

Let $\psi : W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ be the C^1-functional defined by

$$\psi(u) = \frac{1}{p}\|u'\|_p^p - \frac{\mu}{p}\|u\|_p^p \quad \text{for all } u \in W^{1,p}_{\text{per}}(0,b).$$

We consider the following homotopy

$$h_1(\tau, u) = (1-\tau)\varphi(u) + \tau\psi(u) \quad \text{for all } (\tau,u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b).$$

Claim 2. There exist $a \in \mathbb{R}$ and $\delta > 0$ such that for each $\tau \in [0,1]$ one has

$$h_1(\tau,u) \leq a \;\Rightarrow\; \|(h_1)'_u(\tau,u)\| \geq \delta. \tag{4.25}$$

Proceeding indirectly, suppose that Claim 2 is not true. Then we can admit that there are sequences $\{\tau_n\}_{n\geq 1} \subset [0,1]$ and $\{u_n\}_{n\geq 1} \subset W^{1,p}_{\text{per}}(0,b)$ such that $\tau_n \to \tau \in [0,1]$, $\|u_n\| \to \infty$,

$$(h_1)'_u(\tau_n,u_n) \to 0 \text{ in } W^{1,p}_{\text{per}}(0,b)^* \quad \text{and} \quad h_1(\tau_n,u_n) \to -\infty. \tag{4.26}$$

From the convergence $(h_1)'_u(\tau_n,u_n) \to 0$ in $W^{1,p}_{\text{per}}(0,b)^*$ we have

$$\left|\langle A(u_n), h\rangle - (1-\tau_n)\int_0^b f(t,u_n)h\, dt - \tau_n\mu\int_0^b |u_n|^{p-2}u_n h\, dt\right| \leq \varepsilon_n\|h\| \tag{4.27}$$

for all $h \in W^{1,p}_{\text{per}}(0,b)$, with $\varepsilon_n \downarrow 0$.

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then, from (4.27), we have

$$\left| \langle A(y_n), h\rangle - (1-\tau_n)\int_0^b \frac{f(t,u_n)}{\|u_n\|^{p-1}} h\, dt - \tau_n \mu \int_0^b |y_n|^{p-2} y_n h\, dt \right| \\ \leq \frac{\varepsilon_n}{\|u_n\|^{p-1}} \|h\|. \tag{4.28}$$

Since $\|y_n\| = 1$ for all $n \geq 1$, we may assume that

$$y_n \xrightarrow{\text{w}} y \text{ in } W^{1,p}_{\text{per}}(0,b) \quad \text{and} \quad y_n \to y \text{ in } C(T) \text{ as } n \to \infty. \tag{4.29}$$

In (4.28) we choose $h = y_n - y \in W^{1,p}_{\text{per}}(0,b)$. Passing to the limit as $n \to \infty$ and using (4.29), we obtain $\lim\limits_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0$, so

$$y_n \to y \text{ in } W^{1,p}_{\text{per}}(0,b), \text{ hence } \|y\| = 1. \tag{4.30}$$

From the proof of Proposition 3.2, we know that

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{\text{w}} g|y|^{p-2}y \text{ in } L^{p'}(T), \beta(t) \leq g(t) \leq \lambda_2 \text{ a.e. on } T. \tag{4.31}$$

So, if in (4.28) we pass to the limit as $n \to \infty$ and use (4.31), we get

$$\begin{cases} -(|y'(t)|^{p-2}y'(t))' = [(1-\tau)g(t) + \tau\mu]|y(t)|^{p-2}y(t) \quad \text{a.e. on } T, \\ y(0) = y(b), \; y'(0) = y'(b). \end{cases} \tag{4.32}$$

If $\tau \neq 0$, then $\lambda_1 < (1-\tau)g(t) + \tau\mu < \lambda_2$ a.e. on T and so from (4.32) and Proposition 2 of [1], we have $y = 0$, a contradiction to (4.30).

