Multiple solutions to a nonlinear Schrödinger equation with Aharonov–Bohm magnetic potential

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Abstract. We consider the magnetic nonlinear Schrödinger equations

$$\begin{aligned} (-i\nabla + sA)^2 u + u &= |u|^{p-2} u, \quad p \in (2, 6), \\ (-i\nabla + sA)^2 u &= |u|^4 u, \end{aligned}$$

in $\Omega = \mathcal{O} \times \mathbb{R}$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^2 \setminus \{0\}$, $s \in \mathbb{R}$, and $A : \Omega \rightarrow \mathbb{R}^3$ is the Aharonov–Bohm magnetic potential

$$A(x_1, x_2, x_3) := \frac{1}{x_1^2 + x_2^2}(-x_2, x_1, 0).$$

We prove multiplicity results and describe the symmetry properties of the solutions obtained.

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1. Introduction

In quantum mechanics the Hamiltonian for a non-relativistic charged particle in an electromagnetic field is given by $(-i\nabla + A)^2 + V$, where $V : \mathbb{R}^N \supset \Omega \rightarrow \mathbb{R}$ is the electric (or scalar) potential and $A : \mathbb{R}^N \supset \Omega \rightarrow \mathbb{R}^N$ is the magnetic (or vector) potential which is a source for the magnetic field $B = \text{curl } A$. By definition, if $A = (A_1, \dots, A_N)$ then $\text{curl } A$ is the $N \times N$ skew-symmetric matrix with entries $B_{jk} = \partial_j A_k - \partial_k A_j$, or in the language of differential forms, if A is the 1-form $A_1 dx_1 + \dots + A_N dx_N$ then $B = dA$.

Consider the magnetic nonlinear Schrödinger problem

$$(\wp_{\Omega,s}) \quad \begin{cases} (-i\nabla + sA)^2 u + u = |u|^{p-2} u, \\ u \in H_{sA,0}^1(\Omega, \mathbb{C}), \end{cases}$$

where $\Omega = \mathcal{O} \times \mathbb{R}$ with $\mathcal{O}$ an open subset of $\mathbb{R}^2 \setminus \{0\}$, $s \in \mathbb{R}$, $p \in (2, 6)$, and $A : \Omega \rightarrow \mathbb{R}^3$ is the Aharonov–Bohm magnetic potential

$$A(x_1, x_2, x_3) := \frac{1}{x_1^2 + x_2^2}(-x_2, x_1, 0).$$

The space $H_{sA,0}^1(\Omega, \mathbb{C})$ is the closure of $C_c^\infty(\Omega, \mathbb{C})$ with respect to the norm

$$\|u\|_{sA} := \left(\int_{\Omega} (|\nabla_{sA} u|^2 + |u|^2) \right)^{1/2},$$

where $\nabla_{sA} u := \nabla u + isAu$.

We note that $\text{curl } A = 0$ in Ω while $\int_{\gamma} sA = 2\pi s$ if γ is a properly oriented simple closed curve enclosing the x_3 -axis. This integral is called the magnetic flux and describes the influence of a magnetic potential on a charged quantum-mechanical particle moving in a region where the magnetic field is 0 (the so-called Aharonov–Bohm effect, see e.g. [23]). In our setting the particle is confined to a region Ω outside a thin solenoid extending along the x_3 -axis.

If $\mathcal{O}$ is the annulus

$$\mathcal{O}_{a,b} := \{(x_1, x_2) : a < x_1^2 + x_2^2 < b\}, \quad 0 \leq a < b \leq \infty,$$

we write $\Omega_{a,b} := \mathcal{O}_{a,b} \times \mathbb{R}$. We shall prove the following results.

Theorem 1.1. *Let $\Omega = \Omega_{a,b}$ with $0 \leq a < b \leq \infty$. For every $n \in \mathbb{Z}$ there exists a nontrivial solution u_n of problem $(\wp_{\Omega,s})$ with the following properties:*

- (a) $u_0 = |u_0|$ and $|u_n| > 0$ in Ω ,
- (b) $|u_n| \neq |u_m|$ if $|s + m| \neq |s + n|$,
- (c) $u_n(g(x_1, x_2), x_3) = g^n u_n(x_1, x_2, x_3)$ for every $g \in SO(2)$, $x \in \Omega_{a,b}$,
- (d) $u_n(x_1, x_2, x_3) = u_n(x_1, x_2, -x_3)$ for every $x \in \Omega_{a,b}$,
- (e) $\|u_m\|_{sA} < \|u_n\|_{sA}$ if $|s + m| < |s + n|$,
- (f) $\lim_{|n| \rightarrow \infty} \|u_n\|_{sA} = \infty$.

Here $SO(2)$ denotes the group of rotations of the plane $\mathbb{R}^2 \equiv \mathbb{C}$ or, equivalently, the group of unit complex numbers acting by multiplication on $\mathbb{C}$. Let G be a closed subgroup of $SO(2)$. A subset $\mathcal{O}$ of $\mathbb{R}^2$ is said to be G -invariant if $gx \in \mathcal{O}$ for every $g \in G$ and $x \in \mathcal{O}$, and a function $u : \Omega \rightarrow \mathbb{C}$ is said to be G -invariant if $u(g(x_1, x_2), x_3) = u(x_1, x_2, x_3)$ for every $g \in G$.

The closed subgroups of $SO(2)$ consist of the cyclic groups $G_n := \{e^{2\pi i k/n} : k = 0, \dots, n-1\}$ generated by the rotation by angle $2\pi/n$, $n \in \mathbb{N}$, and $G_\infty := SO(2)$ itself. Note that the solutions given by Theorem 1.1 are symmetry breaking: if $n \neq 0$, then u_n is $G_{|n|}$ -invariant but it is not G_m -invariant for any $m > |n|$. Moreover, properties (a) and (c) assert that the map

$$\frac{u_n}{|u_n|} : \Omega_{a,b} \rightarrow \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$$

is well defined and has degree n in the sense that, for an appropriate orientation of $\Omega_{a,b}$, the induced homomorphism of the fundamental groups $\pi_1(\Omega_{a,b}) \rightarrow \pi_1(\mathbb{S}^1)$ is multiplication by n .

Theorem 1.2. *Let $G_n = \{e^{2\pi i k/n} : k = 0, \dots, n-1\}$, $n \in \mathbb{N}$. If $\mathcal{O}$ is bounded and G_n -invariant, then problem $(\wp_{\Omega,s})$ has at least n nontrivial solutions $u_0, \dots, u_{n-1}$ which satisfy*

$$u_m(g(x_1, x_2), x_3) = g^m u_m(x_1, x_2, x_3) \quad \forall g \in G, \quad x \in \Omega, \quad m = 0, \dots, n-1.$$

The above result applied to the trivial group $G_1 = \{1\}$ asserts the existence of at least one solution for every domain $\Omega = \mathcal{O} \times \mathbb{R}$ with $\mathcal{O}$ bounded and every $s \in \mathbb{R}$.

We also consider the problem with critical nonlinearity

$$(\wp_{\Omega,s}^*) \quad \begin{cases} (-i\nabla + sA)^2 u = |u|^4 u, \\ u \in D_{sA,0}^{1,2}(\Omega, \mathbb{C}), \end{cases}$$

where $D_{sA,0}^{1,2}(\Omega, \mathbb{C})$ is the closure of the space $C_c^\infty(\Omega, \mathbb{C})$ with respect to the norm

$$\|u\|_{sA,*} := \left(\int_{\Omega} |\nabla_{sA} u|^2 \right)^{1/2}.$$

We prove the following.

Theorem 1.3. *If either $\Omega = \Omega_{0,\infty}$ or $\Omega = \Omega_{a,b}$ with $0 < a < b < \infty$ then for every $n \in \mathbb{Z}$ there exists a nontrivial solution u_n of problem $(\wp_{\Omega,s}^*)$ with the following properties:*

- (a) $u_0 = |u_0|$ and $|u_n| > 0$ in Ω ,
- (b) $|u_n| \neq |u_m|$ if $|s+m| \neq |s+n|$,
- (c) $u_n(g(x_1, x_2), x_3) = g^n u_n(x_1, x_2, x_3)$ for every $g \in SO(2)$, $x \in \Omega_{a,b}$,
- (d) $u_n(x_1, x_2, x_3) = u_n(x_1, x_2, -x_3)$ for every $x \in \Omega_{a,b}$,
- (e) $\|u_m\|_{sA,*} < \|u_n\|_{sA,*}$ if $|s+m| < |s+n|$.

