Superlinear Ambrosetti–Prodi problem for the p -Laplacian operator

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Abstract. Based on a new Liouville theorem, we study a superlinear Ambrosetti–Prodi problem for the p -Laplacian operator, $1 < p < N$. For this, we use the sub and supersolution method, blow up technique and the Leray–Schauder degree theory.

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1. Introduction

The main purpose of this work is to investigate the existence of multiple solutions of the problem

$$\begin{cases} -\Delta_p u = f(x, u) + t\phi + h, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (P_t)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator and $1 < p < N$. We assume that $\phi, h \in L^\infty(\Omega)$, with $\phi \succ 0$ in Ω and $t \in \mathbb{R}$ is a parameter, where $\succ$ will be defined later. Denoting λ_1 the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ we consider $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous function satisfying the following conditions:

- (H1) $\limsup_{u \rightarrow -\infty} \frac{f(x, u)}{|u|^{p-2}u} = \mu < \lambda_1$, uniformly $x \in \Omega$,
 - (H2) There exists $\sigma > 0$ such that $f(x, s) + \sigma|s|^{p-2}s$ is increasing in s , $f(x, 0) = 0$.
 - (H3) $\lim_{u \rightarrow +\infty} \frac{f(x, u)}{u^\alpha} = a(x)$, $a \in C(\bar{\Omega})$, $a \geq 1$, $p - 1 < \alpha < p^* - \frac{N}{N-p}$, where $p^* = \frac{Np}{N-p}$,
- ((H3) is known as p -superlinear condition at infinity and simply superlinear when $p = 2$).

This problem belongs to a class of problems known as the Ambrosetti–Prodi type. For the case $p = 2$ with different variants and formulations, it has been extensively studied by several authors. We shall quote here the original work [1], as well as the subsequent developments [3–5, 14] and the references therein. For the superlinear case see for example [6, 9–11].

More recently, Ambrosetti–Prodi type results for p -Laplacian operator, with $p > 1$ have been studied, but few results are really known. We can cite Perera [19] which used the linking and homotopy theory to study a similar problem with zero Dirichlet condition on the boundary and asymmetric nonlinearities. Mawhin in [18] studied the existence of periodic solutions for the ODE case with $\phi = 1$. Arcoya and Ruiz in [2] treated problem (P_t) with f growing as $|u|^{p-2}u$, that is, the p -linear case, which was also studied by Koizumi and Schmitt in [15]. Our main contribution is to study this problem where f has a p -superlinear behavior.

Motivated by results in [2] we prove the following theorem:

Theorem 1.1. *Suppose (H1) – (H3) hold. There exist $-\infty < t_* \leq t^* < +\infty$ such that problem (P_t) has:*

- i) *at least, two solutions provided that $t < t_*$,*
- ii) *at least, one solution provided that $t \leq t^*$,*
- iii) *no solution provided that $t > t^*$.*

We conjecture that $t^* = t_*$, but this question is still open. In order to prove Theorem 1.1, we use the sub and supersolution method and topological arguments. Then we need a priori bounds on the eventual solutions of (P_t) to apply the Leray–Schauder degree.

In order to use the sub and supersolution method we only need the hypothesis (H1), and therefore the first solution is obtained arguing as in [2]. We can also cite the work of de Figueiredo, Gossez and Ubilla in [7] that use a variational approach to the method of upper–lower solutions. However to get the second solution we need the a priori estimates of eventual solutions of (P_t) . We remark that Arcoya and Ruiz’s proof of this fact does not apply for our case. To overcome this fact, we follow [8], obtaining not only a priori bound on negative part of u , but also on t , such that (P_t) has a solution and in this way we obtain an a priori bound for u . The main difficulty is that the operator studied in [8] is linear, then the a priori bound on t follows easily from the linearity of the operator and from ABP estimate. For our case, since our operator is not linear, we have to adapt a Dong’s result [12] that use a comparison principle. For the boundedness of u we use the blow up technique, which only was possible to use thanks to a recent paper of Lorca [17], where he proves a Liouville type result for a half space. To obtain the second solution we will use the Leray–Schauder degree theory, following the results of [2] and [5].

This paper is organized as follows. Section 2 is devoted to finding the first solution. In Sect. 3, we obtain a priori estimates for eventual solutions of (P_t) using the blow up method and in Sect. 4 we apply the Leray–Schauder degree theory to obtain the second solution.

All along this work we will use the following notations:

- i) $\|\cdot\|$ represents the L^∞ -norm in Ω ,
- ii) $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}); u(x) = 0, \forall x \in \partial\Omega\}$. Similarly, we define $C_0^1(\overline{\Omega})$ and $C_0^{1,\alpha}(\overline{\Omega})$.
- iii) for any $h_1, h_2 \in L^\infty(\Omega)$, we say that $h_1 \prec h_2$ if, for any compact subset $K \subset \Omega$, there exists $\varepsilon > 0$ such that $h_1(x) + \varepsilon < h_2(x)$ for almost every $x \in K$.
- iv) for any $u, v \in C_0^1(\overline{\Omega})$; we will say that $u \ll v$ in Ω if $u(x) < v(x)$ for $x \in \Omega$ and $\frac{\partial u}{\partial \eta}(x) < \frac{\partial v}{\partial \eta}(x)$ for all $x \in \partial\Omega$, where $\eta(x)$ denotes the inward normal to $\partial\Omega$ at x .

2. First solution

Our main tool to find the first solution will be the sub and supersolution method. We will find now the sub and supersolutions for (P_t) :

Proposition 2.1. *For every $w \in C_0^1(\overline{\Omega})$, $t \in \mathbb{R}$, there exists $\underline{u} \in C_0^{1,\alpha}(\overline{\Omega})$ such that $\underline{u} \ll w$ and*

$$-\Delta_p \underline{u} \leq f(x, \underline{u}) + t\phi + h, \quad \text{in } \Omega.$$

Proof. Let $w \in C_0^1(\overline{\Omega})$, $t \in \mathbb{R}$ be fixed, but arbitrary. By (H1) we have that for $\varepsilon \in (0, \lambda_1 - \mu)$, there exists constant $C > 0$ such that

$$f(x, u) \geq (\mu + \varepsilon)|u|^{p-2}u - C,$$

for all $u < 0$. Let $u \in W_0^{1,p}(\Omega)$ be solution of

$$-\Delta_p u = (\mu + \varepsilon)|u|^{p-2}u - C + t\phi + h, \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

where without loss of generality we take C large enough to get $-C + t\phi + h < 0$.

