Homoclinic orbits for an unbounded superquadratic

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Abstract. We consider the following nonperiodic diffusion systems

$$\begin{cases} \partial_t u - \Delta_x u + b(t, x) \nabla_x u + V(x)u = G_v(t, x, u, v), \\ -\partial_t v - \Delta_x v - b(t, x) \nabla_x v + V(x)v = G_u(t, x, u, v), \end{cases} \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $b \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$, $G \in C^1(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{2m}, \mathbb{R})$ and $z := (u, v) : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^m \times \mathbb{R}^m$. Suppose that the potential V is positive constant and $G(t, x, z)$ is superquadratic in z as $|z| \rightarrow \infty$. By applying a generalized linking theorem for strongly indefinite functionals, we obtain homoclinic solutions z satisfying $z(t, x) \rightarrow 0$ as $|(t, x)| \rightarrow \infty$.

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1. Introduction and main results

1.1. The problem

In this paper we study the existence of homoclinic solutions for the system

$$(HS) \quad \begin{cases} \partial_t u - \Delta_x u + b(t, x) \nabla_x u + V(x)u = G_v(t, x, u, v), \\ -\partial_t v - \Delta_x v - b(t, x) \nabla_x v + V(x)v = G_u(t, x, u, v), \end{cases} \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Here $V \in C(\mathbb{R}^N, \mathbb{R})$, $b := (b_1, \dots, b_N) \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ and $G \in C^1(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{2m}, \mathbb{R})$, $z := (u, v) : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^m \times \mathbb{R}^m$. Such problems arise in optimal control of systems governed by partial differential equations (cf. Lions [27]) and are related to the Schrödinger equations (cf. Nagasawa [30]). If we let

$$\mathcal{J}_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \mathcal{S} := -\Delta_x + V, \quad (1.1)$$

and

$$A = \mathcal{J}_0 S + \mathcal{J} b \cdot \nabla_x, \quad (1.2)$$

then the system (HS) can be rewritten as

$$\mathcal{J} \frac{d}{dt} z + Az = G_z(t, x, z). \quad (1.3)$$

Thus the system (HS) can be regarded as an unbounded Hamiltonian system (cf. Barbu [7]), or an infinite-dimensional Hamiltonian system in $L^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$ (cf. [4, 16, 17]). In this paper, we are interested in the case that

$$G(t, x, z) = -\frac{1}{2}B(t, x)z \cdot z + H(t, x, z),$$

where $B(t, x)$ is $2m \times 2m$ symmetric bounded value matrix and $H(t, x, z)$ is superquadratic in z as $|z| \rightarrow \infty$.

For the case of bounded domain, if $b \equiv 0$ and $V \equiv 0$, there are a number of papers considering the stationary solution of the system

$$(S) \quad \begin{cases} -\Delta u = F_v(x, u, v) & \text{in } \Omega \subset \mathbb{R}^N, \\ -\Delta v = F_u(x, u, v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

For example, see [12, 13, 19, 23, 24, 28, 29] and the references therein. Particularly, in the interesting paper [29], the authors considered the case that $F(u, v) = \lambda|u|^2 + |u|^{p+1} + \mu|v|^2 + |v|^{q+1}$, where λ, μ are real numbers, and $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}$ ($N \geq 4$). If $\lambda = \mu$, $p = q$ and $u = v$, the system (S) reduces to the scalar semilinear elliptic equation $-\Delta u = \lambda u + |u|^{p-1}u$ with critical exponent $p = \frac{N+2}{N-2}$. This leads to the lack of compactness for the corresponding critical functional. In order to circumvent this difficulty, they reduce the strongly indefinite critical functional to the mountain pass geometric case. Recently, De Figueiredo and Ding [20] studied the system

$$\begin{cases} -\Delta u = H_u(x, u, v) & \text{in } \Omega \subset \mathbb{R}^N, \\ -\Delta v = -H_v(x, u, v) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where H satisfies $H(x, u, v) \sim |u|^p + |v|^q + R(x, u, v)$ with $\lim_{|u|+|v| \rightarrow \infty} \frac{R(x, u, v)}{|u|^p + |v|^q} = 0$, $1 < p < 2^* = \frac{2N}{N-2}$ and $q > 1$. They proved the existence of solutions for the above system. More recently, de Figueiredo, do Ó and Ruf [21] treated the following autonomous system via an Orlicz space approach

$$\begin{cases} -\Delta u = g(v) & u(x) > 0 & \text{in } \Omega, \\ -\Delta v = f(u) & v(x) > 0 & \text{in } \Omega, \\ u(x) = v(x) = 0, & & \text{on } \partial\Omega. \end{cases}$$

Under some superlinear conditions on f and g , the authors obtained one solution for the system. There is not much work on nonstationary solutions of

systems like (HS) . Brézis and Nirenberg [8] considered the following parabolic system

$$\begin{cases} \partial_t u - \Delta_x u = -v^5 + f \\ -\partial_t v - \Delta_x v = u^3 + g \end{cases} \quad \forall (t, x) \in (0, T) \times \Omega,$$

where $f, g \in L^\infty$ and Ω is a smoothly bounded domain in $\mathbb{R}^N$. Moreover, u and v subject to the boundary conditions $u = v = 0$ on $(0, T) \times \partial\Omega$ and $u(0, x) = v(T, x) = 0$ on Ω . Using Schauder's fixed point theorem they obtained a solution (u, v) with $u \in L^4$ and $v \in L^6$. Clément et al. [11] investigated the following system

$$(\mathcal{J}) \quad \begin{cases} \partial_t u - \Delta_x u = |v|^{p-1}v & \text{in } \mathbb{R} \times \Omega, \\ -\partial_t v - \Delta_x v = |u|^{q-1}u & \text{in } \mathbb{R} \times \Omega, \\ u(t, \cdot)|_{\partial\Omega} = v(t, \cdot)|_{\partial\Omega} = 0 & \text{on } \mathbb{R} \times \Omega, \end{cases}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^N$ and $N/(N+2) < 1/p + 1/q < 1$. By using a mountain pass argument, they proved that there exists $T_0 > 0$ such that for each $T > T_0$, the system $(\mathcal{J})$ has at least one positive solution. Moreover, by passing to the limit as $T \rightarrow \infty$, they showed that the system $(\mathcal{J})$ has at least one positive solution satisfying

$$\lim_{|t| \rightarrow \infty} u(t, x) = 0 = \lim_{|t| \rightarrow \infty} v(t, x) \quad \text{uniformly in } x \in \Omega.$$

In the paper [10], Clément and Van der Vorst discussed the optimality of the condition $N/(N+2) < 1/p + 1/q < 1$ on star-shaped domain.

Up to now, there are several authors considering this problem in $\mathbb{R}^N$. Suppose $b \equiv 0$ and $V \equiv 1$, G is radial in x and is even in z . Bartsch and De Figueiredo [9] proved that the system, without the depending on time variable, admits infinitely many radial as well as non-radial stationary solutions. For periodic setting, assuming $b \equiv 0$, and under some conditions on G , Bartsch and Ding [4] established a proper variational framework and proved that the system has infinitely many homoclinic orbits. Their results were later generalized by Schechter and Zou [31]. Nearly, Ding et al. [16] considered (HS) with $b(t, x) \not\equiv 0$ via variational methods. Under some periodic assumptions on $V(x)$, $b(t, x)$ and $G(t, x, z)$, the existence of infinitely many solutions were obtained in their paper [16] for both superquadratic or asymptotically linear cases. For nonperiodic case, there has not much works done up to know. More recently, Zhao and Ding [32] considered the nonperiodic asymptotically linear case, and they proved the existence and multiplicity of solutions for (HS) . Up to present, an interesting problem of existence of homoclinic orbits for (HS) with the nonperiodic superquadratic nonlinearity is still left open. Motivated by the present works, in this paper, we first devote to study such a case. The main difficulty here is the lack of compactness of the Sobolev's embedding, since the domain is the whole space. In order to overcome this difficulty, we use the methods which was used in [1, 14, 15, 29] to reduce the strongly indefinite critical functional to the mountain pass geometric case. Therefore, we can

construct a linking level for the energy functional, and show that it has the local compactness.

For stating our main results, we first give some notations and definitions. For two given $2m \times 2m$ symmetric real matrix functions $L_0(t, x)$ and $L_2(t, x)$, we write $L_0(t, x) < L_2(t, x)$ if and only if

$$\max_{\xi \in \mathbb{R}^{2m} |\xi|=1} (L_0(t, x) - L_2(t, x)\xi \cdot \xi) < 0.$$

Define similarly $L_0(t, x) \leq L_2(t, x)$. We denote $B(\infty) := \lim_{|(t,x)| \rightarrow \infty} B(t, x)$ when it exists. Set $\lambda_1 = \max\{\lambda > 0 : \lambda \in \sigma(B(\infty))\}$ and $\lambda_2 = \min\{\lambda < 0 : \lambda \in \sigma(B(\infty))\}$, where $\sigma(B(\infty))$ denotes the eigenvalues of the matrix $B(\infty)$. Let $\tilde{H}(t, x, z) := \frac{1}{2}H_z(t, x, z)z - H(t, x, z)$, we assume further on $b(t, x)$, V and $H(t, x, z)$ that the following conditions are satisfied:

(V₀) $V(x) \equiv V_0 > 0$ and $b(t, x)$ is a constant vector, i.e. $b(t, x) \equiv b_0$;

(H₀) $B(\infty)$ exists, $B(t, x) \leq B(\infty)$ and $B_L := B(\infty) - B(t, x)$ such that

$$0 < \lambda = \max_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} \max_{|\xi|=1} (B_L)\xi \cdot \xi < V_0 - \mu_\infty$$

where $\mu_\infty := \max\{\lambda_1, |\lambda_2|\}$;

(H₁) $H(t, x, z) \in C^1(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{2m}, \mathbb{R})$, $H(t, x, z) \geq 0$ and $H_z(t, x, z) = o(|z|)$ as $|z| \rightarrow 0$ uniformly in (t, x) ;

(H₂) $H(t, x, z)|z|^{-2} \rightarrow \infty$ as $|z| \rightarrow \infty$ uniformly in (t, x) ;

(H₃) $\inf_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} \tilde{H}(t, x, z) > 0$ if $z \neq 0$, and there exist $r_1 > 0$, $c_1 > 0$, $\mu > 1$ if $N = 1$, $\mu > 1 + \frac{N}{2}$ if $N \geq 2$ such that $|H_z(t, x, z)|^\mu \leq c_1 H(t, x, z)|z|^\mu$ if $|z| \geq r_1$;

(H₄) There exists $H_\infty(z) \in C^2(\mathbb{R}^{2m}, \mathbb{R})$ with $H_\infty(0) = 0$ and $H'_\infty(z) = o(|z|)$ as $|z| \rightarrow 0$ such that $H(t, x, z) \geq H_\infty(z)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, and

(i) $H_z(t, x, z) \rightarrow H'_\infty(z)$ as $|(t, x)| \rightarrow \infty$ uniformly in bounded set of z ;

(ii) $H'_\infty(w)(w + 2y) \leq H''_\infty(w)(w + y) \cdot (w + y)$ for all $w, y \in \mathbb{R}^{2m}$.

