Remarks on regularity and uniqueness of the Dirac–Klein–Gordon equations in one space dimension

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Abstract. In recent work, the authors extended the local and global well-posedness theory for the 1D Dirac–Klein–Gordon equations, but the uniqueness of the solutions was only known in the contraction spaces (of Bourgain–Klainerman–Machedon type). Here we prove some unconditional uniqueness results [that is, uniqueness in the larger space $C([0, T]; X_0)$, where X_0 denotes the data space]. We also prove a result about persistence of higher regularity, which is stronger than the standard version obtained from the contraction argument, since our result allows to independently increase the regularity of the spinor and scalar fields, whereas in the standard result they must be increased by the same amount.

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1. Introduction

We are interested in the coupled Dirac–Klein–Gordon (DKG) equations in one space dimension. The resulting system reads

$$\begin{cases} (-i\gamma^0\partial_t - i\gamma^1\partial_x + M)\psi = \phi\psi & (M \geq 0), \\ (-\square + m^2)\phi = \psi^\dagger\gamma^0\psi & (\square = -\partial_t^2 + \partial_x^2, \quad m \geq 0), \end{cases} \quad (1.1)$$

where $\phi : \mathbb{R}^{1+1} \rightarrow \mathbb{R}$ is a scalar field and $\psi : \mathbb{R}^{1+1} \rightarrow \mathbb{C}^2$ is the Dirac spinor field, regarded as a column vector $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$, whose conjugate transpose is denoted $\psi^\dagger$. For the sake of definiteness we shall consider only the massless case $M = m = 0$ in the rest of the paper, but our arguments can easily be modified to handle massive cases as well. We choose the representation $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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and $\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of the Dirac matrices, resulting in the system

$$\begin{cases} (D_t + D_x)u = \phi v, \\ (D_t - D_x)v = \phi u, \\ -\square\phi = \bar{u}v + u\bar{v}, \end{cases} \quad (1.2)$$

where we write $D_t = -i\partial_t$ and $D_x = -i\partial_x$. We prescribe initial data

$$u|_{t=0} = u_0 \in H^s, \quad v|_{t=0} = v_0 \in H^s, \quad (\phi, \partial_t\phi)|_{t=0} = (\phi_0, \phi_1) \in H^r \times H^{r-1}, \quad (1.3)$$

where $r, s \in \mathbb{R}$ will be specified. Here $H^s = (1 - \partial_x^2)^{-s/2} L^2(\mathbb{R})$ is the standard Sobolev space.

Corresponding to the operators $D_t + D_x$, $D_t - D_x$ and $\square$, we define the following spaces of Bourgain–Klainerman–Machedon type.

Definition 1.1. For $s, r, \sigma, \rho \in \mathbb{R}$, let $X_+^{s,\sigma}$, $X_-^{s,\sigma}$ and $H^{r,\rho}$ be the completions of the Schwartz space $\mathcal{S}(\mathbb{R}^{1+1})$ with respect to the norms

$$\begin{aligned} \|u\|_{X_+^{s,\sigma}} &= \|\langle \xi \rangle^s \langle \tau + \xi \rangle^\sigma \tilde{u}(\tau, \xi)\|_{L_{\tau,\xi}^2}, \\ \|v\|_{X_-^{s,\sigma}} &= \|\langle \xi \rangle^s \langle \tau - \xi \rangle^\sigma \tilde{v}(\tau, \xi)\|_{L_{\tau,\xi}^2}, \\ \|\phi\|_{H^{r,\rho}} &= \|\langle \xi \rangle^r \langle |\tau| - |\xi| \rangle^\rho \tilde{\phi}(\tau, \xi)\|_{L_{\tau,\xi}^2}, \end{aligned}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\tilde{u}(\tau, \xi) = \int e^{-i(t\tau + x\xi)} u(t, x) dt dx$ is the space–time Fourier transform. The restrictions of these spaces to a time-slab $S_T = (-T, T) \times \mathbb{R}$ are denoted $X_\pm^{s,\sigma}(S_T)$, etc.

We recall the following well-known fact (see, e.g., [4, Lemma 5]): if

$$(D_t \pm D_x)w = F \quad \text{on } S_T, \quad w|_{t=0} = w_0,$$

where $F \in X_\pm^{s,b-1}(S_T)$ and $w_0 \in H^s$ for some $s \in \mathbb{R}$, $b > 1/2$ and $T > 0$, then

$$\|w\|_{X_\pm^{s,b}(S_T)} \leq C_{T,b} \left(\|w_0\|_{H^s} + \|F\|_{X_\pm^{s,b-1}(S_T)} \right), \quad (1.4)$$

where $C_{T,b}$ is bounded as $T \rightarrow 0$.