Claim 2.2. *We can take $C > 0$ large enough such that $u \ll w$.*

In fact, let $\{C_n\}$ be a sequence such that $C_n \rightarrow \infty$ (without loss of generality assume that $C_n > 1$) and $\{u_n\}$ a sequence of solutions of

$$-\Delta_p u_n = (\mu + \varepsilon)|u_n|^{p-2}u_n - C_n + t\phi + h, \quad \text{in } \Omega, \quad u_n = 0, \text{ on } \partial\Omega. \quad (2.1)$$

Thus, if $v_n = \frac{u_n}{(C_n)^{\frac{1}{p-1}}}$, then we have that v_n is solution of

$$-\Delta_p v_n = (\mu + \varepsilon)|v_n|^{p-2}v_n - 1 + \frac{t\phi + h}{C_n}, \quad \text{in } \Omega, \quad v_n = 0, \quad \text{on } \partial\Omega.$$

By Lemma 2.2 in [2] we obtain $v_n \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, and $\|v_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq M$. Thus by the compact imbedding $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^{1,\beta}(\overline{\Omega})$, $0 \leq \beta < \alpha$ we have that, up to a subsequence, $v_n \rightarrow v$ in $C^{1,\beta}(\overline{\Omega})$. By Lemma 2.3 in [2] we get $v_n \rightarrow v$ which is solution of

$$\begin{cases} -\Delta_p v = (\mu + \varepsilon)|v|^{p-2}v - 1, & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

By Maximum Principle it follows that $v \leq 0$ and by Vazquez Maximum Principle we get $v \ll 0$.

Note that $v_n = \frac{u_n}{(C_n)^{\frac{1}{p-1}}} \rightarrow v \ll 0$, thus $u_n < 0$, $\frac{\partial u_n}{\partial \eta} < 0$ and $\frac{\partial u_n}{\partial \eta}, u_n \rightarrow -\infty$ as $n \rightarrow \infty$. Since $u_n, w \in C_0^1(\bar{\Omega})$ we have that for n large enough, $u_n \ll w$.

Let n_0 such that $u_{n_0} \ll w$ and $C_{n_0} > C$. Then, defining $\underline{u} = u_{n_0}$, since u_{n_0} is solution of (2.1) we have,

$$\begin{aligned} -\Delta_p \underline{u} &= (\mu + \varepsilon)|\underline{u}|^{p-2}\underline{u} - C_{n_0} + t\phi + h \\ &< (\mu + \varepsilon)|\underline{u}|^{p-2}\underline{u} - C + t\phi + h \\ &\leq f(x, \underline{u}) + t\phi + h, \end{aligned}$$

that is, $\underline{u}$ is subsolution of (P_t) that satisfies $\underline{u} \ll w$. $\square$

Proposition 2.3. *Given a function $w \in C_0^1(\bar{\Omega})$, there exists $\underline{t}$ such that for every $t \leq \underline{t}$, there exists a strict supersolution $\bar{u} \in C_0^1(\bar{\Omega})$ for (P_t) with $\bar{u} \ll w$.*

Proof. Fix an open set Ω_0 such that $\Omega_0 \subset \subset \Omega$. Define

$$M > \max_{x \in \bar{\Omega}, s \in [-1, 0]} |f(x, s)| + \|h\|.$$

For each $n > 0$, let $u_n \in W_0^{1,p}(\Omega)$ be the solution of

$$-\Delta_p u_n = g_n(x), \quad \text{in } \Omega, \quad u_n = 0, \quad \text{on } \partial\Omega,$$

where

$$g_n(x) = \begin{cases} -n^{p-1}, & x \in \Omega_0 \\ M, & x \in \Omega \setminus \Omega_0. \end{cases}$$

Note that $\frac{g_n}{n^{p-1}} \rightarrow -\chi_{\Omega_0}$ in $L^\infty(\Omega)$, as n goes to ∞ , where χ_{Ω_0} is the characteristic function of Ω_0 . Define now $v_n = \frac{u_n}{n}$. By Lemma 2.3 in [2], we deduce that the sequence $v_n \rightarrow v$ in $C^{1,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, v solution of

$$\begin{cases} -\Delta_p v = -\chi_{\Omega_0}, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

By Maximum Principle and Vazquez Maximum Principle we have that $v \ll 0$. Arguing as Claim 2.2 we obtain that for n_0 large enough, $u_{n_0} \ll w$ and $u_{n_0} < 0$ in Ω . We can assume that $u_{n_0} < -1$ in Ω_0 .

Define $\bar{u} = u_{n_0}$ and

$$\bar{t} < -\frac{(n_0^{p-1} + \max_{x \in \bar{\Omega}, s \in [\min \bar{u}, -1]} |f(x, s)| + \|h\|)}{\inf_{\bar{u}^{-1}((-\infty, -1])} \phi}.$$

Take into account that, since $\phi \succ 0$, $\inf_{\bar{u}^{-1}((-\infty, -1])} \phi > 0$ (here the infimum is considered in an essential way).

For $x \in \Omega$ there are two possibilities: either $\bar{u}(x) \in [-1, 0)$, or $\bar{u}(x) < -1$. In the first case we have that $x \in \Omega \setminus \Omega_0$, since $\bar{u} < -1$ in Ω_0 , and then

$$f(x, \bar{u}) + \bar{t}\phi(x) + h(x) < M = g_{n_0} = -\Delta_p \bar{u}.$$

On the other hand, if $\bar{u}(x) < -1$, then, by definition of $\bar{t}$,

$$\begin{aligned} f(x, \bar{u}) + \bar{t}\phi(x) + h(x) &\leq \max_{y \in \bar{\Omega}, s \in [\min \bar{u}, -1]} |f(y, s)| + \bar{t} \inf_{\bar{u}^{-1}((-\infty, -1])} \phi + \|h\| \\ &< -n_0^{p-1} = g_{n_0} = -\Delta_p \bar{u}. \end{aligned}$$

Therefore $\bar{u}$ is supersolution of $(P_{\bar{t}})$ and for all $t \leq \bar{t}$, $\bar{u}$ is supersolution of (P_t) with $\bar{u} \ll w$, which concludes the proof. $\square$

The next result is the well known sub and supersolution method, which give us the first solution, through the Schauder fixed point theorem.

Theorem 2.4. *Let $\underline{u}, \bar{u} \in C^1(\bar{\Omega})$ be the sub and supersolution of (P_t) , respectively, such that $\underline{u} \leq \bar{u}$ in Ω , with $\underline{u} \leq 0 \leq \bar{u}$ on $\partial\Omega$. Then there exists $u \in C^1(\bar{\Omega})$ solution of (P_t) such that $\underline{u} \leq u \leq \bar{u}$ in Ω and $u = 0$ on $\partial\Omega$.*

Proof. Define the operators $N_t : C_0^1(\bar{\Omega}) \rightarrow L^\infty(\Omega)$ and $T : L^\infty(\Omega) \rightarrow C_0^1(\bar{\Omega})$ by

$$N(v) = f(x, v) + \sigma|v|^{p-2}v + t\phi + h, \quad v \in C_0^1(\bar{\Omega})$$

and

$$T(v) = w \quad \text{iff} \quad -\Delta_p w + \sigma|w|^{p-2}w = v, \quad v \in L^\infty(\Omega).$$

Define also $Q_t : C_0^1(\bar{\Omega}) \rightarrow C_0^1(\bar{\Omega})$ by $Q_t = T \circ N_t$.

