Description of periodic p -harmonic functions on Cayley tree

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Abstract. We show that any periodic (with respect to normal subgroups of finite index of the group representation of the Cayley tree) p -harmonic function on a Cayley tree is a constant. For some normal subgroups of infinite index we describe a class of (non-constant) periodic p -harmonic functions. We also prove that linear combinations of the p -harmonic functions described for normal subgroups of infinite index are also p -harmonic.

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1. Introduction

A tree is connected acyclic graph. One special case of this is a Cayley tree, i.e., an infinite tree in which each vertex has exactly $k + 1$ incident edges. The Cayley tree can be represented as the group G_k which is the free product of $k + 1$ second-order cyclic groups [2, 3, 10]. The group representation of the Cayley tree was used in [2, 3, 9–12] to study models of statistical mechanics and describe the sets of periodic Gibbs measures, also to study random walk trajectories in a random medium on the Cayley tree. These problems are related to description of harmonic functions on trees. In [9], a natural generalization of the notion of harmonic functions on a Cayley tree is introduced. Using some properties of G_k a set of harmonic functions is described.

Harmonic functions play an important role in probability theory, dynamical systems, statistical mechanics, and theory of electrical circuits.

Note that p -harmonic functions are zeros of the p -Laplacian defined by

$$\Delta_p u = \operatorname{div} (|\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty.$$

If $p \neq 2$, then p -harmonicity is non-linear.

A good reference for the study of p -harmonic functions in R^n is the book [4].

We consider the equation $\Delta_p u = 0$ on a (discrete) Cayley tree. There are many papers devoted to p -harmonic functions on trees (e.g. see [1, 5, 7, 13]). In

[1], the asymptotic behaviour of harmonic and p -harmonic functions ($1 < p < \infty$) on trees, obtaining estimates about the Hausdorff dimension of “radial” limits are studied. In [13], the behaviour of harmonic functions on a homogeneous tree from the point of view of the tangential boundary convergence are studied. Most relevant to our paper is the paper by H. Kurata [7], where he discusses the p -harmonicity of the linear combination of p -harmonic functions in the Euclidean space and on a tree.

Since there is no theory of functional equations with the unknown functions defined on a tree, the problem of finding all solution of $\Delta_p u = 0$ on tree is by no means easy. On the other hand, the theory of functional equations is not very developed even for case in which the unknown function is defined on R , only the solutions of functional equations of some special forms are known (e.g. see [6, 8]). Thus, it is natural to find periodic (simple) solutions of $\Delta_p u = 0$ first.

The main goal of this paper is to describe the set of p -harmonic functions which are periodic with respect to subgroups of the group G_k .

In Sect. 2, we give some preliminary definitions and results which needed when describing p -harmonic (periodic) functions. Section 3 devoted to description of periodic p -harmonic functions (with respect to normal subgroups of finite index). We prove that only constant functions can be such periodic. Section 4 devoted to the problem in the case of infinite index. We find (explicitly) some p -harmonic functions which are periodic with respect to a subgroup of infinite index. In the last section we show that linear combinations of these functions are also p -harmonic.

2. Definitions and statement of the problem

2.1. Cayley tree

A Cayley tree is an infinite tree in which each vertex has exactly $k + 1$ incident edges (the Cayley tree of order $k \geq 1$). Let $\Gamma^k = (V, L)$ be the Cayley tree of order $k \geq 1$, where V and L are the vertex set and the edge set, respectively, of Γ^k .

If $x, y \in V$ are the endpoints of an edge $l \in L$, then x and y are said to be adjacent; in this case, we write $l = \langle x, y \rangle$. The Cayley tree is equipped with a distance $d(x, y)$, $x, y \in V$, given by the formula

$$d(x, y) = \min\{d : \exists x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V \text{ where } \langle x_0, x_1 \rangle, \dots, \langle x_{d-1}, x_d \rangle \text{ are adjacent}\}.$$

The sequence $\pi = \{x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V\}$ at which the minimum is attained is called the path from x to y . Let G_k be the group representation of the Cayley tree; i.e., G_k is the free product of $k + 1$ second-order cyclic groups with generators $a_1, \dots, a_{k+1}$ such that $a_i^2 = e$, $i = 1, \dots, k + 1$, where $e \in G_k$ is the identity element. Let $x \in G_k$. We denote by

$$S(x) = \{y \in G_k : \langle x, y \rangle\}$$

the set of vertices adjacent to x .

