Regularity of weak solutions to the Navier–Stokes equations in exterior domains

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Abstract. Let u be a weak solution of the Navier–Stokes equations in an exterior domain $\Omega \subset \mathbb{R}^3$ and a time interval $[0, T[$, $0 < T \leq \infty$, with initial value u_0 , external force $f = \operatorname{div} F$, and satisfying the strong energy inequality. It is well known that global regularity for u is an unsolved problem unless we state additional conditions on the data u_0 and f or on the solution u itself such as Serrin’s condition $\|u\|_{L^s(0,T;L^q(\Omega))} < \infty$ with $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. In this paper, we generalize results on local in time regularity for bounded domains, see Farwig et al. (Indiana Univ Math J 56:2111–2131, 2007; J Math Fluid Mech 11:1–14, 2008; Banach Center Publ 81:175–184, 2008), to exterior domains. If e.g. u fulfills Serrin’s condition in a left-side neighborhood of t or if the norm $\|u\|_{L^{s'}(t-\delta,t;L^q(\Omega))}$ converges to 0 sufficiently fast as $\delta \rightarrow 0+$, where $\frac{2}{s'} + \frac{3}{q} > 1$, then u is regular at t . The same conclusion holds when the kinetic energy $\frac{1}{2}\|u(t)\|_2^2$ is locally Hölder continuous with exponent $\alpha > \frac{1}{2}$.

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1. Introduction and main results

In this paper, $\Omega \subset \mathbb{R}^3$ is an exterior domain, i.e. an open, connected subset having a nonempty, compact complement in $\mathbb{R}^3$, with smooth boundary $\partial\Omega \in C^{2,1}$, and $[0, T[$, $0 < T \leq \infty$, denotes the time interval. In $[0, T[\times \Omega$ we consider the instationary Navier–Stokes equations

$$\begin{aligned} u_t - \nu \Delta u + u \cdot \nabla u + \nabla p &= f && \text{in }]0, T[\times \Omega \\ \operatorname{div} u &= 0 && \text{in }]0, T[\times \Omega \\ u &= 0 && \text{on }]0, T[\times \partial\Omega \\ u &= u_0 && \text{at } t = 0 \end{aligned} \tag{1.1}$$

with constant viscosity $\nu > 0$ (fixed throughout this paper), external force $f = \operatorname{div} F = (\sum_{i=1}^3 \partial_i F_{i,j})_{j=1}^3$ and initial value u_0 . First we recall the definition of weak and strong solutions. The space of test functions is defined to be

$$C_0^\infty([0, T[; C_{0,\sigma}^\infty(\Omega)) := \{u \mid_{[0, T[\times \Omega}; u \in C_0^\infty(\cdot - 1, T[\times \Omega); \operatorname{div} u = 0\}.$$

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and let $u_0 \in L_\sigma^2(\Omega)$, $f = \operatorname{div} F$ with $F \in L_{\operatorname{loc}}^1([0, T[; L^2(\Omega))$ where $0 < T \leq \infty$. Then a vector field $u \in LH_T$, where LH_T denotes the *Leray–Hopf class*

$$LH_T := L_{\operatorname{loc}}^\infty([0, T[; L_\sigma^2(\Omega)) \cap L_{\operatorname{loc}}^2([0, T[; W_{0,\sigma}^{1,2}(\Omega)), \quad (1.2)$$

is called *weak solution* (in the sense of *Leray–Hopf*) of the instationary Navier–Stokes system (1.1) with data f, u_0 , if the following identity is satisfied for all test functions $w \in C_0^\infty([0, T[; C_{0,\sigma}^\infty(\Omega))$:

$$\begin{aligned} & \int_0^T (-\langle u, w_t \rangle_\Omega + \nu \langle \nabla u, \nabla w \rangle_\Omega + \langle u \cdot \nabla u, w \rangle_\Omega) dt \\ &= \langle u_0, w(0) \rangle_\Omega - \int_0^T \langle F, \nabla w \rangle_\Omega dt. \end{aligned} \quad (1.3)$$

As a consequence of (1.2), (1.3), $u : [0, T[\rightarrow L_\sigma^2(\Omega)$ is—after a possible redefinition on a set of Lebesgue measure 0—weakly continuous and the initial value u_0 is attained in the sense

$$\langle u(t), \phi \rangle \rightarrow \langle u_0, \phi \rangle, \quad t \rightarrow 0+ \quad \forall \phi \in L_\sigma^2(\Omega).$$

Moreover, there exists a distribution p , called an associated pressure, such that the equality

$$u_t - \nu \Delta u + u \cdot \nabla u + \nabla p = f$$

holds in the sense of distributions on $]0, T[\times \Omega$, see [14, V.1.7].

A weak solution of (1.1) is called a *strong solution* if there exist exponents s, q with $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$ such that additionally *Serrin's condition*

$$u \in L^s(0, T; L^q(\Omega)) \quad (1.4)$$

is satisfied. By Hölder's inequality, such a strong solution u satisfies $u \otimes u \in L_{\operatorname{loc}}^2([0, T[; L^2(\Omega))$. Moreover, by Serrin's Uniqueness Theorem [14, V. Theorems 1.5.1, 1.4.1], a weak solution with (1.4) is unique within the class of weak solutions satisfying the energy inequality, i.e., fulfilling (1.5) below with $s = 0$. Finally, $u : [0, T[\rightarrow L_\sigma^2(\Omega)$ is strongly continuous and satisfies the energy identity (1.14) below.

For sufficiently smooth Ω, f, u_0 a strong solution u has the regularity property

$$u \in C^\infty(\bar{\Omega}, T[\times \bar{\Omega}), \quad p \in C^\infty(\bar{\Omega}, T[\times \bar{\Omega}),$$

see [14, Theorem V.1.8.2], and therefore a strong solution is also called a *regular solution*. We call a weak solution u of (1.1) *regular at t* , if there exists a $\delta = \delta(t) > 0$ with $u \in L^s(t - \delta, t + \delta; L^q(\Omega))$ where s, q satisfy $\frac{2}{s} + \frac{3}{q} = 1$.

Now let $\Omega \subset \mathbb{R}^3$ be an exterior domain with smooth boundary. We know, see [13], that there exists at least one weak solution u of (1.1) satisfying the *strong energy inequality*

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_s^t \|\nabla u\|_2^2 d\tau \leq \frac{1}{2} \|u(s)\|_2^2 - \int_s^t \langle F, \nabla u \rangle_\Omega d\tau \quad (1.5)$$

for almost all $s \in [0, T[$ and all $t \in [s, T[$.

Our first main theorem states that if u fulfills the Serrin condition in a left-side neighborhood of t then u is regular at t . Furthermore, it gives conditions depending on $\|u\|_{L^{s'}(0, T; L^q(\Omega))}$ with $\frac{2}{s'} + \frac{3}{q} > 1$ to imply regularity of u at t ; note that in this case, the integrability of u is weaker than in Serrin's condition.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$ and let $1 \leq s' \leq s$. Assume that $f = \operatorname{div} F$ with $F \in L^s(0, T; L^r(\Omega)) \cap L^4(0, T; L^2(\Omega))$, $u_0 \in L_\sigma^2(\Omega)$, $0 < T < \infty$, and let $u \in LH_T$ be a weak solution of the Navier–Stokes equations (1.1) satisfying the strong energy inequality (1.5). Then we have:*

1. Left-side $L^{s'}(L^q)$ -condition. For $t \in]0, T[$ the condition

$$\liminf_{\delta \rightarrow 0+} \frac{1}{\delta^{1-\frac{s'}{s}}} \int_{t-\delta}^t \|u(\tau)\|_q^{s'} d\tau = 0 \quad (1.6)$$

is necessary and sufficient for regularity of u at t .