The first existence results for problems of this type with a nonsingular magnetic potential were obtained by Esteban and Lions in [12]. In particular, for $\mathcal{A}(x_1, x_2, x_3) = (-x_2, x_1, 0)$ they proved the existence of a sequence of solutions to equation $(-i\nabla + \mathcal{A})^2 u + u = |u|^{p-2} u$ in $\mathbb{R}^3$. Further existence results for nonsingular magnetic potentials may be found for example in [2, 5, 10, 22]. Existence and multiplicity of semiclassical solutions are given e.g. in [3, 6–9, 16]. Very recently, Abatangelo and Terracini obtained existence results for problems with critical nonlinearity and singular magnetic and electric potentials [1].

Problems $(\wp_{\Omega,s})$ and $(\wp_{\Omega,s}^*)$ do not always have a ground state in $H_{sA,0}^1(\Omega, \mathbb{C})$ and $D_{sA,0}^{1,2}(\Omega, \mathbb{C})$ respectively, see Propositions 2.1, 2.2 and 3.4. However, they may have ground states on appropriate subspaces. In particular, the solution u_0 in Theorems 1.1 and 1.3 will be obtained by constrained minimization in the subspaces of $H_{sA,0}^1(\Omega, \mathbb{C})$ and $D_{sA,0}^{1,2}(\Omega, \mathbb{C})$ consisting of $SO(2)$ -invariant functions. The other solutions are obtained as a consequence

of the gauge invariance of problems $(\wp_{\Omega,s})$ and $(\wp_{\Omega,s}^*)$. Observe that, unlike the case of nonsingular magnetic potentials in $\mathbb{R}^N$ whose curl is 0, the Aharonov–Bohm potential A is not the gradient of a function defined on Ω . However, it is the gradient of a function which is defined locally. Namely, if $\theta(x_1, x_2, x_3)$ denotes the polar angle of $x_1 + ix_2$ then $\nabla\theta = A$. The function $e^{i\theta}$ is uniquely defined (while θ is only unique up to an integer multiple of 2π) and a direct computation shows that if v_{s+n} is a solution of $(\wp_{\Omega,s+n})$ then $e^{in\theta}v_{s+n}$ is a solution of $(\wp_{\Omega,s})$, and similarly for $(\wp_{\Omega,s}^*)$. We use this fact to construct u_n by taking v_{s+n} to be an $SO(2)$ -invariant solution of $(\wp_{\Omega,s+n})$ and setting $u_n := e^{in\theta}v_{s+n}$. A more detailed study of gauge invariance in domains which are not simply connected may be found in [10]. In what follows

$$\theta = \theta(x_1, x_2, x_3) \text{ denotes the polar angle of } x_1 + ix_2. \quad (1.1)$$

Semiclassical solutions having the kind of symmetry described by (c) in the theorems above have been recently considered in [7]. We will show that the solution u_n in Theorems 1.1 and 1.2 is a ground state for problem $(\wp_{\Omega,s})$ in the subspace of $H_{sA,0}^1(\Omega, \mathbb{C})$ consisting of functions which satisfy the symmetry property (c). A similar statement holds for the solution u_n in Theorem 1.3, see Propositions 2.6 and 3.8 below.

This paper is organized as follows. In Sect. 2 we discuss the subcritical case and prove Theorems 1.1 and 1.2. In Sect. 3 we discuss the critical case and prove Theorem 1.3.

2. The subcritical case

2.1. Preliminaries

Let $\Omega = \mathcal{O} \times \mathbb{R}$ with $\mathcal{O}$ an open subset of $\mathbb{R}^2 \setminus \{0\}$, $s \in \mathbb{R}$, and let $A : \Omega \rightarrow \mathbb{R}^3$ be the Aharonov–Bohm magnetic potential

$$A(x_1, x_2, x_3) := \frac{1}{x_1^2 + x_2^2}(-x_2, x_1, 0).$$

Throughout this section we fix $p \in (2, 6)$.

For $0 \leq a < b \leq \infty$ we set

$$\mathcal{O}_{a,b} := \{(x_1, x_2) : a < x_1^2 + x_2^2 < b\} \quad \text{and} \quad \Omega_{a,b} := \mathcal{O}_{a,b} \times \mathbb{R}.$$

We start by recalling some useful inequalities. If $u \in H_{sA,0}^1(\Omega, \mathbb{C})$, then the absolute value $|u|$ of u is in $H_0^1(\Omega, \mathbb{R})$ and

$$|\nabla |u| (x)| \leq |\nabla_{sA} u(x)| \quad \text{for a.e. } x \in \Omega. \quad (2.1)$$

This is called the *diamagnetic inequality* [18]. Together with the Sobolev inequality it yields

$$|u|_q := \left(\int_{\Omega} |u|^q \right)^{1/q} \leq \tilde{C}_q \|u\|_{sA} \quad \forall q \in [2, 6], \quad (2.2)$$

where $\tilde{C}_q > 0$ is a constant depending on q . Hence, $H_{sA,0}^1(\Omega, \mathbb{C})$ is continuously embedded in $L^q(\Omega, \mathbb{C})$ for all $q \in [2, 6]$.

Inequalities (2.1) and (2.2) hold for more general vector potentials. For our specific A more is true. It was shown in [17] that the Aharonov–Bohm potential in $\mathbb{R}^2$ satisfies

$$\int_{\mathbb{R}^2} \frac{|\varphi|^2}{x_1^2 + x_2^2} \leq C \int_{\mathbb{R}^2} \left| \nabla \varphi + i \frac{(-x_2, x_1)}{x_1^2 + x_2^2} \varphi \right|^2 \quad \forall \varphi \in C_c^\infty(\mathcal{O}_{0,\infty}, \mathbb{C})$$

for some positive constant C . Following [1], we integrate this inequality with respect to x_3 and obtain

$$\int_{\mathbb{R}^3} \frac{|\varphi|^2}{x_1^2 + x_2^2} \leq C \int_{\mathbb{R}^3} |\nabla_A \varphi|^2 \quad \forall \varphi \in C_c^\infty(\Omega_{0,\infty}, \mathbb{C}).$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \varphi|^2 &\leq c \left(\int_{\mathbb{R}^3} |\nabla_{sA} \varphi|^2 + \int_{\mathbb{R}^3} |sA\varphi|^2 \right) \\ &= c \left(\int_{\mathbb{R}^3} |\nabla_{sA} \varphi|^2 + s^2 \int_{\mathbb{R}^3} \frac{|\varphi|^2}{x_1^2 + x_2^2} \right) \\ &\leq C_s \int_{\mathbb{R}^3} |\nabla_{sA} \varphi|^2 \quad \forall \varphi \in C_c^\infty(\Omega_{0,\infty}, \mathbb{C}). \end{aligned} \quad (2.3)$$

This inequality implies that

$$\begin{aligned} \|\varphi\|^2 &:= \int_{\Omega} (|\nabla \varphi|^2 + |\varphi|^2) \\ &\leq C_s \int_{\Omega} (|\nabla_{sA} \varphi|^2 + |\varphi|^2) = C_s \|\varphi\|_{sA}^2 \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{C}). \end{aligned}$$

If $0 \notin \overline{\Omega}$ then $A \in L^\infty(\Omega)$ and both norms are equivalent. But in general they are not. However, this last inequality asserts that $H_{sA,0}^1(\Omega, \mathbb{C})$ is continuously embedded in the nonmagnetic Sobolev space $H_0^1(\Omega, \mathbb{C})$. As has been observed in [1], these spaces are in general not equal. This can be seen by taking $u(x) = \chi(x)|x|^{-1}$, where $\chi \in C_c^\infty(\mathbb{R}^3, [0, 1])$ is such that $\chi = 0$ for $|x| \leq 1$ and $\chi = 1$ for $2 \leq |x| \leq 3$. Then $u \in H^1(\mathbb{R}^3, \mathbb{C})$. Note that, since a line in $\mathbb{R}^3$ has capacity 0, $H_0^1(\Omega_{0,\infty}, \mathbb{C}) = H^1(\mathbb{R}^3, \mathbb{C})$ (see Theorem 4.7.2 in [13] and Theorem 9.2.3 in [20]). On the other hand, $u \notin H_{sA,0}^1(\Omega_{0,\infty}, \mathbb{C})$ unless $s = 0$. The same conclusion remains valid for the spaces $D_{sA,0}^{1,2}(\Omega_{0,\infty}, \mathbb{C})$ and $D^{1,2}(\mathbb{R}^3, \mathbb{C})$.