2.2. Normal subgroups of G_k

Let $\omega_x(a_i)$ be the number of occurrences of the letter $a_i, i = 1, \dots, k+1$, in the reduced word $x \in G_k$. Let $A \subseteq N_k = \{1, \dots, k+1\}$. The set $H_A = \{x \in G_k : \sum_{i \in A} \omega_x(a_i) \text{ is even}\}$ is a normal subgroup of index 2 of G_k . Some set of normal subgroups with index 2^m can be obtained by intersection $H_{A_1} \cap H_{A_2} \cap \dots \cap H_{A_m}$ for suitable $A_1, \dots, A_m \subseteq N_k$ (see [3]). The following theorem gives possible values of the (finite) index of normal subgroups.

Theorem 1. [9] *The group G_k does not have normal subgroups of odd index ($\neq 1$). It has a normal subgroup of arbitrary even index.*

Also there are normal subgroups of infinite index. Some of them can be described as following [11]. Fix $M \subseteq N_k$ such that $|M| > 1$. $|\bullet|$ is the cardinality of $\bullet$. Let the mapping $\pi_M : \{a_1, \dots, a_{k+1}\} \longrightarrow \{a_i, i \in M\} \cup \{e\}$ be defined by

$$\pi_M(a_i) = \begin{cases} a_i, & \text{if } i \in M \\ e, & \text{if } i \notin M. \end{cases}$$

Denote by G_M the free product of cyclic groups $\{e, a_i\}, i \in M$. Consider

$$f_M(x) = f_M(a_{i_1} a_{i_2} \dots a_{i_m}) = \pi_M(a_{i_1}) \pi_M(a_{i_2}) \dots \pi_M(a_{i_m}).$$

Then it is easy to see that f_M is a homomorphism and hence $H_M = \{x \in G_k : f_M(x) = e\}$ is a normal subgroup of infinite index.

2.3. p -Harmonic functions

The resistance r is a positive function on L such that $r(y, x) = r(x, y)$. Denote $\mathcal{R} = \{r(x, y) : (x, y) \in L\}$. We define the discrete ∇u and the discrete p -Laplacian $\Delta_p u$ for a function u on $V (= G_k)$ by

$$\begin{aligned} \nabla u(x, y) &= r(x, y)^{-1} (u(y) - u(x)), \\ \Delta_p u(x) &= \sum_{y \in S(x)} |\nabla u(x, y)|^{p-2} \nabla u(x, y), \end{aligned}$$

where $1 < p < \infty$. Let $D \subset V$. If $\Delta_p u = 0$ in D , then we say that u is p -harmonic in D [7, 14–16]. Let $\{u_1, \dots, u_m\}$ be an m -tuple of p -harmonic functions in D . If $p \neq 2$, then the linear combination of p -harmonic functions $u_1, \dots, u_m$ need not be p -harmonic. The m -tuple of p -harmonic functions in D has a linear relation if every linear combination $\sum_{j=1}^m t_j u_j$ is p -harmonic in D . Also we say that $\{u_1, \dots, u_m\}$ has a partial linear relation in D if $\sum_{j=1}^m t_j u_j$ is p -harmonic in D for some $t_1, \dots, t_m \in R \setminus \{0\}$.

Definition 1. Let $H \subset G_k$ be a subgroup. A function $\{u(x), x \in G_k\}$ is said to be H -periodic if $u(x) = u(yx)$ for any $x \in G_k$ and $y \in H$.

If H is a normal subgroup of finite index, then the description of H -periodic solutions of $\Delta_p u$ can be reduced to solving a system of equations with finitely many unknowns.

The main goal of this paper is to describe H -periodic p -harmonic functions for any normal subgroup H and $\forall p \in (1, \infty)$.

3. The case of finite index

Let G_k^* be a normal subgroup of (finite) index $m \geq 1$ for G_k . Denote by $\mathcal{H}_{p,k,m}(G_k^*, \mathcal{R})$ the set of all G_k^* -periodic, p -harmonic functions for given $p \in (1, \infty)$, $\mathcal{R}$, $m \geq 1$ and $k \geq 1$. The following theorem completely describes the set of all periodic p -harmonic functions for finite $m \geq 1$.