Note that u is a fixed point of Q_t if and only if u is a solution of (P_t) . We will show that Q_t is compact: since $Q_t = T \circ N_t$ and N_t is continuous, it is enough to show that T is compact. In fact, let $\{u_n\}$ be a bounded sequence in $L^\infty(\Omega)$ and $w_n = T(u_n)$, thus

$$-\Delta_p w_n + \sigma|w_n|^{p-2}w_n = u_n.$$

By Lemma 2.2 in [2] $w_n \in C^{1,\alpha}(\bar{\Omega})$ with $\|w_n\|_{C^{1,\alpha}} < C$. By the compact embedding $C^{1,\alpha}(\bar{\Omega}) \hookrightarrow C^{1,\beta}(\bar{\Omega})$ it follows that, up to a subsequence, $w_n \rightarrow w$ in $C^{1,\beta}(\bar{\Omega})$, therefore T is compact.

Now, define

$$\mathcal{O} = \{u \in C_0^1(\bar{\Omega}); \underline{u}(x) \leq u(x) \leq \bar{u}(x)\}.$$

Note that $\mathcal{O}$ is closed and convex. Still, by the comparison principle (see Tolksdorf [21]) we have that $Q_t(\mathcal{O}) \subset \mathcal{O}$, and by Lemma 2.2 in [2] $Q_t(\mathcal{O})$ is bounded. Thus, by Schauder fixed point theorem there exists $u \in C_0^1(\bar{\Omega})$ which is a fixed point of Q_t , that is, there exists $u \in C_0^1(\bar{\Omega})$ solution of (P_t) such that $\underline{u} \leq u \leq \bar{u}$ in Ω . $\square$

3. A priori bounds

In order to obtain the second solution using the degree theory, we need to get a priori estimates. In this section we start getting a estimate for the negative part of u , and later we show that if u is solution of (P_t) for t bigger than a certain negative constant, then $t < C(1 + \|u\|^{p-1})$. Finally, we prove an a priori bound on u using the blow up technique, as introduced by Gidas–Spruck [13]. We remark here that to use the blow up technique we need the Liouville type results for $\mathbb{R}^N$ and half space, which had been gotten by [20] and [17], respectively. For that matter, in our hypothesis (H3) we need to impose that $p - 1 < \alpha < p^* - \frac{N}{N-p}$ to be able to use those results. For the reader's convenience we enunciate these results:

Theorem 3.1 (Theorem III [20]). Assume g is subcritical and $g(u) > 0$ for all $u > 0$. Then every bounded solution of

$$\Delta_p u + g(u) = 0, \quad u \geq 0 \quad x \in \mathbb{R}^N, \quad p < N,$$

is trivial.

Theorem 3.2 (Theorem 3.1 [17]). Consider the problem

$$u^m \leq -\Delta_p u \leq Cu^m, \quad x \in \mathbb{R}_+^N, \quad (3.1)$$

where $C \geq 1$. Assume that $p - 1 < m < p^* - \frac{N}{N-p}$. Then, there is no positive solution of (3.1) in $C^1(\mathbb{R}_+^N)$.

To prove the boundedness of the negative part, we will need the following result due to Ladyzhenskaya and Ural'tseva [16]:

Theorem 3.3 (Theorem 5.1 [16]). Suppose $u \in W^{1,m}(\Omega) \cap L^q(\Omega)$ has bounded essential $\max_{\partial\Omega} u(x)$ and that for $k \geq \hat{k} \geq \text{essential } \max_{\partial\Omega} u(x)$ satisfies the inequality

$$\int_{A_k} |\nabla u|^m \leq \hat{\gamma} \left(\int_{A_k} |u - k|^l \right)^{\frac{m}{l}} + \hat{\gamma} \sum_{i=1}^{\bar{M}} k^{\alpha_i} (\text{mes}(A_k))^{1 - \frac{m}{N} + \varepsilon_i},$$

where $A_k = \{x \in \Omega; u(x) \geq k\}$, $\hat{\gamma}, l, \alpha_i$ and $\bar{M}$ are positive constants. Suppose also that

$$l \leq \frac{Nm}{N-m}, \quad \varepsilon_i > 0, \quad m \leq \alpha_i < \varepsilon_i q + m.$$

Then, essential $\max_{\Omega} u(x)$ does not exceed some constant that depends on $\hat{\gamma}, l, \alpha_i, \varepsilon_i$ (for $i = 1, \dots, \bar{M}$), $m, N, \hat{k}, q, \text{mes}(\Omega)$, and either the norm $\|u\|_{L^1(A_{\hat{k}})}$ (for $q < \frac{Nm}{N-m}$) or the norm $\|u\|_{L^q(A_{\hat{k}})}$ (for $q \geq \frac{Nm}{N-m}$).

Next, arguing as [2], we will prove the boundedness of negative part of a eventual solution of (P_t) :

Theorem 3.4. For any $C_0 \in \mathbb{R}_+$, there exists constant $M > 0$ such that for all $t \geq -C_0$, if u is solution of (P_t) then $\|u^-\| \leq M$.

Proof. Let u be a solution of (P_t) . Considering u^- as a test function we get

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla u^- = \int_{\Omega} (f(x, u) + t\phi + h) u^-.$$

Now, if $\Omega^- = \{x \in \Omega; u(x) \leq 0\}$, since $\nabla u^- = -\nabla u$ in Ω^- and $u^- = -u$ in Ω^- , we have that

$$\int_{\Omega^-} |\nabla u|^p = \int_{\Omega^-} f(x, u) u + t\phi u + hu.$$

Since f satisfies (H1) we have that for all $\varepsilon \in (0, \lambda_1 - \mu)$, there exists $C > 0$ large enough such that $f(x, u) \geq (\mu + \varepsilon)|u|^{p-2}u - C$, for all $u < 0$, that is $f(x, u)u \leq (\mu + \varepsilon)|u|^p - Cu$, for all $u < 0$. Then,

$$\int_{\Omega^-} |\nabla u|^p \leq \int_{\Omega^-} (\mu + \varepsilon)|u|^p - Cu + t\phi u + hu.$$

Thus, by the variational characterization of λ_1 and by Holder inequality it follows that

$$\left(1 - \frac{(\mu + \varepsilon)}{\lambda_1}\right) \int_{\Omega^-} |\nabla u|^p \leq \left(\int_{\Omega^-} |-C - C_0\phi + h|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\Omega^-} |u|^p\right)^{\frac{1}{p}},$$

which implies

$$\|\nabla u^-\|_p^{p-1} \leq M_2, \quad (3.2)$$

where M_2 is independent of $t \geq -C_0$. Now, consider $\hat{k} = 1$ and for $k \geq \hat{k}$ define the set $A_k = \{x \in \Omega; u^-(x) > k\}$, then

$$\int_{A_k} |\nabla u^-|^p = - \int_{A_k} |\nabla u|^{p-2} \nabla u \nabla (u^- - k) = - \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla (u^- - k)^+.$$

Since u is a solution of (P_t) and again by $(H1)$ we have

$$\begin{aligned} - \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla (u^- - k)^+ &\leq - \int_{A_k} (f(x, u) - C_0\phi + h)(u^- - k) \\ &\leq \int_{A_k} (\mu + \varepsilon) |u^-|^{p-2} u^- (u^- - k) + \int_{A_k} (C + C_0\|\phi\| + \|h\|)(u^- - k) \\ &= I_1 + I_2. \end{aligned}$$