Here, $H'_\infty(z) := \nabla_z H_\infty(z)$, $H''_\infty(z) := \Delta_z H_\infty(z)$ and $\tilde{H}_\infty(z) := \frac{1}{2}H'_\infty(z)z - H_\infty(z)$.

As in [4], we introduce the anisotropic space

$$\begin{aligned} B_r &:= B_r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m}) \\ &:= W^{1,r}(\mathbb{R}, L^r(\mathbb{R}^N, \mathbb{R}^{2m})) \cap L^r(\mathbb{R}, W^{2,r}(\mathbb{R}^N, \mathbb{R}^{2m})) \cap W_0^{1,r}(\mathbb{R}^N, \mathbb{R}^{2m}) \end{aligned}$$

equipped with the norm

$$\|z\|_{B_r} := \left(\int_{\mathbb{R} \times \mathbb{R}^N} \left(|z|^r + |\partial_t z|^r + \sum_{j=1}^N |\partial_{x_j}^2 z|^r \right) \right)^{\frac{1}{r}},$$

where $r \geq 1$. It is easy to see that B_r is the completion of $C_0^\infty(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$ with respect to the norm $\|\cdot\|_{B_r}$. Moreover, if $r = 2$, B_2 is a Hilbert space.

For a solution z of (HS), we denote the associated action integral by

$$\Phi(z) := \int_{\mathbb{R} \times \mathbb{R}^N} \left[\frac{1}{2} \left(\mathcal{J} \frac{d}{dt} z \cdot z + Az \cdot z \right) - G(t, x, z) \right].$$

Set

$$c_{\min} := \inf\{\Phi(z) \mid z \neq 0 \text{ is a solution (HS)}\}.$$

A solution $z_0 \neq 0$ with $\Phi(z_0) = c_{\min}$ is said to be a least action solution. Let $S_{\min}$ denote the set of all least action solutions of (HS).

Theorem 1.1. *Let $(\mathcal{V}_0)$, $(\mathcal{H}_0)$ – $(\mathcal{H}_4)$ be satisfied. Then*

- (i) *(HS) has at least one nontrivial homoclinic orbit;*
- (ii) *$S_{\min}$ is compact in $B_2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$.*

Remark 1.1. Let $\hat{x} = (t, x)$. A simple example of matrix functions satisfying $(\mathcal{H}_0)$ is the matrix function

$$B(\hat{x}) = \begin{pmatrix} \alpha(\hat{x}) & 0 \\ 0 & -\beta(\hat{x}) \end{pmatrix},$$

where $\alpha(\hat{x})$ and $\beta(\hat{x})$ such that $\alpha_0 = \lim_{|\hat{x}| \rightarrow \infty} \alpha(\hat{x})$ and $\beta_0 = \lim_{|\hat{x}| \rightarrow \infty} \beta(\hat{x})$ exist, and $\max\{|\alpha_0|, |\beta_0|\} < V_0$ and

$$\begin{aligned} 0 &\leq \alpha_0 - \alpha(\hat{x}) < \min\{V_0 - |\alpha_0|, V_0 - |\beta_0|\}, \\ 0 &\leq \beta(\hat{x}) - \beta_0 < \min\{V_0 - |\alpha_0|, V_0 - |\beta_0|\} \end{aligned}$$

for all $\hat{x}$.

Remark 1.2. The following function satisfies $(\mathcal{H}_1)$ – $(\mathcal{H}_4)$.

Ex1. $H(t, x, z) = \frac{1}{2}a(t, x)((|z|^2 - 1)\ln(1 + |z|) + \frac{|z|}{2}(2 - |z|))$, where $a(t, x) > 0$, $a_0 := \lim_{|(t, x)| \rightarrow \infty} a(t, x)$ exists and $a(t, x) \geq a_0$ for all (t, x) .

2. Preliminary works

2.1. Variational setting

In this section, we first study the spectrum of the operator $L := \mathcal{J}\partial_t + A$, where $A = \mathcal{J}_0\mathcal{S} + \mathcal{J}b \cdot \nabla_x$. By Lemma 2.2 of [32], we know that L is a self-adjoint operator in $\mathcal{H} := L^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$ with the domain $\mathcal{D}(L) = B_2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$. Set $L_B := L + B(t, x)$ and $L_\infty := L + B(\infty)$, then by $(\mathcal{H}_0)$, we know that $\mathcal{D}(L) = \mathcal{D}(L_\infty) = \mathcal{D}(L_B) = B_2$. Let $\sigma_d(K)$, $\sigma_e(K)$ denote, respectively, the discrete spectrum and the essential spectrum of the operator K . Let $|\cdot|_q$ denote the usual $L^q(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$ -norm and $(\cdot, \cdot)_2$ is the usual $L^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$ -inner product, c or c_i stands for different positive constants. As a special case of Lemma 2.1 of [16], we have

Lemma 2.1. *Let $(\mathcal{V}_0)$ be satisfied. Then*

- (i) $\sigma(L) = \sigma_e(L)$, i.e., L has only essential spectrum;
- (ii) $\sigma(L) \subset \mathbb{R} \setminus (-V_0, V_0)$;
- (iii) $\sigma(L)$ is symmetric with respect to origin, that is, $\sigma(L) \cap (0, \infty) = -\sigma(L) \cap (-\infty, 0)$;
- (iv) $\underline{\lambda} := \inf\{\sigma_e(L) \cap (0, \infty)\} = V_0$.

Lemma 2.2. *Under the conditions $(\mathcal{V}_0)$ and $(\mathcal{H}_0)$, one has*

- (i) $\sigma(L_\infty) \subset \mathbb{R} \setminus (-(V_0 - \mu_\infty), (V_0 - \mu_\infty))$;

$$(ii) \quad \sigma(L_B) \subset \mathbb{R} \setminus (-(V_0 - \mu_\infty - \lambda), (V_0 - \mu_\infty - \lambda)).$$

Proof. (i) By (ii) of Lemma 2.1, we have

$$\begin{aligned} |L_\infty z|_2 &= |(L + B_\infty)z|_2 \geq |Lz|_2 - |B(\infty)z|_2 \\ &\geq V_0|z|_2 - \mu_\infty|z|_2 = (V_0 - \mu_\infty)|z|_2, \end{aligned}$$

$$\text{thus } \sigma(L_\infty) \subset \mathbb{R} \setminus (-(V_0 - \mu_\infty), (V_0 - \mu_\infty)).$$

(ii) Similarly, since $L_\infty = L_B + (B(\infty) - B(t, x))$, then one has

$$\begin{aligned} |L_B z|_2 &= |(L_\infty + (B(t, x) - B(\infty)))z|_2 \geq |L_\infty z|_2 - |(B(t, x) - B(\infty))z|_2 \\ &\geq (V_0 - \mu_\infty)|z|_2 - \lambda|z|_2 = (V_0 - \mu_\infty - \lambda)|z|_2. \end{aligned}$$

$$\text{Thus we have } \sigma(L_B) \subset \mathbb{R} \setminus (-(V_0 - \mu_\infty - \lambda), (V_0 - \mu_\infty - \lambda)). \quad \square$$

Now we consider the operator $L_0 := \mathcal{J}(\partial_t + b_0 \cdot \nabla_x) + \mathcal{J}_0(-\Delta_x + 1)$. This is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(L) = \mathcal{D}(L_0)$. By Lemma 4.3 in Bartsch and Ding [4] and Lemma 2.4 of [16], we have

Lemma 2.3. *Let (V_0) be satisfied. Then one has*

- (i) $\sigma(L_0) = \sigma_e(L_0)$, i.e., L_0 has only essential spectrum, and $\sigma(L_0) \subset \mathbb{R} \setminus (-1, 1)$;
- (ii) For every $r \geq 1$, there exist constants $d_1, d_2 > 0$ such that

$$d_1 \|z\|_{B_r}^r \leq |L_0 z|_r^r = \int_{\mathbb{R} \times \mathbb{R}^N} |L_0 z|^r \leq d_2 \|z\|_{B_r}^r.$$

for all $z \in B_r$. Consequently, $L_0 : B_r \rightarrow L^r$ is an isomorphism, $r \geq 1$.