The nontrivial solutions of $(\varphi_{\Omega,s})$ are in one-to-one correspondence with the critical points of the functional

$$I_s(u) := \|u\|_{sA}^2$$

on the manifold

$$\Sigma_{\Omega,s,p} := \left\{ u \in H_{sA,0}^1(\Omega, \mathbb{C}) : |u|_p = 1 \right\}.$$

More precisely, $u \in \Sigma_{\Omega,s,p}$ is a critical point of I_s iff $I_s(u)^{\frac{1}{p-2}}u$ is a nontrivial solution of $(\wp_{\Omega,s})$. Set

$$S_{\Omega,s,p} := \inf_{u \in \Sigma_{\Omega,s,p}} I_s(u) = \inf_{\substack{u \in H_{sA,0}^1(\Omega, \mathbb{C}) \\ u \neq 0}} \frac{\|u\|_{sA}^2}{|u|_p^2}.$$

It follows from inequality (2.2) that $S_{\Omega,s,p} > 0$. Observe that $\overline{\nabla_{sA}u} = \nabla_{-sA}\bar{u}$, where $\bar{u}$ stands for the complex conjugate of u . Hence $I_s(u) = I_{-s}(\bar{u})$ and, consequently, $S_{\Omega,s,p} = S_{\Omega,-s,p}$.

2.2. Existence and nonexistence of ground states

Next we investigate whether $S_{\Omega,s,p}$ is attained. Let $\omega \in H^1(\mathbb{R}^3, \mathbb{R})$ be the positive radially symmetric ground state of problem

$$\begin{cases} -\Delta v + v = |v|^{p-2}v, \\ v \in H^1(\mathbb{R}^3, \mathbb{R}). \end{cases} \quad (2.4)$$

Let $S_p := \frac{\|\omega\|^2}{|\omega|_p^2}$. The following holds.

Proposition 2.1. *If $\mathcal{O} \supset \mathcal{O}_{a,\infty}$ for some $a \in (0, \infty)$ then $S_{\Omega,s,p} = S_p$.*

Proof. Let $\psi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ be radially symmetric and such that $0 \leq \psi \leq 1$, $\psi(x_1, x_2) = 0$ if $x_1^2 + x_2^2 \leq a^2$ and $\psi(x_1, x_2) = 1$ if $x_1^2 + x_2^2 \geq 2a^2$. Define

$$\begin{aligned} \omega_n(x_1, x_2, x_3) &:= \omega(x_1 - n, x_2, x_3), \\ v_n(x_1, x_2, x_3) &:= \psi(x_1, x_2)\omega_n(x_1, x_2, x_3). \end{aligned}$$

Since v_n is real-valued, using the fact that $|A(x_1, x_2, x_3)| \rightarrow 0$ as $x_1^2 + x_2^2 \rightarrow \infty$, we obtain

$$\begin{aligned} \|v_n\|_{sA}^2 &= \int_{\Omega} (|\nabla v_n|^2 + |sAv_n|^2 + |v_n|^2) \\ &= \int_{\mathbb{R}^3} (|\psi \nabla \omega_n + \omega_n \nabla \psi|^2 + \psi^2 |sA\omega_n|^2 + \psi^2 \omega_n^2) \\ &= \int_{\mathbb{R}^3} \psi^2 (|\nabla \omega_n|^2 + \omega_n^2) - \int_{\mathbb{R}^3} (\psi \Delta \psi) \omega_n^2 + \int_{\mathbb{R}^3} \psi^2 |sA|^2 \omega_n^2 \\ &\leq \int_{\mathbb{R}^3} (|\nabla \omega|^2 + \omega^2) + o(1), \\ |v_n|_p^p &= \int_{\mathbb{R}^3} |\omega|^p + \int_{\mathbb{R}^3} (\psi^p - 1) |\omega_n|^p = \int_{\mathbb{R}^3} |\omega|^p + o(1). \end{aligned}$$

Therefore,

$$\frac{\|v_n\|_{sA}^2}{|v_n|_p^2} \leq \frac{\|\omega\|^2 + o(1)}{|\omega|_p^2 + o(1)} = S_p + o(1).$$

On the other hand, the diamagnetic inequality (2.1) yields

$$S_p \leq \frac{\|v\|_{sA}^2}{|v|_p^2} \leq \frac{\|v\|_{sA}^2}{|v|_p^2} \quad \forall v \in H_{sA,0}^1(\Omega, \mathbb{C}) \setminus \{0\}. \quad (2.5)$$

We conclude that $S_{\Omega,s,p} = S_p$. $\square$

As we have mentioned in the introduction, the function θ given by (1.1) is defined only locally in $\Omega_{0,\infty}$ while if $m \in \mathbb{Z}$ then the function $x \mapsto e^{im\theta(x)}$ is globally defined. A straightforward computation shows that

$$e^{im\theta} \nabla_{(s+m)A}(\varphi) = \nabla_{sA}(e^{im\theta}\varphi) \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{C}), \quad m \in \mathbb{Z}. \quad (2.6)$$

Proposition 2.2. *Assume $S_{\Omega,s,p} = S_p$. Then $S_{\Omega,s,p}$ is attained if and only if $\Omega = \Omega_{0,\infty}$ and $s \in \mathbb{Z}$.*

Proof. If $\Omega = \Omega_{0,\infty}$ then, as we have noticed earlier, $H_0^1(\Omega, \mathbb{C}) = H^1(\mathbb{R}^3, \mathbb{C})$. So the unique positive radially symmetric solution ω of (2.4) belongs to $H_0^1(\Omega, \mathbb{C})$. For $m \in \mathbb{Z}$ set $u_{-m} := e^{-im\theta}\omega$ where $\theta(x_1, x_2, x_3)$ is the polar angle of $x_1 + ix_2$. It follows from (2.6) that $u_{-m} \in H_{mA,0}^1(\Omega_{0,\infty}, \mathbb{C})$ and

$$\frac{\|u_{-m}\|_{mA}^2}{|u_{-m}|_p^2} = \frac{\|\omega\|^2}{|\omega|_p^2} = S_p = S_{\Omega,m,p}.$$

Next assume there exists $v \in \Sigma_{\Omega,s,p}$ such that $I_s(v) = S_{\Omega,s,p} = S_p$. Then the diamagnetic inequality

$$S_p \leq \| |v| \|^2 \leq \|v\|_{sA}^2 = S_p$$

becomes an equality. So, up to translation, $S_p^{1/(p-2)} |v| = \omega$. In particular, $\Omega = \Omega_{0,\infty}$ and $|v(x)| > 0$ for all x . The equality $\| |v| \|^2 = \|v\|_{sA}^2$ is equivalent to

$$sA = -\operatorname{Im} \frac{\nabla v}{v} \quad (2.7)$$

(cf. the proof of the diamagnetic inequality in [18] and [2], and also the argument of Theorem 1.1 in [2]). Let $\gamma(t) := (\cos t, \sin t, 0)$, $t \in [0, 2\pi]$, and $u(t) := v(\gamma(t))$. Identifying $(x_1, x_2, 0)$ with $z := x_1 + ix_2$ we may consider u as a curve in $\mathbb{C} \setminus \{0\}$. Since

$$\frac{u'(t)}{u(t)} = \frac{\nabla v(\gamma(t))}{v(\gamma(t))} \cdot \gamma'(t),$$

it follows from (2.7) that

$$2\pi s = \int_\gamma sA = -\operatorname{Im} \int_\gamma \frac{\nabla v}{v} = -\operatorname{Im} \int_0^{2\pi} \frac{u'(t)}{u(t)} dt = -\operatorname{Im} \int_\gamma \frac{dz}{z} = -2\pi m,$$

where m is the winding number of u with respect to 0. Hence $s = -m \in \mathbb{Z}$. $\square$

Note that the above result can be rephrased by saying that a necessary and sufficient condition for the existence of a ground state in $H_{sA,0}^1(\Omega, \mathbb{C})$ is that $\Omega = \Omega_{0,\infty}$ and the magnetic potential can be gauged away.