We need also the analogous estimate for the wave equation (see, e.g., [4] and the references therein): if

$$\square w = F \quad \text{on } S_T, \quad (w, \partial_t w)|_{t=0} = (w_0, w_1),$$

where $F \in H^{r-1,b-1}(S_T)$, $w_0 \in H^r$ and $w_1 \in H^r$ for some $r \in \mathbb{R}$, $b > 1/2$ and $T > 0$, then

$$\begin{aligned} &\|w\|_{H^{r,b}(S_T)} + \|\partial_t w\|_{H^{r-1,b}(S_T)} \\ &\leq C_{T,b} \left(\|w_0\|_{H^r} + \|w_1\|_{H^{r-1}} + \|F\|_{H^{r-1,b-1}(S_T)} \right), \end{aligned} \quad (1.5)$$

where $C_{T,b}$ is bounded as $T \rightarrow 0$.

The first global well-posedness result for 1D DKG was obtained by Chadam [3]. Since the work of Bournaveas [1], there has been a lot of progress

on lowering the initial data regularity required for global or local well-posedness; see [2, 5–10]. Our aim here is to improve the statements of uniqueness and persistence of higher regularity in the existing local well-posedness theory, which we recall in the following theorem:

Theorem 1.2. (See [9].) *Suppose (s, r) belongs to the convex region*

$$\Omega = \left\{ (s, r) \in \mathbb{R}^2 : s > -\frac{1}{4}, \quad r > 0, \quad |s| \leq r \leq 1 + s \right\}. \quad (1.6)$$

Then:

- (i) (*Local existence.*) *There exists a time $T > 0$, depending continuously on the norms of the initial data (1.3), and there exists a solution*

$$(u, v, \phi, \partial_t \phi) \in C([-T, T]; H^s \times H^s \times H^r \times H^{r-1}),$$

of (1.2), (1.3) on $S_T = (-T, T) \times \mathbb{R}$. The solution depends continuously on the data, in the sense that $(u_0, v_0, \phi_0, \phi_1) \mapsto (u, v, \phi, \partial_t \phi)(t)$ is continuous from $H^s \times H^s \times H^r \times H^{r-1}$ to itself, locally uniformly in t .

- (ii) (*Uniqueness.*) *The solution belongs to, and is unique in, the class*

$$(u, v, \phi, \partial_t \phi) \in X_+^{s, \sigma}(S_T) \times X_-^{s, \sigma}(S_T) \times H^{r, \rho}(S_T) \times H^{r-1, \rho}(S_T),$$

for suitably chosen $\sigma, \rho \in (1/2, 1)$. In fact,

$$\sigma = \tilde{\sigma}(s, r) \equiv \begin{cases} 1/2 + \varepsilon & \text{if } s \geq 0, \\ 1/2 + \varepsilon & \text{if } s \in (-1/4, 0) \quad \text{and } r < 1 + 2s, \\ 1 - \varepsilon & \text{if } s \in (-1/4, 0) \quad \text{and } 1 + 2s \leq r, \end{cases}$$

$$\rho = 1/2 + \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small, depending on s and r .

- (iii) (*Persistence of higher regularity.*) *If the data belongs to the higher order Sobolev space*

$$(u_0, v_0, \phi_0, \phi_1) \in H^{s+\delta} \times H^{s+\delta} \times H^{r+\delta} \times H^{r-1+\delta},$$

for some $\delta > 0$, then the solution persists in that space as a continuous function of time:

$$(u, v, \phi, \partial_t \phi) \in C([-T, T]; H^{s+\delta} \times H^{s+\delta} \times H^{r+\delta} \times H^{r-1+\delta}),$$

with T as in part (i).

Remark 1.3. In part (i), when we say that (u, v, ϕ) is a solution of (1.2), we mean in the sense of distributions on $S_T = (-T, T) \times \mathbb{R}$. However, it is not obvious that the bilinear terms in (1.2) are well-defined in the sense of distributions: indeed, s can be negative, in which case $u(t)$ and $v(t)$ may not be functions. Nevertheless, the bilinear terms do make sense. For the terms ϕu and ϕv , this can be seen from the Sobolev product inequality (in one space dimension)

$$\|fg\|_{H^{-c}} \leq C \|f\|_{H^a} \|g\|_{H^b}, \quad (1.7)$$

which holds if (and only if)

$$a + b + c \geq \frac{1}{2}, \quad (1.8)$$

$$\min(a + b, a + c, b + c) \geq 0, \quad (1.9)$$

$$(1.8) \text{ and } (1.9) \text{ are not both equalities.} \quad (1.10)$$

Since $\phi(t) \in H^r$ and $u(t) \in H^s$, where $(r, s) \in \Omega$, we conclude that, for any $\varepsilon > 0$,

$$\phi(t)u(t) \in \begin{cases} H^{s-(1/2-r)-\varepsilon} & \text{if } r \leq 1/2, \\ H^s & \text{if } r > 1/2, \end{cases} \quad (1.11)$$

continuously in t , and similarly for ϕv . The terms $\bar{u}v$ and $u\bar{v}$ are more delicate; they make sense because $u \in X_+^{s,\sigma}$ and $v \in X_-^{s,\sigma}$, permitting the use of the bilinear estimate in Lemma 2.4.

