Grow-up of critical solutions for a non-local porous medium problem with Ohmic heating source

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Abstract. We investigate the behaviour of solution $u = u(x, t; \lambda)$ at $\lambda = \lambda^*$ for the non-local porous medium equation $u_t = (u^n)_{xx} + \lambda f(u) / (\int_{-1}^1 f(u) dx)^2$ with Dirichlet boundary conditions and positive initial data. The function f satisfies: $f(s), -f'(s) > 0$ for $s \geq 0$ and $s^{n-1}f(s)$ is integrable at infinity. Due to the conditions on f , there exists a critical value of parameter λ , say λ^* , such that for $\lambda > \lambda^*$ the solution $u = u(x, t; \lambda)$ blows up globally in finite time, while for $\lambda \geq \lambda^*$ the corresponding steady-state problem does not have any solution. For $0 < \lambda < \lambda^*$ there exists a unique steady-state solution $w = w(x; \lambda)$ while $u = u(x, t; \lambda)$ is global in time and converges to w as $t \rightarrow \infty$. Here we show the global grow-up of critical solution $u^* = u(x, t; \lambda^*)$ ($u^*(x, t) \rightarrow \infty$, as $t \rightarrow \infty$ for all $x \in (-1, 1)$).

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1. Introduction

In the present work we prove the global “grow-up” of critical solutions for the following non-local problem:

$$u_t - (u^n)_{xx} = \frac{\lambda f(u)}{\left(\int_{-1}^1 f(u) dx\right)^2}, \quad x \in \Omega = (-1, 1), \quad t > 0, \quad (1a)$$

$$u(1, t) = 0, \quad u(-1, t) = 0, \quad t > 0, \quad (1b)$$

$$u(x, 0) = u_0(x) > 0, \quad x \in \Omega. \quad (1c)$$

Here $\lambda > 0$ and u is a (classical) solution ($u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$, for some $T > 0$) to (1), provided that $u_0(x) \in C(\overline{\Omega})$ where $\Omega_T = (-1, 1) \times (0, T)$.

On the other hand, the function $u : [0, T] \rightarrow L^1(\Omega)$, is called a weak or L^1 -solution to problem (1) if $u \in C([0, T]; L^1(\Omega) \cap L^\infty(\Omega_T))$ for some $T > 0$ and satisfies:

$$P(u; \eta) \equiv \int_{\Omega_T} [u\eta + u^n \eta_{xx} + \lambda \frac{\eta f(u)}{(\int_{\Omega} f(u) dx)^2}] dx dt + \int_{\Omega} u_0(x) \eta(x, 0) dx = 0, \quad (2)$$

where $\eta = \eta(x, t) \in C_0^\infty(\overline{\Omega_T})$.

The corresponding steady-state problem to (1) is:

$$(w^n)'' + \mu f(w) = 0, \quad x \in (-1, 1), \quad (3a)$$

$$w(1) = w(-1) = 0, \quad (3b)$$

where $w = w(x) = w(x; \mu)$ and $\mu = \lambda / (\int_{-1}^1 f(w) dx)^2$ is referred as the local parameter, while λ as the non-local one.

Setting $z = w^n$, w is called a classical solution (in the sense $w \in C^2(\Omega) \cap C(\overline{\Omega})$) to (2) if z is a classical solution to

$$z_{xx} + \mu \phi(z) = 0, \quad x \in (-1, 1),$$

$$z(1) = z(-1) = 0,$$

where $\phi(z) = f(z^{1/n})$.

The nonlinear function $f(s) = \phi(s^n)$ satisfies:

$$f(s) > 0, \quad f'(s) < 0, \quad s \geq 0, \quad (4a)$$

$$\int_0^\infty s^{n-1} f(s) ds < \infty, \quad (4b)$$

$n \geq 1$. The parameter λ is positive and there exists a critical value (due to 4b, [15]), say λ^* , such that to each $\lambda > \lambda^*$ the solution $u = u(x, t; \lambda)$ blows-up in finite time and there is not any steady-state solution w for $\lambda \geq \lambda^*$, while to each $\lambda < \lambda^*$ there exists a unique steady-state solution w and the solution u is global in time. The complete response μ -diagram (local) and λ -diagram (non-local) to problem (3) can be obtained by quadratures, see [15, p.448].

By “global grow-up” of critical solutions u^* we mean that the solution $u^* = u(x, t; \lambda^*)$ to problem (1) is global in time and diverges at infinity globally in space, in the sense that $u^*(x, t) \rightarrow \infty$ as $t \rightarrow \infty$ for all $x \in (-1, 1)$.

Condition (4a) permits us to use direct comparison methods, while (4b) implies bounded λ -spectrum ($0 < \lambda < \lambda^* < \infty$) for the corresponding non-local steady-state problem, see [8, 9, 13, 14]. On the other hand, if $f'(s) > 0$ then a direct comparison does not hold, so for comparison purposes we have to use lower-upper solution pair, see [10]. We denote the solution to (1) by $u = u(x, t) = u(x, t; \lambda)$. For simplicity we assume $u_0(x)$ to be continuous, $u_0(1) = u_0(-1) = 0$ and $u_0(x) > 0$ for $x \in (-1, 1)$, the latter requirement is needed in order to take strictly positive solutions and to avoid degeneracy of problem (1).

Our motivation to address (1), concerning the conduction term Δu^n (or $\nabla \cdot u^{n-1} \nabla u$) comes from [15]. In [15], the “plasma heating” equation

$u_t = (u^3 u_x)_x + \lambda f(u) / (\int_{-1}^1 f(u) dx)^2$ is discussed; more precisely the conduction term $(u^4)_{xx}$ or $(u^3 u_x)_x$ is introduced, where the term u^3 accounts for heat transport dominated by thermal radiation by assuming the Stefan-Boltzmann law for emission of thermal radiation. Actually, in this view Eq. (1) is a generalization of the plasma heating equation.

On the other hand, if u represents the fluid density of a compressible fluid through a porous medium, then Darcy's law can lead to the equation:

$$u_t = \nabla \cdot \sigma(u) \nabla u, \quad \sigma(u) = K'(u), \quad (5)$$

which is the filtration equation. For the latter equation specializing to an isothermal perfect gas, we can obtain the porous medium equation:

$$u_t = \nabla \cdot (u^n \nabla u), \quad n > 0, \quad \text{or} \quad u_t = \Delta u^n, \quad n > 1. \quad (6)$$

The Eq. (6) also models the thickness u of a viscous drop spreading under gravity over horizontal surface as well as the horizontal spreading of highly fissured volcanos, see for details [12, p.216].