2.3. Existence of G -invariant ground states

If $\mathcal{O}$ is invariant under the action of some closed subgroup G of $SO(2)$ and $\Omega = \mathcal{O} \times \mathbb{R}$, then G acts on $H_{sA,0}^1(\Omega, \mathbb{C})$ by $(g, u) \mapsto u_g$, where

$$u_g(x_1, x_2, x_3) := u(g^{-1}(x_1, x_2), x_3). \quad (2.8)$$

This is an orthogonal action, that is,

$$\langle u_g, v_g \rangle_{sA} = \langle u, v \rangle_{sA} \quad \forall g \in G, \quad u, v \in H_{sA,0}^1(\Omega, \mathbb{C}),$$

where

$$\langle u, v \rangle_{sA} := \operatorname{Re} \int_{\Omega} (\nabla_{sA} u \cdot \overline{\nabla_{sA} v} + u \bar{v})$$

is the inner product associated to the norm $\|u\|_{sA}$. Moreover,

$$|u_g|_p = |u|_p \quad \forall g \in G, \quad u \in L^p(\Omega, \mathbb{C}).$$

Therefore, $\Sigma_{\Omega,s,p}$ is G -invariant, i.e. $u_g \in \Sigma_{\Omega,s,p}$ iff $u \in \Sigma_{\Omega,s,p}$, and the functional I_{sA} satisfies $I_{sA}(u_g) = I_{sA}(u)$. Set

$$\begin{aligned} H_{sA,0}^1(\Omega, \mathbb{C})^G &:= \{u \in H_{sA,0}^1(\Omega, \mathbb{C}) : u_g = u \quad \forall g \in G\}, \\ \Sigma_{\Omega,s,p}^G &:= \Sigma_{\Omega,s,p} \cap H_{sA,0}^1(\Omega, \mathbb{C})^G. \end{aligned}$$

By the principle of symmetric criticality [21] (see also Theorem 1.28 in [26]), the critical points of the restriction of I_{sA} to $\Sigma_{\Omega,s,p}^G$ correspond to the nontrivial G -invariant solutions to problem $(\wp_{\Omega,s})$, i.e. to solutions u of $(\wp_{\Omega,s})$ which satisfy

$$u(g(x_1, x_2), x_3) = u(x_1, x_2, x_3) \quad \forall g \in G, \quad x \in \Omega. \quad (2.9)$$

Set

$$S_{\Omega,s,p}^G := \inf_{u \in \Sigma_{\Omega,s,p}^G} I_s(u) = \inf_{\substack{u \in H_{sA,0}^1(\Omega, \mathbb{C})^G \\ u \neq 0}} \frac{\|u\|_{sA}^2}{|u|_p^2}.$$

Proposition 2.3. *I_s attains its minimum on $\Sigma_{\Omega,s,p}^G$ for every $s \in \mathbb{R}$ if one of the following two conditions holds:*

- (D1) $\mathcal{O}$ is bounded and G is an arbitrary closed subgroup of $SO(2)$.
- (D2) $\mathcal{O} = \mathcal{O}_{a,\infty} := \{(x_1, x_2) : x_1^2 + x_2^2 > a\}$, $a \in [0, \infty)$, and $G = SO(2)$.

Proof. Let (v_n) be a sequence in $\Sigma_{\Omega,s,p}^G$ such that $\lim_{n \rightarrow \infty} I_s(v_n) = S_{\Omega,s,p}^G$. The diamagnetic inequality (2.1) implies that $(|v_n|)$ is bounded in $H_0^1(\Omega, \mathbb{R})$, so if

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^3} \int_{B_1(x)} |v_n|^2 = 0$$

then by P.L. Lions' lemma [19] (see also Lemma 1.21 in [26]) we would have that $v_n \rightarrow 0$ in $L^p(\mathbb{R}^3)$ which is impossible because $|v_n|_p = 1$. Hence there exist $\delta > 0$ and $\xi_n \in \mathbb{R}^3$ such that, after passing to a subsequence,

$$\int_{B_1(\xi_n)} |v_n|^2 \geq \delta.$$

Here $B_r(\xi) := \{x \in \mathbb{R}^3 : |x - \xi| < r\}$. Write $\xi_n = (\xi_{n1}, \xi_{n2}, \xi_{n3})$ and define

$$w_n(x_1, x_2, x_3) := v_n(x_1, x_2, x_3 + \xi_{n3}).$$

Then $\|w_n\|_{sA}^2 = \|v_n\|_{sA}^2$, $|w_n|_p^p = |v_n|_p^p = 1$, and

$$\int_{B_1(\xi_{n1}, \xi_{n2}, 0)} |w_n|^2 \geq \delta.$$

We claim that the sequence $(\xi_{n1}, \xi_{n2}, 0)$ is bounded. This is trivially true if $\mathcal{O}$ is bounded. So assume that $G = SO(2)$ and $\mathcal{O} = \mathcal{O}_{a,\infty}$. Let $m = m(n)$ be the largest number of elements $g_1, \dots, g_m \in SO(2)$ such that

$$B_1(g_j(\xi_{n1}, \xi_{n2}), 0) \cap B_1(g_k(\xi_{n1}, \xi_{n2}), 0) = \emptyset \quad \text{if } j \neq k.$$

Since $w_n(g(x_1, x_2), x_3) = w_n(x_1, x_2, x_3)$ for all $g \in SO(2)$, we have that

$$\|w_n\|_{sA}^2 > \int_{\Omega} |w_n|^2 \geq m \int_{B_1(\xi_{n1}, \xi_{n2}, 0)} |w_n|^2 \geq m\delta.$$

If $\xi_{n1}^2 + \xi_{n2}^2 \rightarrow \infty$, then $m \rightarrow \infty$, contradicting the fact that (w_n) is bounded in $H_{sA,0}^1(\Omega, \mathbb{C})$. Consequently, if either (D1) or (D2) holds, there exists $R > 0$ such that

$$\int_{B_R(0)} |w_n|^2 \geq \delta. \quad (2.10)$$

Now, after passing to a subsequence, we have that

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{weakly in } H_{sA,0}^1(\Omega, \mathbb{C})^G, \\ w_n(x) &\rightarrow w(x) \quad \text{a.e. in } \Omega, \\ w_n &\rightarrow w \quad \text{strongly in } L_{loc}^2(\Omega, \mathbb{C}). \end{aligned}$$

Inequality (2.10) implies that $w \neq 0$.

Since $|w_n|_p = 1$, using the Brézis–Lieb lemma [4] (see also Lemma 1.32 in [26]) we obtain $1 - \lim_{n \rightarrow \infty} |w_n - w|_p^p = |w|_p^p$ and hence

$$\begin{aligned} S_{\Omega,s,p}^G &= \lim_{n \rightarrow \infty} \|w_n\|_{sA}^2 = \lim_{n \rightarrow \infty} \|w_n - w\|_{sA}^2 + \|w\|_{sA}^2 \\ &\geq S_{\Omega,s,p}^G \left[\lim_{n \rightarrow \infty} \left(|w_n - w|_p^p \right)^{2/p} + \left(|w|_p^p \right)^{2/p} \right] \\ &= S_{\Omega,s,p}^G \left[(1 - |w|_p^p)^{2/p} + (|w|_p^p)^{2/p} \right] \\ &\geq S_{\Omega,s,p}^G (1 - |w|_p^p + |w|_p^p)^{2/p} = S_{\Omega,s,p}^G. \end{aligned}$$

Since $w \neq 0$, this implies that $\lim_{n \rightarrow \infty} |w_n - w|_p^p = 0$. Hence, $|w|_p^p = 1$ and

$$S_{\Omega,s,p}^G = S_{\Omega,s,p}^G |w|_p^2 \leq \|w\|_{sA}^2 \leq \lim_{n \rightarrow \infty} \|w_n\|_{sA}^2 = S_{\Omega,s,p}^G.$$

Consequently, $w \in \Sigma_{\Omega,s,p}^G$ and $I_s(w) = S_{\Omega,s,p}^G$. □

2.4. Proof of Theorems 1.1 and 1.2

Next we prove our main results for the subcritical case.