Therefore $\tau = 0$ and so from (4.32) we obtain

$$\begin{cases} -(|y'(t)|^{p-2}y'(t))' = g(t)|y(t)|^{p-2}y(t) \quad \text{a.e. on } T, \\ y(0) = y(b), \; y'(0) = y'(b). \end{cases} \tag{4.33}$$

Again the case $g(t) \leq \lambda_2$ a.e. on T, strictly on a set of positive measure, should be excluded since it leads to $y = 0$. Hence it is sufficient to consider that $g(t) = \lambda_2$ a.e. on T and so $y \in C^1(T)$ is an eigenfunction corresponding to the eigenvalue $\lambda_2 > 0$. In particular, $y(t) \neq 0$ a.e. on T.

Recall from (4.26) that $h_1(\tau_n, u_n) \to -\infty$ as $n \to \infty$. Then we can find $n_0 \geq 1$ such that

$$\frac{1}{p}\|u_n'\|_p^p - (1-\tau_n)\int_0^b F(t,u_n)\, dt - \tau_n \frac{\mu}{p}\|u_n\|_p^p \leq 0 \quad \text{for all } n \geq n_0,$$

thus

$$\frac{1}{p}\|y_n'\|_p^p - (1-\tau_n)\int_0^b \frac{F(t,u_n)}{\|u_n\|^p}\, dt - \tau_n \frac{\mu}{p}\|y_n\|_p^p \leq 0 \quad \text{for all } n \geq n_0. \tag{4.34}$$

We know that $\tau_n \to 0$ and from the proof of Proposition 3.2 [see (3.11) and (3.12)], we have

$$\frac{pF(\cdot, u_n(\cdot))}{\|u_n\|^p} \xrightarrow{\text{w}} \xi|y|^p, \quad \text{with } \xi \in L^\infty(T)_+, \; \xi(t) \leq \eta(t) \text{ a.e. on } T. \tag{4.35}$$

So, if we pass to the limit as $n \to \infty$ in (4.34) and use (4.35), we obtain

$$\frac{1}{p}\|y'\|_p^p \le \frac{1}{p}\int_0^b \xi |y|^p \, dt < \frac{\lambda_2}{p}\|y\|_p^p,$$

a contradiction. This proves Claim 2. Similarly for $h_2(\tau, u) := h_1(1-\tau, u)$.

Let $\beta < \inf\{h_1(\tau, u), h_2(\tau, u) : \tau \in [0,1], \|u\| \le R\}$. Reasoning as in the proof of Proposition 4.1, we show that the level sets φ^β and ψ^β are of the same homotopy type, hence

$$H_k(W, \varphi^\beta) = H_k(W, \psi^\beta) \quad \text{for all integers } k \ge 0,$$

where $W = W^{1,p}_{\mathrm{per}}(0,b)$, which ensures

$$C_k(\varphi, \infty) = C_k(\psi, \infty) \quad \text{for all integers } k \ge 0. \tag{4.36}$$

Because $\mu \in (\lambda_1, \lambda_2)$, we see that 0 is the only critical point of ψ, thereby

$$C_k(\psi, \infty) = C_k(\psi, 0) \quad \text{for all integers } k \ge 0. \tag{4.37}$$

By definition

$$C_k(\psi, 0) = H_k(\psi^0, \psi^0 \setminus \{0\}) \quad \text{for all integers } k \ge 0. \tag{4.38}$$

Note that $\hat{L} \subset \psi^0 \setminus \{0\}$ and let $* \in \hat{L}$. From the reduced exact homology sequence (see [17, p. 388]) we see that

$$\begin{aligned} \ldots H_k(\psi^0, *) \xrightarrow{i_*} H_k(\psi^0, \psi^0 \setminus \{0\}) \xrightarrow{\partial_*} H_{k-1}(\psi^0 \setminus \{0\}, *) \ldots \\ \text{for all integers } k \ge 0, \end{aligned} \tag{4.39}$$

where i_* is the group homomorphism resulting from the corresponding inclusion map and ∂_* is the boundary homomorphism. Since the functional ψ is p-homogeneous, it follows that ψ^0 is contractible and so

$$H_k(\psi^0, *) = 0 \quad \text{for all integers } k \ge 0. \tag{4.40}$$

Then from (4.39) and (4.40) we infer that $\operatorname{im} i_* = \ker \partial_* = 0$. This means that ∂_* is an isomorphism of $H_k(\psi^0, \psi^0 \setminus \{0\}) = C_k(\psi, 0)$ (see (4.38)) with a subgroup of $H_{k-1}(\psi^0 \setminus \{0\}, *)$. Note that

$$\hat{L} = \dot{\psi}^0 := \{u \in W^{1,p}_{\mathrm{per}}(0,b) : \psi(u) < 0\}.$$