Concerning the non-local reaction term of problem (1), this comes from modelling Ohmic heating phenomena as well as in shear bands of metals which are being deformed at high strain rates (see [2–4]), in the theory of gravitational equilibrium of polytropic stars [7], in the investigation of the fully turbulent behaviour of flows, using invariant measures for the Euler equation [5].

In the plasma heating problem $((u^n)_{xx})$ for $n = 4$, see [15], for $0 < \lambda < \lambda^*$ there exists a unique global in time solution $u = u(x, t; \lambda) \rightarrow w(x; \lambda)$, for any $x \in (-1, 1)$ as $t \rightarrow \infty$, where $w(x; \lambda)$ is the corresponding unique steady-state solution to problem (1). While, for $\lambda \geq \lambda^*$ there is no any solution $w = w(x; \lambda)$, actually $w(x; \lambda) \rightarrow \infty$ as $\lambda \rightarrow \lambda^* -$ for all $x \in (-1, 1)$, see the corresponding response (bifurcation) diagram in [15]. The solution $u = u(x, t; \lambda)$ blows up globally in finite time for $\lambda > \lambda^*$, i.e. $u(x, t) \rightarrow \infty$ for all $x \in (-1, 1)$ and $u_x(\pm 1, t) \rightarrow \mp \infty$, as $t \rightarrow t^* - < \infty$. All the results in [15], given for $n = 4$, can be generalized straightforward by the same way for any $n > 1$, $n \in \mathbb{N}$.

Our main purpose in this article is to prove the global “grow-up” of solution $u^* = u(x, t; \lambda^*, u_0^*)$ to problem (1). Therefore, we make a number of comparisons, namely we get: $u^{*n}(x, t) < U^n(x, t, \lambda^*, u_0) = z(x, t) < \bar{u}(x, t) = \hat{u}(x, \tau)$, where u^*, z are lower solutions to U , $\bar{u}$ —problems, respectively, while $\hat{u}(x, \tau)$, with rescaling time $t = t(\tau)$, satisfies the non-local heat equation for which we know that it is a global in time solution. This result is given in the following section.

2. Grow-up of critical solutions ($\lambda = \lambda^*$)

Firstly we define (classical) upper and lower solutions to problem (1). An upper (lower) solution to problem (1) is any function $\bar{u}$ ($\underline{u}$) which satisfies (1) by substituting $\geq$ ($\leq$) for $=$.

We also need to impose an extra condition on f , (see [6]). Thus we define:

$$f(s) = f(\sigma^{1/n}) = \phi(\sigma), \quad s = \sigma^{1/n} \quad \text{and} \quad \Psi(\sigma) = \frac{\phi(\sigma) \sqrt{\mu(\sigma)}}{\Phi(\sigma)},$$

where $\int_0^\infty s^{n-1} f(s) ds = \frac{1}{n} \int_0^\infty \phi(\sigma) d\sigma < \infty$ (due to (4b)) and

$$\Phi(\sigma) = \int_\sigma^\infty \phi(p) dp, \quad \mu(\sigma) = \frac{1}{2} \left(\int_0^\sigma \frac{dp}{\sqrt{\Phi(p) - \Phi(\sigma)}} \right)^2.$$

Finally the function ϕ satisfies:

$$\liminf_{\sigma \rightarrow \infty} \Psi(\sigma) > C > 0, \quad \text{and} \quad \lim_{\sigma \rightarrow \infty} \mu(\sigma) \phi(\sigma) = C_0 > 0, \quad (7a)$$

$$\text{or} \quad \lim_{\sigma \rightarrow \infty} \frac{\mu(\sigma) \phi(\sigma)}{\sigma} = C_1 > 0. \quad (7b)$$

For instance, a function satisfying (7a) is: $\phi(\sigma) = e^{-\sigma}$, while for (7b) is $\phi(\sigma) = b(1 + \sigma)^{-1-b}$, $C_1 \leq 2$, $b > 0$; for these type of conditions, see also [6, 11].

Now we state our main theorem. Actually we have the following:

Theorem 1. *If f satisfies (4), (7) (in the case of (7b) we prove this theorem only for $\phi(\sigma) = b(1 + \sigma)^{-1-b}$) and $u_0^*(x) > 0$ then $u^* = u(x, t; \lambda^*) > 0$ is a unique global-in-time solution to (1) and $u^*(x, t) \rightarrow \infty$ as $t \rightarrow \infty$ for all $x \in (-1, 1)$, $u_x^*(\pm 1, t) \rightarrow \mp \infty$ as $t \rightarrow \infty$, i.e. u^* grows up globally.*

The proof is given in the following lemmas and propositions:

Proposition 2. (Comparison). *Let $u, v \in C^2(\Omega_T) \cap C(\bar{\Omega}_T)$, and f satisfies (3) with*

$$u_t - (u^n)_{xx} - \frac{\lambda f(u)}{\left(\int_{-1}^1 f(u) dx\right)^2} < v_t - (v^n)_{xx} - \frac{\lambda f(v)}{\left(\int_{-1}^1 f(v) dx\right)^2}, \quad \text{in } \Omega_T, \quad (8a)$$

$$0 \leq u(\pm 1, t) < v(\pm 1, t), \quad t > 0, \quad (8b)$$

$$0 < u(x, 0) < v(x, 0), \quad x \in (-1, 1) \quad (8c)$$

then $u(x, t) < v(x, t)$ for $(x, t) \in [-1, 1] \times [0, T]$. In addition if f is locally Lipschitz continuous and again $u(x, 0) > 0$ then the proposition is true by substituting all $\leq$ for $<$.

Proof. We prove the proposition in two steps. Actually we give the proof for $\Omega \in \mathbb{R}^N$, $N \geq 1$ and for the Filtration equation i.e. instead of u^n we work with $K(u)$.