Lemma 2.4. *Let (V_0) be satisfied. Then for all $z \in B_2$, we have*

$$c_1 |L_0 z|_2^2 \leq |Lz|_2^2 \leq c_2 |L_0 z|_2^2.$$

Consequently, for all $z \in B_2$

$$c_3 \|z\|_{B_2}^2 \leq |Lz|_2^2 \leq c_4 \|z\|_{B_2}^2.$$

Proof. Observing that $L = L_0 + \mathcal{J}_0(V_0 - 1)$, we have

$$|L_0 z|_2^2 \leq 2|Lz|_2^2 + 2|\mathcal{J}_0(V_0 - 1)z|_2^2 \leq h|Lz|_2^2,$$

where $h > 0$. Thus there is $c_1 > 0$ such that

$$c_1 |L_0 z|_2^2 \leq |Lz|_2^2.$$

This yields the left inequality. We now prove the right inequality. Clearly,

$$|Lz|_2^2 \leq 2|L_0 z|_2^2 + 2|\mathcal{J}_0(V_0 - 1)z|_2^2 \leq c_2 |L_0 z|_2^2.$$

This proves the lemma. $\square$

Lemma 2.5. *Let (V_0) and $(\mathcal{H}_0)$ be satisfied. Then for all $z \in B_2$, we have*

$$d_1 |Lz|_2^2 \leq |L_\infty z|_2^2 \leq d_2 |Lz|_2^2.$$

Consequently, for all $z \in B_2$

$$d_3 \|z\|_{B_2}^2 \leq |L_\infty z|_2^2 \leq d_4 \|z\|_{B_2}^2.$$

Proof. Since $L_\infty = L + B(\infty)$, by Lemma 2.1, it follows that

$$\begin{aligned} |L_\infty z|_2 &= |(L + B(\infty))z|_2 \geq |Lz|_2 - |B(\infty)z|_2 \\ &\geq |Lz|_2 - \frac{\mu_\infty}{V_0} |Lz|_2 = \left(1 - \frac{\mu_\infty}{V_0}\right) |Lz|_2. \end{aligned}$$

Recall that $\frac{\mu_\infty}{V_0} < 1$. Then there is a positive constant $d_1 > 0$ such that

$$|L_\infty z|_2^2 \geq d_1 |Lz|_2^2.$$

On the other hand,

$$|L_\infty z|_2^2 \leq 2|Lz|_2^2 + 2|B(\infty)z|_2^2 \leq d_2 |Lz|_2^2.$$

Now the conclusion follow. $\square$

Remark 2.1. In virtue of Lemmas 2.3–2.5, we write $\mathcal{D}(L) = \mathcal{D}(L_\infty) = B_2$ with the graph norm $\|z\|_L := |L_\infty z|_2$, which is equivalent to the norm $\|z\|_{B_2}$. By the embedding theorem on B_2 we see that $\mathcal{D}(L_\infty)$ is continuously embedded in $L^r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$ for r satisfying $2 \leq r < \infty$ if $N = 1$, and $0 \leq (\frac{1}{2} - \frac{1}{r})(1 + \frac{N}{2}) \leq 1$ if $N \geq 2$. $\mathcal{D}(L_\infty)$ embeds compactly in L_{loc}^r for all $r \geq 2$ if $N = 1$, and if $N \geq 2$ for all $r \geq 2$ satisfying $(\frac{1}{2} - \frac{1}{r})(1 + \frac{N}{2}) < 1$ (see [5, 6]).

It follows from Lemma 2.2 that $\mathcal{H}$ possesses the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+, \quad z = z^- + z^+$$

such that L_∞ is negative (resp. positive) definite in $\mathcal{H}^-$ (resp. $\mathcal{H}^+$).

Let $E := \mathcal{D}(|L_\infty|^{\frac{1}{2}})$. Then E is a Hilbert space under the inner product

$$(z_1, z_2) := (|L_\infty|^{\frac{1}{2}} z_1, |L_\infty|^{\frac{1}{2}} z_2)_2,$$

with the induced norm $\|z\| = (z, z)^{\frac{1}{2}}$. Moreover, E has the following orthogonal decomposition

$$E = E^- \oplus E^+, \quad \text{where } E^\pm = E \cap \mathcal{H}^\pm.$$

It is clear that

$$\|z^\pm\|^2 \geq (V_0 - \mu_\infty) |z^\pm|_2^2 \quad \text{for all } z \in E. \quad (2.1)$$

Let $N^* := \infty$ if $N = 1$ and $N^* := \frac{2(N+2)}{N}$ if $N \geq 2$. Then we have

Lemma 2.6. *E is continuously embedded in L^r for any $r \geq 2$ if $N = 1$, and for $r \in [2, N^*]$ if $N \geq 2$. Moreover, E is compactly embedded in L_{loc}^r for all $r \in [2, N^*)$.*

Proof. The proof was given in [4, 16]. We state it here for reader's convenience. We only consider the case $N \geq 2$. It follows by (complex) interpolation that

$$E = \mathcal{D}(|L_\infty|^{\frac{1}{2}}) \cong [\mathcal{D}(L_\infty), \mathcal{H}]_{\frac{1}{2}}.$$

By Remark 2.1, the embeddings

$$E = \mathcal{D}(|L_\infty|^{\frac{1}{2}}) \cong [\mathcal{D}(L_\infty), \mathcal{H}]_{\frac{1}{2}} \hookrightarrow [L^r, \mathcal{H}]_{\frac{1}{2}} \hookrightarrow L^r$$

are continuous for $r = \infty$ if $N = 2$, and for $r = \frac{2(N+2)}{N-2}$ if $N \geq 3$, and q satisfying $\frac{1}{q} = \frac{1}{2}(\frac{1}{2} + \frac{1}{r})$, i.e., $q = \frac{2(N+2)}{N}$. For $r \in (2, q)$, the Hölder inequality implies

$$|z|_r \leq |z|_2^{1-\beta} |z|_q^\beta \quad \text{with } \beta = \frac{q(r-2)}{r(q-2)}.$$

Therefore, E embeds continuously into L^r for any $r \in [2, N^*]$ if $N \geq 2$. Similarly, we see that E is compactly embedded in L^r_{loc} for all $r \in [2, N^*)$. $\square$

Set

$$\Psi(z) = \int_{\mathbb{R} \times \mathbb{R}^N} H(t, x, z).$$

By assumptions and Lemma 2.6, $\Psi(z) \in C^1(E, \mathbb{R})$ and

$$\Psi'(z)v = \int_{\mathbb{R} \times \mathbb{R}^N} H_z(t, x, z)v, \quad \forall z, v \in E.$$

Now, let us consider the functional

$$\Phi(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}^N} B_L z \cdot z - \Psi(z), \quad \forall z = z^- + z^+ \in E, \quad (2.2)$$

where B_L is defined in $(\mathcal{H}_0)$. Then $\Phi \in C^1(E, \mathbb{R})$, and the critical points of Φ are weak solutions of (HS) .

Using the spectrum decomposition of L_B one may give Φ another representation as follows. Note that, by Lemma 2.2, $E = \mathcal{D}(|L_B|^{\frac{1}{2}})$ is a Hilbert space with the equivalent inner product

$$(z, v)_B := (|L_B|^{\frac{1}{2}}z, |L_B|^{\frac{1}{2}}v)_2$$

and norm $\|z\| := (z, z)_B^{\frac{1}{2}}$. Then as above there is a decomposition

$$E = E_B^- \oplus E_B^+$$

with

$$\|z^\pm\|_B^2 \geq (V_0 - \mu_\infty - \lambda)|z^\pm|_2^2, \quad \forall z^\pm \in E_B^\pm. \quad (2.3)$$

Now Φ can be represented as

$$\Phi(z) = \frac{1}{2}\|z^+\|_B^2 - \frac{1}{2}\|z^-\|_B^2 - \Psi(z), \quad \forall z = z^- + z^+ \in E_B^- \oplus E_B^+. \quad (2.4)$$

Now we study the regularity result of the solutions for (HS) .

Lemma 2.7. *Suppose that the assumptions of Theorem 1.1 hold. Then we have that the critical points of Φ are element of $B_r(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m})$ for all $r \geq 2$.*

Proof. If $B(t, x) \equiv 0$, the proof was given in [4, 16]. Therefore, in our case, the matrix $B(t, x) \neq 0$ is bounded. The idea of the proof is the same as in Lemma 4.7 of [4], so we outline it for reader's convenience. By [5], we know that the following embedding holds:

$$B_q \hookrightarrow L^r \quad \text{is continuous for } q > 1, \quad 0 \leq \frac{1}{q} - \frac{1}{r} \leq \frac{2}{N+2}. \quad (2.5)$$

Set

$$\psi(q) := \begin{cases} (N+2)q/(N+2-2q) & \text{if } 0 < q < (N+2)/2 \\ \infty & \text{if } q \geq (N+2)/2. \end{cases}$$

So $B_q \hookrightarrow L^r$ is continuous for $1 < q \leq r < \psi(q)$ and also for $r = \psi(q)$ if $\psi(q) < \infty$.

Now let $z \in E$ be a weak solution of (HS) . Set $w = \mathcal{J}_0(1 - V_0)z + B(t, x)z + H_z(\cdot, \cdot, z)$. Then z is a weak solution of $L_0 z = w$, hence

$$z = L_0^{-1}w = L_0^{-1}(\mathcal{J}_0(1 - V_0)z + B(t, x)z + H_z(\cdot, \cdot, z))$$

Now we define $\chi_z : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\chi_z(t, x) := \begin{cases} 1 & \text{if } |z(t, x)| < 1; \\ 0 & \text{if } |z(t, x)| \geq 1, \end{cases}$$

and set

$$w_1(t, x) = \mathcal{J}_0(1 - V_0)z + B(t, x)z + H_z(t, x, \chi_z(t, x)z(t, x))$$

and

$$w_2(t, x) = H_z(t, x, (1 - \chi_z(t, x))z(t, x)).$$

Then we have $w(t, x) = w_1(t, x) + w_2(t, x)$. From our assumptions on $B(t, x)$ and H , it follows that

$$|w_1(t, x)| \leq d|z(t, x)| \quad (2.6)$$

and

$$|w_2(t, x)| \leq \begin{cases} 0 & \text{if } |z(t, x)| < 1; \\ d|z(t, x)|^{\alpha-1} & \text{if } |z(t, x)| \geq 1, \end{cases} \quad (2.7)$$

where $\alpha := \frac{2\mu}{\mu-1} \in (2, N^*)$ and μ is given in $(\mathcal{H}_3)$. Thus $w_1 \in L^r$ for $r \in [2, r_1]$ where $r_1 := N^*$, and $w_2 \in L^r$ for $r \in [1, q_1]$ where $q_1 := \frac{r_1}{\alpha-1}$. Here we used that

$$\text{meas}(\{(t, x) \in \mathbb{R} \times \mathbb{R}^N : |z(t, x)| \geq 1\}) \leq \int_{\mathbb{R} \times \mathbb{R}^N} |z|^2 < \infty.$$

Now by Lemma 2.3, we obtain

$$z_1 := L_0^{-1}w_1 \in B_r \quad r \in [2, r_1] \quad (2.8)$$

and

$$z_2 := L_0^{-1}w_2 \in B_r \quad r \in [1, q_1] \quad (2.9)$$

We distinguish two cases.