2. Global $L^{s'}(L^q)$ -condition. There exists a constant $\epsilon_* = \epsilon_*(q, s', \Omega) > 0$, independent of f, u_0, T with the following property: If $u_0 \in L_\sigma^2(\Omega) \cap L_\sigma^q(\Omega)$, $u \in L^{s'}(0, T; L_\sigma^q(\Omega))$ and the conditions

$$\int_0^T \|F(\tau)\|_r^s d\tau \leq \epsilon_* \nu^{2s-1} \quad \text{and} \quad \int_0^T \|u(\tau)\|_q^{s'} d\tau \leq \epsilon_* \frac{\nu^{s-1}}{\|u_0\|_q^{s-s'}} \quad (1.7)$$

are satisfied, then $u \in L^s(0, T; L^q(\Omega))$.

Remark. (1) We note that in the case $s' = s$ condition (1.6) can be written in the simpler form

$$u \in L^s(t - \delta, t; L_\sigma^q(\Omega)) \quad \text{for some } 0 < \delta = \delta(t) < t.$$

- (2) The necessity of (1.6) for a solution u to be regular at t is trivial by Hölder's inequality.

The following theorem states that Hölder continuity of the kinetic energy with exponent $\alpha \in]\frac{1}{2}, 1[$ implies regularity of u at t . In the case $\alpha = \frac{1}{2}$ we need a smallness condition for the corresponding Hölder term under which we can prove regularity of u at t .

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega \in C^{2,1}$, let $0 < T < \infty$ and let u be a weak solution of the Navier–Stokes equations (1.1) satisfying the strong energy inequality (1.5) with initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f = \operatorname{div} F$ which will be specified below. Furthermore, we assume that the kinetic energy $E(t) := \frac{1}{2} \|u(t)\|_2^2$ is a continuous function of $t \in [0, T[$. Then we have:*

1. *Let $\alpha \in]\frac{1}{2}, 1[$, $2 < s' < 4\alpha$, $3 < q < 6$, $\frac{2}{s'} + \frac{3}{q} = \frac{3}{2}$, $\frac{2}{s} + \frac{3}{q} = 1$, let $f \in L^{\frac{s}{s'}}(0, T; L^2(\Omega))$ and $F \in L^4(0, T; L^2(\Omega)) \cap L^s(0, T; L^r(\Omega))$, where $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, and let u satisfy at $t \in]0, T[$ the left-side condition*

$$\sup_{t-\mu < t' < t} \frac{|E(t) - E(t')|}{|t - t'|^\alpha} < \infty \quad (1.8)$$

with a $\mu > 0$. Then u is regular at t .

2. *(The case $\alpha = \frac{1}{2}$) Let $f \in L^2(0, T; L^2(\Omega))$, $F \in L^4(0, T; L^2(\Omega))$. Then there exists a constant $\gamma_* = \gamma_*(\Omega)$ such that the left-side condition*

$$\sup_{t-\mu < t' < t} \frac{|E(t) - E(t')|}{|t - t'|^{\frac{1}{2}}} \leq \gamma_* \nu^{\frac{5}{2}} \quad (1.9)$$

with a $\mu > 0$ implies regularity of u at t .

Remark. (1) The proof of Theorem 1.3, in particular see (4.8), will yield the following regularity criteria using the dissipation energy: If

$$\alpha \in]\frac{1}{2}, 1[\quad \text{and} \quad \liminf_{\delta \rightarrow 0+} \frac{1}{\delta^\alpha} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 d\tau < \infty, \quad (1.10)$$

or

$$\liminf_{\delta \rightarrow 0+} \frac{1}{\delta^{\frac{1}{2}}} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 d\tau \leq \gamma_* \nu^{\frac{3}{2}} \quad (1.11)$$

then u is regular at t .

- (2) In the case $\alpha = \frac{1}{2}$ a smallness condition as in (1.9) and (1.11) is necessary. Indeed, if $f = 0$ and $]0, t[$ is a maximal regularity interval of u , then

$$\|\nabla u(\tau)\|_2 \geq \frac{c_0}{(t - \tau)^{\frac{1}{4}}}, \quad 0 < \tau < t,$$

where $c_0 = c_0(\Omega) > 0$, see [8]. Hence

$$\liminf_{\delta \rightarrow 0+} \frac{1}{\delta^{\frac{1}{2}}} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 d\tau \geq 2c_0^2 > 0,$$

and due to the strong energy inequality (1.5) it holds for all $\mu > 0$

$$\sup_{t-\mu < t' < t} \frac{|E(t) - E(t')|}{|t - t'|^{\frac{1}{2}}} \geq 2\nu c_0^2 > 0.$$

The proofs of the regularity criteria formulated in this paper are based on a local or global identification of a weak solution with a very weak solution, a concept described in Definition 2.3 below. The following key result, Theorem 1.4, gives conditions under which a given very weak solution is also a weak solution in the sense of Leray–Hopf and, therefore, yields under certain

smallness conditions on the data f and u_0 the existence of a unique strong solution of (1.1) on $[0, T[\times \Omega$.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$ and let $\frac{1}{3} + \frac{1}{q} = \frac{1}{q^*}$. Then there exists a constant $\epsilon_* = \epsilon_*(q, \Omega) > 0$ with the following property: Given $0 < T < \infty$ and data $u_0 \in L^2_\sigma(\Omega) \cap L^q_\sigma(\Omega)$ and $f = \operatorname{div} F$ with $F \in L^s(0, T; L^{q^*}(\Omega)) \cap L^4(0, T; L^2(\Omega))$ satisfying the following two conditions:*

$$\int_0^T \|F(\tau)\|_{q^*}^s d\tau \leq \epsilon_* \nu^{2s-1}, \quad (1.12)$$

$$\int_0^T \|e^{-\nu\tau A_q} u_0\|_q^s d\tau \leq \epsilon_* \nu^{s-1}. \quad (1.13)$$

In this case, there exists a unique weak solution $u \in LH_T$ of (1.1) satisfying the Serrin condition $u \in L^s(0, T; L^q(\Omega))$. After a possible redefinition on a set of Lebesgue measure 0, we get that $u : [0, T[\rightarrow L^2_\sigma(\Omega)$ is strongly continuous and it holds the energy identity

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_0^t \|\nabla u\|_2^2 d\tau = \frac{1}{2} \|u_0\|_2^2 - \int_0^t \langle F, \nabla u \rangle_\Omega d\tau \quad (1.14)$$

for all $t \in [0, T[$.

The proof of this crucial result is the content of Sect. 3 and differs from the case of bounded domains, see [4, 6], where the trivial inclusion $L^q(\Omega) \subset L^r(\Omega)$, $q > r$, yielding also better embedding properties of fractional powers of the Stokes operator A_q , was used several times. The main idea of the proof is to construct a very weak solution $v \in L^s(0, T; L^q_\sigma(\Omega))$ for the given data u_0, f and to identify u and v by Serrin's Uniqueness Theorem; for this reason, we have to show that v lies in the Leray–Hopf class LH_T .

After some preliminaries to be discussed in Sect. 2 we prove Theorem 1.4 in Sect. 3. Finally, Sect. 4 deals with the proofs of the main results Theorem 1.2 and 1.3.