First step: Let that u, v satisfy (8), we shall show that $u(x, t) < v(x, t)$ for all $(x, t) \in \bar{\Omega} \times [0, T]$, for some $T > 0$. Let $d = d(x, t) = v(x, t) - u(x, t)$ (see also [1]). We assume that the conclusion is false; that is, there exists a first $\bar{t} \in (0, T]$ such that $d(\bar{x}, \bar{t}) = 0$ for some $\bar{x} \in \Omega_0 \subseteq \Omega$ and $d(x, \bar{t}) > 0$ for some $x \in \Omega_+ \subseteq \Omega$, $\Omega_0 \cup \Omega_+ = \bar{\Omega}$. Also, we have that $d(x, t) > 0$ for $(x, t) \in \bar{\Omega} \times [0, \bar{t})$ and $d_t(\bar{x}, \bar{t}) \leq 0$, moreover, $d(\bar{x}, \bar{t})$ attains its minimum at $x = \bar{x}$, so $\nabla d(\bar{x}, \bar{t}) = \nabla v(\bar{x}, \bar{t}) - \nabla u(\bar{x}, \bar{t}) = 0$ and $\Delta d(\bar{x}, \bar{t}) \geq 0$.

Thus, due to (8a), we have at $(\bar{x}, \bar{t})$:

$$\begin{aligned}
 0 &\geq d_t(\bar{x}, \bar{t}) = v_t(\bar{x}, \bar{t}) - u_t(\bar{x}, \bar{t}) \\
 &> K'(v)\Delta v + K''(v)|\nabla v|^2 - K'(u)\Delta u - K''(u)|\nabla u|^2 + NLTs \\
 &= K'(u)\Delta d(\bar{x}, \bar{t}) + K''(u)(|\nabla v|^2 - |\nabla u|^2) + \lambda \left(\frac{f(v(\bar{x}, \bar{t}))}{(\int_{\Omega} f(v(x, \bar{t}))dx)^2} \right. \\
 &\quad \left. - \frac{f(u(\bar{x}, \bar{t}))}{(\int_{\Omega} f(u(x, \bar{t}))dx)^2} \right) \geq f(u(\bar{x}, \bar{t})) \frac{(\int_{\Omega} f(u(x, \bar{t}))dx)^2 - (\int_{\Omega} f(v(x, \bar{t}))dx)^2}{((\int_{\Omega} f(v(x, \bar{t}))dx)^2 \int_{\Omega} f(u(x, \bar{t}))dx)^2} \\
 &= f(u(\bar{x}, \bar{t})) \frac{[\int_{\Omega} (f(u(x, \bar{t})) + f(v(x, \bar{t})))dx][\int_{\Omega} (f(u(x, \bar{t})) - f(v(x, \bar{t})))dx]}{[\int_{\Omega} f(v(x, \bar{t}))dx \int_{\Omega} f(u(x, \bar{t}))dx]^2} \geq 0,
 \end{aligned}$$

where $NLTs$ is the difference of the non-local terms. The term with Laplacian is non-negative, the term with the gradient is zero and the difference of the non-local terms is non-negative due to the monotonicity of f , ($f' < 0$). Hence, $0 \geq d_t(\bar{x}, \bar{t}) > 0$, which is a contradiction.

Second step: Let us now assume that u, v satisfy (8) but substituting $<$ for $\leq$. We shall show that $u(x, t) \leq v(x, t)$ for all $(x, t) \in \bar{\Omega} \times [0, T]$, for some $T > 0$. Since $f \in C^1$, it is Lipschitz continuous and also one side Lipschitz continuous i.e. $f(a+b) - f(b) \leq La$ where L is a positive constant and $0 < a < R$ for some R . Let $v^\varepsilon = v + \varepsilon e^{\sigma t} > v$ for some $\varepsilon > 0$, actually we get $0 < \varepsilon \ll 1$, $\sigma \gg 1$ and $\varepsilon e^{\sigma t} < \varepsilon e^{\sigma T} = R$ and L fixed, then we have:

$$\begin{aligned}
 S(v^\varepsilon) &\equiv v_t^\varepsilon - \Delta K(v^\varepsilon) - \lambda \frac{f(v^\varepsilon)}{(\int_{\Omega} f(v^\varepsilon)dx)^2} \\
 &= v_t + \varepsilon \sigma e^{\sigma t} - K''(v^\varepsilon)|\nabla v|^2 - K'(v^\varepsilon)\Delta v - \lambda \frac{f(v^\varepsilon)}{(\int_{\Omega} f(v^\varepsilon)dx)^2} \\
 &\geq v_t - \Delta K(v) - \lambda \frac{f(v^\varepsilon)}{(\int_{\Omega} f(v^\varepsilon)dx)^2} + \lambda \frac{f(v^\varepsilon)}{(\int_{\Omega} f(v)dx)^2} - \lambda \frac{f(v) + L\varepsilon e^{\sigma t}}{(\int_{\Omega} f(v)dx)^2} \\
 &\quad + \varepsilon \sigma e^{\sigma t} + (K''(v) - K''(v^\varepsilon))|\nabla v|^2 + (K'(v) - K'(v^\varepsilon))\Delta v \\
 &= S(v) + \varepsilon e^{\sigma t} \left[\sigma - \lambda f(v^\varepsilon) \frac{[\int_{\Omega} (f(v^\varepsilon) + f(v))dx][\int_{\Omega} f'(s)L\varepsilon e^{\sigma t}dx]}{[\int_{\Omega} (f(v^\varepsilon)dx)]^2 [\int_{\Omega} f(v)dx]^2} \right. \\
 &\quad \left. - \lambda \frac{L}{(\int_{\Omega} f(v)dx)^2} - K'''(v)|\nabla v|^2 - K''(v)\Delta v \right] + O(\varepsilon^2) \\
 &> S(v) \geq S(u),
 \end{aligned}$$

on the above we have used $f(v^\varepsilon) \leq f(v) + L\varepsilon e^{\sigma t}$. The last inequality is due to the fact that, σ can be taken sufficient large ($\sigma \gg 1$) while R and L are fixed and the function v is bounded in $C^{2,1}(\bar{\Omega}_T)$. Then from first step we derive $v^\varepsilon > u$ and $v^\varepsilon > v$ for $\varepsilon \ll 1$. Now we have $v^\varepsilon = v + \varepsilon e^{\sigma t} > u$ for any $0 < \varepsilon \ll 1$ and taking $\varepsilon \rightarrow 0$ we get $v \geq u$. This completes the proof. $\square$

A similar to Proposition 2 holds for weak upper (lower) solution to problem (1). A weak upper (lower) solution to problem (1) is any function $\bar{u}$ ($\underline{u}$) which satisfies (2) by substituting $\geq$ ($\leq$) for $=$.

