Nonlinear Differ. Equ. Appl. 17 (2010), 195–212

1021-9722/10/020195-18
published online November 18, 2009
DOI 10.1007/s00030-009-0048-3

Nonlinear Differential Equations and Applications NoDEA

On quasilinear parabolic equations involving weighted p-Laplacian operators

Cung The Anh and Tran Dinh Ke

Abstract. In this paper we consider the initial boundary value problem for a class of quasilinear parabolic equations involving weighted p-Laplacian operators in an arbitrary domain, in which the conditions imposed on the non-linearity provide the global existence, but not uniqueness of solutions. The long-time behavior of the solutions to that problem is considered via the concept of global attractor for multi-valued semiflows. The obtained results recover and extend some known results related to the p-Laplacian equations.

Mathematics Subject Classification (2000). 35B41, 35K65, 35D05.

Keywords. Degenerate parabolic equation, m-semiflow, Global solution, Global attractor, Compact embedding, Weighted p-Laplacian operator.

1. Introduction

Parabolic equations of p-Laplacian type arise in many applications in the fields of mechanics, physics and biology (non-Newtonian fluids, gas flows in porous media, spread of biological populations, etc.). One of the most interesting problems concerning these equations is to understand the asymptotic behavior of solutions when time grows to infinity. The study of the asymptotic behavior of the equation is giving us relevant information about "the structure" of the phenomenon described in the model.

In this paper we study the asymptotic behavior of solutions to the following problem

$$\begin{cases} u_t - \operatorname{div}(\sigma(x)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u - f(x,u), & x \in \Omega,\ t > 0, \\ u|_{t=0} = u_0(x) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $2 \leqslant p < N$ and Ω is an arbitrary (bounded or unbounded) domain in $\mathbb{R}^N$.

Denote

$$\begin{aligned} L_{p,\sigma}u &= -\mathrm{div}(\sigma(x)|\nabla u|^{p-2}\nabla u),\\ p' &= \frac{p}{p-1}, \text{ the conjugate exponent of } p,\\ p^*_\gamma &= \frac{pN}{N-p+\gamma}, \quad \text{for } \gamma \in \mathbb{R}^+,\\ a\wedge b &= \max(a,b), \quad \text{for } a,b\in\mathbb{R},\\ \mathcal{I}[p,q] &= \{tp+(1-t)q : 0\leqslant t\leqslant 1\}. \end{aligned}$$

We assume the following conditions:

(H1) The function $\sigma : \Omega \to \mathbb{R}$ satisfies the following assumptions: when the domain Ω is bounded,

($\mathcal{H}_\alpha$) $\sigma \in L^1_{loc}(\Omega)$ and for some $\alpha \in (0,p)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$,

and when the domain Ω is unbounded,

($\mathcal{H}^\infty_{\alpha,\beta}$) σ satisfies condition ($\mathcal{H}_\alpha$) and $\liminf_{|x|\to\infty} |x|^{-\beta}\sigma(x) > 0$ for some $\beta > p + \frac{N}{2}(p-2)$.

(H2) $f : \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, i.e. $f(.,u)$ is measurable and $f(x,.)$ is continuous, and satisfies

(1) $|f(x,u)| \leqslant C_1|u|^{q-1} + h_1(x)$,

(2) $uf(x,u) \geqslant C_2|u|^q - h_2(x)$,

where $q > 1$ and C_1, C_2 are positive constants; $h_1 \in L^{q'}(\Omega), h_2 \in L^1(\Omega)$ are non-negative functions.

(H3) $\lambda < \lambda_1$ if $q \leqslant p$, where λ_1 is the first eigenvalue of $L_{p,\sigma}$ with the Dirichlet boundary condition (the existence of λ_1 is ensured by Proposition 2.3 below).

(H4) $[1, p^*_\alpha)\cap\mathcal{I}[p',q'] \neq \emptyset$, where α is given in ($\mathcal{H}_\alpha$), if Ω is a bounded domain; $(p^*_\beta \wedge 1, p] \cap \mathcal{I}[p',q'] \neq \emptyset$, where α,β are given in ($\mathcal{H}^\infty_{\alpha,\beta}$), if Ω is an unbounded domain.

The degeneracy of problem (1.1) is considered in the sense that the measurable, non-negative diffusion coefficient $\sigma(x)$ is allowed to have at most a finite number of (essential) zeroes at some points. The physical motivation of the assumption ($\mathcal{H}_\alpha$) is related to the modeling of reaction diffusion processes in composite materials, occupying a bounded domain Ω, in which at some points they behave as *perfect insulator*. Following [13, p. 79], when at some points the medium is perfectly insulating, it is natural to assume that $\sigma(x)$ vanishes at these points. On the other hand, when condition ($\mathcal{H}^\infty_{\alpha,\beta}$) is satisfied, it is easy to see that the diffusion coefficient has to be unbounded. Physically, this situation corresponds to a non-homogeneous medium, occupying the unbounded domain Ω, which behaves as a perfect conductor in $\Omega \setminus B_R(0)$ (see [13], p. 79), and as a perfect insulator in a finite number of points in $B_R(0)$. Note that in various diffusion processes, the equation involves diffusion $\sigma(x) \sim |x|^\alpha, \alpha \in (0,p)$, in the case of a bounded domain, and $\sigma(x) \sim |x|^\alpha + |x|^\beta, \alpha \in (0,p), \beta > p$, in the case of an unbounded domain.

It is noticed that a type of weight which is close to the weight σ above is the one of Caffarelli–Kohn–Nirenberg type. We refer the reader to works [2,3] on Caffarelli–Kohn–Nirenberg inequalities associated to the study quasilinear elliptic equations.

Let us now give some comments about remaining conditions. The nonlinearity f is assumed satisfying the polynomial type growth and a standard dissipative condition. A typical example of f which satisfies condition $(H2)$ is that

$$f(u) \sim u|u|^{q-2}, \quad q > 1.$$

The condition $(H3)$ is necessary for the global existence of weak solutions to problem (1.1) (see Proposition 3.1 below), while $(H4)$ is a technical condition (see the proof of Lemma 3.3).

The existence and long-time behavior of solutions to problem (1.1) in the case $p = 2$, the semilinear case, have been studied in [16,17] and recently in [1]. The aim of this paper is to study the existence and long-time behavior of solutions to problem (1.1) in the case $2 \leqslant p < N$. Noting that the conditions imposed on f provide the global existence of a weak solution to problem (1.1), but not uniqueness. Thus, when studying the long-time behavior of solutions, in order to handle non-uniqueness of solutions, we need to use the theory of global attractor for multi-valued semiflows.