Case 1: $q_1 \geq N^*$. Then $z_2 \in L^r$ for all $r \in [q_1, \infty)$ as a consequence of (2.5). By interpolation we get $z_2 \in L^r$ for $r \geq 2$. Since $r_1 > q_1 \geq (N+2)/2$ we similarly obtain $z_1 \in L^r$ for $r \geq 2$.

Case 2: $q_1 < N^*$. In this case we define inductively $r_{k+1} := \psi(q_k)$ and $q_{k+1} := r_{k+1}/(\alpha-1) < r_{k+1}$. Suppose $z_1 \in B_r$ for $r \in [2, r_k]$, and $z_2 \in B_r$ for $r \in [2, q_k]$. Then $z_1 \in L^r$ for $r \in [2, \psi(r_k)]$, and $z_2 \in L^r$ for $r \in [2, \psi(q_k)]$. Moreover, since $\psi(r_k) > r_{k+1}$, then $z \in L^r$ for $r \in [2, r_{k+1}]$. This implies

$w_1 \in L^r$ for $r \in [2, r_{k+1}]$, and $w_2 \in L^r$ for $r \in [2, q_{k+1}]$. We claim that there exists $k_0 \geq 1$ with $q_{k_0} \geq (N+2)/2$. Then we are back in *Case 1* and therefore done.

By induction one proves that

$$r_k = \frac{2(N+2)}{N(\alpha-1)^{k-1} - 4\sum_{i=1}^{k-2}(\alpha-1)^{k-2}} = \frac{2(N+2)(\alpha-2)}{(\alpha-1)^k(N(\alpha-2)-4)+4}$$

Since $2 < \alpha < 2(N+2)/N = 2 + 4/N$ we see that $\psi(q_{k-1}) < 0$ for k large enough. This implies $q_{k-1} \geq (N+2)/2$ as required. $\square$

Lemma 2.8. *Assume that $(\mathcal{V}_0)$ and $(\mathcal{H}_0)$ – $(\mathcal{H}_4)$ are satisfied. If $z \in E$ is a solution of (HS) such that*

$$|\Phi(z)| \leq C_1 \quad \text{and} \quad \|z\|_2 \leq C_2,$$

then for any $q \geq 2$, we have

$$z \in B_q(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^{2m}) \quad \text{and} \quad \|z\|_{B_q} \leq \Lambda_q,$$

where Λ_q depends only on C_1, C_2 and q .

Proof. The proof is a standard bootstrap argument (see, e.g., [22, Proposition 3.2], or [4], Lemma 4.7 for the iterative steps), so the details are omitted here. $\square$

2.2. The abstract critical point theorem

In order to study the critical points of Φ , we now recall some generalized linking theorem developed recently in [3]. For related results, see [6, 25] and the references therein.

Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space with direct sum decomposition $\mathbb{E} = X \oplus Y$ and P_X, P_Y denote the projections onto X, Y , respectively. For a functional $\Phi \in C^1(\mathbb{E}, \mathbb{R})$, we write $\Phi_a := \{z \in \mathbb{E} : \Phi(z) \geq a\}$, $\Phi^b := \{z \in \mathbb{E} : \Phi(z) \leq b\}$ and $\Phi_a^b = \Phi_a \cap \Phi^b$. Next, let's us recall some definitions:

- (i) Φ is said to be weakly sequentially upper semi-continuous, i.e., $z_n \rightharpoonup z$ in $\mathbb{E}$ one has $\Phi(z) \geq \liminf_{n \rightarrow \infty} \Phi(z_n)$;
- (ii) Φ' is said to be weakly sequentially continuous, i.e., $\lim_{n \rightarrow \infty} \Phi'(z_n)w = \Phi'(z)w$ for each $w \in \mathbb{E}$;
- (iii) A sequence $\{z_n\} \subset \mathbb{E}$ is said to be a $(C)_c$ -sequence if $\Phi(z_n) \rightarrow c$ and $(1 + \|z_n\|)\Phi'(z_n) \rightarrow 0$. Φ is said to satisfy the $(C)_c$ -condition if any $(C)_c$ -sequence has a convergent subsequence.

From now on we assume that the Banach space X is separable and reflexive, and fix a countable dense subset $B \subset X^*$. For each $b \in B$ we define a semi-norm on $\mathbb{E}$ by

$$P_b : \mathbb{E} = X \oplus Y \rightarrow \mathbb{R}, \quad P_b(x+y) = q_b(x) + \|y\|, \quad \text{for } x+y \in X \oplus Y,$$

where $q_b(x) = |(x, b)_{X, X^*}| = |b(x)|$. We denote by $\mathcal{T}_B$ the induced topology. Let w^* denote the *weak**-topology on $\mathbb{E}^*$.

Assume:

$(\mathcal{A}_0)$ For any $c \geq 0$, there exists $\xi > 0$ such that $\|z\| < \xi \|P_Y z\|$ for all $z \in \Phi_c$.

($\mathcal{A}_1$) For any $c \in \mathbb{R}$, Φ_c is $\mathcal{T}_B$ -closed, and $\Phi' : (\Phi_c, \mathcal{T}_B) \rightarrow (\mathbb{E}^*, w^*)$ is continuous.

($\mathcal{A}_2$) There exists $\varrho > 0$ with $\kappa := \inf \Phi(S_\varrho Y) > 0$ where $S_\varrho Y := \{z \in Y : \|z\| = \varrho\}$.

Then we have the following theorem

Theorem 2.9. *Let $(\mathcal{A}_0)$ – $(\mathcal{A}_2)$ be satisfied and suppose there are $R > \varrho > 0$ and $e \in Y$ with $\|e\| = 1$ such that $\sup \Phi(\partial M) \leq \kappa$ where $M := \{z = x + se; s \geq 0, x \in X, \|z\| \leq R\}$. Then Φ has a $(C)_c$ -sequence with $\kappa \leq \Phi(z) \leq \tilde{c} := \sup \Phi(M)$. Moreover, if Φ satisfies the $(C)_c$ -condition for all $c \leq \tilde{c}$, then Φ has a critical point z with $\kappa \leq \Phi(z) \leq \tilde{c}$.*

This theorem is a special case of the theorem 4.4 of [3]. In the following, we will use Theorem 2.9 to prove our Theorem 1.1. The key is to verify the above conditions $(\mathcal{A}_0)$ – $(\mathcal{A}_2)$, and the following lemma is useful to verify $(\mathcal{A}_1)$ (see [18] or [3]).

Lemma 2.10. *Let*

$$\Phi(u) = \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 - \Psi(u), \text{ for } u = x + y \in \mathbb{E} = X \oplus Y.$$

Suppose

- (1) $\Psi \in C^1(\mathbb{E}, \mathbb{R})$ is bounded from below;
- (2) $\Psi : (\mathbb{E}, \mathcal{T}_w) \rightarrow \mathbb{R}$ is sequentially lower semicontinuous, that is, $u_n \rightharpoonup u$ in $\mathbb{E}$ implies $\Psi(u) \leq \liminf_{n \rightarrow \infty} \Psi(u_n)$;
- (3) $\Psi' : (\mathbb{E}, \mathcal{T}_w) \rightarrow (\mathbb{E}, \mathcal{T}_{w^*})$ is sequentially continuous;
- (4) $\nu : \mathbb{E} \rightarrow \mathbb{R}$, $\nu(u) := \|u\|^2$ is C^1 and $\nu' : (\mathbb{E}, \mathcal{T}_w) \rightarrow (\mathbb{E}, \mathcal{T}_{w^*})$ is sequentially continuous.

Then Φ satisfies $(\mathcal{A}_1)$.

3. The limit equation

In this section we study the following autonomous equation

$$\mathcal{J} \frac{d}{dt} z + \mathcal{J}_0 S z + \mathcal{J} b \cdot z + B(\infty) z = H_\infty(z). \quad (3.1)$$

where H_∞ is given in $(\mathcal{H}_4)$.

In the following for notational simplicity, we write $\hat{x} = (t, x)$. For $0 \neq k = (k_0, k_1, \dots, k_N) \in \mathbb{Z}^{N+1}$, define $k * z(t, x) := z(t + k_0, x_1 + k_1, \dots, x_N + k_N)$. By the virtue of $(\mathcal{H}_1)$ – $(\mathcal{H}_4)$, we first have the following lemma.

Lemma 3.1. *The limit H_∞ possesses the following properties.*

- (i) $H_\infty(0) = 0$, $H_\infty(z) > 0$ if $z \neq 0$, and $H'_\infty(z) = o(z)$ as $|z| \rightarrow 0$.
- (ii) $H_\infty(z)|z|^{-2} \rightarrow \infty$ as $|z| \rightarrow \infty$.
- (iii) $\tilde{H}_\infty(z) > 0$ if $z \neq 0$, and $|H'_\infty(z)|^\mu \leq c_1 \tilde{H}_\infty(z)|z|^\mu$ if $|z| \geq r_1$, where μ is defined in $(\mathcal{H}_3)$.
- (iv) $H_\infty(z)$ is strictly convex in $\mathbb{R}^{2m} \setminus \{0\}$.

Proof. Both $H_\infty(0) = 0$ and $H'_\infty(z) = o(z)$ as $|z| \rightarrow 0$ are assumed in $(\mathcal{H}_4)$. By $(\mathcal{H}_3)$ one has $\tilde{H}(t, x, z) \geq 0$, which together with $(\mathcal{H}_1)$ implies that $\frac{1}{2}H_z(t, x, z)z \geq H(t, x, z) \geq 0$, hence $H(t, x, 0) = 0$. Since $\tilde{H}(t, x, z) > 0$ if $z \neq 0$, we see that $H(t, x, z) = \int_0^1 H_z(t, x, \tau z)z d\tau > 0$ if $z \neq 0$. Observe that

$$0 \leq H(t, x, z) - H_\infty(z) = \int_0^1 (H_z(t, x, \tau z) - H'_\infty(\tau z))z d\tau.$$

It follows from $(\mathcal{H}_4)$ that $H(t, x, z) \rightarrow H_\infty(z)$ uniformly in bounded sets of z as $|\hat{x}| \rightarrow \infty$. This, jointly with that $\inf_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} \tilde{H}(t, x, z) > 0$ if $z \neq 0$, implies

$$\tilde{H}_\infty(z) > 0 \quad \text{if } z \neq 0. \quad (3.2)$$

Hence $H'_\infty(z)z > 0$ and

$$H_\infty(z) = \int_0^1 H'_\infty(\tau z)z d\tau > 0 \quad \text{if } z \neq 0,$$

which shows (i).