Proposition 3. (*Comparison weak-form*) Let $u, v \in C([0, T]; L^1(\Omega) \cap L^\infty(\Omega_T))$, and f satisfies (3a), with $P(u; \eta) < P(v; \eta)$, then $u(x, t) < v(x, t)$ a.e. in Ω_T . If in addition f is locally Lipschitz continuous and again $u(x, 0) > 0$ then the proposition is true by substituting all $\leq$ for $<$.

Lemma 4. (*Local Existence*). If f satisfies (4a) and $u_0(x) > 0$ then there exists a unique weak solution $u(x, t; \lambda)$ to problem (1) to each $\lambda > 0$ in Ω_T for some $T > 0$. Moreover $u \in L^\infty(\Omega_T)$ and hence is classical.

Proof. Due to the fact that $f, -f' > 0$, a maximum principle holds for problem (1), this implies (Proposition 2) that if $u_0(x) > 0$ then $u(x, t) > 0$. Moreover, using comparison methods we can construct an increasing (decreasing) sequence of lower solutions $\underline{u}_n$ ($\bar{u}_n$), via an iteration scheme, such that $\underline{u}_1 < \dots < \underline{u}_n < \dots \leq \underline{u} \leq \bar{u} \leq \dots < \bar{u}_n < \dots < \bar{u}_1$ where $\underline{u}_n \nearrow \underline{u}$ and $\bar{u}_n \searrow \bar{u}$ as $n \rightarrow \infty$. The iteration scheme is defined by substituting u_n for u at the left hand-side and u_{n-1} for u at the right hand-side of the Eq. (1a), for more details see [10]. In addition, since f is Lipschitz continuous we can prove (again by using maximum principle) that $\underline{u} \geq \bar{u}$ and hence $\underline{u} = u = \bar{u}$ where u is the local in time unique solution to (1). This guarantees the local existence and uniqueness of u . $\square$

- Remark 5.** (i) The above lemma can be obtained by regularization of the solution u_δ which satisfies problem (1) where the term $(u_\delta^n)_{xx} = (u^n + \delta u)_{xx}$ is substituted for $(u^n)_{xx}$. The u_δ -problem does not create degeneracy.
- (ii) The degeneracy of problem (1) takes place only on the boundary $x = \pm 1$ and the solution is classical in Ω , provided that u is bounded.
- (iii) Also the proof of Lemma 4 can be obtained by using the L^1 -solution. It is well known that an L^1 -solution which belongs to $L^\infty(\Omega_T)$ is also classical.

Lemma 6. *The following statements hold:*

- (a) If $u_0(x)$ is a symmetric (even) function then the solution $u(x, t)$ to problem (1) is symmetric too.
- (b) If u_0 is a strict lower solution to (3) then $u_t = u_t(x, t; \lambda^*, u_0) > 0$.
- (c) If $u_0 \geq u_0^* > 0$ then $U = u(x, t; \lambda^*, u_0) \geq u(x, t; \lambda^*, u_0^*) = u^* > 0$ where U, u^* are solutions to (1).

Proof. (a) This is an immediate consequence of the symmetry of the problem.

- (b) On using standard comparison arguments we get by Proposition 2 that $u(x, t) \geq u_0(x)$. Defining now $u^\delta(x, t) = u(x, t + \delta)$, then u^δ is a solution to problem (1) and so is an upper solution with $u^\delta(x, 0) = u(x, t) \geq u_0(x)$. Hence $u^\delta(x, t) \geq u(x, t)$ see also [1].

- (c) This is a direct consequence of Proposition 2. $\square$

Lemma 7. *Let that all conditions in Theorem 1 hold, then $u^*(x, t) \rightarrow \infty$ for all $x \in (-1, 1)$ and $u_x(\pm 1, t) \rightarrow \mp \infty$ as $t \rightarrow t^* - \leq \infty$, for some $t^* > 0$, where t^* is the maximal existence time of u^* .*

Proof. It is enough to construct a lower solution $\underline{v}(x, t)$ to u^* -problem (problem (1) with $u(x, 0) = u_0^*(x)$ and $\lambda = \lambda^*$) such that $\underline{v}(x, t) \rightarrow \infty$ for all $x \in (-1, 1)$ as $t \rightarrow T^* - \leq \infty$ for some $T^* \geq t^* > 0$, where T^* is the maximal existence time of $\underline{v}$. Therefore we get, $\underline{v}(x, t) = \underline{w}(x; \mu(t))$ where $\underline{w}$ satisfies (3), see also [8, 9, 15]. On using maximum principle we get $w_\mu > 0$ (differentiating (3a) with respect to μ and setting $z = w^n$ in (3), then $z_\mu = nw^{n-1}w_\mu$ and $z''_\mu + \mu z_\mu f'(w)/(nw^{n-1}) < 0$ with $z_\mu(\pm 1, t) = 0$ which implies $z_\mu \geq 0$). Now

$$\begin{aligned} & \underline{v}_t - (\underline{v}^n)_{xx} - \lambda^* \frac{f(\underline{v})}{\left(\int_{-1}^1 f(\underline{v}) dx\right)^2} \\ &= \underline{w}_\mu \left\{ \dot{\mu} - \frac{f(\underline{w})}{\underline{w}_\mu \left(\int_{-1}^1 f(\underline{w}) dx\right)^2} \left[\lambda^* - \mu \left(\int_{-1}^1 f(\underline{w}) dx\right)^2 \right] \right\} \\ &\leq \underline{w}_\mu \left\{ \dot{\mu} - \left[\lambda^* - \mu \left(\int_{-1}^1 f(\underline{w}) dx\right)^2 \right] \right\} \frac{1}{\left(\int_{-1}^1 f(\underline{w}) dx\right)^2} \inf_{(-1, 1)} \frac{f(\underline{w})}{\underline{w}_\mu} \leq 0 \end{aligned}$$

where $\mu = \mu(t)$ satisfy:

$$0 < \dot{\mu} \leq \lambda^* - \mu \left(\int_{-1}^1 f(\underline{w}) dx\right)^2.$$

On taking $\underline{w}(x; \mu(0)) \leq u_0^*(x)$, $\underline{v}$ is a lower solution to u^* -problem with $\underline{v}_t = \underline{w}_\mu \dot{\mu} > 0$, implying $\underline{v}(x, t) = \underline{w}(x; \mu(t)) \rightarrow \infty$ for all $x \in (-1, 1)$, $\mu(t) \rightarrow \infty$ (or $\lambda(t) \rightarrow \lambda^* -$) as $t \rightarrow T^* \leq \infty$, see also [9, p.204] and the response μ , λ -diagram in [15, p.448]. Hence $u(x, t) \rightarrow \infty$ as $t \rightarrow t^* \leq \infty$ uniformly in compact subsets of $(-1, 1)$ and also $u_x(\pm 1, t) \rightarrow \mp \infty$ as $t \rightarrow t^* \leq \infty$. This completes the proof of the proposition. $\square$