2. Preliminaries

Given $1 \leq q \leq \infty, k \in \mathbb{N}$ we need the usual Lebesgue and Sobolev spaces, $L^q(\Omega), W^{k,q}(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{k,q}(\Omega)} = \|\cdot\|_{k,q}$, respectively. For two measurable functions f, g with the property $f \cdot g \in L^1(\Omega)$, where $f \cdot g$ means the usual scalar product of vector or matrix fields, we set

$$\langle f, g \rangle_\Omega := \int_\Omega f(x) \cdot g(x) dx.$$

Note that the same symbol $L^q(\Omega)$ etc. will be used for spaces of scalar-, vector or matrix-valued functions. Let $C^m(\Omega), m = 0, 1, \dots, \infty$ denote the usual space of functions for which all partial derivatives of order $|\alpha| \leq m$ exist and are continuous. As usual, $C^m_0(\Omega)$ is the set of all functions from $C^m(\Omega)$ with

compact support in Ω . Further we need the space of smooth solenoidal vector fields

$$C_{0,\sigma}^\infty(\Omega) := \{v \in C_0^\infty(\Omega)^3; \operatorname{div} v = 0\}$$

and define the spaces

$$L_\sigma^q(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}, \quad W_{0,\sigma}^{1,2}(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}}.$$

For $1 \leq q \leq \infty$ let $q' \in [1, \infty]$ denote its dual exponent. It is well known that $L_\sigma^q(\Omega)' = L_\sigma^{q'}(\Omega)$ using the standard pairing $\langle \cdot, \cdot \rangle_\Omega$. Moreover, let us write $[d, v]_\Omega$ for the application of a distribution $d \in C_0^\infty(\Omega)'$ on a test function $v \in C_0^\infty(\Omega)$.

Given a Banach space X and an interval $[0, T]$, $0 < T \leq \infty$, we denote by $L^p(0, T; X)$, $1 \leq p \leq \infty$, the space of all equivalence classes of strongly measurable functions $f : [0, T] \rightarrow X$ such that

$$\|f\|_p := \left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} < \infty$$

if $p < \infty$, and $\|f\|_\infty := \operatorname{ess\,sup}_{[0, T]} \|f(\cdot)\|_X$, if $p = \infty$. Moreover, we define the set of *locally integrable* L^p -functions on $[0, T[$ as

$$L_{\operatorname{loc}}^p([0, T[; X) := \{u : [0, T[\rightarrow X \text{ strongly measurable,} \\ u \in L^p(0, T'; X) \text{ for all } 0 < T' < T\}.$$

When $X = L^q(\Omega)$, $1 \leq q \leq \infty$, we denote the norm in $L^p(0, T; L^q(\Omega))$ by $\|\cdot\|_{q,p,\Omega;T}$. For $1 < p, q < \infty$ it holds

$$L^p(0, T; L^q(\Omega))' = L^{p'}(0, T; L^{q'}(\Omega))$$

and we define

$$\langle f, g \rangle_{\Omega, T} := \int_0^T \int_\Omega f(t, x) \cdot g(t, x) dx dt$$

for $f \in L^p(0, T; L^q(\Omega))$, $g \in L^{p'}(0, T; L^{q'}(\Omega))$.

Given an exterior domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^{2,1}$ and $1 < q < \infty$, there exists a bounded, linear projection $P_q : L^q(\Omega) \rightarrow L_\sigma^q(\Omega)$ with range $\mathcal{R}(P_q) = L_\sigma^q(\Omega)$ and nullspace $N(P_q) = \{\nabla p \in L^q(\Omega); p \in L_{\operatorname{loc}}^q(\overline{\Omega})\}$. The operator P_q is called *Helmholtz projection* and is *consistent* in the sense that

$$P_q f = P_r f \quad \forall f \in L^q(\Omega) \cap L^r(\Omega). \quad (2.1)$$

Furthermore, we get $P_q' = P_{q'}$ for the dual operator, i.e.,

$$\langle P_q f, g \rangle_\Omega = \langle f, P_{q'} g \rangle_\Omega \quad \forall f \in L^q(\Omega) \quad \forall g \in L^{q'}(\Omega). \quad (2.2)$$

For $1 < q < \infty$ we define the *Stokes operator* A_q on $L_\sigma^q(\Omega)$ by

$$\mathcal{D}(A_q) = L_\sigma^q(\Omega) \cap W_{0,\sigma}^{1,q}(\Omega) \cap W^{2,q}(\Omega), \quad (2.3)$$

$$A_q u := -P_q \Delta u, \quad u \in \mathcal{D}(A_q). \quad (2.4)$$

The Stokes operator is *consistent* in the sense that for $1 < q, r < \infty$ it holds

$$A_q u = A_r u \quad \forall u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_r). \quad (2.5)$$

In general, $\mathcal{D}(A_q)$ will be equipped with the graph norm $\|u\|_{\mathcal{D}(A_q)} := \|u\|_q + \|A_q u\|_q$ for $u \in \mathcal{D}(A_q)$ which makes $\mathcal{D}(A_q)$ to a Banach space since the Stokes operator is closed. For simplicity, we use the notation $A = A_2$.

For $\alpha \in [-1, 1]$ the fractional power $A_q^\alpha : \mathcal{D}(A_q^\alpha) \rightarrow L_\sigma^q(\Omega)$ with dense domain $\mathcal{D}(A_q^\alpha) \subseteq L_\sigma^q(\Omega)$ is a well defined, injective, closed operator such that

$$(A_q^\alpha)^{-1} = A_q^{-\alpha}, \quad \mathcal{R}(A_q^\alpha) = \mathcal{D}(A_q^{-\alpha}) \quad \text{and} \quad (A_q^\alpha)' = A_{q'}^\alpha.$$

It holds $\mathcal{D}(A_q^{1/2}) = W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$ for $1 < q < 3$, and the estimate

$$\|\nabla u\|_{q,\Omega} \leq c \|A_q^{1/2} u\|_{q,\Omega} \quad \text{for } 1 < q < 3, u \in \mathcal{D}(A_q^{1/2}), \quad (2.6)$$

with a constant $c = c(\Omega, q) > 0$. Moreover,

$$\|u\|_{\gamma,\Omega} \leq c \|A_q^\alpha u\|_{q,\Omega} \quad \text{where } 0 \leq \alpha \leq \frac{1}{2}, 1 < q < 3, 2\alpha + \frac{3}{\gamma} = \frac{3}{q}, \quad (2.7)$$

for all $u \in \mathcal{D}(A_q^\alpha)$ with a constant $c = c(\Omega, q, \gamma) > 0$. It is well known that $-A_q$ generates a uniformly bounded analytic semigroup $\{e^{-tA_q} : t \geq 0\}$ on $L_\sigma^q(\Omega)$ satisfying the decay estimate

$$\|A_q^\alpha e^{-tA_q}\|_q \leq ct^{-\alpha} \quad \forall t > 0, \quad (2.8)$$

where $\alpha \geq 0, 1 < q < \infty$ and $c = c(\Omega, q, \alpha) > 0$. We refer to [7, 9–11] for these properties of the Stokes operator in exterior domains.

Lemma 2.1. *Let $d \in C_0^\infty(\Omega)'$ be a distribution, well defined for all $v \in \mathcal{D}(A_{q'}^\alpha)$ where $1 < q < \infty, 0 < \alpha \leq 1$. We assume that there exists a constant $c \geq 0$, independent of $v \in \mathcal{D}(A_{q'}^\alpha)$, such that*

$$|[d, v]_\Omega| \leq c \|A_{q'}^\alpha v\|_{q',\Omega} \quad \forall v \in \mathcal{D}(A_{q'}^\alpha). \quad (2.9)$$

Then there exists a unique element $\tilde{d} \in L_\sigma^q(\Omega)$, to be denoted by $A_q^{-\alpha} P_q d$, with the properties

$$\langle A_q^{-\alpha} P_q d, A_{q'}^\alpha v \rangle_\Omega = [d, v]_\Omega \quad \forall v \in \mathcal{D}(A_{q'}^\alpha) \quad \text{and} \quad \|A_q^{-\alpha} P_q d\|_q \leq c \quad (2.10)$$

with the constant c from (2.9). In particular, if $F \in L^q(\Omega)$, and $\frac{3}{2} < q < \infty$, then $A_q^{-\frac{1}{2}} P_q \operatorname{div} F \in L_\sigma^q(\Omega)$ and

$$\|A_q^{-\frac{1}{2}} P_q \operatorname{div} F\|_q \leq c \|F\|_q. \quad (2.11)$$