For I_2 we have that

$$I_2 \leq (C + C_0\|\phi\| + \|h\|) \int_{A_k} u^- k^{p-p}.$$

In order to estimate these integrals, we will use the following Young's inequality

$$ab \leq \delta a^{\frac{1}{s}} + \left(\frac{s}{\delta}\right)^{\frac{s}{1-s}} (1-s)b^{\frac{1}{1-s}}, \quad \forall a, b \geq 0, \quad s \in (0, 1), \quad \delta > 0.$$

Setting $\tilde{\gamma} = C + C_0\|\phi\| + \|h\|$, by Young's inequality with $s = \frac{1}{p} \in (0, 1)$ and $\delta = \frac{\lambda_1 k^\gamma}{4\tilde{\gamma}} > 0$ where $\gamma \in (p(\frac{N-p^2}{N-p}), p)$ for I_2 , and for I_1 taking $s = \frac{p-1}{p} \in (0, 1)$ and $\delta = \frac{\lambda_1}{4(\mu+\varepsilon)} > 0$ we obtain,

$$\begin{aligned} \int_{A_k} |\nabla u^-|^p &\leq I_1 + I_2 \\ &\leq \frac{\lambda_1}{4} \int_{A_k} |u^-|^p + \left(\frac{4(\mu + \varepsilon)(p-1)}{p\lambda_1}\right)^{p-1} \frac{\mu + \varepsilon}{p} \int_{A_k} |u^- - k|^p \\ &\quad + \frac{\lambda_1 k^\gamma}{4} \int_{A_k} |u^-|^p k^{-p^2} + \tilde{\gamma} \left(\frac{4\tilde{\gamma}}{p\lambda_1 k^\gamma}\right)^{\frac{1}{p-1}} \frac{p-1}{p} k^{\frac{p^2}{p-1}} \text{mes}(A_k). \end{aligned}$$

By the variational characterization of λ_1 , it follows

$$\underbrace{\left(1 - \frac{1}{4} - \frac{k^{\gamma-p^2}}{4}\right)}_{\nu_1} \int_{A_k} |\nabla u^-|^p \leq \underbrace{\left(\frac{4(\mu + \varepsilon)(p-1)}{p\lambda_1}\right)^{p-1} \frac{\mu + \varepsilon}{p}}_{\nu_2} \int_{A_k} |u^- - k|^p \\ + \underbrace{\tilde{\gamma} \left(\frac{4\tilde{\gamma}}{p\lambda_1}\right)^{\frac{1}{p-1}} \frac{p-1}{p} \text{mes}(A_k) k^{\frac{p^2-\gamma}{p-1}}}_{\nu_3}.$$

It follows from definition of γ that $k^{\gamma-p^2} < 1$ for all $k > \hat{k}$ and then $\nu_1 > \frac{1}{2}$. We have also that $\nu_2, \nu_3 > 0$. Take $\bar{\gamma} = 2 \max\{\nu_2, \nu_3\}$, then

$$\int_{A_k} |\nabla u^-|^p \leq \bar{\gamma} \left(\int_{A_k} |u^- - k|^p \right) + \bar{\gamma} k^{\frac{p^2-\gamma}{p-1}} \text{mes}(A_k).$$

Since $1 < p < N$, we have by Sobolev imbedding and by (3.2) that $\|u^-\|_{L^{p^*}(\Omega)} \leq C\|u^-\|_{W_0^{1,p}(\Omega)} \leq \bar{C}$, where $\bar{C} > 0$ is independent of $t \geq -C_0$. By Theorem 3.3 with $q = p^*$, $m = l = p$, $\bar{M} = 1$, $\varepsilon_1 = \frac{p}{N}$, $\alpha_1 = \frac{p^2-\gamma}{p-1} \in (p, p^* \frac{p}{N} + p)$, $\hat{\gamma} = \bar{\gamma}$ and $\hat{k} = 1$, we obtain that $\|u^-\| \leq M$, where M depends only of $p, \gamma, N, 1, \text{mes}(\Omega), \bar{C}$. $\square$

Theorem 3.5. *For any $C_0 \in \mathbb{R}_+$, there exists constant C_1 such that for all $t \geq -C_0$, if u is solution of (P_t) then $t \leq C_1(1 + \|u\|^{p-1})$.*

Proof. Let $C_0 \in \mathbb{R}_+$ and $t \geq -C_0$.

If $-C_0 \leq t \leq 0$, then let $C_1 \geq 0$ be an arbitrary constant, thus

$$t \leq C_1 \leq C_1(1 + \|u\|^{p-1})$$

and the theorem is proved.

Suppose that $t > 0$. Let u be solution of (P_t) . By Theorem 3.4 we have that $\|u^-\| \leq M$, hence by (H2) we have

$$f(x, u) + \sigma|u|^{p-2}u \geq f(x, -M) + \sigma|-M|^{p-2}(-M).$$

Define

$$k_1 = \inf_{\Omega} f(x, -M) + \sigma|-M|^{p-2}(-M)$$

(note that k_1 does not depend on t). Let $\Omega_0 \subset \Omega$ be a compact set and $\delta > 0$ such that $\phi > \delta$ in Ω_0 , then

$$-\Delta_p u + \sigma|u|^{p-2}u = f(x, u) + \sigma|u|^{p-2}u + t\phi + h \\ > k_1 + t\delta - \|h\|,$$

in Ω_0 . If

$$t \leq 2 \left| \frac{-k_1 + \|h\|}{\delta} \right|, \quad (3.3)$$

then defining $C_1 = 2 \left| \frac{-k_1 + \|h\|}{\delta} \right|$ it follows that

$$t \leq C_1 \leq C_1(1 + \|u\|^{p-1})$$

and the theorem is proved.

Assume

$$t > 2 \left| \frac{-k_1 + \|h\|}{\delta} \right|.$$

Thus we have that $t\delta > 2|-k_1 + \|h\|| \geq -k_1 + \|h\|$ and hence $k_1 - \|h\| + t\delta > 0$.

Define $w_1 = \left(\frac{k_1 + t\delta - \|h\|}{2} \right)^{\frac{1}{p-1}} v$, where v is the solution of

$$-\Delta_p v + \sigma|v|^{p-2}v = 1 \text{ in } \Omega_0, \quad v = 0 \text{ on } \partial\Omega_0.$$

Then we have

$$\begin{aligned} -\Delta_p w_1 + \sigma|w_1|^{p-2}w_1 &= \left(\frac{k_1 + t\delta - \|h\|}{2} \right) (-\Delta_p v + \sigma|v|^{p-2}v) \\ &= \left(\frac{k_1 + t\delta - \|h\|}{2} \right) \\ &< -\Delta_p u + \sigma|u|^{p-2}u, \end{aligned}$$

in Ω_0 . Since $u \in C^1(\Omega)$, it follows that $u \geq \inf_{\partial\Omega_0} u(x) \geq -M$. Now we will analyze the two cases: $\inf_{\partial\Omega_0} u(x) \geq 0$ and $\inf_{\partial\Omega_0} u(x) < 0$.