Now, due to Lemma 6, for any $u_0^*(x)$ it is enough for Theorem 1 to hold, to choose $u_0 > u_0^*$ and prove that $U(x, t) = u(x, t; \lambda^*, u_0)$ is a global in time solution of u^* . In addition we assume u_0 to be a positive, symmetric and a lower solution to (3) (for instance such a u_0 is: $u_0(x) = k > 0$, for $|x| \leq \delta$, k is a constant and $u_0(x) = w(y(x))$ with $y(x) = (|x| - \delta)/(1 - \delta)$ for $0 < \delta \leq |x| \leq 1$ where $w(y(x))$ satisfies problem (3)).

On taking now U to be a solution to (1) and make the “pressure transformation”: $0 < z = z(x, t) = U^n(x, t)$, then $nU_t = z_t z^{-q}$ where $0 < q = 1 - 1/n$. Thus problem (1) becomes the “pressure problem”:

$$z_t = nz^q \left(z_{xx} + \lambda^* \frac{\phi(z)}{\left(\int_{-1}^1 \phi(z) dx\right)^2} \right), \quad x \in (-1, 1), \quad t > 0, \quad (9a)$$

$$z(1, t) = z(-1, t) = 0, \quad t > 0, \quad (9b)$$

$$z(x, 0) = z_0(x) > 0, \quad x \in (-1, 1), \quad (9c)$$

where the function ϕ is defined: $\phi(z) = f(z^{1/n}) = f(U) > 0$ with $\phi'(z) < 0$, for $z \geq 0$. The solution $z(x, t)$ of (9) satisfies: $z(x, t)$ is symmetric with respect

to x , $\max_x z(x, t) = z(0, t) = M(t) = U^n(0, t)$ and $z_t > 0$ since $U_t > 0$, due to Lemma 6.

Moreover due to (4b) we get:

$$\int_0^\infty \phi(z) \, dz = \int_0^\infty f(z^{1/n}) \, dz = n \int_0^\infty U^{n-1} f(U) \, dU < \infty. \quad (10)$$

Finally, the “pressure equation” gives:

$$z_t \leq nM^q \left(z_{xx} + \lambda^* \frac{\phi(z)}{\left(\int_{-1}^1 \phi(z) \, dx \right)^2} \right). \quad (11)$$

Now we introduce the $\bar{u}$ -problem:

$$\bar{u}_t = nM^q \left(\bar{u}_{xx} + \lambda^* \frac{\phi(\bar{u})}{\left(\int_{-1}^1 \phi(\bar{u}) \, dx \right)^2} \right), \quad x \in (-1, 1), \quad t > 0, \quad (12a)$$

$$\bar{u}(1, t) = \bar{u}(-1, t) = 0, \quad t > 0, \quad (12b)$$

$$\bar{u}(x, 0) = \bar{u}_0(x) > z_0(x) = u_0^n(x) > 0, \quad x \in (-1, 1), \quad (12c)$$

where $\bar{u}_0$ is taken to be symmetric and a lower solution to (12a), (12b); hence $\bar{u}_t > 0$. Then z is a lower solution to $\bar{u}$ -problem with $z(x, t) < \bar{u}(x, t)$ and thus $M(t) < \bar{M}(t) = \max_x \bar{u}(x, t) = \bar{u}(0, t)$.

We proceed by rescaling the time so that:

$$\frac{d\tau}{dt} = nM^q(t), \quad (13)$$

and $\bar{u}(x, t) = \hat{u}(x, \tau)$ with $\bar{u}(0, t) = \bar{M}(t) = \hat{u}(0, \tau) = \widehat{M}(\tau)$. Thus we have concluded to the following $\hat{u}$ -problem:

$$\hat{u}_\tau = \hat{u}_{xx} + \lambda^* \frac{\phi(\hat{u})}{\left(\int_{-1}^1 \phi(\hat{u}) \, dx \right)^2}, \quad x \in (-1, 1), \quad \tau > 0, \quad (14a)$$

$$\hat{u}(1, \tau) = \hat{u}(-1, \tau) = 0, \quad \tau > 0, \quad (14b)$$

$$\hat{u}(x, 0) = \hat{u}_0(x), \quad x \in (-1, 1). \quad (14c)$$

Proposition 8. *If $\phi(s) = f(s^{1/n})$ satisfies (4), (7) (in the case of (7b) we have this proposition only for $\phi(\sigma) = b(1 + \sigma)^{-1-b}$) and $\hat{u}_0(x) > 0$ then $\hat{u}(x, \tau) > 0$ is a unique global-in-time solution to (14) and $\hat{u}(x, \tau) \rightarrow \infty$ as $\tau \rightarrow \infty$ for all $x \in (-1, 1)$, $\hat{u}_x(\pm 1, \tau) \rightarrow \mp \infty$ as $\tau \rightarrow \infty$ i.e. $\hat{u}$ diverges globally in x or equivalently $\hat{u}$ grows up globally.*

Proof. For the proof see [6, p. 795, Prop. 2],[11]. □

Remark 9. The method used to prove Proposition 8 in [6], is based on the construction of an upper solution to u^* which is global in time. It seems that this method works only for the Δu and not for Δu^n . Any such construction, for the case Δu^n , leads to an upper solution which blows-up in finite time and then it is useless for our purpose. This is the reason why, in this work, we introduce a different method.

Hence relation (13) and Proposition 8 imply:

$$\hat{u}(x, \tau) = \bar{u}(x, t) \rightarrow \infty, \quad (15)$$

for each $x \in (-1, 1)$ as $\tau \rightarrow \infty$ and as $t \rightarrow t^* \leq \infty$.