By $(\mathcal{H}_2)$, for any $K > 0$ there is $b_K > 0$ such that $H(t, x, z)|z|^{-2} \geq K$ if $|z| \geq b_K$. Observe that

$$\begin{aligned} \frac{H_\infty(z)}{|z|^2} &= \frac{H_\infty(z) - H(t, x, z)}{|z|^2} + \frac{H(t, x, z)}{|z|^2} \\ &\geq \frac{H_\infty(z) - H(t, x, z)}{|z|^2} + K \end{aligned}$$

for all $|z| \geq b_K$. Letting $|\hat{x}| \rightarrow \infty$, we get that $H_\infty(z)|z|^{-2} \geq K$ for all $|z| \geq b_K$, proving (ii).

Note that by $(\mathcal{H}_3)$ for all $|z| \geq r_1$, we have

$$\begin{aligned} \frac{|H'_\infty(z)|}{|z|} &\leq \frac{|H'_\infty(z) - H_z(t, x, z)|}{|z|} + \frac{|H_z(t, x, z)|}{|z|} \\ &\leq \frac{|H'_\infty(z) - H_z(t, x, z)|}{|z|} + c_1^{\frac{1}{\mu}} (\tilde{H}(t, x, z))^{\frac{1}{\mu}}. \end{aligned}$$

Fixed arbitrarily $|z| \geq r_1$, letting $|\hat{x}| \rightarrow \infty$ yields

$$\frac{|H'_\infty(z)|}{|z|} \leq c_1^{\frac{1}{\mu}} \tilde{H}_\infty(z)^{\frac{1}{\mu}}.$$

This together with (3.2) implies the conclusion (iii).

Finally, by (3.2) again, $H'_\infty(z)z > 0$ if $z \neq 0$. Using the conclusion (ii) of $(\mathcal{H}_4)$ with $w \neq 0$, $y = 0$ one gets $H''_\infty(w)w \cdot w \geq H'_\infty(w)w > 0$, this proves (iv). $\square$

As a consequence one sees easily that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ satisfying

$$\frac{|H'_\infty(z)|}{|z|} \leq \varepsilon + C_\varepsilon \tilde{H}_\infty(z)^{\frac{1}{\mu}}, \quad \text{for all } z \neq 0. \quad (3.3)$$

In the following, we consider $\frac{2\mu}{\mu-1} \leq p < N^*$. Note that by the assumption $(\mathcal{H}_3)$, one has $2 < \frac{2\mu}{\mu-1} < N^*$ if $N \geq 2$, and

$$|H'_\infty(z)| \leq \varepsilon|z| + C_\varepsilon|z|^{p-1} \quad \text{and} \quad |H_\infty(z)| \leq \varepsilon|z|^2 + C_\varepsilon|z|^p \quad (3.4)$$

for all $z \in E$.

Set

$$\Psi_\infty(z) = \int_{\mathbb{R} \times \mathbb{R}^N} H_\infty(z),$$

and define the functional

$$F(z) := \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \Psi_\infty(z), \quad \forall z = z^- + z^+ \in E.$$

It follows from the assumptions on $H_\infty(z)$ that $F \in C^1(E, \mathbb{R})$ and its critical points are solutions of (3.1). Moreover, by Lemmas 2.6 and 3.1, we have the following results.

Lemma 3.2. *Under the assumptions of Theorem 1.1, one has*

- (1) Ψ_∞ is weakly sequentially lower semicontinuous and F' is weakly sequentially continuous;
- (2) For any finite dimensional subspace $Z \subset E^+$,

$$F(z) \rightarrow -\infty \quad \text{as } z \in E^- \oplus Z, \quad \|z\| \rightarrow \infty;$$

- (3) There are $\rho > 0$ and $\kappa > 0$ such that

$$F(z)|_{K_\rho \cap E^+} \geq 0 \quad \text{and} \quad F(z)|_{\partial K_\rho \cap E^+} \geq \kappa,$$

where $K_\rho := \{z \in E; \|z\| \leq \rho\}$.

Proof. The proof was given in [16, 18], here we omit the details. $\square$

Lemma 3.3. *Let $\{z_j\}$ be a $(C)_c$ -sequence for F . Then it is bounded and $c \geq 0$. Moreover, there is a subsequence, denoted again by $\{z_j\}$, satisfying $z_j \rightharpoonup z$, $F(z_j - z) \rightarrow c - F(z)$ and $F'(z_j - z) \rightarrow 0$.*

Proof. First, by Lemma 3.1, a standard argument shows that $\{z_j\}$ is bounded (see, e.g., [18]). Next, by the conclusion (iii) of Lemma 3.1, it holds that

$$F(z_j) - \frac{1}{2}F'(z_j)z_j = \int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}_\infty(z_j) \geq 0.$$

Taking the limit yields $c \geq 0$.

Finally, by Lemma 5.4 of [16], we know that $F(z_j - z) \rightarrow c - F(z)$ and $F'(z_j - z) \rightarrow 0$. $\square$

Let $\mathcal{K} := \{z \in E : F'(z) = 0\}$ be the critical set of F .

Lemma 3.4. $\mathcal{K} \setminus \{0\} \neq \emptyset$, $\mathcal{K} \subset \cap_{r \geq 2} B_r$ and

$$\hat{C} := \inf\{F(z) : z \in \mathcal{K} \setminus \{0\}\} > 0$$

Proof. In the following, we will use Theorem 2.9 to prove that $\mathcal{K} \setminus \{0\} \neq \emptyset$. First we verify that the functional F satisfies the conditions $(\mathcal{A}_0)$ – $(\mathcal{A}_2)$. Let $X = E^-$ and $Y = E^+$, one has $E = X \oplus Y$. Let $F_0 = \{z \in E : F(z) \geq 0\}$. Since $H_\infty(z) \geq 0$, for $z \in F_0$ one has

$$0 \leq \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int_{\mathbb{R}^N} H_\infty(z) \leq \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2,$$

which implies $\|z^-\|^2 \leq \|z^+\|^2$. Then there exists $\xi > \sqrt{2}$ such that $\|z\| \leq \xi\|z^+\|$, and F satisfies $(\mathcal{A}_0)$. By virtue of Lemma 2.10 and the conclusion (1) of Lemma 3.2, it follows that $(\mathcal{A}_1)$ holds. From the conclusion (3) of Lemma 3.2, we know that $(\mathcal{A}_2)$ holds. Let $e \in E^+$ with $\|e\| = 1$. By the conclusion (2) of Lemma 3.2, we have that there exist $R > \rho > 0$ such that $F|_{\partial M} \leq 0$, where

$$M = \{z = z^- + se : z^- \in E^-, s \geq 0, \|z\| \leq R\}.$$

Thus all of the conditions of Theorem 2.9 hold. Then we have that there is a $(C)_c$ -sequence $\{z_j\}$ for F with $\kappa \leq c \leq \sup F|_M$. By Lemma 3.3 one sees that $\{z_j\}$ is bounded in E . By the concentration principle [26], a standard argument shows that $\{z_j\}$ is non-vanishing, that is, there exist $r, \eta > 0$ and $\{a_j\} \subset \mathbb{Z}^{1+N}$ such that $\limsup_{j \rightarrow \infty} \int_{D(a_j, r)} |z_j|^2 \geq \eta$, where $D(a_j, r) := \{\hat{x} \in \mathbb{R} \times \mathbb{R}^N : |\hat{x} - a_j| \leq r\}$. Set $v_j = z_j * a_j$. It follows the invariance of the norm and the functional under the $*$ -action that $\|v_j\| = \|z_j\| \leq C$ and $F(v_j) \rightarrow c \geq \kappa$, $F'(v_j) \rightarrow 0$. Therefore, $v_j \rightharpoonup v$ with $v \neq 0$ and $F'(v) = 0$, that is, v is a nontrivial solution of (3.1). Thus,

$$\mathcal{K} \setminus \{0\} \neq \emptyset.$$

As in Lemma 2.7 one sees that $\mathcal{K} \subset B_r$ for $r \geq 2$.

If $z \in \mathcal{K}$, one has

$$F(z) - \frac{1}{2}F'(z)z = \int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}_\infty(z) \geq 0.$$

For proving $\hat{C} > 0$, assume by contradiction that $\hat{C} = 0$. Let $z_j \in \mathcal{K} \setminus \{0\}$ be such that $F(z_j) \rightarrow 0$. Then $\{z_j\}$ is a $(C)_0$ -sequence, hence it is bounded by Lemma 3.3. We can assume that $z_j \rightharpoonup z \in \mathcal{K}$. Then

$$F(z_j) = \int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}_\infty(z_j) \rightarrow 0.$$

By (3.3) and Hölder inequality ($\frac{1}{\mu} + \frac{1}{\mu'} = 1$, $\mu = \frac{p}{p-2}$), one sees that for any $\varepsilon > 0$

$$\begin{aligned} \|z_j\|^2 &= \int_{\mathbb{R} \times \mathbb{R}^N} H'_\infty(z_j)(z_j^+ - z_j^-) \\ &\leq \varepsilon |z_j|_2^2 + C_\varepsilon \int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}_\infty(z_j)^{\frac{1}{\mu}} |z_j| |z_j^+ - z_j^-| \\ &\leq \varepsilon |z_j|_2^2 + C_\varepsilon \left(\int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}_\infty(z_j) \right)^{\frac{1}{\mu}} |z_j|_p \left(\int_{\mathbb{R} \times \mathbb{R}^N} |z_j^+ - z_j^-|^p \right)^{\frac{1}{p}} \\ &\leq c\varepsilon \|z_j\|^2 + cC_\varepsilon F(z_j)^{\frac{1}{\mu}} \|z_j\|^2. \end{aligned}$$

Hence $1 \leq c\varepsilon + o(1)$, a contradiction.