In the last years, there have been some theories for which one can treat multi-valued semiflows and their asymptotic behavior, such as generalized semiflows theory of Ball [4–6] and theory of multi-valued semiflows of Melnik and Valero [19]. A comparison of these two theories can be found in [7]. We note also that the theory of trajectory attractors of Chepyzhov and Vishik [9,10] has also been fruitfully applied to treat equations without uniqueness. Thanks to these theories, the asymptotic behavior of equations without uniqueness of the Cauchy problems has been studied by several authors in the last years. There are two important reasons which justify the interest of the researches in such type of equations. On the one hand, they contain important models coming from Mathematical Physics, as we can see in the example of the relevant three-dimensional Navier–Stokes equations. On the other hand, they allow to weaken the conditions imposed on the non-linear functions involved in the equations, which are in many cases very restrictive. In this way we can extend the class of equations for which the asymptotic behavior of solutions can be studied. Several results concerning the existence of global attractors in the case of non-uniqueness have been proved for parabolic problems. However, most of them have been devoted to the existence of global attractors for semilinear non-degenerate parabolic equations and systems (see, e.g. [14,15,20,22] and references therein). In this paper we prove the existence of a global attractor for a class of quasilinear degenerate parabolic equations in an arbitrary domain. Here the condition (H_α) in the case of bounded domains and $(H^{\infty}_{\alpha,\beta})$ in the case of unbounded domain ensure the compactness of some necessary embeddings (see Proposition 2.1 for details).

Let us explain the methods used in the paper. First, by the compactness method and monotonicity method [18, Chapters 1, 2] we prove the global existence of a weak solution. Let $D(u_0)$ be the set of all global weak solutions of the problem (1.1) with the initial data u_0. We define the multi-valued map $\mathcal{G} : \mathbb{R}^+ \times L^2(\Omega) \longmapsto 2^{L^2(\Omega)}$ as follows

$$\mathcal{G}(t, u_0) = \{u(t) : u(\cdot) \in D(u_0)\}.$$

We show that $\mathcal{G}$ is a strict multi-valued semiflow and then use the theory of global attractor for multi-valued semiflows of Melnik and Valero [19] to prove the existence of a compact global attractor in $L^2(\Omega)$ for the multi-valued semiflow $\mathcal{G}$. Our main results can be summarized in the following theorem.

Theorem 1.1. *Under the conditions (H1)–(H4), problem (1.1) defines a strict multivalued semiflow $\mathcal{G} : \mathbb{R}^+ \times L^2(\Omega) \longmapsto 2^{L^2(\Omega)}$, which possesses an invariant compact global attractor $\mathcal{A}$ in $L^2(\Omega)$.*

The rest of the paper is organized as follows. In Sect. 2, to study problem (1.1) we introduce the weighted Sobolev space related to the functional formulation of the problem and prove some compactness results, which are generalizations of the results in the case $p = 2$ of [12]. In Sects. 3 and 4, we only consider the case of an unbounded domain for the sake of clarity and because it is more complicated. Section 3 is devoted to prove the global existence of a weak solution to problem (1.1) using compactness and monotonicity methods. In Section 4, the existence of an invariant compact global attractor for the m-semiflow generated by problem (1.1) is proved. In the last section, we give some remarks on similar results in the case of a bounded domain and results in the case of uniqueness, which generalize some known results related to the p-Laplacian equation with uniqueness in a bounded domain.

2. Preliminaries

In order to study the problem (1.1), we introduce the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega, \sigma)$ defined as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|v\|_{\mathcal{D}_0^{1,p}(\Omega,\sigma)} = \left(\int_\Omega \sigma |\nabla v|^p dx\right)^{\frac{1}{p}} \tag{2.1}$$

and denote by $\mathcal{D}^{-1,p'}(\Omega, \sigma)$ the dual space of $\mathcal{D}_0^{1,p}(\Omega, \sigma)$.

We now prove some compactness results, which are generalizations of the results in the case $p = 2$ of Caldiroli and Musina [12].

Proposition 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N, N \geqslant 2$, and σ satisfies the hypothesis $(\mathcal{H}_\alpha)$. Then the following embeddings hold:*

(i) $\mathcal{D}_0^{1,p}(\Omega, \sigma) \subset W_0^{1,\beta}(\Omega)$ *continuously if* $1 \leqslant \beta < \frac{pN}{N+\alpha}$,
(ii) $\mathcal{D}_0^{1,p}(\Omega, \sigma) \subset L^r(\Omega)$ *compactly if* $1 \leqslant r < p_\alpha^*$, *where* $p_\alpha^* = \frac{pN}{N-p+\alpha}$.

Proof. The Hölder inequality yields

$$\int_\Omega |\nabla u|^\beta \leqslant \left(\int_\Omega \sigma|\nabla u|^p\right)^{\frac{\beta}{p}} \left(\int_\Omega \sigma^{-\frac{\beta}{p-\beta}}\right)^{\frac{p-\beta}{p}}. \tag{2.2}$$

From the assumption $(\mathcal{H}_\alpha)$, it is easy to see that there exist a finite set $\{x_1, ..., x_m\}$ in $\overline{\Omega}$ and $r > 0$ such that

$$\sigma(x) \geqslant \delta|x - x_i|^\alpha, \text{ for } x \in B_r(x_i) \cap \Omega, \quad i = 1, ..., m;$$
$$\sigma(x) \geqslant \delta \text{ for } x \in \Omega\backslash \bigcup_i B_r(x_i)$$

for some $\delta > 0$. On the other hand, $|x - x_i|^{-\frac{\alpha\beta}{p-\beta}} \in L^1(\Omega)$ if $\beta < \frac{pN}{N+\alpha}$. Hence we get the conclusion (i) and we can take $p^*_\alpha = \frac{pN}{N-p+\alpha}$. The second conclusion follows from (i) and the Rellich–Kondrachov embedding theorem. □

Proposition 2.2. *Let Ω be an unbounded domain in $\mathbb{R}^N$, $N \geqslant 2$, and let σ satisfy assumption $(\mathcal{H}^\infty_{\alpha,\beta})$. Then the embedding $\mathcal{D}^{1,p}_0(\Omega, \sigma) \subset L^r(\Omega)$ holds compactly for every $r \in (p^*_\beta \wedge 1, p^*_\alpha)$.*

Proof. Let $\{u_m\}$ be a sequence in $\mathcal{D}^{1,p}_0(\Omega, \sigma)$ such that $u_m \rightharpoonup 0$ in $\mathcal{D}^{1,p}_0(\Omega, \sigma)$. For any fixed $r \in (p^*_\beta \wedge 1, p^*_\alpha)$, we have to prove that $u_m \to 0$ strongly in $L^r(\Omega)$. For $R > 0$, write B_R for the ball centered at 0 with radius R. Using Proposition 2.1, we see that $\mathcal{D}^{1,p}_0(B_R, \sigma) \subset L^r(B_R)$ compactly. Then $\|u_m\|_{L^r(B_R)} \to 0$ as $m \to \infty$. On the contrary, we assume that $\|u_m\|_{L^r(\Omega)} \not\to 0$. Then $\|u_m\|_{L^r(\Omega\backslash B_R)} \not\to 0$, so there exist $\eta > 0$ and a subsequence of u_m, still denoted by u_m, such that

$$\int_{\Omega\backslash B_R} |u_m|^r \geqslant \eta, \quad \text{for all } R > 0. \tag{2.3}$$

Choose a function $\varphi \in C^\infty(\mathbb{R}^N)$ such that

- $0 \leqslant \varphi \leqslant 1$,
- $\varphi = 0$ in B_R and $\varphi = 1$ in $\Omega\backslash B_{2R}$.