Proof of Theorem 1.2. Let $G := \{e^{2\pi i k/n} : k = 0, \dots, n-1\}$, $n \in \mathbb{N}$. Since $\mathcal{O}$ is bounded, Proposition 2.3 asserts that $S_{\Omega,t,p}^G$ is attained for every $t \in \mathbb{R}$. Let $w_t \in \Sigma_{\Omega,t,p}^G$ be such that $\|w_t\|_{tA}^2 = S_{\Omega,t,p}^G$. Then $v_t := (S_{\Omega,t,p}^G)^{\frac{1}{p-2}} w_t$ is a solution of $(\wp_{\Omega,t})$. For $m = 0, \dots, n-1$ define $u_m := e^{im\theta} v_{s+m}$ where θ is given by (1.1). Using equality (2.6) it is easy to see that u_m is a nontrivial solution

of $(\wp_{\Omega,s})$. Note that $\theta(e^{2\pi ik/n}(x_1, x_2), x_3) = \theta(x_1, x_2, x_3) + 2\pi k/n$. Therefore, since v_{s+m} is G -invariant, we have that

$$u_m(g(x_1, x_2), x_3) = g^m u_m(x_1, x_2, x_3) \quad \forall g \in G, x \in \Omega. \quad (2.11)$$

We claim that these solutions are all different. Indeed, if $u_k = u_m$, then equality (2.11) implies that $g^{k-m} u_k(x) = u_k(x)$ for all $g \in G$ and all $x \in \Omega$. Hence $k = m$. $\square$

In the remaining part of this subsection we make the following

Assumption. $G = SO(2)$ and $\Omega = \Omega_{a,b}$ with $0 \leq a < b \leq \infty$.

If $\varphi \in C_c^\infty(\Omega, \mathbb{C})$ satisfies (2.9), then φ is constant on every circle $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 = r^2, x_3 = c\}$. Therefore,

$$\nabla \varphi(x) \cdot A(x) = 0 \quad \forall x \in \Omega.$$

It follows that $|\nabla \varphi + isA\varphi|^2 = |\nabla \varphi|^2 + |sA\varphi|^2$, so

$$\|u\|_{sA}^2 = \|u\|^2 + \int_{\Omega} |sA|^2 |u|^2 \quad \forall u \in H_{sA,0}^1(\Omega, \mathbb{C})^G. \quad (2.12)$$

Let $H_{sA,0}^1(\Omega, \mathbb{R})$ be the subspace of real-valued functions in $H_{sA,0}^1(\Omega, \mathbb{C})$ and

$$\Sigma_{\Omega,s,p,\mathbb{R}}^G := \Sigma_{\Omega,s,p}^G \cap H_{sA,0}^1(\Omega, \mathbb{R})$$

be the submanifold of all real-valued functions in $\Sigma_{\Omega,s,p}^G$.

Lemma 2.4. *One has that*

$$\inf_{u \in \Sigma_{\Omega,s,p,\mathbb{R}}^G} I_s(u) = \inf_{u \in \Sigma_{\Omega,s,p}^G} I_s(u) =: S_{\Omega,s,p}^G. \quad (2.13)$$

Moreover, if u is a minimizer of I_s on $\Sigma_{\Omega,s,p}^G$, then $|u|$ is a minimizer of I_s on $\Sigma_{\Omega,s,p,\mathbb{R}}^G$.

Proof. Since $\Sigma_{\Omega,s,p,\mathbb{R}}^G \subset \Sigma_{\Omega,s,p}^G$, we have that

$$\inf_{u \in \Sigma_{\Omega,s,p,\mathbb{R}}^G} I_s(u) \geq \inf_{u \in \Sigma_{\Omega,s,p}^G} I_s(u).$$

On the other hand, applying the diamagnetic inequality (2.1) and (2.12) we obtain

$$\begin{aligned} \|u\|_{sA}^2 &= \int_{\Omega} |\nabla |u||^2 + (|sA|^2 + 1) |u|^2 \\ &\leq \int_{\Omega} |\nabla u|^2 + (|sA|^2 + 1) |u|^2 = \|u\|_{sA}^2. \end{aligned}$$

Therefore,

$$\inf_{u \in \Sigma_{\Omega,s,p,\mathbb{R}}^G} I_s(u) \leq \inf_{u \in \Sigma_{\Omega,s,p}^G} I_s(u).$$

Thus, equality (2.13) holds. Moreover, if $u \in \Sigma_{\Omega,s,p}^G$ and $I_s(u) = S_{\Omega,s,p}^G$, then $I_s(|u|) = S_{\Omega,s,p}^G$. Therefore, $|u|$ is a minimizer of I_s on $\Sigma_{\Omega,s,p,\mathbb{R}}^G$. $\square$

Lemma 2.5. (a) $S_{\Omega,s,p}^G < S_{\Omega,t,p}^G$ if $s^2 < t^2$,

(b) The set $\{S_{\Omega,s,p}^G : s \geq 0\}$ is unbounded.

Proof. (a) By Proposition 2.3, for every $s \in \mathbb{R}$ there exists $v_s \in \Sigma_{\Omega,s,p}^G$ such that $I_s(v_s) = S_{\Omega,s,p}^G$. Formula (2.12) yields

$$\|v_t\|_{tA}^2 = \|v_t\|_{sA}^2 + (t^2 - s^2) \int_{\Omega} |A|^2 |v_t|^2 > \|v_t\|_{sA}^2 \quad \text{if } t^2 > s^2.$$

Hence, $S_{\Omega,s,p}^G < S_{\Omega,t,p}^G$ if $t^2 > s^2$.

(b) Arguing by contradiction, assume that $\sup_{s \geq 0} S_{\Omega,s,p}^G =: \hat{c} \in \mathbb{R}$. Let $v_n \in \Sigma_{\Omega,n,p}^G$ be such that $\|v_n\|_{nA}^2 = S_{\Omega,n,p}^G \leq \hat{c}$. Arguing as in the proof of Proposition 2.3 we conclude that there exist $R > 0$ and $z_n \in \mathbb{R}$ such that, after passing to a subsequence, we have

$$\int_{B_R(0,0,z_n)} |v_n|^2 \geq \delta > 0.$$

It follows that

$$\int_{B_R(0,0,z_n)} |A|^2 |v_n|^2 \geq \left(\inf_{x \in B_R(0,0,z_n)} |A(x)|^2 \right) \int_{B_R(0,0,z_n)} |v_n|^2 \geq \frac{\delta}{R^2}.$$

Therefore

$$S_{\Omega,n,p}^G = \|v_n\|_{nA}^2 = \|v_n\|^2 + n^2 \int_{\Omega} |A|^2 |v_n|^2 \geq n^2 \frac{\delta}{R^2}$$

and $\sup_{s \geq 0} S_{\Omega,s,p}^G = \infty$, contradicting our assumption. $\square$

Proof of Theorem 1.1. Let w_s be a positive minimizer of I_s on $\Sigma_{\Omega,s,p,\mathbb{R}}^G$. Applying the moving plane method [14] (see also Appendix C in [26]), we may assume that $w_s(x_1, x_2, x_3) = w_s(x_1, x_2, -x_3)$. Set $v_s := (S_{\Omega,s,p}^G)^{\frac{1}{p-2}} w_s$. Note that, $v_s \neq v_t$ iff $s^2 \neq t^2$. Otherwise, $v = v_s = v_t$ would satisfy

$$-\Delta v + (|sA|^2 + 1)v = |v|^{p-2}v = -\Delta v + (|tA|^2 + 1)v,$$

which implies $s^2 = t^2$. Define

$$u_n := e^{in\theta} v_{s+n}, \quad n \in \mathbb{Z}.$$

Then $|u_n| = v_{s+n}$ satisfies (a) and (b). Clearly, (c) and (d) hold.

Using equality (2.6) it is easy to see that u_n is a solution of problem $(\wp_{\Omega,s})$ satisfying

$$\|u_n\|_{sA}^2 = \|v_{s+n}\|_{(s+n)A}^2 = S_{\Omega,s+n,p}^G. \quad (2.14)$$

Lemma 2.5 yields (e) and (f). $\square$

Equality (2.14) and Lemma 2.5 say that, among the solutions u_n of $(\wp_{\Omega,s})$ given by Theorem 1.1, the minimal value of $I_s(\frac{u_n}{|u_n|_p})$ is $S_{\Omega,\sigma,p}^G$ where $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$ and $\sigma - s \in \mathbb{Z}$. So

$$I_s \left(\frac{u_{\sigma-s}}{|u_{\sigma-s}|_p} \right) = S_{\Omega,\sigma,p}^G \geq S_{\Omega,s,p} \geq S_p.$$

Propositions 2.1 and 2.2 assert that $S_{\Omega,\sigma,p}^G > S_{\Omega,s,p} = S_p$ if either $a \neq 0$ and $b = \infty$ or $a = 0$, $b = \infty$ and $s \notin \mathbb{Z}$. If $a = 0$ and $b = \infty$ then $S_{\Omega,0,p} = S_p$ is

attained at a radial function. Therefore $S_{\Omega,0,p}^G = S_p$. Since $\sigma = 0$ for $s \in \mathbb{Z}$ it follows that $S_{\Omega,\sigma,p}^G = S_{\Omega,s,p} = S_p$ if $a = 0$, $b = \infty$ and $s \in \mathbb{Z}$.