Finally, we show that there is $\{z_j\} \in \mathcal{K} \setminus \{0\}$ with $F(z_j) \rightarrow \hat{C}$. Then, as before, $\{z_j\}$ is bounded, and by applying the concentration principle one may assume $z_j \rightarrow z \in \mathcal{K} \setminus \{0\}$. Now

$$\hat{C} = \lim_{j \rightarrow \infty} F(z_j) = \lim_{j \rightarrow \infty} \int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}_\infty(z_j) \geq \int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}_\infty(z) = F(z) \geq \hat{C},$$

that is, $F(z) = \hat{C}$. □

In the following, we follow the idea of [14, 15] to establish the linking levels for the functional $\Phi(z)$. Following Ackermann [1], for fixed $u \in E^+$, we introduce the functional $\psi_u : E^- \rightarrow \mathbb{R}$ by

$$\psi_u(v) := F(u+v) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|v\|^2 - \Psi_\infty(u+v).$$

One has

$$\begin{aligned} \psi_u''(v)[w, w] &= -\|w\|^2 - \Psi_\infty''(u+v)[w, w] \\ &= -\|w\|^2 - \int_{\mathbb{R} \times \mathbb{R}^N} H_\infty''(u+v)w \cdot w \end{aligned}$$

for all $v, w \in E^-$, which implies that $\psi_u(\cdot)$ is strictly concave. Moreover,

$$\psi_u(v) \leq \frac{1}{2}(\|u\|^2 - \|v\|^2) \rightarrow -\infty, \quad \text{as } \|v\| \rightarrow \infty.$$

Clearly, it follows from Lemma 3.2 that ψ_u is weak sequential upper semicontinuous. Thus there is a unique strict maximum point $f(u)$ for $\psi_u(\cdot)$ which is also the only critical point of ψ_u on E^- and satisfies:

$$v \neq f(u) \Leftrightarrow F(u+v) < F(u+f(u)), \quad (3.5)$$

$$v = f(u) \Leftrightarrow \psi_u'(f(u))w = 0, \quad \text{i.e.,}$$

$$(f(u), w) + \int_{\mathbb{R} \times \mathbb{R}^N} H_\infty'(u+f(u))w = 0 \quad (3.6)$$

for all $u \in E^+$ and $v, w \in E^-$.

By Lemma 5.6 of [1], we have the following results.

Lemma 3.5. *Under the conditions of Theorem 1.1, we have:*

- (i) f is $\mathbb{R} \times \mathbb{R}^N$ -invariant, i.e., $f(a * u) = f(u)$ for all $a \in \mathbb{R} \times \mathbb{R}^N$;
- (ii) $f \in C^1(E^+, E^-)$ and $f(0) = 0$;
- (iii) f is a bounded map;
- (iv) If $u_n \rightarrow u$, then $f(u_n) - f(u_n - u) \rightarrow f(u)$ and $f(u_n) \rightarrow f(u)$. The same is true for $|f(u)|_2^2$.

Now we define the reduce functional $I(u) := F(u + f(u)) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|f(u)\|^2 - \Psi_\infty(u + f(u))$. By (3.6), we have

$$\begin{aligned} I'(u)v &= (u, v) - (f(u), f'(u)v) - \int_{\mathbb{R} \times \mathbb{R}^N} H'_\infty(u + f(u))(v + f'(u)v) \\ &= F'(u + f(u))(v + f(v)) \\ &= (u, v) - (u, f(v)) - \int_{\mathbb{R} \times \mathbb{R}^N} H'_\infty(u + f(u))(v + f(v)) \end{aligned}$$

for all $u, v \in E^+$. Then we know that the critical points of I and F are in one to one correspondence via the injective map $u \mapsto u + f(u)$ from E^+ into E^- . That is, letting

$$\mathcal{K}^+ := \{u \in E^+ : I'(u) = 0\},$$

one has

$$\mathcal{K} := \{u + f(u) : u \in \mathcal{K}^+\}.$$

In particular,

$$\hat{C} = \inf\{I(u) : u \in \mathcal{K}^+ \setminus \{0\}\}. \quad (3.7)$$

Lemma 3.6. *If $\{u_j\} \subset E^+$ is a $(C)_c$ -sequence for I , then $z_j := u_j + f(u_j) \in E$ is a $(C)_c$ -sequence for F . Additionally, along a subsequence, $u_j \rightharpoonup u \in \mathcal{K}^+$, $I(u_j - u) \rightarrow c - I(u)$ and $I'(u_j - u) \rightarrow 0$.*

Proof. It is clear by definition that

$$F(z_j) = F(u_j + f(u_j)) = I(u_j) \rightarrow c,$$

by (3.6), for any $w = w^- + w^+ \in E$ one has

$$\begin{aligned} F'(z_j)w &= (u_j, w^+) - (f(u_j), w^-) - \int_{\mathbb{R} \times \mathbb{R}^N} H'_\infty(z_j)(w^-) - \int_{\mathbb{R} \times \mathbb{R}^N} H'_\infty(z_j)(w^+) \\ &= (u_j, w^+) - \int_{\mathbb{R} \times \mathbb{R}^N} H'_\infty(z_j)(w^+) \\ &= I'(u_j)w^+. \end{aligned}$$

Thus $(1 + \|z_j\|)F'(z_j) \rightarrow 0$. The last conclusion follows from Lemmas 3.3 and 3.5. $\square$

In the next, we discuss the mountain pass geometry of functional I . Obviously, one has the following

Lemma 3.7. *I possesses the mountain pass geometry:*

- (1) *There is $\rho > 0$ such that $\inf I(E^+ \cap \partial K_\rho) > 0$, where $\partial K_\rho = \{z \in E : \|z\| = \rho\}$;*
- (2) *For any finite dimensional subspace $X \subset E^+$, $I(u) \rightarrow -\infty$ as $u \in X$, $\|u\| \rightarrow \infty$.*

Proof. (1) By (3.4), for $u \in E^+$ one has

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\|f(u)\|^2 - \int_{\mathbb{R} \times \mathbb{R}^N} H_\infty(u + f(u)) \\ &= \frac{1}{2}\|u\|^2 + (F(u + f(u)) - F(u)) - \int_{\mathbb{R} \times \mathbb{R}^N} H_\infty(u) \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R} \times \mathbb{R}^N} H_\infty(u) \\ &\geq \frac{1}{2}\|u\|^2 - \varepsilon|u|_2^2 - C_\varepsilon|u|_p^p, \end{aligned}$$

and the result follows from Lemma 2.6.

- (2) Arguing indirectly, assume that for some $u_j \in X$ with $\|u_j\| \rightarrow \infty$. There is $K > 0$ such that $I(u_j) \geq -K$ for all j . Let $v_j := u_j + f(u_j)$, then it follows from $\|u_j\| \leq \|v_j\|$ that $\|v_j\| \rightarrow \infty$ as $j \rightarrow \infty$. Setting $w_j := \frac{v_j}{\|v_j\|}$, by Lemma 3.5, we have that $\|w_j\| = 1$, $w_j \rightharpoonup w$, $w_j^- \rightharpoonup w^-$, $w_j^+ \rightharpoonup w^+$ and

$$-\frac{K}{\|v_j\|^2} \leq \frac{I(u_j)}{\|v_j\|^2} = \frac{F(v_j)}{\|v_j\|^2} = \frac{1}{2}\|w_j^+\|^2 - \frac{1}{2}\|w_j^-\|^2 - \int_{\mathbb{R} \times \mathbb{R}^N} \frac{H_\infty(v_j)}{\|v_j\|^2}. \quad (3.8)$$

We claim that $w^+ \neq 0$. Indeed, if not then it follows from (3.8) that

$$0 \leq \frac{1}{2}\|w_j^-\|^2 + \int_{\mathbb{R} \times \mathbb{R}^N} \frac{H_\infty(v_j)}{\|v_j\|^2} \leq \frac{1}{2}\|w_j^+\|^2 + \frac{K}{\|v_j\|^2} \rightarrow 0.$$

This implies that $\|w_j^-\| \rightarrow 0$, hence $1 = \|w_j\|^2 \rightarrow 0$, a contradiction.

Since X is a finite subspace of E^+ , then there exists a best constant $\omega > 0$ such that $\|w^+\|^2 \leq \omega|w^+|_2^2$. By the conclusion (ii) of Lemma 3.1, then there is $R > 0$ such that $H_\infty(z) \geq 2\omega|z|^2$ if $|z| \geq R$. Observing that

$$\|w^+\|^2 - \|w^-\|^2 - 2\omega|w|_2^2 \leq -(\omega|w^+|_2^2 + \|w^-\|^2 + 2\omega|w^-\|_2^2) < 0,$$

hence, there is a bounded set $D \subset \mathbb{R} \times \mathbb{R}^N$ such that

$$\|w^+\|^2 - \|w^-\|^2 - 2\omega \int_D |w|_2 < 0.$$

Then

$$\begin{aligned} \frac{I(u_j)}{\|v_j\|^2} &\leq \frac{1}{2}(\|w_j^+\|^2 - \|w_j^-\|^2) - \int_D \frac{H_\infty(v_j)}{\|v_j\|^2} \\ &= \frac{1}{2}(\|w_j^+\|^2 - \|w_j^-\|^2 - 2\omega \int_D |w_j|_2) - \int_D \frac{H_\infty(v_j) - \omega|v_j|^2}{\|v_j\|^2} \\ &\leq \frac{1}{2}(\|w_j^+\|^2 - \|w_j^-\|^2 - 2\omega \int_D |w_j|_2) + \frac{\omega R^2|D|}{\|v_j\|^2}. \end{aligned}$$

Hence, letting $j \rightarrow \infty$ yields

$$0 \leq \|w^+\|^2 - \|w^-\|^2 - 2\omega \int_D |w|_2 < 0,$$

which is contradiction. $\square$

Lemma 3.4 implies 0 is an isolated critical point of I . Therefore there is $k > 0$ such that $\|w\| \geq k$ for all nontrivial critical points w of I . Below we set

$$\mathcal{N}^+ := \{u \in E^+ \setminus \{0\} : I'(u)u = 0\}.$$

Lemma 3.8. *For each $u \in E^+ \setminus \{0\}$, there is a unique $s = s(u) > 0$ such that $su \in \mathcal{N}^+$.*