First case: If $\inf_{\partial\Omega_0} u(x) \geq 0$, then $w_1 = 0 \leq \inf_{\partial\Omega_0} u(x) \leq u$ on $\partial\Omega_0$ and by the Comparison Principle we obtain $u \geq w_1$ in Ω_0 . Let $x_0 \in \Omega_0$ such that $v(x_0) = \|v\|_{L^\infty(\Omega_0)}$, thus

$$\|u\| \geq u(x_0) \geq \left(\frac{k_1 + t\delta - \|h\|}{2} \right)^{\frac{1}{p-1}} v(x_0)$$

that is

$$\begin{aligned} t &\leq \frac{2\|u\|^{p-1}}{\delta\|v\|_{L^\infty(\Omega_0)}^{p-1}} + \frac{\|h\| - k_1}{\delta} \leq \frac{2\|u\|^{p-1}}{\delta\|v\|_{L^\infty(\Omega_0)}^{p-1}} + \left| \frac{\|h\| - k_1}{\delta} \right| \\ &< \frac{2\|u\|^{p-1}}{\delta\|v\|_{L^\infty(\Omega_0)}^{p-1}} + \frac{t}{2}, \end{aligned}$$

or

$$t < \frac{4\|u\|^{p-1}}{\delta\|v\|_{L^\infty(\Omega_0)}^{p-1}}. \quad (3.4)$$

Second case: If $\inf_{\partial\Omega_0} u(x) < 0$, then $z = v - \tilde{M}$, where $\tilde{M} = \frac{M}{\left(\frac{k_1 + t\delta - \|h\|}{2} \right)^{\frac{1}{p-1}}}$

satisfies, in Ω_0 ,

$$\begin{aligned} -\Delta_p z + \sigma|z|^{p-2}z &= -\Delta_p v + \sigma|v - \tilde{M}|^{p-2}(v - \tilde{M}) \\ &< -\Delta_p v + \sigma|v|^{p-2}v = 1, \end{aligned}$$

and

$$z = v - \tilde{M} = -\tilde{M}, \quad \text{on } \partial\Omega_0.$$

Thus defining $w_2 = \left(\frac{k_1 + t\delta - \|h\|}{2}\right)^{\frac{1}{p-1}} z$, it follows that

$$\begin{aligned} -\Delta_p w_2 + \sigma |w_2|^{p-2} w_2 &= \left(\frac{k_1 + t\delta - \|h\|}{2}\right) (-\Delta_p z + \sigma |z|^{p-2} z) \\ &< -\Delta_p u + \sigma |u|^{p-2} u \end{aligned}$$

in Ω_0 . Still

$$\begin{aligned} w_2 &= \left(\frac{k_1 + t\delta - \|h\|}{2}\right)^{\frac{1}{p-1}} z = -\left(\frac{k_1 + t\delta - \|h\|}{2}\right)^{\frac{1}{p-1}} \tilde{M} \\ &= -M \leq u, \end{aligned}$$

on $\partial\Omega_0$. By the Comparison Principle we have $u \geq w_2$ on Ω_0 .

Let $x_0 \in \Omega_0$ as above, thus

$$\begin{aligned} \|u\| &\geq u(x_0) \geq w_2(x_0) \\ &= \left(\frac{k_1 + t\delta - \|h\|}{2}\right)^{\frac{1}{p-1}} (v(x_0) - \tilde{M}) \\ &= \left(\frac{k_1 + t\delta - \|h\|}{2}\right)^{\frac{1}{p-1}} \|v\|_{L^\infty(\Omega_0)} - M, \end{aligned}$$

that is,

$$t \leq \frac{2(\|u\| + M)^{p-1}}{\delta \|v\|_{L^\infty(\Omega_0)}^{p-1}} + \frac{\|h\| - k_1}{\delta} < \frac{2^{p-1}(\|u\|^{p-1} + M^{p-1})}{\delta \|v\|_{L^\infty(\Omega_0)}^{p-1}} + \frac{t}{2},$$

or

$$t < \frac{2^p(\|u\|^{p-1} + M^{p-1})}{\delta \|v\|_{L^\infty(\Omega_0)}^{p-1}}. \quad (3.5)$$

Define

$$C_1 = \max \left\{ \frac{4}{\delta \|v\|_{L^\infty(\Omega_0)}^{p-1}}, \frac{2^p}{\delta \|v\|_{L^\infty(\Omega_0)}^{p-1}}, \frac{2^p M^{p-1}}{\delta \|v\|_{L^\infty(\Omega_0)}^{p-1}}, 2 \left| \frac{\|h\| - k_1}{\delta} \right| \right\}.$$

Then by (3.3), (3.4) and (3.5) we have $t \leq C_1(1 + \|u\|^{p-1})$, which concludes the proof. $\square$

We also need the following theorem:

Theorem 3.6. *There exists a constant C_2 such that for all $t \geq 1$ and for all solution u of (P_t) with this t we have $\|u\| \leq C_2 t^{\frac{1}{\alpha}}$.*

Proof. Consider $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(x, s) = f(x, s) - a(x)(s^+)^{\alpha}.$$

Note that g is continuous and by (H3) we have

$$\lim_{s \rightarrow \infty} \frac{|g(x, s)|}{s^{\alpha}} = 0. \quad (3.6)$$

Suppose that the conclusion of theorem is false, then there exists sequences $\{t_n\}$ such that $t_n \geq 1$ and $\{u_n\}$ solution of (P_{t_n}) such that

$$\|u_n\|^\alpha > nt_n. \quad (3.7)$$

Since u_n is solution of (P_{t_n}) , it follows from definition of g that

$$-\Delta_p u_n = a(x)(u_n^+)^{\alpha} + g(x, u_n) + t_n \phi + h. \quad (3.8)$$

By Theorem 3.4 we have that $\|u_n^-\| \leq M$, thus $\|u_n^+\| = \|u_n\|$ for $n \geq n_0$ large enough. Define $\lambda_n = \|u_n\|^{-\frac{1}{q}}$, where $q = \frac{p}{\alpha - (p-1)} > 0$. Then it follows that

$$\lambda_n = \|u_n\|^{-\frac{1}{q}} \leq (nt_n)^{-\frac{1}{q\alpha}} \leq n^{-\frac{1}{q\alpha}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Without loss of generality we can assume that $\lambda_n^q < 1$ for $n \geq n_0$.

Consider also, for $n \geq n_0$ a function v_n defined by

$$v_n(x) = \lambda_n^q u_n(\lambda_n x + x_n) + M,$$

where $u_n(x_n) = \|u_n^+\| = \|u_n\|$. Thus, for $n \geq n_0$,

$$v_n(0) = 1 + M \quad \text{and } 0 \leq v_n(x) \leq 1 + M.$$

On the other hand (3.8) and a direct computation show that v_n satisfies

$$\begin{aligned} -\Delta_p v_n(x) &= \lambda_n^{q(p-1)+p} a(y_n)(u_n^+(y_n))^{\alpha} + \lambda_n^{q(p-1)+p} t_n \phi(y_n) \\ &\quad + \lambda_n^{q(p-1)+p} [g(y_n, u_n(y_n)) + h(y_n)] \\ &= \lambda_n^{q(p-1)+p-q\alpha} a(y_n)[v_n(x) - M + \lambda_n^q u_n^-(y_n)]^{\alpha} + \lambda_n^{q(p-1)+p} t_n \phi(y_n) \\ &\quad + \lambda_n^{q(p-1)+p} [g(y_n, u_n(y_n)) + h(y_n)], \end{aligned}$$

where $y_n = \lambda_n x + x_n$ for $x \in \Omega_n = \frac{\Omega - x_n}{\lambda_n}$ and $n \geq n_0$.