Proof. We define for $w \in \mathcal{R}(A_{q'}^\alpha)$

$$[\tilde{d}, w]_\Omega := [d, v]_\Omega, \quad \text{where } w = A_{q'}^\alpha v, v \in \mathcal{D}(A_{q'}^\alpha).$$

By the density of $\mathcal{R}(A_{q'}^\alpha)$ in $L_\sigma^{q'}(\Omega)$, we extend $\tilde{d}$ to a functional defined on $L_\sigma^{q'}(\Omega)$. We use $L_\sigma^{q'}(\Omega)' = L_\sigma^q(\Omega)$ to obtain a unique element $A_q^{-\alpha} P_q d \in L_\sigma^q(\Omega)$ satisfying the identity in (2.10). For the proof of (2.11) we exploit (2.6) with q replaced by $q' \in]1, 3[$. $\square$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $1 < q, s < \infty$ and $0 < T < \infty$. Furthermore, let $f \in L^s(0, T; L_\sigma^q(\Omega))$ and $u_0 \in L_\sigma^q(\Omega)$ such that $\int_0^\infty \|A_q e^{-tA_q} u_0\|_{q,\Omega}^s dt < \infty$. Then the instationary Stokes system*

$$\begin{aligned} u_t + \nu A_q u &= f \quad \text{in } (0, T) \\ u(0) &= u_0 \end{aligned} \quad (2.12)$$

has a unique strong solution $u \in L^s(0, T; D(A_q))$ with $u_t \in L^s(0, T; L_\sigma^q(\Omega))$ and $u \in C([0, T]; L_\sigma^q(\Omega))$. Moreover, u satisfies the maximal regularity estimate

$$\begin{aligned} &\|u_t\|_{q,s,\Omega;T} + \|\nu A_q u\|_{q,s,\Omega;T} \\ &\leq c \left(\left(\int_0^T \|\nu A_q e^{-\nu t A_q} u_0\|_{q,\Omega}^s dt \right)^{\frac{1}{s}} + \|f\|_{q,s,\Omega;T} \right) \end{aligned} \quad (2.13)$$

with a constant $c = c(\Omega, q, s)$ independent of T and ν . It holds the representation

$$u(t) = e^{-\nu t A_q} u_0 + \int_0^t e^{-\nu(t-\tau) A_q} f(\tau) d\tau \quad (2.14)$$

for all $t \in [0, T[$. In the case $T = \infty$ we get a unique strong solution $u \in L_{\text{loc}}^s(0, \infty; D(A_q))$ of (2.12) satisfying $u_t \in L^s(0, \infty; L_\sigma^q(\Omega))$ and $u \in C([0, \infty]; L_\sigma^q(\Omega))$ and it holds the estimate (2.13) and the representation (2.14) for all $t \in [0, \infty[$.

Proof. See [10, Theorem 4.2]. □

A major tool for the proof of Theorem 1.4 is the theory of very weak solutions. In this context we refer to [3] for exterior domains and to [4] for bounded domains. In the following definition let

$$\begin{aligned} C_0^1([0, T[; C_{0,\sigma}^2(\bar{\Omega})) &:= \{ w \mid_{[0,T] \times \bar{\Omega}} \text{ with } w \in C_0^{1,2}(-)1, T[\times \mathbb{R}^3; \\ &\quad \text{div } w = 0, w|_{\partial\Omega} = 0 \text{ for all } t \in [0, T[\} \end{aligned}$$

denote the space of test functions and let

$$\begin{aligned} \mathcal{J}^{q,s}(\Omega) &:= \{ u_0 \in C_0^\infty(\Omega)' ; \\ &\quad A_q^{-1} P_q u_0 \in L_\sigma^q(\Omega), \int_0^\infty \|A_q e^{-tA_q} (A_q^{-1} P_q u_0)\|_{q,\Omega}^s dt < \infty \} \end{aligned}$$

denote the space of initial values.

Definition 2.3. Let $\Omega \subset \mathbb{R}^3$ be an exterior domain, let $F \in L^s(0, T; L^r(\Omega))$ and $u_0 \in \mathcal{J}^{q,s}(\Omega)$ where $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. Then $u \in L^s(0, T; L_\sigma^q(\Omega))$ is called *very weak solution* of the instationary Navier–Stokes equations (1.1) if

$$\int_0^T \langle -u, w_t \rangle_\Omega - \nu \langle u, \Delta w \rangle_\Omega - \langle u \otimes u, \nabla w \rangle_\Omega dt = [u_0, w(0)]_\Omega - \int_0^T \langle F, \nabla w \rangle_\Omega dt \quad (2.15)$$

holds for all test functions $w \in C_0^1([0, T[; C_{0,\sigma}^2(\bar{\Omega}))$.

In the corresponding definition of very weak solutions to the linear, instationary Stokes system where the nonlinear term $u \cdot \nabla u$ is absent, we may omit in Definition 2.3 the restriction $\frac{2}{s} + \frac{3}{q} = 1$, and in (2.15) the term $\langle u \otimes u, \nabla w \rangle_{\Omega, T}$ is absent. A proof of the following Theorem can be found in [3, 12].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$ and let $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. Then there exists a constant $c = c(q, \Omega) > 0$ with the following property: For data $f = \operatorname{div} F$ with $F \in L^s(0, T; L^r(\Omega))$ and $u_0 \in \mathcal{J}^{q,s}(\Omega)$, satisfying the condition*

$$\left(\int_0^T \| \nu A_q e^{-\nu t A_q} (A_q^{-1} P_q u_0) \|_{q, \Omega}^s dt \right)^{\frac{1}{s}} + \| F \|_{r, s, \Omega; T} \leq c \nu^{1+\alpha} \quad (2.16)$$

with $\alpha := \frac{3}{2q} + \frac{1}{2} = 1 - \frac{1}{s}$, there exists a unique very weak solution $u \in L^s(0, T; L_\sigma^q(\Omega))$ of the instationary Navier–Stokes system (1.1). Moreover, u has the representation $u = E + \tilde{u}$, where $E \in L^s(0, T; L_\sigma^q(\Omega))$ is the unique very weak solution of the linear Stokes system with data f, u_0 and $\tilde{u}$ is the unique solution in $L^s(0, T; L_\sigma^q(\Omega))$ of the nonlinear fixed point equation

$$\tilde{u}(t) = - \int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \operatorname{div} ((\tilde{u}(\tau) + E(\tau)) \otimes (\tilde{u}(\tau) + E(\tau))) d\tau \quad (2.17)$$

for almost all $t \in [0, T[$.

Finally we recall the Hardy–Littlewood inequality [14, II Lemma 3.3.2]. Let $0 < \alpha < 1$, $1 < r < q < \infty$ with $\alpha + \frac{1}{q} = \frac{1}{r}$ and let $f \in L^r(\mathbb{R})$. Then the integral

$$u(t) := \int_{\mathbb{R}} |t - \tau|^{\alpha-1} f(\tau) d\tau$$

converges absolutely for almost all $t \in \mathbb{R}$ and it holds

$$\|u\|_{L^q(\mathbb{R})} \leq c \|f\|_{L^r(\mathbb{R})} \quad (2.18)$$

with a constant $c = c(\alpha, q) > 0$.