We can now state the two new results to be proved here.

The first result improves the uniqueness statement in Theorem 1.2, for a slightly smaller range of Sobolev exponents.

Theorem 1.4. *If (s, r) is in the convex region*

$$R = \left\{ (s, r) : s > -\frac{1}{4}, \quad \max\left(\frac{1}{6}, \frac{1}{6} - \frac{2s}{3}, -2s\right) < r \leq 1 + s, \quad r \geq s \right\},$$

then uniqueness for (1.2), (1.3) holds in the class

$$(u, v, \phi, \partial_t \phi) \in C([-T, T]; H^s \times H^s \times H^r \times H^{r-1}).$$

Remark 1.5. Again it is not obvious that the terms $\bar{u}v$ and $u\bar{v}$ appearing in the third equation in (1.2) are meaningful as distributions, if $s < 0$. But in the proof of Theorem 1.4 we show, using just the first two equations in (1.2), that u and v belong to $X_+^{a,\alpha}$ and $X_-^{a,\alpha}$ for suitable a, α , making it possible to apply Lemma 2.4, and thus make sense of $\bar{u}v$ and $u\bar{v}$.

Our second result improves the statement of persistence of higher regularity in Theorem 1.2, allowing us to increase the Sobolev exponents s and r by independent amounts δ and γ , provided we stay inside the region Ω .

Theorem 1.6. *Let $\delta, \gamma \geq 0$, and suppose that both (s, r) and $(s + \delta, r + \gamma)$ belong to the convex region Ω defined in Theorem 1.2. Consider the solution*

$$(u, v, \phi, \partial_t \phi) \in C([-T, T]; H^s \times H^s \times H^r \times H^{r-1})$$

of (1.2), (1.3), as in Theorem (1.2)(i). If the data belongs to the higher order space

$$(u_0, v_0, \phi_0, \phi_1) \in H^{s+\delta} \times H^{s+\delta} \times H^{r+\gamma} \times H^{r-1+\gamma},$$

then the solution persists in that space as a continuous function of time:

$$(u, v, \phi, \partial_t \phi) \in C([-T, T]; H^{s+\delta} \times H^{s+\delta} \times H^{r+\gamma} \times H^{r-1+\gamma}).$$

FIGURE 1. Sketch of the regions Ω and R (shaded) appearing in Theorems 1.2 and 1.4

The first global result was proved in [3]. In fact, for $s \geq 0$, any local well-posedness result can be extended to a global one by using the conservation of charge; see [1, 2, 5–7]. Global well-posedness for a range of negative s was obtained first in [8], and then for a larger range of r (but the same range of s) in [10].

In the next section, we discuss some bilinear and null form estimates needed for the proofs of Theorems 1.4 and 1.6, which are given in Sect. 3 and 4, respectively.

In estimates we use $X \lesssim Y$ as shorthand for $X \leq CY$, where $C \gg 1$ is a constant (which may depend on quantities which are considered fixed).

2. Bilinear and null form estimates

Proofs of the first three lemmas below can be found in [9].

Lemma 2.1. *Suppose $A + B + C > 1/2$ and $A + B, A + C, B + C \geq 0$. Suppose further that $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma > 1/2$. Then*

$$\|wz\|_{H^{-C, -\gamma}} \lesssim \|w\|_{H^{A, \alpha}} \|z\|_{H^{B, \beta}}.$$

Lemma 2.2. *Given $\tau, \lambda, \xi, \eta \in \mathbb{R}$, define*

$$\Gamma = |\tau| - |\xi|, \quad \Theta_+ = \lambda + \eta, \quad \Sigma_- = \tau - \lambda - (\xi - \eta).$$

Then

$$\min(|\eta|, |\xi - \eta|) \lesssim \max(|\Gamma|, |\Theta_+|, |\Sigma_-|).$$

Lemma 2.3. *For any $\alpha > 1/2$,*

$$\|wz\|_{L^2} \lesssim \|w\|_{X_+^{0, \alpha}} \|z\|_{X_-^{0, \alpha}}.$$

The latter is what we may call a null form estimate, the null structure being due to the difference in signs, without which the estimate would fail. The same comment applies to the next two lemmas.