Then, from the second deformation theorem and Granas–Dugundji [17, p. 407], we see that for $\varepsilon > 0$ small

$$\psi^0 \setminus \{0\} \text{ is homotopy equivalent to } \psi^{-\varepsilon}, \tag{4.41}$$

$$\hat{L} \text{ is homotopy equivalent to } \psi^{-\varepsilon}. \tag{4.42}$$

Properties (4.41) and (4.42) guarantee that $\psi^0 \setminus \{0\}$ is homotopy equivalent to $\hat{L}$ and, as a consequence,

$$H_k(\psi^0 \setminus \{0\}, *) = H_k(\hat{L}, *) \quad \text{for all integers } k \ge 0. \tag{4.43}$$

But from Claim 1 we know that $\hat{L}$ is path-connected, which implies that $H_0(\hat{L}, *) = 0$. Taking into account (4.43), we arrive at

$$H_0(\psi^0 \setminus \{0\}, *) = 0. \tag{4.44}$$

Because $H_k(\psi^0, \psi^0 \setminus \{0\}) = C_k(\psi, 0)$ is isomorphic to a subgroup of $H_{k-1}(\psi^0 \setminus \{0\}, *)$, we have

$$C_0(\psi, 0) = 0 \tag{4.45}$$

since $H_{-1}(\psi^0 \setminus \{0\}, *) = 0$, and by (4.44),

$$C_1(\psi, 0) = 0. \tag{4.46}$$

Combining (4.36), (4.37), (4.45) and (4.46), we achieve the desired conclusion. □

Next we compute the critical groups of φ at the two constant sign solutions $u_0 \in \operatorname{int} \hat{C}_+$ and $v_0 \in -\operatorname{int} \hat{C}_+$ obtained in Proposition 3.5.

Proposition 4.5. *If hypotheses* (H) *and* $2 \le p < \infty$ *hold, then* $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ *for all integers* $k \ge 0$.

Proof. We start by computing the critical groups of φ_+^ε at u_0 and of φ_-^ε at v_0. Subsequently, exploiting the homotopy invariance of the critical groups, we will be able to compute the critical groups of φ at those points.

Claim. $C_k(\varphi_+^\varepsilon, u_0) = C_k(\varphi_-^\varepsilon, v_0) = \delta_{k,1}\mathbb{Z}$ for all integers $k \ge 0$.

As before, we do the proof for the pair $\{\varphi_+^\varepsilon, u_0\}$, the proof for the pair $\{\varphi_-^\varepsilon, v_0\}$ being similar. From the proof of Proposition 3.5 we know that $\varphi_+^\varepsilon(0) = 0 < \eta_\rho^+ \le \varphi_+^\varepsilon(u_0)$. Let $\theta < 0 < \mu < \eta_\rho^+$ and consider the following triple of sets

$$(\varphi_+^\varepsilon)^\theta \subset (\varphi_+^\varepsilon)^\mu \subset W = W^{1,p}_{\mathrm{per}}(0, b).$$

For this triple we consider the long exact sequence

$$\dots H_k(W, (\varphi_+^\varepsilon)^\theta) \xrightarrow{i_*} H_k(W, (\varphi_+^\varepsilon)^\mu) \xrightarrow{\partial_*} H_{k-1}((\varphi_+^\varepsilon)^\mu, (\varphi_+^\varepsilon)^\theta) \dots \tag{4.47}$$

As before, i_* is the group homomorphism resulting from the corresponding inclusion map and ∂_* is the boundary homomorphism.