3. Proof of Theorem 1.4

Before proving Theorem 1.4 we discuss the nonlinear term arising in the nonlinear fixed point problem (2.17). We denote by $\operatorname{div}(u \otimes u)$ the functional defined for suitable vector fields w by

$$[\operatorname{div}(u \otimes u), w]_\Omega := - \langle u \otimes u, \nabla w \rangle_\Omega.$$

The following lemma is technical but essential for Lemma 3.2 below.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega \in C^{2,1}$, let $3 < q < \infty$, $r \in [\frac{q}{2}, q]$ and $\beta := \frac{3}{q} - \frac{3}{2r} + \frac{1}{2}$.*

1. *There exists a constant $c = c(\Omega, q, r) > 0$ such that for all $u \in L_\sigma^q(\Omega)$*

$$\|A_r^{-\beta} P_r \operatorname{div}(u \otimes u)\|_{r, \Omega} \leq c \|u\|_{q, \Omega}^2. \quad (3.1)$$

2. For $2 < s < \infty, 3 < q < \infty, 0 < T \leq \infty$ there exists a constant $c = c(\Omega, q, r) > 0$ such that for all $u \in L^s(0, T; L^q_\sigma(\Omega))$

$$\|A_r^{-\beta} P_r \operatorname{div}(u \otimes u)\|_{r, \frac{s}{2}, \Omega; T} \leq c \|u\|_{q, s, \Omega; T}^2. \quad (3.2)$$

Proof. The assumptions of the lemma imply

$$2 \left(\beta - \frac{1}{2} \right) + \frac{3}{\left(\frac{q}{2}\right)'} = \frac{3}{r'} \quad \text{with } 1 < r' < 3, \frac{1}{2} \leq \beta < 1. \quad (3.3)$$

Then we get for arbitrary $w \in \mathcal{D}(A_{r'}^\beta)$ by (2.6) using $1 < \left(\frac{q}{2}\right)' < 3$, (2.7) and (2.5) (applied to $A^{1/2}$ instead of A)

$$\begin{aligned} |[\operatorname{div}(u \otimes u), w]| &= | - \langle u \otimes u, \nabla w \rangle | \\ &\leq \|u \otimes u\|_{\frac{q}{2}} \|\nabla w\|_{\left(\frac{q}{2}\right)'} \\ &\leq c \|u\|_q^2 \|A_{(q/2)'}^{1/2} w\|_{\left(\frac{q}{2}\right)'} \\ &\leq c \|u\|_q^2 \|A_{r'}^{(\beta - \frac{1}{2})} (A_{(q/2)'}^{1/2} w)\|_{r'} \\ &\leq c \|u\|_q^2 \|A_{r'}^\beta w\|_{r'}. \end{aligned}$$

It is possible to choose the constant $c > 0$ in the above estimate depending only on Ω, q and r . For the second assertion we use (3.1), which holds for almost all $t \in [0, T[$, and integrate over $[0, T]$. $\square$

Lemma 3.2. Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $0 < T \leq \infty$, $2 < s < \infty, \frac{2}{s} + \frac{3}{q} = 1$ and let $u \in L^s(0, T; L^q(\Omega))$. We define for $r \in [\frac{q}{2}, q]$ and $\beta := \frac{3}{q} - \frac{3}{2r} + \frac{1}{2}$ the term $\Lambda^r(u)$ by

$$\Lambda^r u(t) := - \int_0^t A_r^\beta e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau)) d\tau. \quad (3.4)$$

Then the following statements are satisfied.

1. For almost all $t \in [0, T[$ we get

$$\int_0^t \|A_r^\beta e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau))\|_r d\tau < \infty \quad (3.5)$$

and

$$\begin{aligned} &-A_r^\beta \int_0^t e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau)) d\tau \\ &= - \int_0^t A_r^\beta e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau)) d\tau. \end{aligned} \quad (3.6)$$

2. For all $r_1, r_2 \in [\frac{q}{2}, q]$ with $\beta_1 := \frac{3}{q} - \frac{3}{2r_1} + \frac{1}{2}$, $\beta_2 := \frac{3}{q} - \frac{3}{2r_2} + \frac{1}{2}$ it holds

$$\Lambda^{r_1} u(t) = \Lambda^{r_2} u(t) \quad \text{for almost all } t \in [0, T[. \quad (3.7)$$

Therefore, we can denote the expression in (3.4), independently of $r \in [\frac{q}{2}, q]$, by $\Lambda(u)$.

3. For all $q_1 \in [\frac{q}{2}, q]$ with $3 < q_1 < \infty$ and $s_1 > 2$ defined by $\frac{2}{s_1} + \frac{3}{q_1} = 1$ we have

$$\Lambda u \in L^{s_1}(0, T; L^{q_1}(\Omega)). \quad (3.8)$$

4. If $q \in]3, 6[$ then

$$\Lambda u \in L^{\frac{s}{2}}(0, T; L^{q_2}(\Omega)) \quad (3.9)$$

where $q_2 > 3$ satisfies $\frac{1}{3} + \frac{1}{q_2} = \frac{1}{(\frac{q}{2})}$ and consequently $\frac{2}{(\frac{s}{2})} + \frac{3}{q_2} = 1$.

Proof. (1) By (2.8) and (3.1) we know that for all $t \in [0, T[$

$$\begin{aligned} & \int_0^t \|A_r^\beta e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau))\|_r d\tau \\ & \leq c(\Omega, q, r) \nu^{-\beta} \int_0^T |t - \tau|^{-\beta} \|u(\tau)\|_q^2 d\tau. \end{aligned} \quad (3.10)$$

Moreover, as for almost all $t \in [0, T[$ the integral in (3.10) is finite (see the application of the Hardy–Littlewood inequality (2.18) in the proof of part (3) below) and

$$\int_0^t \|e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u \otimes u)\|_r d\tau \leq c \int_0^t \|A_r^{-\beta} P_r \operatorname{div}(u \otimes u)\|_r d\tau < \infty,$$

the closedness of the operator A_r^β yields the identity (3.6).

(2) To prove (3.7) for $t \in (0, T[$ as in (1) let

$$f_t^r(\tau) := A_r^\beta e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau)) \quad \text{for almost all } \tau \in]0, t[,$$

where $\beta = \beta(r) = \frac{3}{q} - \frac{3}{2r} + \frac{1}{2}$. Since for all $\phi \in C_{0,\sigma}^\infty(\Omega)$

$$\int_0^t \langle f_t^{r_1}(\tau), \phi \rangle_\Omega d\tau = - \int_0^t \langle u(\tau) \otimes u(\tau), \nabla e^{-\nu(t-\tau)A_{r'}} \phi \rangle_\Omega d\tau,$$

we see that

$$\int_0^t \langle f_t^{r_1}(\tau), \phi \rangle_\Omega d\tau = \int_0^t \langle f_t^{r_2}(\tau), \phi \rangle_\Omega d\tau;$$

for details of the proof we refer to [12]. A density argument finishes the proof of (3.7).

(3) We consider (3.10) and use the Hardy–Littlewood inequality (2.18) with $(1 - \beta) + \frac{1}{s_1} = \frac{1}{(\frac{s}{2})}$ to conclude with $\Lambda^{q_1} u = \Lambda u$ and (3.2) that

$$\begin{aligned} & \|\Lambda u\|_{q_1, s_1, \Omega; T} \\ & \leq \left(\int_0^T \left(c \nu^{-\beta} \int_0^T |t - \tau|^{-\beta} \|A_{q_1}^{-\beta} P_{q_1} \operatorname{div}(u(\tau) \otimes u(\tau))\|_{q_1} d\tau \right)^{s_1} dt \right)^{\frac{1}{s_1}} \\ & \leq c \nu^{-\beta} \|A_{q_1}^{-\beta} P_{q_1} \operatorname{div}(u(\tau) \otimes u(\tau))\|_{q_1, \frac{s}{2}, \Omega; T} \\ & \leq c(q, q_1, \Omega) \nu^{-\beta} \|u\|_{q, s, \Omega; T}^2 < \infty. \end{aligned}$$

(4) From $2\frac{1}{2} + \frac{3}{q_2} = \frac{3}{(\frac{q}{2})}$ and (2.7) it follows with (3.6) and $\beta = \frac{1}{2}$, $r = \frac{q}{2}$, for almost all $t \in [0, T[$

$$\begin{aligned} \|\Lambda^{q_2} u(t)\|_{q_2} &\leq \|A_{\frac{3}{2}}^{1/2} \Lambda u(t)\|_{\frac{q}{2}} \\ &= \|A_{\frac{q}{2}} \int_0^t e^{-\nu(t-\tau)A_{\frac{q}{2}}} A_{\frac{q}{2}}^{-1/2} P_{\frac{q}{2}} \operatorname{div}(u(\tau) \otimes u(\tau)) d\tau\|_{\frac{q}{2}}. \end{aligned} \quad (3.11)$$