Lemma 2.4. *The bilinear estimate*

$$\|wz\|_{H^{-s_3, b-1}} \lesssim \|w\|_{X_+^{s_1, b'}} \|z\|_{X_-^{s_2, b'}} \quad (2.1)$$

holds if

$$\begin{cases} \frac{1}{2} < b \leq b' \leq 1, \\ s_1 + s_2 + s_3 \geq b - b', \\ s_1 + s_2 \geq b - 1, \\ s_1 + s_3 \geq 0, \\ s_2 + s_3 \geq 0, \end{cases} \quad (2.2)$$

Proof. Set

$$\begin{aligned} F_+(\lambda, \eta) &= \langle \eta \rangle^{s_1} \langle \lambda + \eta \rangle^{b'} |\tilde{w}(\lambda, \eta)|, \\ G_-(\lambda, \eta) &= \langle \eta \rangle^{s_2} \langle \lambda - \eta \rangle^{b'} |\tilde{z}(\lambda, \eta)|. \end{aligned}$$

Then (2.1) is equivalent to

$$J \lesssim \|F_+\|_{L^2} \|G_-\|_{L^2}, \quad (2.3)$$

where

$$J = \left\| \iint \frac{F_+(\lambda, \eta) G_-(\tau - \lambda, \xi - \eta)}{\langle \eta \rangle^{s_1} \langle \xi - \eta \rangle^{s_2} \langle \xi \rangle^{s_3} \langle \Theta_+ \rangle^{b'} \langle \Sigma_- \rangle^{b'} \langle \Gamma \rangle^{1-b}} d\lambda d\eta \right\|_{L_{\tau, \xi}^2},$$

with Γ , Θ_+ and Σ_- defined as in Lemma 2.2.

Case 1: $\max(|\Gamma|, |\Theta_+|, |\Sigma_-|) = |\Gamma|$. By symmetry, we may assume $|\eta| \leq |\xi - \eta|$. Then either $|\xi| \lesssim |\eta| \sim |\xi - \eta|$ or $|\eta| \ll |\xi - \eta| \sim |\xi|$. Using also Lemma 2.2, we get

$$J \lesssim J_1 + J_2,$$

where

$$J_1 = \left\| \iint \chi_{|\xi| \lesssim |\eta| \sim |\xi - \eta|} \frac{F_+(\lambda, \eta) G_-(\tau - \lambda, \xi - \eta)}{\langle \xi \rangle^{s_3} \langle \eta \rangle^{s_1 + s_2 + 1 - b} \langle \Theta_+ \rangle^{b'} \langle \Sigma_- \rangle^{b'}} d\lambda d\eta \right\|_{L_{\tau, \xi}^2},$$

$$J_2 = \left\| \iint \chi_{|\eta| \ll |\xi - \eta| \sim |\xi|} \frac{F_+(\lambda, \eta) G_-(\tau - \lambda, \xi - \eta)}{\langle \eta \rangle^{s_1 + 1 - b} \langle \xi - \eta \rangle^{s_2 + s_3} \langle \Theta_+ \rangle^{b'} \langle \Sigma_- \rangle^{b'}} d\lambda d\eta \right\|_{L_{\tau, \xi}^2}.$$

The estimate (2.3) for J_1 follows from Lemma 2.3. Indeed, depending on whether $s_3 \geq 0$ or $s_3 < 0$, we can replace $\langle \xi \rangle^{s_3} \langle \eta \rangle^{s_1 + s_2 + 1 - b}$ in J_1 by $\langle \eta \rangle^{s_1 + s_2 + 1 - b}$ or $\langle \eta \rangle^{s_1 + s_2 + s_3 + 1 - b}$, respectively; in either case, (2.2) guarantees that the exponent of $\langle \eta \rangle$ is nonnegative. The estimate for J_2 also holds by Lemma 2.3, since [noting that $s_2 + s_3 \geq 0$ by (2.2)] we can replace $\langle \eta \rangle^{s_1 + 1 - b} \langle \xi - \eta \rangle^{s_2 + s_3}$ in J_2 by $\langle \eta \rangle^{s_1 + s_2 + s_3 + 1 - b}$, whose exponent is nonnegative.

Case 2: $\max(|\Gamma|, |\Theta_+|, |\Sigma_-|) = |\Theta_+|$ or $|\Sigma_-|$. Then (2.3) follows from Lemma 2.1 with $(A, B, C) = (s_1 + b', s_2, s_3)$ or $(s_1, s_2 + b', s_3)$ and $(\alpha, \beta, \gamma) = (0, b', 1 - b)$ or $(b', 0, 1 - b)$, and (2.2) guarantees that the hypotheses of Lemma 2.1 are satisfied. $\square$

Lemma 2.5. *The bilinear estimates*

$$\|wz\|_{X_+^{-s_1, -b_1}} \lesssim \|w\|_{H^{s_3, b_3}} \|z\|_{X_-^{s_2, b_2}} \quad (2.4)$$

$$\|wz\|_{X_-^{-s_2, -b_2}} \lesssim \|w\|_{H^{s_3, b_3}} \|z\|_{X_+^{s_1, b_1}} \quad (2.5)$$

hold if

$$\begin{cases} b_1, b_2, b_3 \geq 0, \\ b_1 + b_2, b_1 + b_3, b_2 + b_3 > \frac{1}{2}, \\ s_1 + s_2 + s_3 + \min(b_1, b_2, b_3) > \frac{1}{2}, \\ s_1 + s_2 + \min(b_1, b_2, b_3) \geq 0, \\ s_1 + s_3 \geq 0, \\ s_2 + s_3 \geq 0, \end{cases}$$

Proof. We only prove (2.4), as (2.5) follows by a symmetric argument.