Now putting $\hat{u}_m = \varphi u_m$, we have

$$\int_\Omega \sigma(x)|\nabla \hat{u}_m|^p \leqslant C\left[\int_\Omega \sigma(x)|\nabla u_m|^p + \int_\Omega \sigma(x)|u_m|^p|\nabla\varphi|^p\right]. \tag{2.4}$$

One can rewrite the last integral as

$$\int_\Omega \sigma(x)|u_m|^p|\nabla\varphi|^p = \int_{\Omega\cap(B_{2R}\backslash B_R)} \sigma(x)|u_m|^p|\nabla\varphi|^p.$$

Using Proposition 2.1 again for the bounded domain $\Omega \cap (B_{2R}\backslash B_R)$, we obtain that $u_m \to 0$ a.e. in $\Omega \cap (B_{2R}\backslash B_R)$ and hence

$$\int_\Omega \sigma(x)|u_m|^p|\nabla\varphi|^p = o(1) \text{ as } m \to \infty. \tag{2.5}$$

Taking $\gamma \in (0,p)$ such that $p_\gamma^* = r$ and using the Caffarelli–Kohn–Nirenberg inequality, we have

$$\left(\int_{\Omega\setminus B_{2R}} |u_m|^{p_\gamma^*}\right)^{\frac{p}{p_\gamma^*}} = \left(\int_{\Omega\setminus B_{2R}} |\hat{u}_m|^{p_\gamma^*}\right)^{\frac{p}{p_\gamma^*}} \leqslant \left(\int_{\Omega} |\hat{u}_m|^{p_\gamma^*}\right)^{\frac{p}{p_\gamma^*}} \leqslant \int_{\Omega} |x|^\gamma |\nabla \hat{u}_m|^p. \tag{2.6}$$

Since σ satisfies $(\mathcal{H}_{\alpha,\beta}^\infty)$, one can see that

$$\sigma(x) \geqslant \delta|x|^\beta \geqslant \delta R^{\beta-\gamma}|x|^\gamma \text{ for some } \delta > 0 \text{ and for all } x \in \Omega\setminus B_R$$

with R large enough. Putting this together with (2.6), we arrive at

$$\left(\int_{\Omega\setminus B_{2R}} |u_m|^{p_\gamma^*}\right)^{\frac{p}{p_\gamma^*}} \leqslant \delta^{-1} R^{\gamma-\beta} \int_{\Omega} \sigma(x)|\nabla \hat{u}_m|^p. \tag{2.7}$$

This combining with (2.4) and (2.5) imply that

$$\left(\int_{\Omega\setminus B_{2R}} |u_m|^{p_\gamma^*}\right)^{\frac{p}{p_\gamma^*}} \leqslant C R^{\gamma-\beta} \int_{\Omega} \sigma(x)|\nabla u_m|^p + o(1). \tag{2.8}$$

Taking (2.3) into account, we see that (2.8) leads to a contradiction when R is chosen large enough. The proof is complete. □

Remark 1. In fact, the result in Proposition 2.2 is valid if σ satisfies condition $(\mathcal{H}_{\alpha,\beta}^\infty)$ with $\alpha < p < \beta$. The condition $\beta > p + \frac{N}{2}(p-2)$ is made to ensure that $p_\beta^* < 2$, which is necessary for proving the dissipativeness of the multi-valued semiflow generated by problem (1.1) (see Lemma 4.3).

The following proposition is an immediate consequence of the compactness results established above.

Proposition 2.3. *Assume that σ satisfies assumption (H1). Then*

$$\lambda_1 = \inf\left\{ \frac{\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\sigma)}}{\|u\|^p_{L^p(\Omega)}} \;\middle|\; u \in \mathcal{D}_0^{1,p}(\Omega,\sigma), u \neq 0 \right\}$$

is a positive number and it is attained in $\mathcal{D}_0^{1,p}(\Omega,\sigma)$ by a non-negative and unique (up to a multiplication constant) function u_1, which is a weak solution of the problem

$$\begin{cases} -div(\sigma(x)|\nabla u_1|^{p-2}\nabla u_1) = \lambda_1 |u_1|^{p-2}u_1 & in\ \Omega, \\ u_1 = 0 & on\ \partial\Omega. \end{cases}$$

Proof. The proof is standard and we omit it. □

The next two propositions, which are easily proved by using similar arguments as in [18, Chapter 2], give some important properties of the operator $L_{p,\sigma}$.

Proposition 2.4. *The operator $L_{p,\sigma}$ maps $\mathcal{D}_0^{1,p}(\Omega,\sigma)$ into its dual $\mathcal{D}^{-1,p'}(\Omega,\sigma)$. Moreover,*

i) *$L_{p,\sigma}$ is hemicontinuous, i.e. for all $u, v, w \in \mathcal{D}_0^{1,p}(\Omega,\sigma)$, the map $\lambda \mapsto \langle L_{p,\sigma}(u+\lambda v), w\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.*

(ii) *$L_{p,\sigma}$ is monotone, i.e. $\langle L_{p,\sigma}u - L_{p,\sigma}v, u-v\rangle \geqslant 0$ for all $u, v \in \mathcal{D}_0^{1,p}(\Omega,\sigma)$.*

Proposition 2.5. *Assume that*

- $u_n \rightharpoonup u$ *in* $L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma))$,
- $L_{p,\sigma}u_n \rightharpoonup \psi$ *in* $L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma))$ *and*
- $\overline{\lim_{n\to\infty}} \langle L_{p,\sigma}u_n, u_n\rangle \leqslant \langle \psi, u\rangle$.

Then $\psi = L_{p,\sigma}u$.

3. Existence of a global weak solution

Denote

$$\begin{aligned} Q_T &= \Omega \times (0,T), \\ V &= L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,\sigma)) \cap L^2(Q_T) \cap L^q(Q_T), \\ V^* &= L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma)) + L^2(Q_T) + L^{q'}(Q_T). \end{aligned}$$

In what follows, we assume that $u_0 \in L^2(\Omega)$ is given.