Now, note that by definition of q we have $q(p-1) + p - q\alpha = 0$. Still, by (3.7),

$$\lambda_n^{q(p-1)+p} t_n = \|u_n\|^{-\alpha} t_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, since $\phi \in L^\infty(\Omega)$, we have that $\lambda_n^{q(p-1)+p} t_n \phi \rightarrow 0$ as $n \rightarrow \infty$. Moreover,

$$\lim_{n \rightarrow \infty} \lambda_n^{q(p-1)+p} [g(y_n, u_n(y_n)) + h(y_n)] = \lim_{n \rightarrow \infty} \frac{g(\lambda_n x + x_n, u_n(\lambda_n x + x_n))}{\|u_n\|^\alpha},$$

since $h \in L^\infty(\Omega)$. We know that $\{u_n(\lambda_n x + x_n)\}$ is bounded from below by Theorem 3.4. If $\{u_n(\lambda_n x + x_n)\}$ is bounded from above, for all $x \in \Omega_n$ and $n \in \mathbb{N}$, then since g is continuous, we have that

$$\lim_{n \rightarrow \infty} \frac{g(\lambda_n x + x_n, u_n(\lambda_n x + x_n))}{\|u_n\|^\alpha} = 0.$$

If $\{u_n(\lambda_n x + x_n)\}$ is unbounded from above, then there exists $\{\tilde{x}_n\} \subset \Omega_n$ such that $u_n(\lambda_n \tilde{x}_n + x_n) \rightarrow \infty$ as $n \rightarrow \infty$. Since $\alpha > 0$, by definition of $\{x_n\}$ we have that $(u_n(\lambda_n \tilde{x}_n + x_n))^\alpha \leq (u_n(x_n))^\alpha$ and then by (3.6),

$$\lim_{n \rightarrow \infty} \frac{|g(\lambda_n \tilde{x}_n + x_n, u_n(\lambda_n \tilde{x}_n + x_n))|}{\|u_n\|^\alpha} \leq \lim_{n \rightarrow \infty} \frac{|g(\lambda_n \tilde{x}_n + x_n, u_n(\lambda_n \tilde{x}_n + x_n))|}{(u_n(\lambda_n \tilde{x}_n + x_n))^\alpha} = 0.$$

Hence, since $\overline{\Omega}$ is compact, we have that, for a subsequence, $x_n \rightarrow x_0 \in \overline{\Omega}$. By regularity theorems in [21, 22] for p -Laplacian operators, one can obtain

estimates on $\{v_n\}$ ensuring that, for a subsequence, $v_n \rightarrow v$ locally uniformly, with $v \in C^1(\mathcal{G})$ and v satisfies

$$-\Delta_p v = a(x_0)(v - M)^\alpha \quad \text{in } \mathcal{G},$$

where $\mathcal{G} = \mathbb{R}^N$, if $x_0 \in \Omega$ or $\mathcal{G} = \mathbb{R}_+^N$, if $x_0 \in \partial\Omega$.

Since $v_n(x) - M + \lambda_n^q u_n^-(y_n) = \lambda_n^q u_n^+(y_n) \geq 0$, for all $x \in \Omega_n, n \geq n_0$ and $v_n(\cdot) - M + \lambda_n^q u_n^-(\lambda_n \cdot + x_n) \rightarrow v - M$ locally uniformly, we have that $v - M \geq 0$, for all $x \in \mathcal{G}$. Thus defining $w = v - M$, we have that $w \in C^1(\mathcal{G})$, $w(0) = 1$ and $w(x) \geq 0$, for all $x \in \mathcal{G}$. Moreover, w satisfies

$$-\Delta_p w = a(x_0)w^\alpha \quad \text{in } \mathcal{G}.$$

If $\mathcal{G} = \mathbb{R}^N$, by Theorem 3.1 we have that $w \equiv 0$, which is a contradiction with the fact that $w(0) = 1$. And if $\mathcal{G} = \mathbb{R}_+^N$, in the same way, we obtain a contradiction with the Theorem 3.2. Thus we conclude the proof. $\square$

Remark 3.7. Since $t\phi = t^+\phi - t^-\phi$, we can consider $\tilde{h} = h - t^-\phi$ and suppose $t > 0$, thus repeating the proof of Theorem 3.6 replacing t by $\max\{t, 1\}$, we obtain: for each $C_0 \in \mathbb{R}_+$, there exists a constant C_2 such that for all $t \geq -C_0$ and for all solution of (P_t) with this t we have $\|u\|^\alpha \leq C_2 \max\{t, 1\}$.

From Theorems 3.5 and 3.6 we get the following results:

Theorem 3.8. *Given $t_0 \in \mathbb{R}$, there exist $\tilde{R}, \bar{R} > 0$ such that for all $t \geq t_0$, if u is a solution of (P_t) then $\|u\| \leq \tilde{R}$ and $t \leq \bar{R}$.*

Proof. Let $t_0 \in \mathbb{R}$ and define $C_0 = t_0^- \geq 0$. It follows from Theorem 3.5 that for all $t \geq t_0 \geq -t_0^- = -C_0$, if u is a solution of (P_t) then $t \leq C_1(1 + \|u\|^{p-1})$. By Remark 3.7 we have that

$$t \leq C_1(1 + \|u\|^{p-1}) \leq C_1(1 + (C_2 \max\{1, t\})^{\frac{p-1}{\alpha}}).$$

If $t \geq 1$, then $t \leq C_1(1 + (C_2 t)^{\frac{p-1}{\alpha}})$. Since $\frac{p-1}{\alpha} < 1$ we infer that t is bounded, that is, there exists $\bar{R} \geq 1$ such that $t \leq \bar{R}$. By Remark 3.7 we obtain that $\|u\|^\alpha \leq C_2 \max\{1, t\} \leq \bar{R}C_2$, defining $\tilde{R} = (\bar{R}C_2)^{\frac{1}{\alpha}}$ we obtain the result. $\square$

Theorem 3.9. *Given $t_0 \in \mathbb{R}$, there exist $R, \bar{R} > 0$ such that for all $t \geq t_0$, if u is a solution of (P_t) then $\|u\|_{C^1(\bar{\Omega})} \leq R$ and $t \leq \bar{R}$.*

Proof. Given $t_0 \in \mathbb{R}$, define $C_0 = t_0^- \geq 0$. By Theorem 3.8 we have that if u is a solution of (P_t) then $\|u\| < \tilde{R}$ and $t < \bar{R}$. Now, since u is a solution of (P_t) we have

$$-\Delta_p u = f(x, u) + t\phi + h.$$

Note that $d(x, u) = f(x, u) + t\phi + h$ is a Caratheodory function that satisfies $|d(x, u)| \leq |f(x, u)| + |t||\phi| + \|h\|$. Since $u \in [-M, \tilde{R}]$ ($-M$ given by Theorem 3.4), $|f(x, u)| \leq R_1$ for $(x, u) \in \bar{\Omega} \times [-M, \tilde{R}]$ and then

$$|d(x, u)| \leq R_1 + \bar{R}\|\phi\| + \|h\| = R_2.$$

Thus by Lemma 2.2 in [2] $u \in C^{1,\alpha}(\bar{\Omega})$, for $0 \leq \alpha < 1$ and $\|u\|_{C^{1,\alpha}(\bar{\Omega})} \leq R$. $\square$

4. Second solution

As we said previously, employing Leray–Schauder degree Theory, we complete the proof Theorem 1.1 by finding another solution of (P_t) . Now we are ready to prove it.