By duality, (2.4) is equivalent to

$$\|wz\|_{H^{-s_3, -b_3}} \lesssim \|w\|_{X_+^{s_1, b_1}} \|z\|_{X_-^{s_2, b_2}}. \quad (2.6)$$

Set

$$F_+(\lambda, \eta) = \langle \eta \rangle^{s_1} \langle \lambda + \eta \rangle^{b_1} |\tilde{w}(\lambda, \eta)|,$$

$$G_-(\lambda, \eta) = \langle \eta \rangle^{s_2} \langle \lambda - \eta \rangle^{b_2} |\tilde{z}(\lambda, \eta)|.$$

Then (2.6) is equivalent to

$$I \lesssim \|F_+\|_{L^2} \|G_-\|_{L^2},$$

where

$$I = \left\| \iint \frac{F_+(\lambda, \eta) G_-(\tau - \lambda, \xi - \eta)}{\langle \eta \rangle^{s_1} \langle \xi - \eta \rangle^{s_2} \langle \xi \rangle^{s_3} \langle \Theta_+ \rangle^{b_1} \langle \Sigma_- \rangle^{b_2} \langle \Gamma \rangle^{b_3}} d\lambda d\eta \right\|_{L^2_{\tau, \xi}},$$

with Γ , Θ_+ and Σ_- as in Lemma 2.2, which allows us to reduce to Lemma 2.1 with $(A, B, C) = (s_1 + b_{\min}, s_2, s_3)$ or $(s_1, s_2 + b_{\min}, s_3)$, where $b_{\min} = \min(b_1, b_2, b_3)$, and with $(\alpha, \beta, \gamma) = (b_1, b_2, 0)$, $(b_1, 0, b_3)$ or $(0, b_2, b_3)$. $\square$

3. Proof of uniqueness

Here we prove Theorem 1.4. Assume that $(s, r) \in R$ and that

$$(u, v, \phi, \partial_t \phi) \in C([-T, T]; H^s \times H^s \times H^r \times H^{r-1}) \quad (3.1)$$

is a solution of (1.2), (1.3) on $S_T = (-T, T) \times \mathbb{R}$. It suffices to prove that

$$(u, v, \phi, \partial_t \phi) \in X_+^{s, \sigma}(S_T) \times X_-^{s, \sigma}(S_T) \times H^{r, \rho}(S_T) \times H^{r-1, \rho}(S_T), \quad (3.2)$$

with ρ, σ as in part (ii) of Theorem 1.2, since uniqueness is known in this class.

From (3.1) it is clear that

$$(u, v, \phi, \partial_t \phi) \in X_+^{s, 0}(S_T) \times X_-^{s, 0}(S_T) \times H^{r, 0}(S_T) \times H^{r-1, 0}(S_T). \quad (3.3)$$

To improve this, we shall use the structure of the equations (1.2). Consider first the equations for u and v . By (1.3), (1.4) (with $b = 1$) and (1.11) (which holds also for ϕv) we have

$$(u, v) \in \begin{cases} X_+^{s-(1/2-r)-\varepsilon, 1}(S_T) \times X_-^{s-(1/2-r)-\varepsilon, 1}(S_T) & \text{if } r \leq 1/2, \\ X_+^{s, 1}(S_T) \times X_-^{s, 1}(S_T) & \text{if } r > 1/2, \end{cases} \quad (3.4)$$

hence we split into the cases $r \leq 1/2$ and $r > 1/2$ [assuming $(r, s) \in R$, however].

3.1. The case $r \leq 1/2$

Then $r < 1 + 2s$ (since $s > -1/4$), so $\rho = \sigma = 1/2 + \varepsilon$ [see part (ii) of Theorem 1.2].

Interpolating between (3.3) and (3.4), we get

$$u \in X_+^{s-\theta(1/2-r)-\theta\varepsilon, \theta}(S_T), \quad v \in X_-^{s-\theta(1/2-r)-\theta\varepsilon, \theta}(S_T), \quad (3.5)$$

for all $\theta \in [0, 1]$. Take $\theta = \rho = 1/2 + \varepsilon$, and apply Lemma 2.4 with $s_1 = s_2 = s - \rho(1/2 - r) - \rho\varepsilon$ and $b = b' = \rho$; the condition $r + 2s > 0$ (from the definition of the region R) guarantees that $s_1 + s_2 > \rho - 1$ (for $\varepsilon > 0$ small enough), so the hypotheses of the lemma reduce to the two conditions $s_3 + 2[s - \rho(1/2 - r) - \rho\varepsilon] \geq 0$ and $s_3 + s - \rho(1/2 - r) - \rho\varepsilon \geq 0$. Thus,

$$\bar{u}v \in \begin{cases} H^{s-\rho(1/2-r)-\rho\varepsilon, \rho-1}(S_T) & \text{if } s - \rho(1/2 - r) - \rho\varepsilon > 0, \\ H^{2[s-\rho(1/2-r)-\rho\varepsilon], \rho-1}(S_T) & \text{if } s - \rho(1/2 - r) - \rho\varepsilon \leq 0. \end{cases}$$