Definition 3.1. A function $u(x,t)$ is called a weak solution of (1.1) on $(0,T)$ iff

$$\begin{aligned} &u \in V, \quad \frac{\partial u}{\partial t} \in V^*, \\ &u|_{t=0} = u_0 \text{ a.e. in } \Omega \end{aligned}$$

and

$$\int_{Q_T} \left(\frac{\partial u}{\partial t}\xi + \sigma|\nabla u|^{p-2}\nabla u \nabla \xi + (f(x,u) - \lambda |u|^{p-2}u)\xi \right) dxdt = 0 \tag{3.1}$$

for all test functions $\xi \in V$.

We begin this section by showing that the assumption (H3) is necessary for studying the long-time behavior of solutions to problem (1.1). We will prove that if $q \leqslant p$ and $\lambda > \lambda_1$, the solutions of (1.1) may blow-up in a finite time.

Proposition 3.1. *Let $p > 2$ and $f(x,u) = \mu u$ with $\mu > 0$. Then the solutions of problem (1.1) blow-up in a finite time for some class of initial data u_0.*

Proof. Put

$$\begin{aligned} E(t) &= \frac{1}{p}\int_\Omega \sigma(x)|\nabla u|^p - \frac{\lambda}{p}\int_\Omega |u|^p + \frac{\mu}{2}\int_\Omega u^2, \\ H(t) &= \frac{1}{2}\int_\Omega u^2. \end{aligned}$$

By computations, we have

$$E'(t) = -\|u_t\|^2_{L^2(\Omega)} \text{ in } \mathcal{D}'(\mathbb{R}^+),$$
$$H'(t) + pE(t) = \frac{(p-2)\mu}{2}\int_\Omega u^2 \text{ in } \mathcal{D}'(\mathbb{R}^+).$$

Then

$$H''(t) - (p-2)\mu H'(t) = -pE'(t) \text{ in } \mathcal{D}'(\mathbb{R}^+). \tag{3.2}$$

On the other hand, one has

$$(H'(t))^2 = \left(\int_\Omega uu_t\right)^2 \leqslant \|u\|^2_{L^2(\Omega)}\|u_t\|^2_{L^2(\Omega)} = -2H(t)E'(t).$$

Combining this with (3.2), we obtain

$$\frac{p}{2}(H'(t))^2 \leqslant H(t)[H''(t) - (p-2)\mu H'(t)].$$

Multiplying last inequality by $H^{-1-\frac{p}{2}}$, we arrive at

$$\frac{p}{2}H^{-1-\frac{p}{2}}(H')^2 - H^{-\frac{p}{2}}H'' + (p-2)\mu H^{-\frac{p}{2}}H' \leqslant 0. \tag{3.3}$$

Now putting $k(t) = H^{1-\frac{p}{2}}(t)$, (3.3) turns to

$$k''(t) - (p-2)\mu k'(t) \leqslant 0.$$

It follows that

$$k'(t) - (p-2)\mu k(t) \leqslant \ell,$$

where $\ell = k'(0) - (p-2)\mu k(0)$. An application of the Gronwall inequality yields

$$k(t) \leqslant e^{(p-2)\mu t}\left[k(0) + \frac{\ell}{(p-2)\mu}\right] - \frac{\ell}{(p-2)\mu}.$$

The last inequality implies that, the blow-up of solutions of (1.1) occurs if the following holds

$$\frac{\ell}{\ell + (p-2)\mu k(0)} > 0,$$

or equivalently

$$\frac{k(0)}{k'(0)} < \frac{1}{(p-2)\mu}.$$

This takes place if one has

$$\frac{H(0)}{H'(0)} > -\frac{1}{2\mu}. \tag{3.4}$$

We now testify to (3.4) for a class of initial data. Taking $u_0 = \theta u_1$, where $\theta > 0$ and u_1 is the eigenfunction corresponding to λ_1, we see that

$$H'(0) = \theta^p(\lambda - \lambda_1)\int_\Omega |u_1|^p - \theta^2\mu\int_\Omega u_1^2.$$

It follows that, if θ is taken large enough then $H'(0) > 0$ and (3.4) holds. This completes the proof. $\square$

The following proposition makes the initial condition in problem (1.1) meaningful.

Proposition 3.2. *If $u \in V$ and $\dfrac{du}{dt} \in V^*$ then $u \in C([0,T]; L^2(\Omega))$.*

Proof. We select a sequence $u_n \in C^1([0,T]; \mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^2(\Omega) \cap L^q(\Omega))$ such that

$$\begin{cases} u_n \longrightarrow u \text{ in } V \\ \frac{\partial u_n}{\partial t} \longrightarrow \frac{\partial u}{\partial t} \text{ in } V^*. \end{cases}$$

Then, for all $t, t_0 \in [0,T]$, we have

$$\|u_n(t) - u_m(t)\|^2_{L^2(\Omega)} = \|u_n(t_0) - u_m(t_0)\|^2_{L^2(\Omega)} + 2\int_{t_0}^{t} \langle u_n'(s) - u_m'(s), u_n(s) - u_m(s)\rangle \, ds.$$

We choose t_0 so that

$$\|u_n(t_0) - u_m(t_0)\|^2_{L^2(\Omega)} = \frac{1}{T}\int_0^T \|u_n(t) - u_m(t)\|^2 dt.$$

We have the estimates

$$\begin{aligned} &\int_\Omega |u_n(t) - u_m(t)|^2 dx \\ &= \frac{1}{T}\int_\Omega\int_0^T |u_n(t) - u_m(t)|^2 dt dx \\ &\quad + 2\int_\Omega\int_{t_0}^{t} (u_n'(s) - u_m'(s))\,(u_n(s) - u_m(s))\, ds dx \\ &\leqslant \frac{1}{T}\int_\Omega\int_0^T |u_n(t) - u_m(t)|^2 dt dx + 2\|u_n' - u_m'\|_{V^*}\|u_n - u_m\|_V. \end{aligned}$$

Hence $\{u_n\}$ is a Cauchy sequence in $C([0,T]; L^2(\Omega))$. Thus, the sequence $\{u_n\}$ converges in $C([0,T]; L^2(\Omega))$ to a function $v \in C([0,T]; L^2(\Omega))$. Since $u_n(t) \longrightarrow u(t) \in L^2(\Omega)$ for a.e. $t \in [0,T]$, we deduce that $u = v$ a.e. $t \in [0,T]$. After redefining on a subset of zero-measure, we get $u \in C([0,T]; L^2(\Omega))$. $\square$

Before proving the existence result, we need an auxiliary lemma.

