The boundary integral method for the steady rotating Navier–Stokes equations in exterior domain (I): the existence of solution

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Abstract. In this paper, we apply the boundary integral method to the steady rotating Navier–Stokes equations in exterior domain. Introducing some open ball which decomposes the exterior domain into a finite domain and a infinite domain, we obtain a coupled problem by the steady rotating Navier–Stokes equations in finite domain and a boundary integral equation without using the artificial boundary condition. For the coupled problem, we show the existence of solution in a convex set.

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1. Introduction

Let $\Omega^c \subset \mathbb{R}^3$ be a simply connected bounded domain with smooth boundary Γ . Assume that Ω^c is rotating with the constant angular velocity $\vec{\omega}$ in a viscous incompressible fluid filling the entire space, then the rotating Navier–Stokes equations for the fluid occupying $\Omega = \mathbb{R}^3 \setminus \overline{\Omega^c}$ consist in finding the velocity vector u for the fluid and the pressure p such that

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u - (\vec{\omega} \times x) \cdot \\ \quad \nabla u + \vec{\omega} \times u + \nabla p = f & \text{in } \Omega, \ t > 0, \\ \operatorname{div} u = 0 & \text{in } \Omega, \ t > 0, \\ u = \vec{\omega} \times x & \text{on } \Gamma, \ t > 0, \\ u(x) \longrightarrow 0 & \text{as } |x| \longrightarrow +\infty, \ t > 0. \end{cases} \quad (1)$$

where $\nu > 0$ is the viscous coefficient, f denotes the external forces.

On the existence, uniqueness and regularity of solutions of (1), there are many papers, such as [1–8] and references therein. However, few results about the numerical analysis are developed. The reason is that the coefficient of “ $\vec{\omega} \times x$ ” tends to infinity as $|x| \rightarrow +\infty$. A natural method to overcome this difficulty is the artificial boundary method. The main idea is that the exterior domain or the unbounded domain has to be truncated and the appropriate artificial boundary conditions are introduced. For example, Bonisch and Heuveline [9–11] study the adaptive finite element method for the free fall of a solid in a fluid occupied the whole space. Dunne and Rannacher [12, 13] study the adaptive finite element simulation for the fluid–structure interaction. On these references, we also refer the collected paper [14]. In [14], Bonisch et al. truncate the exterior domain to a bounded domain and introduce artificial “outflow” boundary conditions governed by the Gaussian heat kernel. They analyze the effect of the diameter of the truncated computational domain on the free-fall of the solid. To obtain the satisfactory accuracy, the diameter d_D of the truncated domain must satisfy $d_D \geq 400$ if the length of the solid is $l = 6$ and the width of the solid is $w = 1$. Hence, for the artificial boundary method, the truncated domain must be sufficiently large. A direct effect to the numerical computation is adding the CPU time. The reason that why the truncated domain is sufficiently large is that the artificial “outflow” boundary conditions are not exact on the truncated boundary. To avoid using the large truncated domain, we study the coupled method of the boundary element and the finite element, which also is called the boundary integral method.

The boundary integral method is a good method to study the numerical method for the exterior problems or the unbounded problems. A general references can be found in [15]. The main idea is introducing some open ball which decomposes the exterior domain into a finite domain and a infinite domain, and transforming the equation in infinite domain to the boundary integral equation without using the artificial boundary condition. There are many papers about the boundary integral method for Navier–Stokes equations in exterior domain, such as [16–21] and references therein.

In this paper, we will apply the boundary integral method to the steady rotating Navier–Stokes equations in exterior domain. Firstly, we truncate the exterior domain by a open ball, and obtain a bounded domain and a new exterior domain which is outside the ball. In the bounded domain, we consider the rotating Navier–Stokes equations. But in the new exterior domain, we consider the Stokes problem. Subsequently, we transform the Stokes problem into a boundary integral equation on the boundary of the ball and avoid using the artificial boundary condition. It is noteworthy that the boundary integral equation is exact on the boundary of the ball. Hence, we obtain the continuous formulation of the coupled method of the boundary element and the finite element. The main objective of this paper is to study the well-posedness of the continuous coupled problem (4) and show that the problem (4) admits a solution in a convex set. The finite element approximation of (4) and the numerical simulations will be given in the forthcoming papers.

2. Rotating Navier–Stokes equations

Consider the following steady rotating Navier–Stokes equations:

$$\begin{cases} -\nu\Delta u + u \cdot \nabla u + \vec{\omega} \times u - (\vec{\omega} \times x) \cdot \nabla u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = \vec{\omega} \times x & \text{on } \Gamma, \\ u(x) \longrightarrow 0 & \text{as } x \longrightarrow +\infty, \end{cases} \quad (2)$$

where $f \in L^2_{\text{loc}}(\Omega) \cap L^{\frac{6}{5}}_{\text{loc}}(\Omega)$.