We may assume that $\{0, u_0\}$ are the only critical points of φ_+^ε (otherwise, we have $\hat{u}$ a third critical point of φ_+^ε, distinct from $\{0, u_0\}$, and as in the proof of Proposition 3.5 we show that $\hat{u} \in \operatorname{int}\hat{C}_+$, hence it is a solution of (1.1) and we have a third nontrivial solution). Since 0 is the only critical point of φ_+^ε with critical value in (θ, μ), there holds

$$H_{k-1}((\varphi_+^\varepsilon)^\mu, (\varphi_+^\varepsilon)^\theta) = C_{k-1}(\varphi_+^\varepsilon, 0) \quad \text{for all integers } k \ge 0 \tag{4.48}$$

(see Chang [6, p. 35]). From Proposition 3.3 we know that 0 is a local minimizer of φ_+^ε, therefore

$$C_k(\varphi_+^\varepsilon, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all integers } k \ge 0$$

(see [6, p. 33] and [19, p. 175]). In view of (4.48), we deduce that

$$H_{k-1}\left(\left(\varphi_+^\varepsilon\right)^\mu, \left(\varphi_+^\varepsilon\right)^\theta\right) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \quad \text{for all integers } k \ge 0. \tag{4.49}$$

Also, u_0 is the only critical point of φ_+^ε with critical value above μ. Hence

$$H_k\left(W, \left(\varphi_+^\varepsilon\right)^\mu\right) = C_k\left(\varphi_+^\varepsilon, u_0\right) \quad \text{for all integers } k \ge 0. \tag{4.50}$$

Moreover, from the definition of critical groups of φ_+^ε at infinity (see Sect. 2), we have

$$H_k\left(W, \left(\varphi_+^\varepsilon\right)^\theta\right) = C_k\left(\varphi_+^\varepsilon, \infty\right) \quad \text{for all integers } k \geq 0,$$

which, due to Proposition 4.1, implies

$$H_k\left(W, \left(\varphi_+^\varepsilon\right)^\theta\right) = 0 \quad \text{for all integers } k \geq 0. \tag{4.51}$$

Returning to (4.47) and using (4.49) and (4.51), we see that we must focus on the tail of sequence, for $k = 1$, since the rest is trivial. Then, owing to the exactness of (4.47), we have

$$\operatorname{rank} H_1\left(W, \left(\varphi_+^\varepsilon\right)^\mu\right) = \operatorname{rank}\left(\operatorname{im}\partial_*\right) + \operatorname{rank}\left(\operatorname{im} i_*\right)$$

yielding, through (4.49), (4.50), (4.51), that

$$\operatorname{rank} C_1\left(\varphi_+^\varepsilon, u_0\right) \leq 1 + 0 = 1. \tag{4.52}$$

But u_0, being a point of mountain pass type, by Chang [6, p. 89], satisfies

$$C_1\left(\varphi_+^\varepsilon, u_0\right) \neq 0. \tag{4.53}$$

Combining (4.52) and (4.53), we obtain

$$C_k\left(\varphi_+^\varepsilon, u_0\right) = \delta_{k,1}\mathbb{Z} \quad \text{for all integers } k \geq 0.$$

Similarly, we show that

$$C_k(\varphi_-^\varepsilon, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all integers } k \geq 0.$$

This proves the Claim.

Next we consider the homotopy

$$h_+(\tau, u) = (1-\tau)\varphi(u) + \tau\varphi_+^\varepsilon(u) \quad \text{for all } (\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b).$$

We may assume that there is $r_0 > 0$ such that u_0 is the only critical point of $h_+(\tau, \cdot)$ in $\overline{B}_{r_0}(u_0) = \{u \in W^{1,p}_{\text{per}}(0,b) : \|u - u_0\| \leq r_0\}$ for all $\tau \in [0,1]$. Indeed, if it is not the case, we can find $\{\tau_n\}_{n\geq1} \subset [0,1]$ and $\{u_n\}_{n\geq1} \subset W^{1,p}_{\text{per}}(0,b)$ such that $\tau_n \to \tau \in [0,1]$, $u_n \to u_0$ in $W^{1,p}_{\text{per}}(0,b)$ and

$$(h_+)'_u(\tau_n, u_n) = 0 \quad \text{for all } n \geq 1. \tag{4.54}$$

From equation (4.54) we have

$$\begin{cases} -(|u_n'(t)|^{p-2}u_n'(t))' + \tau_n\varepsilon|u_n(t)|^{p-2}u_n(t) \\ = (1-\tau_n)f(t, u_n(t)) + \tau_n f_+(t, u_n(t)) + \tau_n\varepsilon(u_n^+(t))^{p-1} \quad \text{a.e. on } T, \\ u_n(0) = u_n(b),\ u_n'(0) = u_n'(b). \end{cases} \tag{4.55}$$