Let us denote

$$\mathcal{S} = \{t \in \mathbb{R}; (P_t) \text{ has at least one supersolution}\}.$$

From Proposition 2.3, we have that there exists $\underline{t} \in \mathbb{R}$ such that $(P_{\underline{t}})$ has a supersolution and for all $t \leq \underline{t}$, (P_t) has a supersolution, thus $\mathcal{S}$ is not empty and if $t \in \mathcal{S}$ then $(-\infty, t] \subset \mathcal{S}$.

Define

$$\mathcal{S}_1 = \{t \in \mathbb{R}; (P_t) \text{ has at least one solution}\}.$$

Obviously $\mathcal{S}_1 \subset \mathcal{S}$. Moreover, if $t \in \mathcal{S}$, then (P_t) has one supersolution $\bar{u}$. It follows from Proposition 2.1 that there exists subsolution $\underline{u}$ such that $\underline{u} \ll \bar{u}$. Thus by Theorem 2.4 there exists u solution of (P_t) with $\underline{u} \leq u \leq \bar{u}$ and therefore $t \in \mathcal{S}_1$. Thus $\mathcal{S}_1 = \mathcal{S}$.

By Theorem 3.8 we get that (P_t) does not have solution for all $t \geq \bar{R}$, thus $\mathcal{S}$ is bounded from above and we can define

$$t^* = \sup_{\mathcal{S}} t.$$

It follows from definition of supremum that for all $t \geq t^*$, (P_t) does not have solution and therefore we prove the item (iii) of Theorem 1.1.

Now we will show (ii).

From definition of supremum there exists $\{t_n\} \subset \mathcal{S}$ such that $t_n \rightarrow t^*$. Without loss of generality we can assume that t_n is increasing. Since $t_n \in \mathcal{S}$, there exists u_n solution of (P_{t_n}) , that is, for all $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi = \int_{\Omega} (f(x, u_n) + t_n \phi + h) \varphi. \quad (4.1)$$

Considering $t_0 = t_1$ (the first element of the above sequence), we have by Theorem 3.4 and Theorem 3.8 that $u_n \in [-M, R]$, thus $|f(x, u_n)| \leq K$. Still by Theorem 3.8 it follows that

$$|f(x, u_n) + t_n \phi + h| \leq K + \bar{R} \|\phi\| + \|h\| \leq C$$

thus by Lemma 2.2 in [2] it follows that $\{u_n\}$ is bounded in $C^{1,\alpha}(\bar{\Omega})$ and therefore it has a convergent subsequence $u_{n_k} \rightarrow u^*$ in $C^{1,\beta}(\bar{\Omega})$, $0 \leq \beta < \alpha$. Thus, take a limit as $n_k \rightarrow \infty$ in (4.1) we get

$$\int_{\Omega} |\nabla u^*|^{p-2} \nabla u^* \nabla \varphi = \int_{\Omega} (f(x, u^*) + t^* \phi + h) \varphi,$$

that is, u^* is solution of (P_{t^*}) . Hence, $t^* \in \mathcal{S}$ and $\mathcal{S} = (-\infty, t^*]$, which concludes the item ii).

We will show now the item i). Let u^* be solution of (P_{t^*}) and define

$$\mathcal{T} = \{t \in \mathbb{R}; (P_t) \text{ has at least one supersolution } \bar{u} \ll u^*\}.$$

It follows from Proposition 2.3 that $\mathcal{T}$ is a nonempty interval which is unbounded from below. Consider $t_0 \in \mathcal{T}$. We will show that (P_{t_0}) has at least two solutions. Since $t_0 \in \mathcal{T}$, there exists $\bar{u}$ supersolution of (P_{t_0}) such that $\bar{u} \ll u^*$. Proposition 2.1 and Theorem 2.4 provide us a solution u_0 of (P_{t_0}) . Taking a smaller subsolution, if necessary, u_0 verifies $\underline{u} \ll u_0 \leq \bar{u} \ll u^*$.

Define

$$\mathcal{D} = \{u \in C_0^1(\bar{\Omega}); \quad \underline{u} < u < u^*, \quad \frac{\partial \underline{u}}{\partial \eta} < \frac{\partial u}{\partial \eta} < \frac{\partial u^*}{\partial \eta}, \quad \|u\|_{C^1(\bar{\Omega})} < R_0\},$$

where R_0 is defined later. Define also

$$\tilde{f}(x, \xi) = \begin{cases} f(x, \underline{u}) + \sigma|\underline{u}|^{p-2}\underline{u}, & \xi \leq \underline{u} \\ f(x, \xi) + \sigma|\xi|^{p-2}\xi, & \underline{u} \leq \xi \leq u^* \\ f(x, u^*) + \sigma|u^*|^{p-2}u^*, & \xi \geq u^* \end{cases}$$

and $\tilde{Q}_t : C_0^1(\bar{\Omega}) \rightarrow C_0^1(\bar{\Omega})$ by

$$\tilde{Q}_t(v) = w \quad \text{iff} \quad -\Delta_p w + \sigma|w|^{p-2}w = \tilde{f}(x, v) + t\phi + h.$$

From Theorem 2.4 we have that $\tilde{Q}_t$ is compact. Since $\tilde{f}$ is bounded we have that the solutions of equation above are uniformly bounded in $C^{1,\alpha}(\bar{\Omega})$ and then we can define $R_0 > \sup\{\|\tilde{Q}_{t_0}(v)\|_{C_0^1(\bar{\Omega})}; v \in C_0^1(\bar{\Omega})\}$.

We will show that $\tilde{Q}_{t_0}(\bar{\mathcal{D}}) \subset \bar{\mathcal{D}}$.