But since $r \leq 1 + s$ and $s > -1/4$ [by (1.6)], we see that $r - 1 \leq s - \rho(1/2 - r) - \rho\varepsilon$ and $r - 1 \leq 2[s - \rho(1/2 - r) - \rho\varepsilon]$ for $\varepsilon > 0$ small enough, hence

$$\bar{u}v \in H^{r-1, \rho-1}(S_T), \quad (3.6)$$

which in particular shows that $\bar{u}v$ is well-defined as a distribution, so that the third equation in (1.2) makes sense. Applying (1.5) to that equation, we now get

$$\|\phi\|_{H^{r,\rho}(S_T)} + \|\partial_t \phi\|_{H^{r-1,\rho}(S_T)} \lesssim \|\phi_0\|_{H^r} + \|\phi_1\|_{H^{r-1}} + \|\bar{u}v\|_{H^{r-1,\rho-1}(S_T)} < \infty. \quad (3.7)$$

Thus,

$$\phi \in H^{r,\rho}(S_T), \quad \partial_t \phi \in H^{r-1,\rho}(S_T), \quad (3.8)$$

and now we can use this information, together with (3.5) with $\theta = \rho$, to improve the estimates for ϕu and ϕv , and hence for u and v . In fact, by Lemma 2.5 we get

$$\|\phi v\|_{X_+^{s,\rho-1}(S_T)} \lesssim \|\phi\|_{H^{r,\rho}(S_T)} \|v\|_{X_-^{s-\rho(1/2-r)-\rho\varepsilon,\rho}(S_T)}, \quad (3.9)$$

and similarly with v replaced by u and the signs inverted. The hypotheses of the lemma are satisfied since $r > 1/6$, $r \geq s$ and $r > 1/6 - 2s/3$ (by the definition of R). From (3.9) (as well as its counterpart for ϕu) and (1.4) we finally conclude that

$$u \in X_+^{s,\rho}(S_T), \quad v \in X_-^{s,\rho}(S_T), \quad (3.10)$$

and together with (3.8) this proves (3.2) in the case $r \leq 1/2$ (recall that $\sigma = \rho$ in this case).

3.2. The case $r > 1/2$

Then (3.4) already implies (3.2) for (u, v) , so we only need to prove that (3.8) holds. But the latter reduces, by (3.7), to proving that $\bar{u}v \in H^{r-1,\rho-1}(S_T)$. This follows by Lemma 2.4, however:

$$\|\bar{u}v\|_{H^{r-1,\rho-1}} \lesssim \|\bar{u}\|_{X_+^{s,1}} \|v\|_{X_-^{s,1}} = \|u\|_{X_+^{s,1}} \|v\|_{X_-^{s,1}}.$$

The hypotheses of the lemma are satisfied since $r \leq 1 + s$ and $s > -1/4$ (by (1.6)).

This concludes the proof of Theorem 1.4.

4. Proof of persistence of higher regularity

Here we prove Theorem 1.6. Assume that (s, r) and $(s', r') \equiv (s + \delta, r + \gamma)$ both belong to the convex region Ω . Let $\tilde{\sigma}$ be defined as in part ii of Theorem 1.2, for an $\varepsilon > 0$ to be chosen sufficiently small, and set

$$\sigma = \tilde{\sigma}(s, r), \quad \sigma' = \tilde{\sigma}(s', r'), \quad \rho = \frac{1}{2} + \varepsilon.$$

From Theorem 1.2 we know that

$$(u, v, \phi, \partial_t \phi) \in X_+^{s,\sigma}(S_T) \times X_-^{s,\sigma}(S_T) \times H^{r,\rho}(S_T) \times H^{r-1,\rho}(S_T). \quad (4.1)$$

We shall prove

$$(u, v, \phi, \partial_t \phi) \in X_+^{s', \sigma'}(S_T) \times X_-^{s', \sigma'}(S_T) \times H^{r', \rho}(S_T) \times H^{r'-1, \rho}(S_T), \quad (4.2)$$

which implies $(u, v, \phi, \partial_t \phi) \in C([-T, T]; H^{s'} \times H^{s'} \times H^{r'} \times H^{r'-1})$, as desired.