Without loss of generality, we assume that the origin of coordinates lies in Ω^c . Let $\delta(\Omega^c)$ be the coordinates of Ω^c , that is

$$\delta(\Omega^c) = \sup_{x, y \in \Omega^c} |x - y|.$$

For each $R > \delta(\Omega^c)$, set

$$\Omega_R = \Omega \cap B_R, \quad \Omega^R = \mathbb{R}^3 \setminus \Omega_R,$$

where $B_R = \{x \in \mathbb{R}^3, |x| \leq R\}$.

Firstly, we make the homogeneous of the boundary condition on Γ in (2). Silvestre [4] showed the following lemma:

Lemma 2.1. *For each $R > \delta(\Omega^c)$, there exists a smooth function $\psi \in C_0^\infty(\overline{\Omega})$ such that*

$$\begin{cases} \operatorname{div} \psi = 0 & \text{in } \Omega, \\ \psi = \vec{\omega} \times x & \text{on } \Gamma, \\ ||\psi||_q + ||\nabla \psi||_r + ||D^2 \psi||_s \leq \kappa_1 |\vec{\omega}| & \text{for all } q > \frac{3}{2}, 1 < r \leq 6, 1 \leq s \leq 2, \\ \left| \int_{\Omega_R} u \cdot \nabla \psi \cdot u dx \right| \leq \frac{\nu}{2} ||\nabla u||_{2, \Omega_R}^2, \end{cases}$$

where $\kappa_1 > 0$ depends on q, r, s, ν, Γ .

Let $u = v + \psi$ in (2), then v satisfies

$$\begin{cases} -\nu\Delta v + v \cdot \nabla v + v \cdot \nabla \psi + \psi \cdot \nabla v - (\vec{\omega} \times x) \cdot \nabla v + \vec{\omega} \times v + \nabla p = F \\ \operatorname{div} v = 0 \\ v = 0 \\ v(x) \longrightarrow 0 \end{cases} \quad \begin{matrix} \Omega, \\ \Omega, \\ \Gamma, \\ |x| \longrightarrow +\infty, \end{matrix} \quad (3)$$

where

$$F = \nu\Delta \psi + (\vec{\omega} \times x) \cdot \nabla \psi - \vec{\omega} \times \psi - \psi \cdot \nabla \psi + f.$$

According to Lemma 2.1, we can easily show $F \in L^2_{\text{loc}}(\Omega) \cap L^{\frac{6}{5}}_{\text{loc}}(\Omega)$ and

$$||F||_{\frac{6}{5}} \leq \kappa_2 |\vec{\omega}|,$$

where $\kappa_2 > 0$ depends on ν and Γ .

3. The boundary integral equation

Select $R > \delta(\Omega^c)$ such that Ω_R contains the compact support of f and ψ . The ball B_R decomposes the unbounded domain Ω into a finite domain Ω_R and a exterior domain Ω^R . We will use the following coupled problem to approximate the steady rotating Navier–Stokes equations (3):

$$\left\{ \begin{array}{ll} -\nu \Delta v_R + v_R \cdot \nabla v_R + v_R \cdot \nabla \psi + \psi \cdot \nabla v_R + \vec{\omega} \times v_R \\ \quad -(\vec{\omega} \times x) \cdot \nabla v_R + \nabla p_R = F & \text{in } \Omega_R, \\ \operatorname{div} v_R = 0 & \text{in } \Omega_R, \\ -\nu \Delta v^R + \nabla p^R = 0 & \text{in } \Omega^R, \\ \operatorname{div} v^R = 0 & \text{in } \Omega^R, \\ v_R = 0 & \text{on } \Gamma, \\ v_R = v^R, \lambda_R = \lambda^R & \text{on } \partial B_R, \end{array} \right. \quad (4)$$

where

$$\begin{aligned} v_R &= v|_{\Omega_R}, \quad v^R = v|_{\Omega^R}, \quad p_R = p|_{\Omega_R}, \quad p^R = p|_{\Omega^R}, \\ \lambda_R &= \left(\sum_{j=1}^3 \sigma_{ij}(v_R, p_R) n_j |_{\partial B_R} \right)_i, \quad \lambda^R = \left(\sum_{j=1}^3 \sigma_{ij}(v^R, p^R) n_j |_{\partial B_R} \right)_i, \\ \sigma_{ij}(v_R, p_R) &= -\delta_{ij} p_R + \nu \left(\frac{\partial v_{Ri}}{\partial x_j} + \frac{\partial v_{Rj}}{\partial x_i} \right), \\ \sigma_{ij}(v^R, p^R) &= -\delta_{ij} p^R + \nu \left(\frac{\partial v_i^R}{\partial x_j} + \frac{\partial v_j^R}{\partial x_i} \right), \end{aligned}$$

and (v, p) is the solution of (3). Moreover, $v^R(x) \rightarrow 0$ as $|x| \rightarrow +\infty$.