Lemma 3.3. *Let $\{u_n\}$ be a bounded sequence in $L^p(0,T; \mathcal{D}_0^{1,p}(\Omega,\sigma))$ such that $\{u_n'\}$ is bounded in V^*. If (H1) and (H4) hold, then $\{u_n\}$ converges almost everywhere in Q_T up to a subsequence.*

Proof. By hypothesis (H4) and Proposition 2.2, one can take a number $r \in (p_\beta^* \wedge 1, p] \cap \mathcal{I}[p', q']$ such that

$$\mathcal{D}_0^{1,p}(\Omega,\sigma) \subset\subset L^r(\Omega). \tag{3.5}$$

Since $r' \in \mathcal{I}[p,q]$, we have

$$L^p(\Omega) \cap L^q(\Omega) \cap L^2(\Omega) \subset L^{r'}(\Omega)$$

and therefore

$$L^r(\Omega) \subset L^{p'}(\Omega) + L^{q'}(\Omega) + L^2(\Omega). \tag{3.6}$$

Using Proposition 2.2 again, we see that

$$\mathcal{D}_0^{1,p}(\Omega,\sigma) \subset L^p(\Omega).$$

This and (3.6) follow that

$$L^r(\Omega) \subset W^* := \mathcal{D}^{-1,p'}(\Omega) + L^{q'}(\Omega) + L^2(\Omega).$$

Now with (3.5), we have an evolution triple

$$\mathcal{D}_0^{1,p}(\Omega,\sigma) \subset\subset L^r(\Omega) \subset W^*. \tag{3.7}$$

The boundedness of $\{u_n'\}$ in V^* ensures that $\{u_n'\}$ is also bounded in $L^s(0,T;W^*)$, where $s = \min\{p', q', 2\}$. Thanks to the Compactness Lemma in [18, p. 58], $\{u_n\}$ is precompact in $L^p(0,T;L^r(\Omega))$ and therefore in $L^r(0,T;L^r(\Omega))$ since $r \leq p$. This implies that $\{u_n\}$ has an a.e. convergent subsequence. □

Theorem 3.4. *Under the assumptions (H1)–(H4), for each $u_0 \in L^2(\Omega)$ and $T > 0$ given, problem (1.1) has at least one weak solution on $(0,T)$.*

Proof. Consider the approximating solution $u_n(t)$ in the form

$$u_n(t) = \sum_{k=1}^{n} u_{nk}(t) e_k,$$

where $\{e_j\}_{j=1}^{\infty}$ is a basis of $\mathcal{D}_0^{1,p}(\Omega,\sigma) \cap L^q(\Omega) \cap L^2(\Omega)$, which is orthogonal in $L^2(\Omega)$. We get u_n from solving the problem

$$\begin{aligned}
&\langle \frac{du_n}{dt}, e_k\rangle = -\langle L_{p,\sigma} u_n, e_k\rangle + \langle \lambda |u_n|^{p-2} u_n - f(x,u_n), e_k\rangle,\\
&(u_n(0), e_k) = (u_0, e_k), \quad k = 1, \dots, n.
\end{aligned}$$

Since f is continuous, using the Peano theorem, we get the local existence of u_n. We now establish some *a priori* estimates for u_n and u_n'. We have

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \int_\Omega \sigma |\nabla u_n|^p + \int_\Omega f(x,u_n) u_n = \lambda \int_\Omega |u_n|^p. \tag{3.8}$$

Using assumption (H2), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \int_\Omega \sigma |\nabla u_n|^p + C_2 \int_\Omega |u_n|^q \leqslant \lambda \int_\Omega |u_n|^p + \int_\Omega h_2. \tag{3.9}$$

First we consider the case $q > p$. By virtue of the interpolation inequality and Proposition 2.2, one has

$$\lambda \|u_n\|^p_{L^p(\Omega)} \leqslant C \|u_n\|^{p\theta}_{L^q(\Omega)} \|u_n\|^{p(1-\theta)}_{L^r(\Omega)} \leqslant C \|u_n\|^{p\theta}_{L^q(\Omega)} \|u_n\|^{p(1-\theta)}_{\mathcal{D}_0^{1,p}(\Omega,\sigma)}$$

for some $r \in (p^*_\beta \wedge 1, p)$, $\theta \in (0,1)$ and $C > 0$. Hence

$$\begin{aligned}\lambda\|u_n\|^p_{L^p(\Omega)} &\leqslant \frac{1}{2}\|u_n\|^p_{\mathcal{D}^{1,p}_0(\Omega,\sigma)} + C\|u_n\|^p_{L^q(\Omega)}\\ &\leqslant \frac{1}{2}\|u_n\|^p_{\mathcal{D}^{1,p}_0(\Omega,\sigma)} + \frac{C_2}{2}\|u_n\|^q_{L^q(\Omega)} + C_3,\end{aligned}$$

where we have used the Young inequality.

Combining with (3.9), we infer that

$$\|u_n(t)\|^2_{L^2(\Omega)} + \int_{Q_t}\sigma|\nabla u_n|^p + C_2\int_{Q_t}|u_n|^q \leqslant \|u_n(0)\|^2_{L^2(\Omega)} + 2t\left(C_3 + \int_\Omega h_2\right) \tag{3.10}$$

for any $t \in (0,T]$.

In the case $q \leqslant p$, using hypothesis (H3), we get the following estimate from (3.9)

$$\frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega)} + \left(1 - \frac{\lambda}{\lambda_1}\right)\int_{Q_t}\sigma|\nabla u_n|^p + C_2\int_{Qt}|u_n|^q \leqslant \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} + t\int_\Omega h_2. \tag{3.11}$$

Both of (3.10) and (3.11) lead to

- $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$,
- $\{u_n\}$ is bounded in $L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))$ (and therefore it is bounded in $L^p(Q_T)$),
- $\{u_n\}$ is bounded in $L^q(Q_T)$.

Using Lemma 3.3, we see that $u_n \to u$ a.e. in Q_T. Then $f(x,u_n) \to f(x,u)$ a.e. in Q_T. In addition, the boundedness of $\{u_n\}$ in $L^q(Q_T)$ and hypothesis (H2) follow that $\{f(x,u_n)\}$ is bounded in $L^{q'}(Q_T)$ and hence $f(x,u_n) \rightharpoonup \chi$ in $L^{q'}(Q_T)$. Thus $\chi = f(x,u)$ thanks to Lemma 1.3 in [18].

We rewrite the equation (1.1) in V^* as

$$u_n' = \lambda|u_n|^{p-2}u_n - L_{p,\sigma}u_n - f(x,u_n) \tag{3.12}$$

and perform the following estimate deduced from the Hölder inequality

$$\begin{aligned}\left|\langle L_{p,\sigma}u_n, v\rangle\right| &= \left|\int_0^T dt\int_\Omega \sigma^{\frac{p-1}{p}}|\nabla u_n|^{p-2}\nabla u_n(\sigma^{\frac{1}{p}}\nabla v)\right|\\ &\leqslant \|u_n\|^{\frac{p}{p'}}_{L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))}\|v\|_{L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))}.\end{aligned}$$

Using the boundedness of $\{u_n\}$ in $L^p(0,T;\mathcal{D}^{1,p}_0(\Omega,\sigma))$ again, we infer that

$$\{L_{p,\sigma}u_n\} \text{ is bounded in } L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma)). \tag{3.13}$$

Then it follows that $\{u_n'\}$ is bounded in V^*. Therefore

- $u_n' \rightharpoonup u'$ in V^*,
- $L_{p,\sigma}u_n \rightharpoonup \psi$ in $L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma))$.