2.5. Minimizing properties of the solutions

Next we obtain some additional information on the solution u_m . As we have seen it lies in the space

$$X_m := \{u \in H_{sA,0}^1(\Omega, \mathbb{C}) : u(g(x_1, x_2), x_3) = g^m u(x_1, x_2, x_3) \ \forall g \in G, \ x \in \Omega\}.$$

Moreover, the following holds.

Proposition 2.6. *The solution u_m in Theorems 1.1 and 1.2 satisfies*

$$\frac{\|u_m\|_{sA}^2}{|u_m|_p^2} = \inf_{\substack{u \in X_m \\ u \neq 0}} \frac{\|u\|_{sA}^2}{|u|_p^2}. \quad (2.15)$$

Proof. The proof of Proposition 2.3 shows that

$$\inf_{\substack{u \in X_m \\ u \neq 0}} \frac{\|u\|_{sA}^2}{|u|_p^2} \quad (2.16)$$

is attained. The only difference is that for the sequence (w_n) constructed there we now have $w_n(g(x_1, x_2), x_3) = g^m w_n(x_1, x_2, x_3)$. Since both sides of this equality have the same absolute value, the argument of this proposition remains valid. Hence (2.16) is attained at some $u \in X_m$. Let $v := e^{-im\theta}u$. Recall from the proof of Theorem 1.1 that $u_m := e^{im\theta}v_{s+m}$. Since v is G -invariant,

$$S_{\Omega,s+m,p}^G \leq \frac{\|v\|_{(s+m)A}^2}{|v|_p^2} = \frac{\|u\|_{sA}^2}{|u|_p^2} \leq \frac{\|u_m\|_{sA}^2}{|u_m|_p^2} = \frac{\|v_{s+m}\|_{(s+m)A}^2}{|v_{s+m}|_p^2} = S_{\Omega,s+m,p}^G.$$

Therefore u_m satisfies (2.15). $\square$

3. The critical case

3.1. Preliminaries

Let $D_{sA,0}^{1,2}(\Omega, \mathbb{C})$ be the closure of $C_c^\infty(\Omega, \mathbb{C})$ with respect to the norm

$$\|u\|_{sA,*} := \left(\int_{\Omega} |\nabla_{sA} u|^2 \right)^{1/2}.$$

Inequality (2.3) asserts that

$$\|\varphi\|_* := \left(\int_{\Omega} |\nabla \varphi|^2 \right)^{1/2} \leq C_s \|\varphi\|_{sA,*} \quad \forall \varphi \in C_c^\infty(\Omega, \mathbb{C}).$$

Hence $D_{sA,0}^{1,2}(\Omega, \mathbb{C})$ is continuously embedded in $D_0^{1,2}(\Omega, \mathbb{C})$.

If G is a closed subgroup of $SO(2)$ and $\mathcal{O}$ is G -invariant we define

$$D_{sA,0}^{1,2}(\Omega, \mathbb{C})^G := \left\{ u \in D_{sA,0}^{1,2}(\Omega, \mathbb{C}) : u_g = u \ \forall g \in G \right\},$$

with u_g as in (2.8). Set $I_{s,*}(u) := \|u\|_{sA,*}^2$,

$$\begin{aligned}\Sigma_{\Omega,s,*} &:= \left\{ u \in D_{sA,0}^{1,2}(\Omega, \mathbb{C}) : |u|_6 = 1 \right\}, \quad \Sigma_{\Omega,s,*}^G := \Sigma_{\Omega,s,*} \cap D_{sA,0}^{1,2}(\Omega, \mathbb{C})^G, \\ S_{\Omega,s,*} &:= \inf_{u \in \Sigma_{\Omega,s,*}} I_{s,*}(u) \quad \text{and} \quad S_{\Omega,s,*}^G := \inf_{u \in \Sigma_{\Omega,s,*}^G} I_{s,*}(u).\end{aligned}$$

Note that $6 = \frac{2N}{N-2}$ is the critical Sobolev exponent in dimension $N = 3$.

3.2. Existence of G -invariant ground states

Let $\mathcal{M}(\mathbb{R}^3)$ be the space of finite measures in $\mathbb{R}^3$. A positive measure $\nu \in \mathcal{M}(\mathbb{R}^3)$ is concentrated at a single G -orbit of $\mathbb{R}^3$ if there exists $\zeta \in \mathbb{R}^3$ such that $\nu(G\zeta) = \|\nu\|$ where $G\zeta := \{(g(\zeta_1, \zeta_2), \zeta_3) : g \in G\}$ is the G -orbit of $\zeta = (\zeta_1, \zeta_2, \zeta_3)$. Note that ν is concentrated at a single G -orbit of $\mathbb{R}^3$ iff $\nu(X) \in \{0, \|\nu\|\}$ for every G -invariant open subset X of $\mathbb{R}^3$.

Lemma 3.1 (Concentration compactness). *Let (u_n) be a sequence in $D_{sA,0}^{1,2}(\Omega, \mathbb{C})^G$ such that*

$$\begin{aligned}u_n &\rightharpoonup u \quad \text{weakly in } D_{sA,0}^{1,2}(\Omega, \mathbb{C}), \\ |\nabla_{sA}(u_n - u)|^2 &\rightharpoonup \mu \quad \text{weakly in } \mathcal{M}(\mathbb{R}^3), \\ |u_n - u|^6 &\rightharpoonup \nu \quad \text{weakly in } \mathcal{M}(\mathbb{R}^3), \\ u_n(x) &\rightarrow u(x) \quad \text{a.e. in } \mathbb{R}^3,\end{aligned}$$

and let

$$\mu_\infty := \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \geq R} |\nabla_{sA} u_n|^2, \quad \nu_\infty := \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \geq R} |u_n|^6.$$

Then, the following hold:

- (a) $S_{\Omega,s,*}^G \|\nu\|^{1/3} \leq \|\mu\|$.
- (b) $S_{\Omega,s,*}^G (\nu_\infty)^{1/3} \leq \mu_\infty$.
- (c) $\limsup_{n \rightarrow \infty} \|u_n\|_{sA,*}^2 = \|u\|_{sA,*}^2 + \|\mu\| + \mu_\infty$.
- (d) $\limsup_{n \rightarrow \infty} |u_n|_6^6 = |u|_6^6 + \|\nu\| + \nu_\infty$.
- (e) If $u = 0$ and $S_{\Omega,s,*}^G \|\nu\|^{1/3} = \|\mu\|$ then μ and ν are concentrated at a single finite G -orbit of $\mathbb{R}^3$ or are zero.

Proof. The proof of assertions (a)–(d) is completely analogous to the one given in [2] for the nonequivariant case (see also Lemma 1.40 in [26]). Further, one can show that if $u = 0$ and $S_{\Omega,s,*}^G \|\nu\|^{1/3} = \|\mu\|$ then μ and ν are concentrated at the G -orbit of some point $\zeta \in \mathbb{R}^3$. One needs only to observe that the test functions $h \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ which occur in [2] and [26] must be G -invariant in our case, cf. [25].