It turns out that $\{(|u_n'(\cdot)|^{p-2}u_n'(\cdot))'\}_{n\geq1}$ is bounded in $L^{p'}(T)$, so $\{u_n'\}_{n\geq1}$ is relatively compact in $C(T)$. Hence we may infer that $\{u_n\}_{n\geq1}$ is relatively compact in $C^1(T)$. We find out that along a subsequence $u_n \to u_0$ in $C^1(T)$ as $n \to \infty$. Recalling that $u_0 \in \operatorname{int}\hat{C}_+$, we get that $u_n \in \operatorname{int}\hat{C}_+$ for all $n \geq n_0$. Therefore (4.55) becomes

$$\begin{cases} -(|u_n'(t)|^{p-2}u_n'(t))' = f(t, u_n(t)) \quad \text{a.e. on } T, \\ u_n(0) = u_n(b),\ u_n'(0) = u_n'(b). \end{cases}$$

We have produced a whole sequence of distinct, strictly positive solutions of (1.1).

Thus, we can say that there is $r_0 > 0$ such that u_0 is the only critical point in $\overline{B}_{r_0}(u_0)$ of $h_+(\tau, \cdot)$ for all $\tau \in [0,1]$. Moreover, as before, we verify that $h_+(\tau, \cdot)$ satisfies the PS-condition. Invoking the homotopy invariance of critical groups (see [7]), we have

$$C_k(h_+(0,\cdot), u_0) = C_k(h_+(1,\cdot), u_0) \quad \text{for all integers } k \geq 0.$$

Due to the Claim, this reads as

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all integers } k \geq 0.$$

Similarly, using the homotopy

$$h_-(\tau, u) = \tau\varphi(u) + (1-\tau)\varphi_-^{\varepsilon}(u) \quad \text{for all } (\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b),$$

we show that $C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all integers $k \geq 0$. □

Now, we are ready to produce a third nontrivial solution and have the complete multiplicity result for problem (1.1).

Theorem 4.6. *If hypotheses* (H) *hold and* $2 \leq p < \infty$*, then problem (1.1) admits at least three nontrivial solutions* $u_0 \in \text{int}\hat{C}_+$*,* $v_0 \in -\text{int}\hat{C}_+$ *and* $y_0 \in C^1(T)$*.*

Proof. From Proposition 3.5, we already have two constant sign solutions $u_0 \in \text{int}\hat{C}_+$ and $v_0 \in -\text{int}\hat{C}_+$. Proposition 4.5 implies that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all integers } k \geq 0. \tag{4.56}$$

Also, from Proposition 3.3, we know that 0 is a local minimizer of φ, hence

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all integers } k \geq 0. \tag{4.57}$$

Assume that $\{0, u_0, v_0\}$ are the only critical points of φ. Invoking the Morse relation (2.1) and using (4.56), (4.57) and Proposition 4.4, we have

$$1 + 2t = t^2 \sum_{k\geq 0} \alpha_{k+2} t^k + (1+t) \sum_{k\geq 0} \beta_k t^k,$$

for nonnegative integers α_{k+2} and β_k, $k \geq 0$. By equating the corresponding coefficients we reach a contradiction. This means that φ has a third nontrivial critical point y_0. Then $y_0 \in C^1(T)$ and solves (1.1). □

Remark 4.7. We conclude the paper by mentioning three open problems:

(a) Is it possible to remove the restriction $2 \leq p < \infty$ allowing in this way for differential operators which are singular on the critical set $\{t \in T : u'(t) = 0\}$?

(b) Is it possible to allow double resonance in the spectral interval $[\lambda_1, \lambda_2]$, namely can we have $\beta(t) = \lambda_1$ a.e. on T?

(c) Can we have such a "three nontrivial solutions theorem", when resonance occurs in the spectral interval $[0, \lambda_1]$ or in a general spectral interval $[\lambda_m, \lambda_{m+1}]$, $m \geq 2$?

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