Theorem 2. *For $\forall p \in (1, \infty)$, $\forall m \geq 1$, $\forall k \geq 1$, $\forall G_k^* \subset G_k$ with index m and any fixed $\mathcal{R} = \{r(x, y) > 0 : (x, y) \in L\}$ the set $\mathcal{H}_{p,k,m}(G_k^*, \mathcal{R})$ contains only constant functions.*

Proof. Let $G_k | G_k^* = \{G_k^1, \dots, G_k^m\}$ be the factor group. Any G_k^* -periodic p -harmonic function $u(x)$ has the following form

$$u(x) = u_i \text{ if } x \in G_k^i, \quad i = 1, \dots, m \quad (1)$$

where u_i , $i = 1, \dots, m$ are a solution to the following system

$$\sum_{j=1}^m \sum_{y \in S_j(x)} \frac{|u_j - u_i|^{p-2}}{r^{p-1}(x, y)} (u_j - u_i) \mathbf{1}(S_j(x) \neq \emptyset) = 0, \quad i = 1, \dots, m \quad (2)$$

where $x \in G_k^i$, $S_j(x) = S(x) \cap G_k^j$, $j = 1, \dots, m$. Note that some $S_j(x)$ can be empty set, so

$$\mathbf{1}(S_j(x) \neq \emptyset) = \begin{cases} 1, & \text{if } S_j(x) \neq \emptyset \\ 0, & \text{if } S_j(x) = \emptyset. \end{cases}$$

We shall prove that the system (2) has only solutions $(u_1^*, \dots, u_m^*)$ with $u_1^* = \dots = u_m^*$. Assume that there is a solution $u^* = (u_1^*, \dots, u_m^*)$ with

$$u_i^* \neq u_j^* \quad \text{for some } i, j \in \{1, \dots, m\} \quad (3)$$

Denote $u_{i_0}^* = \max\{u_i^*, i = 1, \dots, m\}$.

From i_0 th equation of (2) we have

$$\begin{aligned} & \sum_{j=1}^m \sum_{y \in S_j(x)} \frac{|u_j^* - u_{i_0}^*|^{p-2}}{r^{p-1}(x, y)} \mathbf{1}(S_j(x) \neq \emptyset) (u_j^* - u_{i_0}^*) \\ &= \sum_{\substack{j=1: \\ S_j(x) \neq \emptyset}}^m \sum_{y \in S_j(x)} \frac{|u_j^* - u_{i_0}^*|^{p-2} (u_j^* - u_{i_0}^*)}{r^{p-1}(x, y)} < 0, \end{aligned} \quad (4)$$

since $u_j^* - u_{i_0}^* \leq 0$ and by (3) $u_j^* - u_{i_0}^* < 0$ for at least one $j \in \{1, \dots, m\} \setminus \{i_0\}$. The inequality (4) contradicts to the assumption that u^* is a solution of (2). Theorem is proved. $\square$

Remark. Theorem 2 consents with the maximum principle.

4. The case of infinite index

In this section, we show that for some subgroups of infinite index there are non-constant p -harmonic functions.

Consider $M = \{i, j\}$, $i \neq j \in N_k$ and $H_M = H_{ij} = \{x \in G_k : f_M(x) = e\}$ (see Sect. 2.2). It is known that H_{ij} is a normal subgroup of infinite index and the corresponding factor group can be written as (see [11]):

$$G_k/H_{ij} = \{\dots, H_{-2}, H_{-1}, H_0, H_1, H_2, \dots\}.$$

Lemma 1. *If $x \in H_n$ then $S(x) \subset H_{n-1} \cup H_n \cup H_{n+1}$. Moreover, $|S(x) \cap H_{n-1}| = 1$, $|S(x) \cap H_n| = k - 1$, $|S(x) \cap H_{n+1}| = 1$.*

Proof. If $x \in H_n$ then $f_M(x) = \underbrace{a_i a_j \dots a_i}_n$ (or $a_i a_j \dots a_j$ depending on n).