In order to prove Theorem 1, it is required to show that $t^* = \infty$. Therefore we assume that $t^* < \infty$ and get a contradiction. For this purpose we need the following lemma:

Lemma 10. (*Flat Blow-up*). *Let $\bar{u}(x, t)$ be the solution to problem (12) which blows up in finite time t^* , then the blow-up is flat on $[-\delta, \delta]$, $0 < \delta < 1$, i.e. $\bar{u}(x, t) \sim \bar{M}(t)$, for $|x| \leq \delta$, $0 < 1 - \delta \ll 1$ and $0 < t^* - t \ll 1$.*

Proof. For the proof we use similar ideas as in [9, p. 215]. We introduce again an auxiliary problem:

$$H(\tilde{u}) \equiv \tilde{u}_t - \tilde{u}_{xx} - \bar{g}(t)\phi(\bar{M}) = 0, \quad x \in (-1, 1), \quad t > t_0 \geq 0, \quad (16a)$$

$$\tilde{u}(1, t) = \tilde{u}(-1, t) = 0, \quad t > t_0, \quad (16b)$$

$$\tilde{u}(x, t_0) = \tilde{u}_0(x) = 0, \quad x \in (-1, 1), \quad (16c)$$

where $\bar{g}(t) = \lambda^* / (\int_{-1}^1 \phi(\bar{u}) dx)^2$ and for some $t_0 \geq 0$.

From (12), since ϕ is decreasing, we get:

$$0 < \bar{M}'(t) \leq h(t)\phi(\bar{M}(t)) \leq h(t)\phi(\bar{M}(t_0)), \quad t \geq t_0 \geq 0, \quad (17)$$

where $h(t) = nM^q(t)\bar{g}(t)$, $\bar{M}(t) = \max_x \bar{u}(x, t) = \bar{u}(0, t)$, since $\bar{u}_t > 0$ and $\bar{u}_{xx} \leq 0$ at any point that $\bar{u}$ takes the maximum value $\bar{M}$. Since $\bar{u}$ blows up, then $\bar{M}(t) \rightarrow \infty$ as $t \rightarrow t^* -$ (if $\liminf_{t \rightarrow t^* -} \bar{M}(t) < \infty$, then $\liminf_{t \rightarrow t^* -} M(t) < \infty$, which implies that there exists t_n such that the lower solution $\underline{v}(0, \mu(t_n)) = w(0, \mu(t_n)) < \infty$ contradicting to Lemma 7), this also implies that $h(t) = nM^q(t)\bar{g}(t)$, $\bar{g}(t) \rightarrow \infty$ as $t \rightarrow t^* -$, (otherwise $\bar{u}$ does not blow-up, since ϕ is decreasing and necessarily bounded), since the blow-up of $\bar{u}$ is global i.e. $\bar{u}(x, t) \rightarrow \infty$ as $t \rightarrow \infty$ for all $x \in (-1, 1)$. Moreover, integrating (17) we get $\int^t h(s)\phi(\bar{M}(s))ds \rightarrow \infty$ and $\int^t h(s)ds \rightarrow \infty$ as $t \rightarrow t^* -$. Similarly we have: $\int^t h(s)\phi(\bar{M}(s))ds = \int^t nM^q(s)\bar{g}(s)\phi(\bar{M}(s))ds \leq \int^t n\bar{M}^q(s)\bar{g}(s)\phi(\bar{M}(s))ds \leq K \int^t \bar{g}(s)ds$, since $q < 1$, $M(t) \leq \bar{M}(t)$, $\bar{M}\phi(\bar{M}) \rightarrow 0$ as $\bar{M} \rightarrow \infty$ and $\bar{M}\phi(\bar{M}) < K$ for $\bar{M} \gg 1$, hence $\int^t \bar{g}(s)ds \rightarrow \infty$ as $t \rightarrow t^* -$.

We also introduce the initial value problem:

$$V'(t) = h(t)\phi(\bar{M}(t)), \quad V(t_0) = 0, \quad (18)$$

this implies $V(t) = \int_{t_0}^t h(s)\phi(\bar{M}(s))ds \rightarrow \infty$ as $t \rightarrow t^* -$ and t_0 is defined (Lemma 7) so that:

$$1 \leq n\widetilde{M}^q(t_0) \leq n\bar{M}^q(t_0) \leq n\bar{M}^q(t), \quad (19)$$

since $\bar{M}(t) \geq M(t)$, $\widetilde{M}(t) = \max_x \tilde{u}(x, t)$ (see below that $\bar{u} \geq \tilde{u}$) and $M'(t) > 0$, $z_t > 0$ for $t_0 < t < t^*$.

Relations (17), (18) also imply:

$$\overline{M}(t) - \overline{M}(t_0) \leq V(t), \quad t \geq t_0, \quad \text{or} \quad \overline{M}(t) \lesssim V(t), \quad (20)$$

for $\overline{M} \gg 1$ or for $0 < t^* - t \ll 1$.

In addition the solution to (12) is an upper solution to (16). Indeed, since $\overline{u}_t > 0$ then $(\overline{u}_{xx} + \overline{g}(t)\phi(\overline{u})) > 0$; hence from (19) and for $x \in (-1, 1)$, $t \geq t_0$ we have:

$$\begin{aligned} H(\overline{u}) &= \overline{u}_t - \overline{u}_{xx} - \overline{g}(t)\phi(\overline{M}) \geq \overline{u}_t - (\overline{u}_{xx} + \overline{g}(t)\phi(\overline{u})) \\ &\geq \overline{u}_t - nM^q(\overline{u}_{xx} + \overline{g}(t)\phi(\overline{u})) = 0 = H(\tilde{u}). \end{aligned}$$

Also $\tilde{u}(\pm 1, t) = \overline{u}(\pm 1, t) = 0$ and $\tilde{u}(x, t_0) = 0 < \overline{u}(x, t_0)$, due to problem (12) and that $u_0(x) > 0$.

Hence $\overline{u}(x, t) \geq \tilde{u}(x, t)$ and therefore we have:

$$\overline{M}(t) \geq \overline{u}(x, t) \geq \tilde{u}(x, t), \quad (x, t) \in (-1, 1) \times (t_0, t^*). \quad (21)$$

On the other hand, for $t > t_0$, $\tilde{u}$ satisfies:

$$\tilde{u}(x, t) = V(t) + \int_{t_0}^t V(s) [G_y(x, 1, t-s) - G_y(x, -1, t-s)] ds, \quad (22)$$

where $G(x, y, t-s)$ is the Green's function for the heat equation in $(-1, 1)$ with Dirichlet boundary conditions. Also we note that $\overline{u}, \tilde{u} \rightarrow \infty$ as $t \rightarrow t^* -$.