Next we show that $G\zeta$ is finite. Let $\varphi \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$. Since $u_n \rightarrow 0$ in $L_{loc}^2(\mathbb{R}^3, \mathbb{C})$ we have that

$$\begin{aligned} S_{\Omega, s, *} \left(\int_{\mathbb{R}^3} |\varphi u_n|^6 \right)^{1/3} &\leq \int_{\mathbb{R}^3} |\nabla_{sA}(\varphi u_n)|^2 = \int_{\mathbb{R}^3} |\varphi \nabla_{sA}(u_n) + u_n \nabla \varphi|^2 \\ &\leq 2 \int_{\mathbb{R}^3} |\varphi \nabla_{sA}(u_n)|^2 + 2 \int_{\text{supp}(\varphi)} |\nabla \varphi|^2 |u_n|^2 \\ &= 2 \int_{\mathbb{R}^3} |\varphi \nabla_{sA}(u_n)|^2 + o(1). \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$ we obtain

$$S_{\Omega, s, *} \left(\int_{\mathbb{R}^3} |\varphi|^6 d\nu \right)^{1/3} \leq 2 \int_{\mathbb{R}^3} |\varphi|^2 d\mu \quad \forall \varphi \in C_c^\infty(\mathbb{R}^3, \mathbb{R}).$$

By Proposition 4.2 in [25] there exist an at most countable set $\{x_j : j \in J\}$ of points in $\mathbb{R}^3$ and numbers $\nu_j \in (0, \infty)$ such that

$$\nu = \sum_{j \in J} \nu_j \delta_{x_j}.$$

Since $\nu\{g(x_1, x_2), x_3\} = \nu\{(x_1, x_2, x_3)\}$ for every $g \in G$, it follows that $G\zeta = \{x_j : j \in J\}$. Therefore $G\zeta$ must be finite. $\square$

Proposition 3.2. *If $\Omega = \Omega_{0, \infty}$ and $G = SO(2)$ then $S_{\Omega, s, *}^G$ is attained.*

Proof. Let (u_n) be a sequence in $\Sigma_{\Omega, s, *}^G$ such that $\|u_n\|_{sA, *}^2 \rightarrow S_{\Omega, s, *}^G$. Set

$$Q_n(r) := \sup_{z \in \mathbb{R}} \int_{B_r(0, 0, z)} |u_n|^6.$$

Then $Q_n(r) \rightarrow 0$ as $r \rightarrow 0$ and $Q_n(r) \rightarrow 1$ as $r \rightarrow \infty$, for each fixed $n \in \mathbb{N}$. Hence, there exist $r_n \in (0, \infty)$ such that $Q_n(r_n) = \frac{1}{2}$ and $z_n \in \mathbb{R}$ such that

$$\sup_{z \in \mathbb{R}} \int_{B_{r_n}(0, 0, z)} |u_n|^6 = \int_{B_{r_n}(0, 0, z_n)} |u_n|^6 = \frac{1}{2}.$$

Let $w_n(x) := r_n^{1/2} u_n(r_n x + (0, 0, z_n))$. Then $w_n \in \Sigma_{\Omega, s, *}^G$, $\|w_n\|_{sA, *}^2 = \|u_n\|_{sA, *}^2$ and

$$\sup_{z \in \mathbb{R}} \int_{B_1(0, 0, z)} |w_n|^6 = \int_{B_1(0)} |w_n|^6 = \frac{1}{2}. \quad (3.1)$$

Recall that $D_{sA, 0}^{1, 2}(\Omega, \mathbb{C}) \subset D_0^{1, 2}(\Omega, \mathbb{C})$. So passing to a subsequence, we have that $w_n \rightharpoonup w$ weakly in $D_{sA, 0}^{1, 2}(\Omega, \mathbb{C})^G$, $w_n(x) \rightarrow w(x)$ a.e. in $\mathbb{R}^3$, $|\nabla_{sA}(w_n - w)|^2 \rightharpoonup \mu$ and $|w_n - w|^6 \rightharpoonup \nu$ weakly in $\mathcal{M}(\mathbb{R}^3)$. Since

$$\lim_{n \rightarrow \infty} \|w_n\|_{sA, *}^2 = S_{\Omega, s, *}^G \quad \lim_{n \rightarrow \infty} |w_n|_6^2,$$

using Lemma 3.1 and the definition of $S_{\Omega, s, *}^G$ we obtain

$$\begin{aligned} \|w\|_{sA, *}^2 + \|\mu\| + \mu_\infty &= S_{\Omega, s, *}^G (|w|_6^6 + \|\nu\| + \nu_\infty)^{1/3} \\ &\leq S_{\Omega, s, *}^G (|w|_6^2 + \|\nu\|^{1/3} + \nu_\infty^{1/3}) \leq \|w\|_{sA, *}^2 + \|\mu\| + \mu_\infty. \end{aligned}$$

Hence

$$(|w|_6^6 + \|\nu\| + \nu_\infty)^{1/3} = |w|_6^2 + \|\nu\|^{1/3} + \nu_\infty^{1/3}.$$

It follows that exactly one of the quantities $|w|_6, \|\nu\|, \nu_\infty$ is 1 and the other two are 0. Equality (3.1) implies that $\nu_\infty \neq 1$, hence $\nu_\infty = 0$. Assume $\|\nu\| = 1$. Then $w = 0$ and $S_{\Omega,s,*}^G \|\nu\|^{1/3} = \|\mu\|$, so ν is concentrated at the G -orbit of a point $\zeta = (0, 0, \zeta_3) \in \mathbb{R}^3$ (other orbits are infinite). But (3.1) yields

$$\frac{1}{2} = \int_{B_1(0)} |w_n|^6 \geq \lim_{n \rightarrow \infty} \int_{B_1(0,0,\zeta_3)} |w_n|^6 = \|\nu\| = 1.$$

This is a contradiction. Therefore $\|\nu\| = 0$, $|w|_6 = 1$ and $\|w\|_{sA,*}^2 = S_{\Omega,s,*}^G$. $\square$

If $\mathcal{O}$ is bounded, Poincaré's inequality holds for Ω . Together with the diamagnetic inequality this yields

$$\int_{\Omega} (|\nabla |u||^2 + |u|^2) \leq C \|u\|_{sA,*}^2 \quad \forall u \in D_{sA,0}^{1,2}(\Omega, \mathbb{C}) \quad (3.2)$$

for some constant $C > 0$. The proof of the following proposition is completely analogous to that of Theorem 1.1 in [11]. We include it here for the reader's convenience.

Proposition 3.3. *If $\Omega = \Omega_{a,b}$ with $0 < a < b < \infty$ and $G = SO(2)$ then $S_{\Omega,s,*}^G$ is attained.*

Proof. Let (u_n) be a sequence in $\Sigma_{\Omega,s,*}^G$ such that $\|u_n\|_{sA,*}^2 \rightarrow S_{\Omega,s,*}^G$. Poincaré's inequality (3.2) asserts that $(|u_n|)$ is bounded in $H_0^1(\Omega)$, so if

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^3} \int_{B_1(x)} |u_n|^6 = 0$$

then by Lemma 2.1 in [24] we would have that $u_n \rightarrow 0$ in $L^6(\mathbb{R}^3)$ which is impossible because $|u_n|_6 = 1$. Hence there exist $\delta > 0$ and $\xi_n \in \mathbb{R}^3$ such that, after passing to a subsequence,

$$\int_{B_1(\xi_n)} |u_n|^6 \geq \delta.$$

Replacing $u_n(x)$ by $u_n(x + (0, 0, \xi_{n,3}))$ we may assume that (ξ_n) is bounded and hence that

$$\int_{B_R(0)} |u_n|^6 \geq \delta \quad (3.3)$$

for $R > 0$ sufficiently large. Passing to a subsequence, we have that $u_n \rightharpoonup u$ weakly in $D_{sA,0}^{1,2}(\Omega, \mathbb{C})^G$, $u_n(x) \rightarrow u(x)$ a.e. in $\mathbb{R}^3$, $|\nabla_{sA}(u_n - u)|^2 \rightharpoonup \mu$ and $|u_n - u|^6 \rightharpoonup \nu$ weakly in $\mathcal{M}(\mathbb{R}^3)$. Arguing as in the proof of Proposition 3.2 we conclude that

$$\begin{aligned} \|u\|_{sA,*}^2 + \|\mu\| + \mu_\infty &= S_{\Omega,s,*}^G (|u|_6^6 + \|\nu\| + \nu_\infty)^{1/3} \\ &\leq S_{\Omega,s,*}^G (|u|_6^2 + \|\nu\|^{1/3} + \nu_\infty^{1/3}) \leq \|u\|_{sA,*}^2 + \|\mu\| + \mu_\infty. \end{aligned}$$

So one of the quantities $|u|_6, \|\nu\|, \nu_\infty$ is 1 and the other two are 0. Inequality (3.3) implies that $\nu_\infty \leq 1 - \delta$, hence $\nu_\infty = 0$. Assume $\|\nu\| = 1$. Then $u = 0$ and $S_{\Omega,s,*}^G \|\nu\|^{1/3} = \|\mu\|$, so by Lemma 3.1 ν is concentrated at a finite G -orbit. But $\bar{\Omega}$ does not contain such orbits. Therefore $\|\nu\| = 0$ and consequently $|u|_6 = 1$ and $\|u\|_{sA,*}^2 = S_{\Omega,s,*}^G$. $\square$

3.3. Nonexistence of G -invariant ground states

Let U be a fixed positive radially symmetric ground state of problem

$$\begin{cases} -\Delta v = v^5, \\ v \in D^{1,2}(\mathbb{R}^3, \mathbb{R}). \end{cases}$$

For $\varepsilon > 0$ and $x_0 \in \mathbb{R}^3$ let $U_{\varepsilon,x_0}(x) := \varepsilon^{-1/2}U(\varepsilon^{-1}(x - x_0))$. Then U_{ε,x_0} is also a ground state. As usual, we denote $S := \frac{\|U\|_*^2}{|U|_6^2}$.