$S(x) = \{x a_s, s = 1, \dots, k + 1\}$. We have

$$f_M(x a_s) = f_M(x) f_M(a_s) = \underbrace{a_i a_j \dots a_i}_n f_M(a_s) = \begin{cases} \underbrace{a_i a_j \dots a_j}_{n-1} & \text{if } a_s = a_i \\ \underbrace{a_i a_j \dots}_{n-1} & \text{if } a_s \neq a_i, a_j \\ \underbrace{a_i a_j \dots a_i a_j}_{n+1} & \text{if } a_s = a_j \end{cases}$$

This relation completes the proof. $\square$

Note that any H_{ij} -periodic function $u(x)$ has the form

$$u(x) = u_n \text{ if } x \in H_n \quad (5)$$

Consider H_{ij} -periodic collection $\mathcal{R}$ of resistance functions $r(x, y)$ i.e. $r(x, y) = r_{nm}$ if $x \in H_n, y \in H_m$. By Lemma 1 the equation $\Delta_p u = 0$ can be written as

$$\frac{|u_{n+1} - u_n|^{p-2}}{r_{n,n+1}^{p-1}}(u_{n+1} - u_n) + \frac{|u_{n-1} - u_n|^{p-2}}{r_{n-1,n}^{p-1}}(u_{n-1} - u_n) = 0 \quad (6)$$

for any $n \in \mathbb{Z}$.

Denote $a_n = u_{n+1} - u_n$ then from (6) we see that a_n, a_{n-1} must have the same sign i.e., $a_n \cdot a_{n-1} > 0$, $n \in \mathbb{Z}$.

Thus u_n , $n \in \mathbb{Z}$ must be a monotone sequence.

Put

$$X_n = \frac{|a_n|^{p-2}}{r_{n,n+1}^{p-1}} a_n.$$

From (6) we get

$$X_n - X_{n-1} = 0, \quad \forall n \in \mathbb{Z}. \quad (7)$$

Hence $X_n = C = \text{Const}$ for any $n \in \mathbb{Z}$.

Since u_n must be monotone, we have two possibilities.

- 1) $u_{n+1} > u_n$ then $C > 0$. From $X_n = C$ we get

$$u_{n+1} = C^{\frac{1}{p-1}} \cdot r_{n,n+1} + u_n, \quad n \in \mathbb{Z}. \quad (8)$$

From (8) we obtain

$$u_n = C^{\frac{1}{p-1}} \sum_{s=-\infty}^{n-1} r_{s,s+1}, \quad n \in \mathbb{Z}. \quad (9)$$

- 2) $u_{n+1} < u_n$ then $-C > 0$. In this case we get

$$u_n = (-C)^{\frac{1}{p-1}} \sum_{s=n}^{+\infty} r_{s,s+1}, \quad n \in \mathbb{Z}. \quad (10)$$

Thus we have proved the following

Theorem 3. *Let $\mathcal{R}$ be H_{ij} -periodic, and $\sum_{s=-\infty}^{\infty} r_{s,s+1} < +\infty$. Then there are two families U_1, U_2 of H_{ij} -periodic p -harmonic functions on the Cayley tree of $k \geq 1$ such that*

$$U_1 = \{u : u(x) = u_n = C^{\frac{1}{p-1}} \sum_{s=-\infty}^{n-1} r_{s,s+1} \text{ if } x \in H_n, n \in \mathbb{Z}, C \geq 0\};$$

$$U_2 = \{u : u(x) = u_n = C^{\frac{1}{p-1}} \sum_{s=n}^{+\infty} r_{s,s+1} \text{ if } x \in H_n, n \in \mathbb{Z}, C \geq 0\}.$$

Remarks. 1. Theorem 3 is also true for more general class of $\mathcal{R}$ than the class of H_{ij} -periodic resistance functions. For example, one can prove Theorem 3 for resistance functions:

$$r(x, y) = \begin{cases} r_{nm}, & \text{if } x \in H_n, y \in H_m, n \neq m \\ r(x, y), & \text{if } x, y \in H_n \end{cases}$$

i.e. r takes arbitrary values on $(x, y) \in L$ such that $x, y \in H_n$.