Now taking (3.12) into account, we obtain the following equation in V^*

$$u' = \lambda |u|^{p-2}u - \psi - f(x,u). \tag{3.14}$$

In order to prove that u is a weak solution of (1.1), it remains to show that $\psi = L_{p,\sigma}u$. Noticing that

$$\begin{aligned}\langle L_{p,\sigma}u_n, u_n\rangle &= \int_0^T dt \int_\Omega \sigma |\nabla u_n|^p dx \qquad (3.15)\\ &= \int_0^T dt \int_\Omega (\lambda |u_n|^p - f(x,u_n)u_n)dx + \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)}.\end{aligned}$$

It follows from the formulation of $u_n(0)$ that $u_n(0) \to u_0$ in $L^2(\Omega)$. Moreover, by the lower semi-continuity of $\|.\|_{L^2(\Omega)}$ we obtain

$$\|u(T)\|_{L^2(\Omega)} \leqslant \liminf_{n\to\infty} \|u_n(T)\|_{L^2(\Omega)}. \tag{3.16}$$

Meanwhile, by the Lebesgue dominated theorem, one can check that

$$\int_0^T dt \int_\Omega (\lambda |u|^p - f(x,u)u)dx = \lim_{n\to\infty} \int_0^T dt \int_\Omega (\lambda |u_n|^p - f(x,u_n)u_n)dx.$$

This fact and (3.15)–(3.16) imply

$$\begin{aligned}\overline{\lim_{n\to\infty}} \langle L_{p,\sigma}u_n, u_n\rangle \leqslant &\int_0^T dt \int_\Omega (\lambda |u|^p - f(x,u)u)dx \\ &+ \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)}. \qquad (3.17)\end{aligned}$$

In view of (3.14), we have

$$\int_0^T dt \int_\Omega (\lambda |u|^p - f(x,u)u)dx + \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} = \langle \psi, u\rangle.$$

This and (3.17) deduce

$$\overline{\lim_{n\to\infty}} \langle L_{p,\sigma}u_n, u_n\rangle \leqslant \langle \psi, u\rangle.$$

The proof is now completed thanks to Proposition 2.5. □

4. Existence of a global attractor

For the convenience of the reader, we recall some basic concepts and results related to the theory of global attractors for multi-valued semiflows [19] which we will use.

Definition 4.1. Let E be a Banach space. The mapping

$$\mathcal{G} : [0, +\infty) \times E \to 2^E$$

is called an m-semiflow if the following conditions are satisfied

(1) $\mathcal{G}(0, w) = w$ for arbitrary $w \in E$;
(2) $\mathcal{G}(t_1 + t_2, w) \subset \mathcal{G}(t_1, \mathcal{G}(t_2, w))$ for all $w \in E, t_1, t_2 \geqslant 0$.

It is called a strict m-semiflow if $\mathcal{G}(t_1 + t_2, w) = \mathcal{G}(t_1, \mathcal{G}(t_2, w))$, for all $w \in E,\ t_1, t_2 \in \mathbb{R}^+$.

Definition 4.2. The set $\mathcal{A}$ is said to be a global attractor of the m-semiflow $\mathcal{G}$ if the following conditions hold

(1) $\mathcal{A}$ is an attracting, i.e. $dist(\mathcal{G}(t,B),\mathcal{A}) \to 0$ as $t \to \infty$ for all bounded subsets $B \subset E$;
(2) $\mathcal{A}$ is negatively semi-invariant, i.e. $\mathcal{A} \subset \mathcal{G}(t,\mathcal{A})$ for arbitrary $t \geqslant 0$;
(3) If $\mathcal{B}$ is an attracting of $\mathcal{G}$ then $\mathcal{A} \subset \bar{\mathcal{B}}$.

The following theorem gives the sufficient conditions for the existence of a global attractor for the m-semiflow $\mathcal{G}$.

Theorem 4.1 ([19]). *Suppose that the m-semiflow $\mathcal{G}$ has the following properties*

(1) *$\mathcal{G}$ is pointwise dissipative, i.e. there exists $K > 0$ such that for $u_0 \in E, u(t) \in \mathcal{G}(t,u_0)$ one has $\|u(t)\|_E \leqslant K$ if $t \geqslant t_0(\|u_0\|_E)$;*
(2) *$\mathcal{G}(t,.)$ is a closed map for any $t \geqslant 0$, i.e. if $\xi_n \to \xi$, $\eta_n \to \eta$, $\xi_n \in \mathcal{G}(t,\eta_n)$ then $\xi \in \mathcal{G}(t,\eta)$;*
(3) *$\mathcal{G}$ is asymptotically upper semicompact, i.e. if B is a bounded set in E such that for some $T(B)$, $\gamma^+_{T(B)}(B)$ is bounded, any sequence $\xi_n \in \mathcal{G}(t_n,B)$ with $t_n \to \infty$ is precompact in E. Here $\gamma^+_{T(B)}(B)$ is the orbits after the time $T(B)$.*

Then $\mathcal{G}$ has a compact global attractor in E. Moreover, if $\mathcal{G}$ is a strict m-semiflow then $\mathcal{A}$ is invariant, i.e. $\mathcal{G}(t,\mathcal{A}) = \mathcal{A}$ for any $t \geqslant 0$.

By Theorem 3.4, we construct the multi-valued mapping as follows

$$\mathcal{G}(t,u_0) = \{u(t) \mid u(.) \quad \text{is the solution of } (1.1), u(0) = u_0\}.$$

We now check that $\mathcal{G}$ is a strict m-semiflow in the sense of Definition 4.1. Assume that $\xi \in \mathcal{G}(t_1+t_2,u_0)$, then $\xi = u(t_1+t_2)$ where $u(t)$ is a solution of problem (1.1). Denoting $v(t) = u(t+t_2)$, we see that $v(.)$ is also in the set of solutions of (1.1) with respect to initial condition $v(0) = u(t_2)$. Therefore $\xi = v(t_1) \in \mathcal{G}(t_1,u(t_2)) \subset \mathcal{G}(t_2,\mathcal{G}(t_2,u_0))$. It is remain to show that $\mathcal{G}(t_1,\mathcal{G}(t_2,u_0)) \subset \mathcal{G}(t_1+t_2,u_0)$. If $\xi \in \mathcal{G}(t_1,\mathcal{G}(t_2,u_0))$ then $\xi = v(t_1)$, where $v(0) \in \mathcal{G}(t_2,u_0)$. One can suppose that $v(0) = u(t_2)$ where $u(0) = u_0$. Set

$$w(\tau) = \begin{cases} u(\tau), & 0 \leqslant \tau < t_2, \\ v(\tau - t_2), & \tau \geqslant t_2. \end{cases}$$

Since u and v are the solutions of (1.1), we obtain that w is a solution of (1.1) with $w(0) = u(0) = u_0$. In addition, by the fact that $\xi = v(t_1) = w(t_1+t_2)$, we have $\xi \in \mathcal{G}(t_1+t_2,u_0)$.