Since by (3.2)

$$A_{\frac{q}{2}}^{-1/2} P_{\frac{q}{2}} \operatorname{div}(u \otimes u) \in L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega)), \quad (3.12)$$

the maximal regularity estimate (2.13) yields the last statement of the lemma. $\square$

Proof of Theorem 1.4. Given the smallness conditions (1.12) and (1.13), there exists by Theorem 2.4 a unique very weak solution $u \in L^s(0, T; L_\sigma^q(\Omega))$ of (1.1). Moreover, we know the representation $u = E + \tilde{u}$, where the linear part E satisfies

$$E(t) = e^{-\nu t A_q} u_0 + A_q \int_0^t e^{-\nu(t-\tau)A_q} (A_q^{-1} P_q \operatorname{div} F(\tau)) d\tau \quad (3.13)$$

in $[0, T[$ and the nonlinear part $\tilde{u} \in L^s(0, T; L_\sigma^q(\Omega))$ solves the fixed point equation

$$\tilde{u}(t) = - \int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \operatorname{div} ((\tilde{u}(\tau) + E(\tau)) \otimes (\tilde{u}(\tau) + E(\tau))) d\tau \quad (3.14)$$

with $\alpha := \frac{3}{2q} + \frac{1}{2}$ for almost all $t \in [0, T[$. Since $F \in L^2(0, T; L^2(\Omega))$ and $u_0 \in L_\sigma^2(\Omega)$ it follows with (2.5) that

$$E(t) = E_1(t) + E_2(t) := e^{-\nu t A} u_0 + A^{1/2} \int_0^t e^{-\nu(t-\tau)A} A^{-1/2} P \operatorname{div} F(\tau) d\tau \quad (3.15)$$

almost everywhere. We use [14, IV Theorems 2.3.1, 2.4.1] to obtain that E lies in the Leray–Hopf class (1.2) and is a weak solution of the linear stationary Stokes system with data f, u_0 . To finish the proof, we want to show that

$$u \in L^8(0, T; L^4(\Omega)). \quad (3.16)$$

The validity of the above property implies

$$u \otimes u \in L^2(0, T; L^2(\Omega)). \quad (3.17)$$

As a consequence of (3.14) and (3.17) we conclude that $\tilde{u}$ lies in the Leray–Hopf class (1.2) and $\tilde{u}$ is the unique weak solution of the linear, stationary Stokes system with the external force $\operatorname{div}(u \otimes u)$ and vanishing initial value, see [14, IV Theorems 2.3.1, 2.4.1]. Furthermore, from these two Theorems and $\langle u \otimes u, \nabla u \rangle(\tau) = 0$ almost everywhere, it follows that u is, after a possible redefinition on a set of Lebesgue measure 0, strongly continuous and satisfies the energy equality (1.14).

Since in the case $q = 4$ (and $s = 8$) there is nothing left to be proved, we may assume in the proof of (3.16) that $q \neq 4$.

Assertion 1: $E = E_1 + E_2 \in L^8(0, T; L^4(\Omega))$.

Proof. In the case $4 < q < \infty$ it is easily seen since $L_\sigma^2(\Omega) \cap L_\sigma^q(\Omega) \subset L_\sigma^4(\Omega)$ that $E_1(t) = e^{-\nu t A} u_0 = e^{-\nu t A_q} u_0 \in L^8(0, T; L^4(\Omega))$. If $3 < q < 4$ we use [11, Theorem 1.2 (ii)] to find a constant $c > 0$, independent of t , such that

$$\|e^{-\nu t A_q} u_0\|_4 \leq c t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{4})} \|u_0\|_q$$

for all $t > 0$. The estimate

$$\int_0^T \|e^{-\nu t A_q} u_0\|_4^8 dt \leq c \|u_0\|_q^8 \int_0^T t^{-12(\frac{1}{q} - \frac{1}{4})} dt < \infty$$

implies $E_1 \in L^8(0, T; L^4(\Omega))$. To get the property $E_2 \in L^8(0, T; L^4(\Omega))$ we estimate for almost all $t \in [0, T[$, using (2.7), (2.8) and (2.11), that

$$\begin{aligned} \|E_2(t)\|_4 &\leq c \|A^{3/8} E_2(t)\|_2 \\ &= c \left\| \int_0^t A^{7/8} e^{-\nu(t-\tau)A} A^{-1/2} P \operatorname{div} F(\tau) d\tau \right\|_2 \\ &\leq c \nu^{-7/8} \int_0^t |t - \tau|^{-7/8} \|F(\tau)\|_2 d\tau. \end{aligned} \quad (3.18)$$

Then an application of the Hardy–Littlewood inequality (2.18) yields

$$\|E_2\|_{4,8,\Omega;T} \leq c \nu^{-\frac{7}{8}} \|F\|_{2,4,\Omega;T} < \infty.$$

□

Assertion 2: Let $3 < q < 4$. Then $\tilde{u} \in L^8(0, T; L^4(\Omega))$.

Proof. We use an iterative argument to improve the regularity in space step by step. Assume that for almost all $t \in [0, T[$ with certain parameters s_k, r_k, β_k

$$\tilde{u}(t) = - \int_0^t A_{r_k}^{\beta_k} e^{-\nu(t-\tau)A_{r_k}} A_{r_k}^{-\beta_k} P_{r_k} \operatorname{div}((\tilde{u} + E) \otimes (\tilde{u} + E)) d\tau, \quad (3.19)$$

$$\tilde{u}, E \in L^{s_k}(0, T; L^{r_k}(\Omega)) \text{ with } 3 < r_k < 4, \frac{2}{s_k} + \frac{3}{r_k} = 1, \beta_k \in \left[\frac{1}{2}, 1\right]. \quad (3.20)$$

For $k = 1$ the iteration starts with $s_1 := s$, $r_1 := q$ and $\beta_1 := \frac{3}{2q} + \frac{1}{2} = \alpha$, see (3.14). We denote by $r_{k+1} > r_k$ the unique element satisfying $\frac{1}{3} + \frac{1}{r_{k+1}} = \frac{1}{r_k/2}$ and set $s_{k+1} := \frac{s_k}{2}$. Then (3.9) implies that

$$\tilde{u} \in L^{s_{k+1}}(0, T; L^{r_{k+1}}(\Omega)). \quad (3.21)$$

We define $\beta_{k+1} := \frac{3}{r_{k+1}} - \frac{3}{2r_{k+1}} + \frac{1}{2} = \frac{3}{2r_{k+1}} + \frac{1}{2}$ and get with (3.7)

$$\tilde{u}(t) = - \int_0^t A_{r_{k+1}}^{\beta_{k+1}} e^{-\nu(t-\tau)A_{r_{k+1}}} A_{r_{k+1}}^{-\beta_{k+1}} P_{r_{k+1}} \operatorname{div}((\tilde{u} + E) \otimes (\tilde{u} + E)) d\tau. \quad (3.22)$$

From the first step of the proof we know that $E \in L^8(0, T; L^4(\Omega))$. There can occur two different possibilities. If $4 \leq r_{k+1} < \infty$ we get by an interpolation argument $\tilde{u}, E \in L^8(0, T; L^4(\Omega))$. Otherwise, if $3 < r_{k+1} < 4$, an interpolation argument yields $E \in L^{s_{k+1}}(0, T; L^{r_{k+1}}(\Omega))$. Looking at (3.21), (3.22), we see that (3.19) and (3.20) are satisfied with the parameters $s_{k+1}, r_{k+1}, \beta_{k+1}$. Therefore, we can start a new step of this iterative argument. Repeating this step finitely many times, we get $\tilde{u} \in L^8(0, T; L^4(\Omega))$ which finishes the proof of Assertion 2. $\square$

Assertion 3: Let $4 < q < \infty$. Then $\tilde{u} \in L^8(0, T; L^4(\Omega))$.