2. The role of p in U_1 and U_2 is not important: one can denote $C^{\frac{1}{p-1}}$ by C .
3. Note that for $M \subset N_k$ the problem of describing of H_M -periodic p -harmonic functions on Cayley tree of order k is equivalent to describing of arbitrary (non-periodic) p -harmonic functions on a Cayley tree of order $|M| - 1$. Thus for $M \subset N_k$ with $|M| \geq 3$ this problem is difficult.
4. By definition a constant is a p -harmonic function and the linear combination of an arbitrary p -harmonic function and a constant is also p -harmonic. Thus the families U_1 and U_2 generate two parameterized families of p -harmonic functions: $U_1 + C_1, U_2 + C_2$ where $U_i + C_i$ means to each functions of U_i a constant C_i is added.

5. Linear relations

In this section we describe p -harmonic linear combinations of p -harmonic functions from U_1, U_2 .

Theorem 4. Let $\{v_1, \dots, v_{q_1}, v_{q_1+1}, \dots, v_q\}$ be a q -tuple of p -harmonic functions with $\{v_1, \dots, v_{q_1}\} \subset U_1$ and $\{v_{q_1+1}, \dots, v_q\} \subset U_2$. Then $\{v_1, \dots, v_q\}$ has a linear relation on the Cayley tree.

Proof. Note that $v = \sum_{i=1}^q t_i v_i$, $t_i \in R \setminus \{0\}$ has the following form

$$v(x) = \sum_{i=1}^{q_1} t_i \left(C_i \sum_{s=-\infty}^{n-1} r_{s,s+1} \right) + \sum_{i=q_1+1}^q t_i \left(C_i \sum_{s=n}^{+\infty} r_{s,s+1} \right) \quad (11)$$

if $x \in H_n$, $n \in \mathbb{Z}$. Hence v is a H_{ij} -periodic function. We shall prove that $\Delta_p v = 0$ i.e. v is p -harmonic.

Denote $\varphi_p(t) = |t|^{p-2}t$. By definition and (11) we have

$$\begin{aligned} \Delta_p v &= \sum_{y \in S(x)} \frac{|v(y) - v(x)|^{p-2}}{r^{p-1}(x, y)} (v(y) - v(x)) = \sum_{y \in S(x)} \frac{\varphi_p(v(y) - v(x))}{r^{p-1}(x, y)} \\ &= \frac{\varphi_p \left(\sum_{i=1}^{q_1} t_i C_i \left(\sum_{s=-\infty}^{n-2} r_{s,s+1} - \sum_{s=-\infty}^{n-1} r_{s,s+1} \right) + \sum_{i=q_1+1}^q t_i C_i \left(\sum_{s=n-1}^{\infty} r_{s,s+1} - \sum_{s=n}^{\infty} r_{s,s+1} \right) \right)}{r_{n-1,n}^{p-1}} \\ &\quad + \frac{\varphi_p \left(\sum_{i=1}^{q_1} t_i C_i \left(\sum_{s=-\infty}^n r_{s,s+1} - \sum_{s=-\infty}^{n-1} r_{s,s+1} \right) + \sum_{i=q_1+1}^q t_i C_i \left(\sum_{s=n+1}^{\infty} r_{s,s+1} - \sum_{s=n}^{\infty} r_{s,s+1} \right) \right)}{r_{n,n+1}^{p-1}} \\ &= \frac{\varphi_p \left(r_{n-1,n} \left(-\sum_{i=1}^{q_1} t_i C_i + \sum_{i=q_1+1}^q t_i C_i \right) \right)}{r_{n-1,n}^{p-1}} + \frac{\varphi_p \left(r_{n,n+1} \left(\sum_{i=1}^{q_1} t_i C_i - \sum_{i=q_1+1}^q t_i C_i \right) \right)}{r_{n,n+1}^{p-1}} \end{aligned} \quad (12)$$

It is easy to see that $\varphi_p(st) = \varphi_p(s)\varphi_p(t)$; $\varphi_p(t) = t^{p-1}$ if $t > 0$ and $\varphi_p(-t) = -\varphi_p(t)$. Using these properties from (12) we get $\Delta_p v = 0$. The theorem is proved. $\square$

Remark. The H_{ij} -periodic functions described in Theorems 3 and 4 do not depend on $i, j \in N_k$, $i \neq j$.

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