In fact, let $v \in \bar{\mathcal{D}}$ and consider $w = \tilde{Q}_{t_0}(v)$. Since $\tilde{f}$ is increasing and $t_0 \leq t^*$ we have that

$$\begin{aligned} -\Delta_p w + \sigma|w|^{p-2}w &= \tilde{f}(x, v) + t_0\phi + h \leq \tilde{f}(x, u^*) + t_0\phi + h \\ &\leq -\Delta_p u^* + \sigma|u^*|^{p-2}u^*. \end{aligned}$$

By Comparison Principle, $w \leq u^*$. We will show that $\frac{\partial w}{\partial \eta} \leq \frac{\partial u^*}{\partial \eta}$. Suppose by contradiction that $\frac{\partial w}{\partial \eta}(x) > \frac{\partial u^*}{\partial \eta}(x)$ for some $x \in \partial\Omega$, since $w(x) = u^*(x) = 0$ and η is an inward normal vector, we have

$$\frac{\partial u^*}{\partial \eta}(x) < \frac{\partial w}{\partial \eta}(x) = \lim_{\lambda \rightarrow 0} \frac{w(x + \lambda\eta) - w(x)}{\lambda} \leq \lim_{\lambda \rightarrow 0} \frac{u^*(x + \lambda\eta) - u^*(x)}{\lambda} = \frac{\partial u^*}{\partial \eta}(x),$$

which is a contradiction. Hence $\frac{\partial w}{\partial \eta} \leq \frac{\partial u^*}{\partial \eta}$. Similarly we obtain $\underline{u} \leq w$ and $\frac{\partial \underline{u}}{\partial \eta} \leq \frac{\partial w}{\partial \eta}$. By definition of R_0 it follows that $w \in \bar{\mathcal{D}}$.

Moreover if Q_t is given by Theorem 2.4, we have that $\tilde{Q}_t = Q_t$ in $\bar{\mathcal{D}}$, u is a solution of (P_t) if and only if u is a fixed point of Q_t and $u_0 \in \mathcal{D}$. If the boundary of $\mathcal{D}$ contains a solution of (P_{t_0}) then the proof is completed. Suppose that $\partial\mathcal{D}$ does not contain any solution of (P_{t_0}) .

Let $\psi \in \mathcal{D}$ and $H_\lambda = H : [0, 1] \times \bar{\mathcal{D}} \rightarrow C_0^1(\bar{\Omega})$ defined by

$$H(\lambda, u) = \lambda Q_{t_0}(u) + (1 - \lambda)\psi.$$

Note that $\mathcal{D}$ is an open, bounded and convex set thus since $Q_{t_0}(\bar{\mathcal{D}}) \subset \bar{\mathcal{D}}$, we have that $H(\lambda, w) \in \mathcal{D}$, for all $(\lambda, w) \in [0, 1] \times \bar{\mathcal{D}}$. Therefore we conclude that $0 \notin (I - H(\lambda, \cdot))(\partial\mathcal{D})$, for all $\lambda \in [0, 1]$, because if this occurred, we would have $H(\lambda_0, v) = v$, for some $v \in \partial\mathcal{D}$ and for some $\lambda_0 \in [0, 1]$. Now, $H(\lambda, v) \in \mathcal{D}$, for all $\lambda \in [0, 1]$, and for $\lambda = 1$ we would have that $H(1, v) = v$, for $v \in \partial\mathcal{D}$, which

is a contradiction since $\partial\mathcal{D}$ does not contain any solution of (P_{t_0}) . Therefore $\deg(I - H_\lambda, \mathcal{D}, 0)$ is well defined.

By the invariance homotopy we have

$$\deg(I - \psi, \mathcal{D}, 0) = \deg(I - H_0, \mathcal{D}, 0) = \deg(I - H_1, \mathcal{D}, 0) = \deg(I - Q_{t_0}, \mathcal{D}, 0).$$

Now, take $y(\lambda) = \lambda\psi$ and $\bar{H}_\lambda = \bar{H} : [0, 1] \times \bar{\mathcal{D}} \rightarrow C_0^1(\bar{\Omega})$ defined by

$$\bar{H}(\lambda, u) = (1 - \lambda)\psi,$$

again by the invariance homotopy we have

$$\begin{aligned} \deg(I - \psi, \mathcal{D}, 0) &= \deg(I - \bar{H}_0, \mathcal{D}, y(0)) = \deg(I - \bar{H}_1, \mathcal{D}, y(1)) \\ &= \deg(I, \mathcal{D}, \psi) = 1. \end{aligned}$$

Therefore it follows that

$$\deg(I - Q_{t_0}, \mathcal{D}, 0) = 1.$$

We will prove that for $r \in \mathbb{R}$ large enough, $\deg(I - Q_{t_0}, B_r(0), 0) = 0$. For t_0 , let $\bar{R}, R > 0$ be given by Theorem 3.9. Consider $r > R$, $\hat{t} > \bar{R}$ and define

$$\tilde{H}_t = \tilde{H} : [t_0, \hat{t}] \times B_r(0) \rightarrow C_0^1(\bar{\Omega}) \quad \text{by } \tilde{H}(t, u) = Q_t(u).$$

Note that $\tilde{H}$ is compact and $0 \notin (I - \tilde{H}(t, \cdot))(\partial B_r(0))$, for all $t \in [t_0, \hat{t}]$ (because on the contrary case there exists $v \in \partial B_r(0)$ such that $Q_t(v) = \tilde{H}(t, v) = v$, for some $t \in [t_0, \hat{t}]$, and then v is a solution of (P_t) with $\|v\| = r > R$, which is a contradiction since for all $t \geq t_0$, the solutions of (P_t) are bounded from above by R). By the invariance homotopy we have

$$\deg(I - \tilde{H}_{t_0}, B_r(0), 0) = \deg(I - \tilde{H}_{\hat{t}}, B_r(0), 0) = 0,$$

since from Theorem 3.8, for $t > \bar{R}$ (P_t) has no solution.

By the excision property of the degree,

$$0 = \deg(I - Q_{t_0}, B_r(0), 0) = \deg(I - Q_{t_0}, \mathcal{D}, 0) + \deg(I - Q_{t_0}, B_r(0) \setminus \mathcal{D}, 0),$$

and hence $\deg(I - Q_{t_0}, B_r(0) \setminus \mathcal{D}, 0) \neq 0$, that is, there exists $v \in B_r(0) \setminus \mathcal{D}$ solution of (P_{t_0}) and thus (P_{t_0}) has two solutions.

Thus we have that for $t \in \mathcal{T}$, there exists at least two solutions for (P_t) . Consider

$$t_* = \sup_{\mathcal{T}} t.$$

Thus (P_t) has at least two solutions for all $t < t_*$, concluding the proof of the theorem.

Remark 4.1. We would like to prove that $t^* = t_*$, but we do not know how to do it. Indeed, we note that $t_* \leq t^*$, thus we have to prove that $t^* \leq t_*$, that is, $t^* \in \mathcal{T}$. Now, since u^* is solution of (P_{t^*}) , in particular, u^* is a supersolution of (P_{t^*}) and $u^* \leq u^*$. By Proposition 2.1 and Theorem 2.4 there is a solution u of (P_{t^*}) that satisfies $\underline{u} \ll u \leq u^*$. But we do not get to assure that $u \ll u^*$. Thus we can not assure that $u \in \mathcal{D}$, only $u \in \bar{\mathcal{D}}$. Hence the reasoning used in the sequence and the Leray–Schauder degree can not be applied.

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