In order to be able to make $\delta, \gamma > 0$ as small as we like, we partition the line segment $L \subset \Omega$ from (s, r) to (s', r') into smaller segments by defining intermediate points $s_n = s + n\delta/N$ and $r_n = r + n\gamma/N$ for $n = 0, \dots, N$, where N is a large number still to be chosen.

We claim that for $\varepsilon > 0$ small enough (depending on L), and for N large enough (depending on ε), we have, for $n = 1, \dots, N$,

$$(u, v, \phi, \partial_t \phi) \in X_+^{s_n, \sigma_n}(S_T) \times X_-^{s_n, \sigma_n}(S_T) \times H^{r_n, \rho}(S_T) \times H^{r_n-1, \rho}(S_T), \quad (4.3)$$

where $\sigma_n = \tilde{\sigma}(s_n, r_n)$. If we can prove this claim, we are done, since (4.3) for $n = N$ gives (4.2).

By (4.1) we know that (4.3) holds for $n = 0$, so by induction it is enough to prove that (4.1) implies (4.2); but the point is now that by choosing N large enough, we may assume without loss of generality that $\delta, \gamma > 0$ are as small as we like, depending on ε . We emphasize that $\varepsilon > 0$ can be chosen uniformly along L (this is clear from the ensuing estimates; ε is chosen small enough so that the hypotheses of Lemmas 2.4 and 2.5 are verified in various situations below).

Write Ω as the disjoint union of the sets

$$\begin{aligned} \Omega_1 &= \{(s, r) \in \Omega : s \geq 0\}, \\ \Omega_2 &= \{(s, r) \in \Omega : s \in (-1/4, 0), \quad r < 1 + 2s\} \\ \Omega_3 &= \{(s, r) \in \Omega : s \in (-1/4, 0), \quad 1 + 2s \leq r\}. \end{aligned}$$

Since $\sigma = \tilde{\sigma}(s, r)$, where $\tilde{\sigma}(s, r) = 1/2 + \varepsilon$ if $(s, r) \in \Omega_1 \cup \Omega_2$ and $\tilde{\sigma}(s, r) = 1 - \varepsilon$ if $(s, r) \in \Omega_3$, we see from Lemma 2.4 that

$$\bar{w}v \in \begin{cases} H^{s, \rho-1}(S_T) & \text{if } (s, r) \in \Omega_1 \cup \Omega_3, \\ H^{2s, \rho-1}(S_T) & \text{if } (s, r) \in \Omega_2. \end{cases} \quad (4.4)$$

We consider the three regions Ω_1 , Ω_3 and Ω_2 separately, in that order. In each case we prove that (4.1) implies (4.2).

4.1. The case $(s, r) \in \Omega_1$

Then (s', r') must also lie in Ω_1 , so

$$\sigma = \sigma' = \rho = \frac{1}{2} + \varepsilon.$$

By (4.4), (1.5) and the assumption $(\phi_0, \phi_1) \in H^{r+\gamma} \times H^{r+\gamma-1}$,

$$\phi \in H^{a, \rho}(S_T), \quad \partial_t \phi \in H^{a-1, \rho}(S_T), \quad (4.5)$$

where

$$a = \min(1 + s, r + \gamma).$$

4.1.1. The subcase $a = 1 + s$. By Lemma 2.5,

$$\|\phi v\|_{X_+^{s+\delta, \rho-1}(S_T)} \lesssim \|\phi\|_{H^{1+s, \rho}(S_T)} \|v\|_{X_-^{s, \rho}(S_T)},$$

and similarly with v replaced by u and the signs inverted. By (1.4) and the initial data assumption $u_0, v_0 \in H^{s+\delta}$, we now get

$$u \in X_+^{s+\delta, \rho}(S_T), \quad v \in X_-^{s+\delta, \rho}(S_T). \quad (4.6)$$

But then Lemma 2.4 implies $\bar{u}v \in H^{s+\delta, \rho-1}(S_T)$, hence (4.5) holds with a replaced by $a' = \min(1 + s + \delta, r + \gamma)$. But $a' = r + \gamma$, since by assumption $(s + \delta, r + \gamma) \in \Omega$, hence $r + \gamma \leq 1 + (s + \delta)$. This concludes the subcase $a = 1 + s$.

4.1.2. The subcase $a = r + \gamma$. By Lemma 2.5,

$$\|\phi v\|_{X_+^{s+\delta, \rho-1}(S_T)} \lesssim \|\phi\|_{H^{r+\gamma, \rho}(S_T)} \|v\|_{X_-^{s, \rho}(S_T)}, \quad (4.7)$$

and similarly with v replaced by u and the signs inverted. Here the condition $s_1 + s_2 + s_3 + \min(b_1, b_2, b_3) > 1/2$ in the lemma reduces to $r + \gamma > \delta + \varepsilon$. But since $r > 0$ (uniformly along L), we may ensure that $3\varepsilon \leq r$, and since we may choose $\delta > 0$ as small as we like, we can ensure that $2\varepsilon + \delta \leq r$. From (4.7) we infer (4.6), and this concludes the subcase $a = r + \gamma$.