In order to show that $\mathcal{G}$ is pointwise dissipative, we need the following lemma (see [21]).

Lemma 4.2. *Assume that $y(t)$ is an absolutely continuous and non-negative function defined for $t > 0$. If there exist $a > 0, b > 0$ such that*

$$\frac{dy(t)}{dt} + ay^{\rho}(t) \leqslant b$$

where $\rho > 1$, *then*

$$y(t) \leqslant \left(\frac{b}{a}\right)^{\frac{1}{\rho}} + \frac{1}{[a(\rho-1)t]^{\frac{1}{\rho-1}}}.$$

Lemma 4.3. *The m-semiflow $\mathcal{G}$ generated by (1.1) is pointwise dissipative.*

Proof. Let $u(t) \in \mathcal{G}(t, u_0)$, one gets

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \int_\Omega \sigma|\nabla u|^p + C_2\|u\|^q_{L^q(\Omega)} \leqslant \lambda\int_\Omega |u|^p + \int_\Omega h_2.$$

Arguing as in proof of Theorem 3.4, we observe that

- In the case $q > p$, we have

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \int_\Omega \sigma|\nabla u(t)|^p + C_2\int_\Omega |u(t)|^q \leqslant 2C_3 + 2\int_\Omega h_2 \qquad (4.1)$$

  for some $C_3 > 0$.
- If $q \leqslant p$, the assumption (H3) allows us to state that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \left(1 - \frac{\lambda}{\lambda_1}\right)\int_\Omega \sigma|\nabla u(t)|^p + C_2\int_\Omega |u(t)|^q \leqslant \int_\Omega h_2. \quad (4.2)$$

Since $p \geqslant 2$ and $p^*_\beta < 2$, we see that $\mathcal{D}^{1,p}_0(\Omega) \subset L^2(\Omega)$. Then, in all cases, we have the following inequality

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + a\|u(t)\|^p_{L^2(\Omega)} \leqslant b,$$

where $a = a(\lambda, \lambda_1)$ and $b = b(\|h_2\|_{L^1(\Omega)})$ are positive numbers.

Applying Lemma 4.2 (or the Gronwall inequality for the case $p = 2$), we complete the proof. □

The following lemma plays an important role in this section.

Lemma 4.4. *$\mathcal{G}(t^*, .) : L^2(\Omega) \to L^2(\Omega)$ is a compact mapping for each $t^* \in (0, T]$.*

Proof. Assume that B is a bounded set in $L^2(\Omega)$ and $\xi_n \in \mathcal{G}(t^*, B)$. By the definition of $\mathcal{G}$, there exists a sequence $\{u_n(t)\}$ such that $u_n(t)$ is the solution of (1.1) with the initial data belongs to B and $u_n(t^*) = \xi_n$.

Then we have

$$\frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega)} + \int_{Q_t} \sigma|\nabla u_n|^p + \int_{Q_t} f(u_n)u_n = \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} + \lambda\int_{Q_t} |u_n|^p,$$

for $t \in (0, T]$. Repeating the arguments as in proof of Theorem 3.4, we infer that

- $u_n \to u$ a.e. in Q_T,
- $u_n(t) \rightharpoonup u(t)$ in $L^2(\Omega)$ for any $t \in [0, T]$,
- $u \in V$ and $u' \in V^*$.

Using Proposition 3.2, we obtain that u_n and u belong to $C([0,T];L^2(\Omega))$. In the case $t = t^*$, one has $u_n(t^*) \rightharpoonup u(t^*)$ in $L^2(\Omega)$. It remains to show that $\|u_n(t^*)\|_{L^2(\Omega)} \to \|u(t^*)\|_{L^2(\Omega)}$.

Let us denote

$$J_n(t) = \|u_n(t)\|^2_{L^2(\Omega)} - Ct(1 + \|h_2\|_{L^1(\Omega)}),$$

$$J(t) = \|u(t)\|^2_{L^2(\Omega)} - Ct(1 + \|h_2\|_{L^1(\Omega)})$$

for some constant $C > 0$. Obviously, J_n and J are decreasing on $[0,T]$ for C chosen large enough. In addition, $J_n(t) \to J(t)$ for a.e. $t \in [0,T]$. Suppose that $\{t_m\}$ is an increasing sequence in $[0,T], t_m \to t^*$ as $m \to \infty$. Then

- $J_n(t_m) \to J_n(t^*)$ as $m \to \infty$,
- $J_n(t_m) \to J(t_m)$ as $n \to \infty$.

So

$$J_n(t^*) - J(t^*) \leqslant J_n(t_m) - J(t^*) = J_n(t_m) - J(t_m) + J(t_m) - J(t^*) < \varepsilon,$$

for $\varepsilon > 0$. Similarly, $J(t^*) - J_n(t^*) < \varepsilon$. Therefore $J_n(t^*) \to J(t^*)$ and then $\|u_n(t^*)\|_{L^2(\Omega)} \to \|u(t^*)\|_{L^2(\Omega)}$ as $n \to \infty$. □

We can now finish the proof of the main result.

Proof of Theorem. 1.1. It suffices to check the hypotheses (2) and (3) in Theorem 4.1. Assume that $\xi_n \in \mathcal{G}(t, \eta_n)$, $\xi_n \to \xi, \eta_n \to \eta$ in $L^2(\Omega)$. Then there exists a sequence $\{u_n\}$ satisfying

$$u_n(t) = \xi_n, \quad u_n(0) = \eta_n.$$

Using the same arguments as in the proof of Theorem 3.4, we have

- $u_n(t) \rightharpoonup u(t)$ in $L^2(\Omega)$ for arbitrary $t \in [0,T]$ (and then $u(0) = \eta$),
- $f(x, u_n) \rightharpoonup f(x, u)$ in $L^{q'}(Q_T)$,
- $u_n' \rightharpoonup u'$ in V^*,
- $L_{p,\sigma} u_n \rightharpoonup L_{p,\sigma} u$ in $L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,\sigma))$

up to a subsequence. Hence, passing to the limit the following equality in V^*

$$u_n' + L_{p,\sigma} u_n = \lambda |u_n|^{p-2} u_n - f(x, u_n)$$

we conclude that $u(t)$ is the solution of (1.1) with respect to initial condition $u(0) = \eta$. Thus, $\xi \in \mathcal{G}(t, \eta)$.