Proof. We follow the idea of the proof [1]. Observe that,

$$\nabla I(u) = \nabla F(u + f(u)) = \mathcal{P}^+ \nabla F(u + f(u))$$

and

$$\nabla^2 I(u) = \nabla^2 F(u + f(u))(I + f'(u)) = \mathcal{P}^+ \nabla F(u + f(u))(I + f'(u))$$

where $\mathcal{P}^\pm : E \rightarrow E^\pm$ denote the orthogonal projector. Fixed $z \in E^+ \setminus \{0\}$ with $I'(z)z = 0$, and set $u = z + f(z)$ and $v = f'(z)z - f(z) \in E^-$. Then $u \neq 0$, by (3.6) and $(\mathcal{H}_4)$, we get

$$\begin{aligned} I''(z)[z, z] &= (\nabla^2 I(z)z, z) \\ &= (\nabla^2 F(u)[z + f'(z)z], z + f'(z)z) \\ &= (\nabla^2 F(u)[u + v], u + v) \\ &= (\nabla^2 F(u)u, u) + 2(\nabla^2 F(u)u, v) + (\nabla^2 F(u)v, v) \\ &= \|u^+\|^2 - \|u^-\|^2 - (\nabla^2 \Psi_\infty(u)u, u) \\ &\quad + 2(u^+ - u^- - \nabla^2 \Psi_\infty(u)u, v) + (-v - \nabla^2 \Psi_\infty(u)v, v) \\ &= (\nabla \Psi_\infty(u) - \nabla^2 \Psi_\infty(u)u, u) + 2(\nabla \Psi_\infty(u) - \nabla^2 \Psi_\infty(u)u, v) \\ &\quad - (\nabla \Psi_\infty(u)v, v) - \|v\|^2 \\ &= (\Psi'_\infty(u)u - \Psi''_\infty(u)[u, u]) + 2(\Psi'_\infty(u)v - \Psi''_\infty(u)[u, v]) \\ &= -\Psi''_\infty(u)[v, v] - \|v\|^2 \\ &= - \int_{\mathbb{R} \times \mathbb{R}^N} (H''_\infty(u)(u + v)(u + v) - H'_\infty(u)(u + 2v)) - \|v\|^2 < 0. \end{aligned} \tag{3.9}$$

Now let $u \in E^+ \setminus \{0\}$ and $h(s) = I(su)$, one has $h(0) = 0$, $h(s) > 0$ for $s > 0$ sufficiently small, and by Lemma 3.7, we know that $h(s) \rightarrow -\infty$ as $s \rightarrow \infty$. Thus there is $s(u) > 0$ such that

$$I(s(u)u) = \sup_{s \geq 0} I(su).$$

Clearly, we have

$$\frac{dI(su)}{ds} \Big|_{s=s(u)} = I'(s(u)u)u = \frac{1}{s(u)} I'(s(u)u)su = 0,$$

and consequently by (3.9)

$$I''(s(u)u)(s(u)u) < 0.$$

One sees that such $s(u) > 0$ is unique. $\square$

Set

$$b_1 := \inf\{I(u) : u \in \mathcal{N}^+\},$$

$$b_2 := \inf_{\gamma \in \mathcal{M}} \sup_{s \in [0,1]} I(\gamma(s))$$

where

$$\mathcal{M} := \{\gamma \in C([0,1], E^+) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Lemma 3.9. $\hat{C} = b_1 = b_2$.

Proof. In the following, we shall show that $b_1 \leq \hat{C} \leq b_2 \leq b_1$.

$b_1 \leq \hat{C}$. This holds because $\mathcal{K}^+ \setminus \{0\} \subset \mathcal{N}^+$ and (3.7) holds.

$\hat{C} \leq b_2$. Let $\{u_j\} \subset E^+$ be a mountain pass sequence, that is, $I(u_j) \rightarrow b_2$ and $(1 + \|u_j\|)I'(u_j) \rightarrow 0$. It is not difficult to check that $\{u_j\}$ is bounded in E . By the concentration compactness principle, a standard argument shows that $\{u_j\}$ is non-vanishing, that is, there exist $k, \eta > 0$ and $\{a_j\} \subset \mathbb{Z}^{N+1}$

$$\limsup_{j \rightarrow \infty} \int_{D(a_j, r)} |u_j|^2 \geq \eta,$$

where $D(a_j, r) = \{\hat{x} \in \mathbb{R} \times \mathbb{R}^N : |\hat{x} - a_j| \leq r\}$. Set $v_j := a_j * u_j$. It follows from the invariance of the norm and of the functional under the $*$ -action that $I(v_j) \rightarrow b_2$ and $(1 + \|v_j\|)I'(v_j) \rightarrow 0$. Then it follows from Lemma 3.6 that, along a subsequence, $v_j \rightharpoonup v \in \mathcal{K}^+ \setminus \{0\}$ with $I(v_j - v) \rightarrow b_2 - I(v) \geq 0$, consequently, $b_2 \geq \hat{C}$.

$b_2 \leq b_1$. For $u \in \mathcal{N}^+$, there exists $s_0 > 1$ such that $I(t_0 u) < 0$. Set $h(s) := I(ss_0 u)$, similar to Lemma 3.7, we know that $h(0) = 0$, $h(s) > 0$ for s sufficiently small, and $h(s) \rightarrow -\infty$ as $s \rightarrow \infty$. Thus there is $s(s_0 u) > 0$ such that $I(s(s_0 u)s_0 u) = \sup_{t \geq 0} I(ss_0 u)$, so $s(s_0 u)s_0 u \in \mathcal{N}^+$ and $I'(s(s_0 u)s_0 u) = 0$. Let $\gamma(s) = ss_0 u$, then $\gamma(s) \in \mathcal{M}$ and

$$b_2 \leq \max_{s \in [0,1]} I(\gamma(s)) = I(s(s_0 u)s_0 u).$$

This complete the proof of the lemma. $\square$

Let $u_0 \in E^+$ be such that $I(u_0) < 0$, and set

$$\mathcal{M}_0 := \{\gamma \in C([0,1], E^+) : \gamma(0) = 0, \gamma(1) = u_0\}.$$

$$b_0 := \inf_{\gamma \in \mathcal{M}_0} \sup_{s \in [0,1]} I(\gamma(s)).$$

Lemma 3.10. *There holds $b_0 = b_2$.*

Proof. Since $\mathcal{M}_0 \subset \mathcal{M}$, it is clear that $b_2 \leq b_0$. Let $\gamma \in \mathcal{M}$. Then as before $I(s\gamma(1))$ and $I(su_0)$ are strictly decreasing for $s \geq 1$, and $I(s\gamma(1)) \rightarrow -\infty$, $I(su_0) \rightarrow -\infty$ as $s \rightarrow \infty$. Let $\nu(s)$ be a curve in the two-dimensional subspace $\text{span}\{\gamma(1), u_0\}$ joining $\gamma(1)$ and u_0 such that $I(\nu(s)) < 0$ for $1 \leq s \leq 2$ (such a curve exists because of the conclusion (2) of Lemma 3.7). Define $\hat{\nu}(s)$ by $\hat{\nu}(s) = \gamma(2s)$ for $s \in [0, 1/2]$ and $\hat{\nu}(s) = \nu(2s)$ for $s \in [1/2, 1]$. Then $\hat{\nu}(s) \in \mathcal{M}_0$ and $\max_{s \in [0,1]} I(\hat{\nu}(s)) = \max_{s \in [0,1]} I(\nu(s))$. Thus $b_0 \leq b_2$. $\square$

Lemma 3.11. *Let $u \in \mathcal{K}^+$ be such that $I(u) = \hat{C}$, and set $E_u = E^- \oplus \mathbb{R}u$. Then*

$$\sup_{w \in E_u} F(w) = I(u).$$

Proof. For any $w = v + su \in E_u$, by (3.5),

$$\begin{aligned} F(w) &= \frac{1}{2} \|su\|^2 - \frac{1}{2} \|v\|^2 - \int_{\mathbb{R} \times \mathbb{R}^N} H'_\infty(v + su) \\ &\leq F(su + f(su)) = I(su). \end{aligned}$$

Since $u \in \mathcal{N}^+$, one has

$$\sup_{w \in E_u} F(w) \leq \sup_{s \geq 0} I(su) = I(u).$$

□

4. Proof of Theorem 1.1

We are now going to complete the proof of the main result. Let $\hat{\mathcal{K}} := \{u \in E : \Phi'(u) = 0\}$ be the critical set of Φ . Recall that

$$c_{\min} := \inf\{\Phi(u) : u \in \hat{\mathcal{K}}\}$$

and

$$S_{\min} := \inf\{u \in \hat{\mathcal{K}} : \Phi(u) = c_{\min}\}.$$

From the conditions $(\mathcal{H}_1)$ – $(\mathcal{H}_4)$, it yields that for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|H_z(\hat{x}, z)| \leq \varepsilon|z| + C_\varepsilon|z|^{p-1} \quad (4.1)$$

and

$$|H(\hat{x}, z)| \leq \varepsilon|z|^2 + C_\varepsilon|z|^p \quad (4.2)$$

for all $(\hat{x}, z)$, where $\frac{2\mu}{\mu-1} \leq p < N^* = \frac{2(N+2)}{N}$.

Now we study the functional Φ defined by (2.2). From Lemma 4.1 of [16], we have

Lemma 4.1. *Ψ is nonnegative and weakly sequentially lower semicontinuous and Φ' is weakly sequentially continuous.*

Lemma 4.2. *There exist $\kappa > 0$ and $\rho > 0$ such that $\Phi|_{M_\rho^+} \geq 0$ and $\Phi|_{\partial M_\rho^+} \geq \kappa$ where $M_\rho^+ := \{u \in E^+ : \|u\| \leq \rho\}$ and $\partial M_\rho^+ := \{u \in E^+ : \|u\| = \rho\}$.*

Proof. By (4.2) and (2.1), for $u \in E^+$ one has

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}^N} B_L z \cdot z - \int_{\mathbb{R} \times \mathbb{R}^N} H(\hat{x}, u) \\ &\geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} |z|_2^2 - \varepsilon |u|_2^2 - C_\varepsilon |u|_p^p \\ &\geq \left(\frac{1}{2} - \frac{\lambda}{2(V_0 - \mu_\infty)} \right) \|u\|^2 - \varepsilon |u|_2^2 - C_\varepsilon |u|_p^p, \end{aligned}$$

where $\lambda > 0$ is defined in $(\mathcal{H}_0)$. Now the desired conclusion follows from Lemma 2.6 and $(\mathcal{H}_0)$. $\square$

Lemma 4.3. *There is $R > 0$ such that, for any $e \in E^+$ with $\|e\| = 1$ and $E_e = E \oplus \mathbb{R}e$,*

$$\Phi(u) < 0, \quad \forall u \in E_e \setminus K_R$$

where $K_R := \{u \in E_e : \|u\| \leq R\}$.