Proposition 3.4. *$S_{\Omega,s,*} = S$ and it is attained if and only if $\Omega = \Omega_{0,\infty}$ and $s \in \mathbb{Z}$.*

Proof. Fix $x_0 \in \Omega$ and $r > 0$ such that $\overline{B_r(x_0)} \subset \Omega$. Choose $\psi \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ if $|x - x_0| \leq \frac{r}{2}$ and $\psi(x) = 0$ if $|x - x_0| \geq r$. Define $u_\varepsilon := \psi U_{\varepsilon,x_0} \in D_{sA,0}^{1,2}(\Omega, \mathbb{C})$. An easy computation shows that

$$\|u_\varepsilon\|_{sA,*}^2 = \|U_{\varepsilon,x_0}\|_*^2 + o(1) \quad \text{and} \quad |u_\varepsilon|_6^6 = |U_{\varepsilon,x_0}|_6^6 + o(1),$$

where $o(1) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence $S_{\Omega,s,*} \leq S$. The diamagnetic inequality (2.1) yields $S_{\Omega,s,*} = S$. It also implies that if $u \in \Sigma_{\Omega,s,*}$ and $\|u\|_{sA,*}^2 = S_{\Omega,s,*}$ then $\|u\|_*^2 = S$. But $S := \inf_{0 \neq v \in D_0^{1,2}(\Omega, \mathbb{R})} \frac{\|v\|_*^2}{|v|_6^2}$ is not attained if $\bar{\Omega} \neq \mathbb{R}^3$. The remaining part of the argument is the same as in Proposition 2.2. $\square$

Let G be a closed subgroup of $SO(2)$ and $\mathcal{O}$ be G -invariant. For $\lambda > 0$ set

$$\mathcal{O}_\lambda := \{z \in \mathbb{R}^2 : \lambda^{-1}z \in \mathcal{O}\} \quad \text{and} \quad \Omega_\lambda := \mathcal{O}_\lambda \times \mathbb{R}.$$

Proposition 3.5. *If there exists $\lambda \in (0, \infty)$ such that $\mathcal{O} \subset \mathcal{O}_\lambda$ and $\mathcal{O}_\lambda \setminus \bar{\mathcal{O}} \neq \emptyset$, then $S_{\Omega_\lambda,s,*}^G$ is not attained. In particular, if $\Omega = \Omega_{0,a}$ or $\Omega = \Omega_{a,\infty}$ with $a \in (0, \infty)$ then $S_{\Omega,s,*}^G$ is not attained.*

Proof. If $u \in D_{sA,0}^{1,2}(\Omega, \mathbb{C})^G$, the rescaled function $u_\lambda(z) := \lambda^{-1/2}u(\lambda^{-1}z)$ satisfies $u_\lambda \in D_{sA,0}^{1,2}(\Omega_\lambda, \mathbb{C})^G$, $\|u_\lambda\|_{sA,*}^2 = \|u\|_{sA,*}^2$ and $|u_\lambda|_6^6 = |u|_6^6$. Therefore $S_{\Omega_\lambda,s,*}^G = S_{\Omega,s,*}^G$. So if $u \in \Sigma_{\Omega,s,*}^G$ satisfies $\|u\|_{sA,*}^2 = S_{\Omega,s,*}^G$, extending u by 0 outside of Ω we obtain a function $\bar{u} \in \Sigma_{\Omega_\lambda,s,*}^G$ which satisfies $\|\bar{u}\|_{sA,*}^2 = S_{\Omega,s,*}^G = S_{\Omega_\lambda,s,*}^G$. Hence $(S_{\Omega,s,*})^{\frac{1}{4}}\bar{u}$ is a solution of $(\wp_{\Omega_\lambda,s}^*)$ which vanishes in the nonempty open set $\Omega_\lambda \setminus \bar{\Omega}$, contradicting the unique continuation theorem [15]. Here we have used the fact that $u \in H_{loc}^2(\Omega_\lambda, \mathbb{C})$ which follows e.g. from the proof of Proposition 2.2 in [10]. $\square$

3.4. Proof of Theorem 1.3

In the remaining part of this section we assume $G = SO(2)$ and either $\Omega = \Omega_{0,\infty}$ or $\Omega = \Omega_{a,b}$ with $0 < a < b < \infty$. Then, as in the subcritical case (2.12), one has that

$$\|u\|_{sA,*}^2 = \|u\|_*^2 + \int_{\Omega} |sA|^2 |u|^2 \quad \forall u \in D_{sA,0}^{1,2}(\Omega, \mathbb{C})^G. \quad (3.4)$$

Let $D_{sA,0}^{1,2}(\Omega, \mathbb{R})$ be the subspace of real-valued functions in $D_{sA,0}^{1,2}(\Omega, \mathbb{C})$ and

$$\Sigma_{\Omega,s,*}^G := \Sigma_{\Omega,s,*}^G \cap D_{sA,0}^{1,2}(\Omega, \mathbb{R}).$$

Lemma 3.6. *One has that*

$$\inf_{u \in \Sigma_{\Omega,s,*}^G} I_{s,*}(u) = \inf_{u \in \Sigma_{\Omega,s,*}^G} I_{s,*}(u) =: S_{\Omega,s,*}^G.$$

Moreover, if u is a minimizer of $I_{s,*}$ on $\Sigma_{\Omega,s,*}^G$, then $|u|$ is a minimizer of $I_{s,*}$ on $\Sigma_{\Omega,s,*}^G$.

Proof. The proof is similar to that of Lemma 2.4. $\square$

Lemma 3.7. $S_{\Omega,s,*}^G < S_{\Omega,t,*}^G$ if $s^2 < t^2$.

Proof. The proof is similar to that of Lemma 2.5 (a) using Propositions 3.2 and 3.3. $\square$

Proof of Theorem 1.3. The proof is analogous to that of Theorem 1.1 using equality (2.6) and Lemmas 3.6 and 3.7. $\square$

It is not clear whether the solutions given by Theorem 1.3 have the property that $\lim_{|n| \rightarrow \infty} \|u_n\|_{sA,*} = \infty$.

3.5. Minimizing properties of the solutions

As in the subcritical case the solution u_m solves a minimization problem. Set

$$X_m^* := \{u \in D_{sA,0}^{1,2}(\Omega, \mathbb{C}) : u(g(x_1, x_2), x_3) = g^m u(x_1, x_2, x_3) \\ \forall g \in SO(2), x \in \Omega\}.$$

The following holds.

Proposition 3.8. *The solution u_m in Theorem 1.3 satisfies $u_m \in X_m^*$ and*

$$\frac{\|u_m\|_{sA}^2}{|u_m|_6^2} = \inf_{\substack{u \in X_m^* \\ u \neq 0}} \frac{\|u\|_{sA}^2}{|u|_6^2}. \quad (3.5)$$

Proof. The arguments of Propositions 3.2 and 3.3 go through unchanged to show that

$$\inf_{\substack{u \in X_m^* \\ u \neq 0}} \frac{\|u\|_{sA}^2}{|u|_6^2} \quad (3.6)$$

is attained, provided an appropriate version of Lemma 3.1 holds. This lemma requires that the sequence (u_n) is invariant with respect to some $SO(2)$ -action on $D_{sA,0}^{1,2}(\Omega, \mathbb{C})$. The appropriate action is obtained by setting

$$u_g(x_1, x_2, x_3) := g^m u(g^{-1}(x_1, x_2), x_3).$$

X_m^* is the fixed point space for this action. Once we have that (3.6) is attained, the rest of the proof goes as that of Proposition 2.6. $\square$

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