For the hypothesis (3), one observes that

$$\mathcal{G}(t_n, B) = \mathcal{G}(t^* + t_n - t^*, B) \subset \mathcal{G}(t^*, \mathcal{G}(t_n - t^*, B)) \subset \mathcal{G}(t^*, B^*),$$

where $t^* > 0$ and B^* is a bounded set in $L^2(\Omega)$. Using Lemma 4.4, we see that, if $\xi_n \in \mathcal{G}(t_n, B)$ then $\{\xi_n\}$ is precompact in $L^2(\Omega)$. □

5. Some further remarks

In this section we discuss the case of a bounded domain $\Omega \subset \mathbb{R}^N, N \geqslant 2$ and the weight function $\sigma(x)$ satisfies condition $(\mathcal{H}_\alpha)$. Under the conditions (H1)–(H4) when Ω is bounded, using Proposition 2.1 and repeating the arguments in the case of an unbounded domain, one can prove that Theorem 3.4 is still true for this case.

It is worth noticing that under some additional conditions on f, for example,

$$(f(x,u) - f(x,v))\,(u - v) \geqslant -C|u - v|^2 \quad \text{for all } x \in \Omega, u, v \in \mathbb{R},$$

one can prove that the weak solution of problem (1.1) is unique. Then the m-semiflow $\mathcal{G}$ turn to be a single-valued one and, by the definition, the compact global attractor $\mathcal{A}$ obtained in Theorem 1.1 is exactly the one for single-valued semiflow in the usual sense [21].

Noting that in the case of a bounded domain and $\sigma(x)$ satisfies condition $(\mathcal{H}_\alpha)$ the problem (1.1) contains some important classes of parabolic equations, such as the semilinear heat equations (when $\sigma = \text{const} > 0, p = 2$), semilinear degenerate parabolic equations (when $p = 2$), the p-Laplacian equations (when $\sigma = 1, p \neq 2$), etc. Thus, in some sense, our results recover and extend some known results on the existence and long-time behavior of solutions to the semilinear heat equation and the p-Laplacian equations in a bounded domain [8,10,11,15,21].

References

[1] Anh, C.T., Hung, P.Q.: Global existence and long-time behavior of solutions to a class of degenerate parabolic equations. Ann. Pol. Math. **93**(3), 217–230 (2008)

[2] Abdelaoui, B., Peral, I.: On quasilinear elliptic equations related to some Caffarelli–Kohn–Nirenberg inequalities. Commun. Pure Appl. Anal. **2**(4), 539–566 (2003)

[3] Abdellaoui, B., Peral, I.: The effect of Harnack inequality on the existence and nonexistence results for quasi-linear parabolic equations related to Caffarelli–Kohn–Nirenberg inequalities. Nonlinear Differ. Equ. Appl. **14**, 335–360 (2007)

[4] Ball, J.M.: On the asymptotic behavior of generalized processes with applications to nonlinear evolution equations. J. Differ. Eq. **27**, 224–265 (1978)

[5] Ball, J.M.: Continuity properties and global attractor of generalized semiflows and the Navier–Stokes equations. J. Nonlinear Sci. **7**, 475–502 (1997)

[6] Ball, J.M.: Global attractor for damped semilinear wave equations. Discrete Contin. Dyn. Syst. **10**, 31–52 (2004)

[7] Caraballo, T., Marin-Rubio, P., Robinson, J.C.: A comparision between to theories for multi-valued semiflows and their asymptotic behaviour. Set Valued Anal. **11**, 297–322 (2003)

[8] Carvalho, A.N., Cholewa, J.W., Dlotko, T.: Global attractors for problems with monotone operators. Boll. Unione. Mat. Ital. Sez. B Artic. Ric. Mat. **2**(8), 693–706 (1999)

[9] Chepyzhov, V.V., Vishik, M.I.: Evolution equations and their trajectory attractor. J. Math. Pures Appl. **76**, 913–964 (1997)

[10] Chepyzhov, V.V., Vishik, M.I.: Attractors for Equations of Mathematical Physics. American Mathematical Society Colloquium Publications, Vol. 49, American Mathematical Society, Providence (2002)

[11] Cholewa, J.W., Dlotko, T.: Global Attractors in Abstract Parabolic Problems. Cambridge University Press, Cambridge (2000)

[12] Caldiroli, P., Musina, R.: On a variational degenerate elliptic problem. Nonlinear Diff. Equ. Appl. **7**, 187–199 (2000)

[13] Dautray, R., Lions, J.L.: Mathematical Analysis and Numerical Methods for Science and Technology, Vol. I: physical origins and classical methods. Springer, Berlin (1985)

[14] Kapustyan, A.V.: Global attractors of a nonautonomous reaction-diffusion equation. Diffrential Equations **38**(10), 1467–1471 (2002) Translation from Differensial'nye Uravneniya **38**(10), 1378–1381 2002

[15] Kapustyan, A.V., Shkundin, D.V.: Global attractor of one nonlinear parabolic equation. Ukrain. Math. Zh. **55**, 446–455 (2003)

[16] Karachalios, N.I., Zographopoulos, N.B.: Convergence towards attractors for a degenerate Ginzburg-Landau equation. Z. Angew. Math. Phys. **56**, 11–30 (2005)

[17] Karachalios, N.I., Zographopoulos, N.B.: On the dynamics of a degenerate parabolic equation: Global bifurcation of stationary states and convergence. Calc. Var. Partial Differ. Equ. **25**(3), 361–393 (2006)

[18] Lions, J.-L.: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Dunod, Paris (1969)

[19] Melnik, V.S., Valero, J.: On attractors of multi-valued semiflows and differential inclusions. Set Valued Anal. **6**(4), 83–111 (1998)

[20] Morillas, F., Valero, J.: Attractors for reaction-diffusion equations in $\mathbb{R}^N$ with continuous nonlinearity. Asymptot. Anal. **44**, 111–130 (2005)

[21] Temam, R.: Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd edn. Springer, Berlin (1997)

[22] Valero, J., Kapustyan, A.V.: On the connectedness and asymptotic behaviour of solutions of reaction-diffusion systems. J. Math. Anal. Appl. **323**, 614–633 (2006)

C. T. Anh, T. D. Ke
Department of Mathematics, Hanoi National University of Education,
136 Xuan Thuy, Cau Giay, Hanoi, Vietnam
e-mail: anhctmath@hnue.edu.vn

T. D. Ke
e-mail: ketd@hn.vnn.vn

Received: 10 March 2009.
Accepted: 30 October 2009.