Proof. Assume that we have for almost all $t \in [0, T[$ with certain parameters s_k, r_k, β_k

$$\tilde{u}(t) = - \int_0^t A_{r_k}^{\beta_k} e^{-\nu(t-\tau)A_{r_k}} A_{r_k}^{-\beta_k} P_{r_k} \operatorname{div}((\tilde{u}+E) \otimes (\tilde{u}+E)) d\tau, \quad (3.23)$$

$$\tilde{u}, E \in L^{s_k}(0, T; L^{r_k}(\Omega)) \quad \text{with } 4 < r_k < \infty, \frac{2}{s_k} + \frac{3}{r_k} = 1, \beta_k \in \left[\frac{1}{2}, 1 \right]. \quad (3.24)$$

Again, for $k = 1$, the iteration starts with $s_1 := s, r_1 := q$ and $\beta_1 := \frac{3}{2q} + \frac{1}{2} = \alpha$, see (3.14). We set $r_{k+1} := \frac{3}{4}r_k$ and $\beta_{k+1} := \frac{3}{r_k} - \frac{3}{2r_{k+1}} + \frac{1}{2} = \frac{1}{r_k} + \frac{1}{2}$. Let $s_{k+1} > 2$ be the unique element which satisfies the relation $\frac{2}{s_{k+1}} + \frac{3}{r_{k+1}} = 1$. Then (3.7) implies that $\tilde{u}$ has the representation (3.22) with the new parameters $s_{k+1}, r_{k+1}, \beta_{k+1}$. From (3.22) we conclude with (3.8) that

$$\tilde{u} \in L^{s_{k+1}}(0, T; L^{r_{k+1}}(\Omega)). \quad (3.25)$$

From the first step of the proof we know that $E \in L^8(0, T; L^4(\Omega))$. There can occur two different possibilities. If $3 < r_{k+1} \leq 4$ we get by an interpolation argument $\tilde{u}, E \in L^8(0, T; L^4(\Omega))$. Otherwise, if $4 < r_{k+1} < \infty$, we use an interpolation argument to get $E \in L^{s_{k+1}}(0, T; L^{r_{k+1}}(\Omega))$. If we look at (3.22), (3.25) we see that Eqs. (3.23) and (3.24) are satisfied with the parameters $s_{k+1}, r_{k+1}, \beta_{k+1}$. Therefore, we can start a new step of this iterative argument. Repeating this step finitely many times, we get $\tilde{u} \in L^8(0, T; L^4(\Omega))$ which finishes the proof of Assertion 3. $\square$

Now the claim (3.16) for $u = \tilde{u} + E$ follows, and the proof of this theorem is complete. $\square$

4. Proof of regularity results

Before proving Theorems 1.2 and 1.3 we need a useful, but technical lemma. In this lemma we assume that u satisfies the strong energy inequality (1.5) to consider the term $u(t)$ for almost all $t \in [0, T]$ as initial value of a local strong

solution which can be identified locally with u . Therefore, the proof will be based on Theorem 1.4. We will use the notation

$$\int_a^b f(x) dx := \frac{1}{b-a} \int_a^b f(x) dx$$

for the mean value of an integral.

Lemma 4.1. *Let Ω, q, s, f, u_0, T satisfy the assumptions of Theorem 1.4, let $1 \leq s' \leq s$, and let u be a weak solution of (1.1) satisfying the strong energy inequality (1.5). Then there exists a constant $\epsilon_* = \epsilon_*(q, s', \Omega) > 0$ with the following property: If $0 < t_0 < t \leq t_1 \leq T$, and if*

$$\int_{t_0}^{t_1} \|F(\tau)\|_{q^*}^s d\tau \leq \epsilon_* \nu^{2s-1}, \quad (4.1)$$

$$\int_{t_0}^t (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} d\tau \leq \epsilon_* \nu^{s' - \frac{s'}{s}}, \quad (4.2)$$

then there exists a $\delta = \delta(t) > 0$ such that $u \in L^s(t - \delta, t_1; L^q(\Omega))$. In particular, if $t_1 > t$, then t is a regular point of u .

Proof. We may assume that $u(\tau) \in L^2(\Omega)$ for all $\tau \in [0, T[$. From (4.2) and the fact that u satisfies the strong energy inequality we find a null set $N \subset]t_0, t[$ such that for $\tau_0 \in]t_0, t[\setminus N$ it holds $u(\tau_0) \in L_\sigma^q(\Omega)$ and

$$\frac{1}{2} \|u(\tau_1)\|_2^2 + \nu \int_{\tau_0}^{\tau_1} \|\nabla u\|_2^2 d\tau \leq \frac{1}{2} \|u(\tau_0)\|_2^2 - \int_{\tau_0}^{\tau_1} \langle F, \nabla u \rangle_\Omega d\tau \quad (4.3)$$

for all τ_1 mit $\tau_0 \leq \tau_1 < T$. Moreover, the condition (4.2) yields the existence of $\tau_0 \in]t_0, t[\setminus N$ which fulfills the inequality

$$(t_1 - \tau_0)^{\frac{s'}{s}} \|u(\tau_0)\|_q^{s'} \leq \int_{t_0}^t (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} d\tau \leq \epsilon_* \nu^{s' - \frac{s'}{s}}.$$

It follows with a constant $c = c(\Omega, q) > 0$ that

$$\begin{aligned} \int_0^{t_1 - \tau_0} \|e^{-\nu\tau A_q} u(\tau_0)\|_q^s d\tau &\leq \int_0^{t_1 - \tau_0} c \|u(\tau_0)\|_q^s d\tau \\ &= c(t_1 - \tau_0) \|u(\tau_0)\|_q^s \leq c \epsilon_*^{\frac{s}{s'}} \nu^{s-1}. \end{aligned}$$

Hence with a new constant $\tilde{\epsilon}_* := (\frac{\epsilon_*}{c})^{\frac{s'}{s}}$, where ϵ_* is the constant from Theorem 1.4, the conditions of Theorem 1.4 are satisfied. We get the existence of a unique weak solution $v \in L^s([\tau_0, t_1]; L_\sigma^q(\Omega))$ to the Navier–Stokes system (1.1) with initial value $v(\tau_0) = u(\tau_0)$. Considering u as a weak solution to the Navier–Stokes system with initial value $u(\tau_0)$ on $[0, t_1 - \tau_0]$, we use Serrin’s Uniqueness Theorem to get that $u = v \in L^s(\tau_0, t_1; L_\sigma^q(\Omega))$. The proof is complete. $\square$

Proof of Theorem 1.2. (1) Because of (1.6) it is possible to choose a $\delta > 0$ such that with $t_0 := t - \delta, t_1 := t + \delta$ the estimate

$$\begin{aligned} \int_{t-\delta}^t (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} d\tau &\leq \frac{1}{\delta} \int_{t-\delta}^t (2\delta)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} d\tau \\ &= \frac{2^{\frac{s'}{s}}}{\delta^{1-\frac{s'}{s}}} \int_{t-\delta}^t \|u(\tau)\|_q^{s'} d\tau \leq \epsilon_* \nu^{s'-\frac{s'}{s}} \end{aligned}$$

holds. This shows (4.2). Furthermore, condition (4.1) on F can be fulfilled as well. Then Lemma 4.1 proves the sufficiency of (1.6) to imply regularity of u at t .