Now, for $x \in \overline{I}_\delta \equiv [-1 + \delta, 1 - \delta]$ and any fixed $0 < \delta < 1$, the following asymptotic estimate holds:

$$\tilde{u}(x, t) \sim V(t), \quad 0 < t^* - t \ll 1, \quad t > t_0, \quad (23)$$

since for any fixed $x \in \overline{I}_\delta$ the second term on the right hand side of (22) is much smaller than the first one, the latter is due to the form of the Green's function, see [9]. This also hold, on taking $\delta = \delta(t) \rightarrow 0$, as long as it goes to zero slowly enough. Finally (20), (21), (23), for $x \in \overline{I}_\delta$, $0 < t^* - t \ll 1$, $t > t_0$, give:

$$\overline{M}(t) \geq \overline{u}(x, t) \geq \tilde{u}(x, t) \sim V(t) \gtrsim \overline{M}(t) \quad (24)$$

and the proof of the lemma is now completed. $\square$

Now using (15), Proposition 8 and Lemma 10 we obtain,

$$\overline{u}(x, t) \sim \overline{M}(t) = \widehat{M}(\tau) \rightarrow \infty \text{ for } 0 < t^* - t \ll 1 \text{ or } 0 < 1/\tau \ll 1, \quad \forall x \in (-1, 1).$$

Hence, for the $\hat{u}$ -problem we get a flat grow-up, which implies:

$$\widehat{M}'(\tau) = \frac{d\widehat{M}(\tau)}{d\tau} = \lambda^* \frac{\phi(\widehat{M})}{\widehat{I}^2(\tau)} = g(\tau; \hat{u})\phi(\widehat{M}) \sim \lambda^* \frac{\phi(\widehat{M})}{4\phi^2(\widehat{M})} = \frac{\lambda^*}{4\phi(\widehat{M}(\tau))}, \quad (25)$$

for $0 < t^* - t \ll 1$ or $0 < 1/\tau \ll 1$. Thus $\phi(\widehat{M})d\widehat{M} \sim \lambda^* d\tau/4$, $\tau \gg 1$, provided that the dominant contribution to the integral does not come from the boundary layers.

Integrating now we derive:

$$\frac{\lambda^*}{4}\tau \sim \int^{\widehat{M}(\tau)} \phi(s)ds < \int_0^\infty \phi(s)ds < \infty.$$

The last relation is a contradiction since τ is unbounded and hence $t^* = \infty$.

Now it remains to calculate the integral in the case that a significant (but dominant) contribution comes from the boundary layers. To find these two contributions from the boundary layers to the integral I , we follow similar analysis as in the case $\lambda > \lambda^*$, see [9, p. 216].

For simplicity, we only consider the boundary layer at $x = 1$. If the boundary layer has width $O(\delta)$, for some $\delta \ll 1$, to compare in pairs the terms of the equation we rescale again as follow, actually we have:

$$y = \frac{x}{\delta}, \quad \frac{dT}{d\tau} = \frac{1}{t^* - t}, \quad \frac{dT}{dt} = \frac{nM^q(t)}{t^* - t} \quad \text{and} \\ \widehat{u}(x, \tau) = \widetilde{U}(y, T), \quad \widehat{M}(\tau) = \widetilde{M}(T) = \sup_y \widetilde{U}(y, T).$$

Then Eq. (1) becomes

$$\widetilde{U}_T \frac{1}{t^* - t} = \frac{1}{\delta^2} \widetilde{U}_{yy} + g(T; \widetilde{U})\phi(\widetilde{U}). \quad (26)$$

Again following the same analysis as in [9] we have:

- (i) If $g(T; \widetilde{U}) = g(t; \bar{u}) \ll (t^* - t)^{-1}$, then $O(\delta) = O((t^* - t)^{1/2}) \ll 1$ and

$$I = \widetilde{I}(T) = O((g(T; \widetilde{U}))^{-1/2}) \gg (t^* - t)^{1/2}.$$

This suggests that the contribution to the integral I from the boundary layer, is $O(\delta) = O((t^* - t)^{1/2}) \ll 1$ and therefore finally the main core dominates as in the previous case. Namely $I \sim 2\phi(\widehat{M})$; hence again we get a contradiction ($\tau < \infty$), and hence $t^* = \infty$.

- (ii) If $g(T; \widetilde{U}) = g(t; \bar{u}) = g(\tau; \widehat{u}) = \widehat{g}(\tau) \sim O((t^* - t)^{-1})$, again we have

$$2\phi(\widehat{M}) < \widehat{I}(\tau) = \sqrt{\lambda^*/\widehat{g}(\tau)},$$

and wherever we have flat grow-up, (25) gives

$$\widehat{M}'(\tau) \sim g(\tau; \widehat{u})\phi(\widehat{M}) \leq \frac{1}{2}(\lambda^* \widehat{g}(\tau))^{1/2} = O((t^* - t)^{-1/2}), \\ \frac{d\overline{M}}{dt} \frac{1}{n\overline{M}} \leq \frac{d\overline{M}}{dt} \frac{1}{n\overline{M}^q} \leq \frac{d\widehat{M}}{d\tau} = \frac{d\overline{M}}{dt} \frac{1}{nM^q} \leq O((t^* - t)^{-1/2}),$$

or (since $M(t) \leq \overline{M}(t)$ and $q < 1$) we get

$$\ln \frac{\overline{M}(t)}{\overline{M}(t_0)} \leq K[(t^* - t_0)^{1/2} - (t^* - t)^{1/2}].$$

On taking now $t \rightarrow t^* - < \infty$ we get again a contradiction and hence $t^* = \infty$.

- (iii) Now if $g(T; \tilde{U}) \gg (t^* - t)^{-1}$, also we have $g(T; \tilde{U}) = g(\tau; \hat{u}) = \delta^{-2} \gg (t^* - t)^{-1}$. These suggest that $\hat{u}_\tau$ is negligible and the key balance is

$$-\hat{u}_{xx} \sim \frac{\lambda^* \phi(\hat{u})}{(\int_{-1}^1 \phi(\hat{u}) dx)^2} = g(\tau; \hat{u}) \phi(\hat{u}), \quad \text{for } \tau \rightarrow \infty, \quad \hat{u} \rightarrow \infty,$$

(which might be thought of as coming from the approximate equation in the boundary layers which occur for $\lambda > \lambda^*$, and these boundary layers having expanded to fill the whole of $-1 < x < 1$ if $\lambda \rightarrow \lambda^+)$.