4.2. The case $(s, r) \in \Omega_3$

Then

$$\sigma = 1 - \varepsilon, \quad \rho = \frac{1}{2} + \varepsilon,$$

hence $\sigma' \leq \sigma$. On account of (4.4) we again get (4.5).

4.2.1. The subcase $a = 1 + s$. By Lemma 2.5,

$$\|\phi v\|_{X_+^{s+\delta, \sigma-1}(S_T)} \lesssim \|\phi\|_{H^{1+s, \rho}(S_T)} \|v\|_{X_-^{s, \sigma}(S_T)}.$$

The condition $s_1 + s_2 + \min(b_1, b_2, b_3) \geq 0$ in the lemma now becomes $\delta \leq \varepsilon$, but this is no problem, since as noted we may choose $\delta > 0$ as small as we like. Using (1.4) and the assumption $u_0, v_0 \in H^{s+\delta}$, we conclude that

$$u \in X_+^{s+\delta, \sigma}(S_T), \quad v \in X_-^{s+\delta, \sigma}(S_T), \quad (4.8)$$

and since $\sigma' \leq \sigma$, this proves (4.2) for (u, v) .

From (4.8) and Lemma 2.4 we get $\bar{u}v \in H^{s+\delta, \rho-1}(S_T)$, hence (4.5) holds with a replaced by $a' = \min(1 + s + \delta, r + \gamma) = r + \gamma$. This concludes the subcase $a = 1 + s$.

4.2.2. The subcase $a = r + \gamma$. By Lemma 2.5,

$$\|\phi v\|_{X_+^{s+\delta, \sigma-1}(S_T)} \lesssim \|\phi\|_{H^{r+\gamma, \rho}(S_T)} \|v\|_{X_-^{s, \sigma}(S_T)}.$$

The condition $s_1 + s_2 + s_3 + \min(b_1, b_2, b_3) > 1/2$ in the lemma reduces to $r + \gamma \geq \delta + 1/2$, but in the present case, $r > 1/2$ (since $r \geq 1 + 2s$ and $s > -1/4$), so we may from the start choose $\delta > 0$ so small that $1/2 + \delta \leq r$. The condition $s_1 + s_1 + \min(b_1, b_2, b_3) \geq 0$ again reduces to $\delta \leq \varepsilon$, which we may also assume. So again we get (4.8), and this concludes the subcase $a = r + \gamma$.

4.3. The case $(s, r) \in \Omega_2$

Then

$$\sigma = \rho = \frac{1}{2} + \varepsilon.$$

Now we have to deal with the possibility that the line segment passes from Ω_2 into Ω_3 , that is $(s', r') \in \Omega_3$, in which case $\sigma' = 1 - \varepsilon > \sigma$. But since $\delta, \gamma > 0$ are small, this can only happen if $r \geq 1/2$ (we use this fact below).

From (4.4) we infer that (4.5) holds with

$$a = \min(1 + 2s, r + \gamma).$$

4.3.1. The subcase $a = 1 + 2s$. By Lemma 2.5,

$$\|\phi v\|_{X_+^{s+\delta, -\varepsilon}(S_T)} \lesssim \|\phi\|_{H^{1+2s, \rho}(S_T)} \|v\|_{X_-^{s, \rho}(S_T)},$$

provided we choose $\delta > 0$ so small that $\delta \leq 2s + 1/2$ and $\delta \leq \varepsilon$. We conclude that

$$u \in X_+^{s+\delta, 1-\varepsilon}(S_T), \quad v \in X_-^{s+\delta, 1-\varepsilon}(S_T), \quad (4.9)$$

and this proves (4.2) for (u, v) . But now Lemma 2.4 implies $\bar{u}v \in H^{s+\delta, \rho-1}(S_T)$, hence (4.5) holds with a replaced by $a' = \min(1 + s + \delta, r + \gamma) = r + \gamma$, hence we are done with the subcase $a = 1 + 2s$.

4.3.2. The subcase $a = r + \gamma$. By Lemma 2.5 we again have (4.7), hence (4.6) also holds, so we are done if $\sigma' = 1/2 + \varepsilon$. Now assume $\sigma' = 1 - \varepsilon$. As noted above, we must then have $r \geq 1/2$, so Lemma 2.5 implies

$$\|\phi v\|_{X_+^{s+\delta, -\varepsilon}(S_T)} \lesssim \|\phi\|_{H^{r+\gamma, \rho}(S_T)} \|v\|_{X_-^{s, \rho}(S_T)},$$

provided $0 < \delta \leq \varepsilon$. Therefore, (4.9) again holds, and we are done.

This completes the proof of Theorem 1.6.

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