Proof. For $z \in E_e$, we have

$$\begin{aligned} \Phi(z) &= \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}^N} B_L z \cdot z - \int_{\mathbb{R} \times \mathbb{R}^N} H(\hat{x}, z) \\ &\leq \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int_{\mathbb{R} \times \mathbb{R}^N} H_\infty(z) = F(z), \end{aligned}$$

Thus, this lemma follows easily from Lemma 3.2. $\square$

In particular, let $U \in \mathcal{K}$ with $F(U) = \hat{C}$. Set $e \equiv U^+$ and $E_e = E \oplus \mathbb{R}e$.

Lemma 4.4. *We have*

$$d := \sup\{\Phi(u) : u \in E_e\} < \hat{C}.$$

Proof. Observe that by Lemma 4.2 and the linking property we have $d \geq \kappa$.

By $(\mathcal{H}_4)$, we see that $\Phi(u) \leq F(u)$ for all $u = v + sU^+$, and

$$F(u) = F(v + sU^+) \leq F(sU^+ + f(sU^+)) \leq \hat{C},$$

hence $d \leq \hat{C}$. Assume by contradiction that $d = \hat{C}$. Let $w_j := v_j + s_j U^+ \in E_e$ be such that $d - \frac{1}{j} \leq \Phi(w_j) \rightarrow d$. It follows from Lemma 4.3 that w_j is bounded and we can assume $w_j \rightharpoonup w$ in E with $v_j \rightharpoonup v$ and $s_j \rightarrow s$. It is clear that $s > 0$ (otherwise there should appear the contradiction that $d = 0$). Then

$$d - \frac{1}{j} \leq \Phi(w_j) \leq F(w_j) - \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}^N} B_L w_j \cdot w_j \leq \hat{C}$$

Taking the limit yields $\hat{C} \leq \hat{C} - \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}^N} B_L w \cdot w$ which implies that $w = 0$, a contradiction. $\square$

Set

$$Q := \{u = u^- + sU^+ : u^- \in E^-, s \geq 0, \|u\| < R\}$$

As a consequence of Lemma 4.4 one has the following

Lemma 4.5. $\sup \Phi(Q) < \hat{C}$.

We now turn to the analysis on $(C)_c$ -sequences. First we have

Lemma 4.6. *Any $(C)_c$ -sequence for Φ is bounded.*

Proof. The proof was given in Lemmas 5.1–5.3 of [16]. $\square$

Lemma 4.7. *Let $\{z_n\} \subset E$ be a $(C)_c$ -sequence. Then either*

- (i) $z_n \rightarrow z_L$, or

- (ii) $c \geq \hat{C}$ and there exists a positive integer $\ell \leq [\frac{c}{\hat{C}}]$, points $\bar{z}_1, \dots, \bar{z}_\ell \in \mathcal{K} \setminus \{0\}$ (not necessarily distinct), a subsequence of denote again by $\{z_n\}$ and sequence $\{k_n^i\} \subset \mathbb{Z}^{N+1}$ ($i = 1, \dots, \ell$) such that

$$\|z_n - z_L - \sum_{i=1}^{\ell} k_n^i * \bar{z}_i\|_{\nu} \rightarrow 0, \quad |k_n^i - k_n^j| \rightarrow \infty \quad (i \neq j) \text{ as } n \rightarrow \infty,$$

and

$$\Phi(z_L) + \sum_{i=1}^{\ell} F(\bar{z}_i) = c.$$

Proof. Remark that, by Lemma 4.6 any $(C)_c$ -sequence is bounded, hence it is a $(PS)_c$ -sequence. The proof is now well known, see for example Alama and Li [2] (see also Lemma 4.8 of [15] or Lemma 4.7 of [14]). Here we outline it as follows.

First observe that $c \geq 0$ which follows by taking the limit in

$$\Phi(z_n) - \frac{1}{2} \Phi'(z_n) z_n = \int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}(\hat{x}, z) \geq 0.$$

Assume (i) is false. It is easy to see that $z_n^1 := z_n - z_L$ is a $(PS)_{c_1}$ -sequence for F with $c_1 = c - \Phi(z_L)$ and $z_n^1 \rightharpoonup 0$. Note that F is invariant under the $*$ -action of $\mathbb{Z}^{N+1}$. A standard argument of concentration compactness principle implies that there exist a sequence $k_n^1 \in \mathbb{Z}^{N+1}$ with $|k_n^1| \rightarrow \infty$ and a critical point $\bar{z}_1 \neq 0$ of F satisfying $k_n^1 * z_n^1 \rightharpoonup \bar{z}_1$ and

$$F(k_n^1 * z_n^1) \rightarrow c - \Phi(z_L) - F(\bar{z}_1) \geq 0$$

Since $\Phi(z_L) \geq 0$ and $F(\bar{z}_1) \geq \hat{C}$, one sees that $c \geq \hat{C}$.

If $k_n^1 * z_n^1 \rightarrow \bar{z}_1$ then we are done. Otherwise, repeating the above argument, after at most finitely many steps we finish the proof. $\square$

Proof of Theorem 1.1. We first prove our existence results. From the form of Φ (see (2.4)) and $H(\hat{x}, z) \geq 0$, we see that Φ satisfies $(\mathcal{A}_0)$. The combination of Lemmas 4.1 and Theorem 2.10 implies that Φ verifies $(\mathcal{A}_1)$. Lemma 4.2 implies that $(\mathcal{A}_2)$ holds, and Lemma 4.3 shows that the linking condition of Theorem 2.9 is satisfied. These together with Lemma 4.5 imply that Φ has a $(C)_c$ -sequence $\{z_n\}$ with $c < \hat{C}$. Now by virtue of Lemma 4.7, $z_n \rightarrow z$ so that $\Phi'(z) = 0$ and $\Phi(z) \geq \kappa$. This proves that $\hat{\mathcal{K}} \setminus \{0\} \neq \emptyset$. By Lemma 2.8, $\hat{\mathcal{K}} \subset \cap_{r \geq 2} B_r$.

As in the proof of Lemma 3.4 one has $c_{\min} \geq 0$. Let $\{u_j\} \subset \hat{\mathcal{K}} \setminus \{0\}$ such that $\Phi(u_j) \rightarrow c_{\min}$. Then $\{u_j\}$ is a $(C)_{c_{\min}}$ -sequence, and since $c_{\min} < \hat{C}$, then Lemma 4.7 implies $u_j \rightarrow u \in S_{\min}$. Note that $\Phi(u_j) = \int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}(\hat{x}, u_j) \rightarrow c_{\min}$, and by assumptions it holds that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ with

$$\frac{|H_z(\hat{x}, u)|}{|u|} \leq \varepsilon + C_\varepsilon \tilde{H}_z(\hat{x}, u)^{\frac{1}{\mu}}.$$

Using the representation (2.4) and the estimate (2.3) we get

$$\begin{aligned} \|u_j\|_B^2 &= \int_{\mathbb{R} \times \mathbb{R}^N} H_z(\hat{x}, u_j)(u_j^+ - u_j^-) \\ &\leq \varepsilon |u_j|_2^2 + C_\varepsilon \left(\int_{\mathbb{R} \times \mathbb{R}^N} \tilde{H}(\hat{x}, u_j) \right)^{\frac{1}{\mu}} |u_j|_p \left(\int_{\mathbb{R} \times \mathbb{R}^N} |u_j^+ - u_j^-|^p \right)^{\frac{1}{p}} \\ &\leq c\varepsilon \|u_j\|_B^2 + c_1 C_\varepsilon (\Phi(u_j))^{\frac{1}{\sigma}} \|u_j\|_B^2, \end{aligned}$$

where $p = \frac{2\mu}{\mu-1}$. Hence $1 \leq c\varepsilon + c_1 C_\varepsilon c_{min}^{\frac{1}{\mu}}$, consequently $c_{min} > 0$.

We now prove that S_{min} is compact in B_2 . Note that, by $(\mathcal{H}_0)$ – $(\mathcal{H}_4)$, it is not difficult to show that S_{min} is bounded in E (cf. [16] or [18]), hence, $|u|_2 \leq C_2$ for all $u \in S_{min}$ and some C_2 . Therefore, as a consequence of Lemma 2.8 we see that, for each $q \in [2, \infty)$, there is $\Lambda_q > 0$ such that

$$\|u\|_{B_q} \leq \Lambda_q \quad \text{for all } u \in S_{min}. \quad (4.3)$$

This together with the Sobolev embedding theorem (see [5]) implies that for some Λ_∞ independent of $u \in S_{min}$ satisfying

$$|u|_\infty \leq \Lambda_\infty \quad \text{for all } u \in S_{min}. \quad (4.4)$$

Let $u_j \in S_{min}$ be such that $u_j \rightharpoonup u$ in B_2 . Since $\Phi(u_j) = c_{min}$ and $\Phi'(u_j) = 0$, $\{u_j\}$ is a $(C)_{c_{min}}$ -sequence. Since $c_{min} < \hat{C}$, it follows from Lemma 4.7 that $u_j \rightarrow u$ (along a subsequence) in E . Clearly, $u \in S_{min}$. By

$$L_\infty u = B_L u + H_z(\hat{x}, u),$$

one has

$$\begin{aligned} |L_\infty(u_j - u)|_2 &\leq |B_L(u_j - u)|_2 + |H_z(\cdot, u_j) - H_z(\cdot, u)|_2 \\ &\leq o(1) + |H_z(\cdot, u_j) - H_z(\cdot, u)|_2. \end{aligned}$$

Since $|u_j|_\infty \leq \Lambda_\infty$, $u_j \rightarrow u$ in E and $|u(\hat{x})| \rightarrow 0$ as $|\hat{x}| \rightarrow \infty$, then we have

$$|H_z(\cdot, u_j) - H_z(\cdot, u)|_2 \rightarrow 0.$$

Therefore, one sees that $|L_\infty(u_j - u)|_2 \rightarrow 0$, i.e., $u_j \rightarrow u$ in B_2 . $\square$

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