To leading order then,

$$\hat{u} \sim W(x; \hat{g}(\tau)), \quad (27)$$

where

$$\frac{d^2 W}{dx^2} + \hat{g}(\tau) \phi(W) = 0, \quad \text{for } -1 < x < 1, \quad W = 0 \text{ at } x = \pm 1. \quad (28)$$

As in the case of $\lambda > \lambda^*$, [9, p. 217]), the contribution from the boundary layers is

$$\sim \frac{2}{\sqrt{\hat{g}(\tau)}} \int_0^\infty \phi(\hat{u}(1 - \sqrt{\sigma/\hat{g}(\tau)}, \tau)) d\sigma = \sqrt{\lambda^*/\hat{g}(\tau)}.$$

This gives

$$\left(\frac{\lambda^*}{\hat{g}(\tau)} \right)^{1/2} \sim 2\phi(\widehat{M}) + \left(\frac{\lambda^*}{\hat{g}(\tau)} \right)^{1/2}, \quad (29)$$

which implies

$$\phi(\widehat{M}) \sim 0 \quad \text{and} \quad \phi(\widehat{M}) = o(\sqrt{\lambda^*/\hat{g}(\tau)})^{1/2}.$$

The latter asymptotic information is not helpful in estimating the integral I , (as it is for the blow-up case when $\lambda > \lambda^*$, [9])

Therefore we use a different method.

- (a) If the function ϕ satisfies (7a), we have $g(\tau; \hat{u})\phi(\widehat{M}) = \mu(\widehat{M})\phi(\widehat{M}) \sim C_0$ for $\tau \gg 1$ or $\widehat{M} \gg 1$. Then from (25) we get:

$$\frac{d\overline{M}}{dt} \frac{1}{n\overline{M}^q(t)} = \frac{d\widehat{M}}{d\tau} \sim g(\tau; \hat{u})\phi(\widehat{M}) \sim C_0$$

or

$$\frac{d\overline{M}}{dt} \lesssim C_0 n \overline{M}^q(t) \leq k \overline{M}(t),$$

($M < \overline{M}$, $q = 1 - 1/n < 1$). On integrating we get:

$$\ln \frac{\overline{M}(t)}{\overline{M}(t_0)} \lesssim k(t - t_0), \quad \overline{M}(t_0) \gg 1,$$

and as $t \rightarrow t^* -$, $\overline{M}(t) \rightarrow \infty$ which again gives a contradiction. Hence $t^* = \infty$.

- (b) If the function ϕ satisfies (7b), we have $\mu(\sigma)\phi(\sigma) \sim C_1\sigma$ for $\sigma \gg 1$. Then, Eq. (28) is written, for $W(x; \widehat{g}(\tau)) \gg 1$ (away from the boundaries):

$$\frac{d^2 W}{dx^2} + C_1 W \sim 0, \text{ for } -1 + \delta < x < 1 - \delta, W = 0 \text{ at } x = \pm 1, \delta > 0. \quad (30)$$

This implies $W(x; \widehat{g}(\tau)) = a(\tau) \cos kx + b(\tau) \sin kx$ where $k^2 = C_1$, $a(\tau) = \widehat{M}(t)$ and $x \in (-1 + \delta, 1 - \delta)$, but the latter contradicts the flat grow-up of $\widehat{u} \sim W(x; \widehat{g}(\tau))$ in the main core. This result holds provided that (27) is true. As it is not trivial to verify the validity of (27) we give two special cases, see also [6].

- Exponential decay ($\phi(s) = e^{-s}$).

For this case we have:

$$\frac{d\widehat{M}}{d\tau} = \widehat{g}(\tau)\phi(\widehat{M}) \sim \frac{\pi^2}{2} e^{\widehat{M}} e^{-\widehat{M}} = \frac{\pi^2}{2},$$

and integrating again we get a contradiction as in (iiia). Hence $t^* = \infty$.

- Power-law decay ($\phi(s) = b(1+s)^{-(1+b)}$, $b > 0$).

Again following the same analysis of [6] we get:

$$\widehat{u}(x, \tau) \sim \widehat{M}(\tau)W(x) \sim c_2 \tau^{2/b} w(x), \quad \text{as } \tau \rightarrow \infty, \quad x \in (-1, 1),$$

contradicting the flat grow-up of $\widehat{u}(x, \tau)$, as in (iiib).

Finally in all above cases we get a contradiction. This fact implies that $\bar{u}$ and u^* are both global-in-time solutions, with u^* the solution of problem (12).

Proof. (of Theorem 1): By Lemma 4 we get a local-in-time unique solution u^* to (1). By Lemma 7, $u^*(x, t)$ is unbounded, actually $u^*(x, t) \rightarrow \infty$ for all $x \in (-1, 1)$ as $t \rightarrow t^* - \leq \infty$. Then by Lemma 7, Proposition 8 and the above asymptotic analysis, we obtain that u^* is global-in-time, actually we have,

$$u^*(x, t) \leq \bar{u}(x, t) = \widehat{u}(x, \tau) \rightarrow \infty, \quad x \in (-1, 1), \quad \text{as } t, \tau \rightarrow \infty.$$

Moreover $u_x^*(\pm 1, t) \rightarrow \mp \infty$ as $t \rightarrow \infty$. This completes the proof of the Theorem. $\square$

3. Discussion

In this problem we know, as in [15], that there exist a critical value λ^* of the parameter λ , such that for $\lambda > \lambda^*$, $u = u(x, t; \lambda)$, the solution to (1), blows-up globally in finite time and there is not any steady-state solution for $\lambda \geq \lambda^*$, while for $\lambda \in (0, \lambda^*)$ the solution $u(x, t; \lambda)$ is a global-in-time bounded solution so that $u(x, t; \lambda) \rightarrow w(x; \lambda)$ as $t \rightarrow \infty$, where w is the unique solution to (3).

In the present work we have shown that for $\lambda = \lambda^*$, the critical solution $u^* = u(x, t; \lambda^*)$, under the assumption of Theorem 1, “grows-up” globally, this means that u^* is a global-in-time unbounded solution which diverges globally ($u(x, t) \rightarrow \infty$ for all $x \in (-1, 1)$, $u_x(\pm 1, t) \rightarrow \mp \infty$ as $t \rightarrow \infty$).

As a future work, it will be interesting to find exactly how this solution u^* diverges (asymptotic behaviour). A similar result has been examined in [6]

and we conjecture that this method can also be applied to our solution u^* . An upcoming work it will be also to study the same problem in N -dimensions.

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