(2) The constant $\epsilon_* = \epsilon_*(q, \Omega) > 0$ will be determined in the proof; therefore, we begin with considering ϵ_* as an arbitrary, fixed positive number. Let $\varepsilon_1 = \varepsilon_1(q, \Omega) > 0$ denote the constant from Theorem 1.4 which in (1.12), (1.13) is called ϵ_* , and let $\varepsilon_2 = \varepsilon_2(s', \Omega)$ be the constant in Lemma 1.5 called ϵ_* in (4.1), (4.2). We assume $\epsilon_* \leq \varepsilon_1$ and $u_0 \neq 0$. It holds

$$\int_0^{\delta_1} \|e^{-\nu\tau A_q} u_0\|_q^s d\tau \leq c\delta_1 \|u_0\|_q^s, \quad c = c(\Omega, q) > 0.$$

We define

$$\delta_1 := \min \left(\frac{\varepsilon_1 \nu^{s-1}}{c \|u_0\|_q^s}, T \right). \quad (4.4)$$

If $\delta_1 = T$, we already know that $u \in L^s(0, T; L^q(\Omega))$. So let us assume that $\delta_1 = \frac{\varepsilon_1 \nu^{s-1}}{c \|u_0\|_q^s}$. With this choice of δ_1 , Theorem 1.4 yields the existence of a unique weak solution $v \in L^s(0, \delta_1; L^q(\Omega))$ of (1.1), which coincides by Serrin's Uniqueness Theorem with u on $[0, \delta_1[$. For an arbitrary $t \in [\frac{\delta_1}{2}, T - \frac{\delta_1}{2}]$, we get with $t_0 := t - \frac{\delta_1}{2}, t_1 := t + \frac{\delta_1}{2}$

$$\begin{aligned} \int_{t_0}^t (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} d\tau &\leq \frac{2}{\delta_1^{1-\frac{s'}{s}}} \int_0^T \|u(\tau)\|_q^{s'} d\tau \\ &\leq 2 \left(\frac{\varepsilon_1 \nu^{s-1}}{c \|u_0\|_q^s} \right)^{\frac{s'}{s}-1} \epsilon_* \frac{\nu^{s-1}}{\|u_0\|_q^{s-s'}} = 2 \left(\frac{\varepsilon_1}{c} \right)^{\frac{s'}{s}-1} \epsilon_* \nu^{s'-\frac{s'}{s}}. \end{aligned} \quad (4.5)$$

From this estimate it follows that we may define

$$\epsilon_* := \min \left(\frac{\varepsilon_2}{2} \left(\frac{\varepsilon_1}{c} \right)^{1-\frac{s'}{s}}, \varepsilon_1, \varepsilon_2 \right). \quad (4.6)$$

We see that ϵ_* depends only on Ω, q, s' . Using Lemma 4.1 we find $\delta = \delta(t) > 0$ such that

$$u \in L^s(t - \delta(t), t + \frac{\delta_1}{2}; L^q(\Omega)). \quad (4.7)$$

With (4.7) and $u \in L^s(0, \delta_1; L^q(\Omega))$ we obtain due to the compactness of the interval $[0, T]$ that $u \in L^s(0, T; L^q(\Omega))$.

Now the theorem is completely proved. $\square$

Proof of Theorem 1.3. By interpolation, in both cases the weak solution u satisfies $u \in L^{s'}(0, T; L^q(\Omega))$. The idea of the proof is to use Lemma 4.1. To control the term in (4.2) we use the interpolation inequality, see [1, Theorem 4.3.1],

$$\|v\|_q \leq c \|v\|_2^{1-\frac{2}{s'}} \|\nabla v\|_2^{\frac{2}{s'}}, \quad v \in H_0^1(\Omega),$$

where $c = c(\Omega, q) > 0$. For $\delta \in]0, \delta_0[$ with a small $\delta_0 > 0$ we get with $t_0 := t - \delta$, $t_1 := t + \delta$ the estimate

$$\begin{aligned} I(\delta) &:= \int_{t-\delta}^t (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} d\tau \\ &\leq c \delta^{\frac{s'}{s}-1} \int_{t-\delta}^t \left(\|u(\tau)\|_2^{1-\frac{2}{s'}} \|\nabla u(\tau)\|_2^{\frac{2}{s'}} \right)^{s'} d\tau \\ &\leq c \delta^{\frac{s'}{s}-1} \|u\|_{2,\infty;T}^{s'-2} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 d\tau \end{aligned} \quad (4.8)$$

with a constant $c = c(\Omega, q) > 0$. Since u is supposed to satisfy the strong energy inequality (1.5), we may proceed for almost all $\delta \in]0, \delta_0[$ as follows:

$$I(\delta) \leq \frac{c}{\nu} \delta^{\frac{s'}{s}-1} \left(|E(t-\delta) - E(t)| + \left| \int_{t-\delta}^t \langle f, u \rangle d\tau \right| \right) \quad (4.9)$$

where the constant c depends on $\|u\|_{2,\infty;T}$ in the case $\alpha > \frac{1}{2}$ and $c = c(\Omega)$ if $\alpha = \frac{1}{2}$. By Hölder's inequality we get that

$$\left| \frac{1}{\delta^{\frac{s'}{4}}} \int_{t-\delta}^t \langle f(\tau), u(\tau) \rangle d\tau \right| \leq \|u\|_{2,\infty;T} \left(\int_{t-\delta}^t \|f\|_2^{\frac{4}{4-s'}} d\tau \right)^{\frac{4-s'}{4}}. \quad (4.10)$$

As $\frac{s'}{s} = \frac{4}{4-s'}$ and consequently $f \in L^{\frac{4}{4-s'}}(0, T; L^2(\Omega))$, the left-hand side in the previous inequality converges to 0 as $\delta \rightarrow 0+$.

First consider the case $\alpha > \frac{1}{2}$ and choose $\epsilon > 0$ with $s' = 4\alpha - \epsilon$. Due to the assumption (1.8) we get with $1 - \frac{s'}{s} = \frac{s'}{4} = \alpha - \frac{\epsilon}{4}$

$$\lim_{\delta \rightarrow 0+} \frac{c}{\nu} \delta^{-\frac{s'}{4}} |E(t-\delta) - E(t)| = \lim_{\delta \rightarrow 0+} \frac{c}{\nu} \delta^{\frac{\epsilon}{4}} \frac{|E(t-\delta) - E(t)|}{\delta^\alpha} = 0. \quad (4.11)$$

Consequently the right hand side of (4.9) converges to 0 as $\delta \rightarrow 0+$. Hence we can fulfill (4.2) and, due to the assumption $F \in L^s(0, T; L^r(\Omega))$, it is also possible to satisfy (4.1). Altogether, Lemma 4.1 yields regularity of u at t .

Secondly, consider the case $\alpha = \frac{1}{2}$ in which $s' = 2$, $s = 4$. We will choose the constant $\gamma_* = \gamma_*(\Omega) > 0$ below. Let $\epsilon_* = \epsilon_*(q) > 0$ denote the constant from Lemma 4.1. The assumption (1.9) implies that for all $0 < \delta < \mu$

$$\frac{1}{\nu} \frac{|E(t-\delta) - E(t)|}{\delta^{\frac{1}{2}}} \leq \gamma_* \nu^{\frac{3}{2}}. \quad (4.12)$$

Then by (4.9), (4.10) and (4.12) we get with a constant $c = c(\Omega) > 0$ for almost all $\delta \in]0, \delta_0[$ that

$$I(\delta) \leq c\gamma_*\nu^{\frac{3}{2}} + \frac{c}{\nu}\|u\|_{2,\infty;T} \left(\int_{t-\delta}^t \|f\|_2^2 d\tau \right)^{\frac{1}{2}}.$$

Now with $\gamma_* := \frac{\epsilon_*}{2c}$ we find $0 < \delta < \mu$ such that $I(\delta) \leq \epsilon_*\nu^{\frac{3}{2}}$, cf. (4.2), and that (4.1) is satisfied. Hence Lemma 4.1 implies regularity of u at t . $\square$

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