Let $(E_{ij}(x, y), P_i(x, y)) (i, j = 1, 2, 3)$ be the fundamental solutions of three dimensional Stokes problem, then they satisfy

$$\left\{ \begin{array}{l} -\nu \Delta E_{ij}(x - y) + \frac{\partial}{\partial x_j} P_i(x - y) = \delta_{ij} \delta(x - y), \\ \frac{\partial}{\partial x_i} E_{ij}(x - y) = 0, \end{array} \right.$$

where δ_{ij} is Kronecker symbol and $\delta(x - y)$ is the Dirac function. It is well known that $(E_{ij}(x, y), P_i(x, y))$ have the following forms

$$\left\{ \begin{array}{l} E_{ij}(x, y) = -\frac{1}{8\pi} \left(\frac{\delta_{ij}}{|x - y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^3} \right), \\ P_i(x, y) = \frac{1}{4\pi} \frac{x_i - y_i}{|x - y|^3}. \end{array} \right.$$

Ladyzhenskaya [22] shows that if (v, p) is a solution of (4) in Ω^R , then for $x \in \Omega^R$, there holds

$$\begin{cases} v_k(x) = - \int_{\partial B_R} v(y) \cdot \sigma(E_k, P_k)(x-y) \cdot n(y) ds(y) \\ \quad + \int_{\partial B_R} E_k(x-y) \cdot \lambda(y) ds(y) & k = 1, 2, 3, \\ p(x) = -2\nu \int_{\partial B_R} \frac{\partial P}{\partial n}(x-y) \cdot v(y) ds(y) \\ \quad + \int_{\partial B_R} P(x-y) \cdot \lambda(y) ds(y), \end{cases}$$

where

$$\begin{cases} E_k(x-y) = (E_{k1}(x-y), E_{k2}(x-y), E_{k3}(x-y)), \\ P(x-y) = (P_1(x-y), P_2(x-y), P_3(x-y)), \\ \sigma_{ij}(v, p) = -\delta_{ij}p + \nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \\ \lambda = \left(\sum_{j=1}^3 \sigma_{ij}(v, p) n_j \right)_{| \partial B_R}_i, \quad i = 1, 2, 3. \end{cases}$$

Moreover, if $x \in \partial B_R$, there holds

$$\begin{aligned} \frac{1}{2}v_k(x) &= - \int_{\partial B_R} v(y) \cdot \sigma(E_k, P_k)(x-y) \cdot n(y) ds(y) \\ &\quad + \int_{\partial B_R} E_k(x-y) \cdot \lambda(y) ds(y) \quad k = 1, 2, 3, \end{aligned} \quad (5)$$

which is the boundary integral equation.

4. The coupled problem

If (v, p) is the solution of (4) in Ω_R , then

$$\begin{cases} -\nu \Delta v + v \cdot \nabla v + v \cdot \nabla \psi + \psi \cdot \nabla v + \vec{\omega} \times v - (\vec{\omega} \times x) \cdot \\ \quad \nabla v + \nabla p = F & \text{in } \Omega_R, \\ \operatorname{div} v = 0 & \text{in } \Omega_R, \\ v = 0 & \text{on } \Gamma. \end{cases} \quad (6)$$

Introduce the Hilbert space

$$V = \{v \in H^1(\Omega_R)^3, \quad v = 0 \text{ on } \Gamma\}$$

with norm

$$\|\nabla v\| = \left(\int_{\Omega_R} |\nabla v|^2 dx \right)^{\frac{1}{2}}.$$

Denote by V_σ the solenoidal subspace of V and

$$M = \{q \in L^2(\Omega_R), \quad \int_{\Omega_R} q dx = 0\} = L_0^2(\Omega_R)$$

with $L^2(\Omega_R)$ -norm. Introduce the following bilinear forms and trilinear form:

$$\begin{cases} a(u, v) = 2\nu \sum_{i,j=1}^3 \int_{\Omega_R} \varepsilon_{ij}(u) \varepsilon_{ij}(v) dx, \\ d(v, p) = \sum_{i=1}^3 \int_{\Omega_R} p \frac{\partial v_i}{\partial x_i} dx, \\ b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega_R} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \\ (f, v) = \sum_{i=1}^3 \int_{\Omega_R} f_i v_i dx, \\ \langle v, \lambda \rangle = \sum_{i=1}^3 \int_{\partial B_R} \lambda_i v_i ds(x), \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial B_R)^3$ and $H^{\frac{1}{2}}(\partial B_R)^3$, $\varepsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$. Then the variational formulation of (6) is

$$\begin{cases} a(v, w) + b(v, v, w) + b(v, \psi, w) + b(\psi, v, w) - b(\vec{\omega} \times x, v, w) \\ \quad + (\vec{\omega} \times v, w) - d(w, p) + \langle w, \lambda \rangle = (F, w) \quad \forall w \in V, \\ d(v, q) = 0 \quad \forall q \in M. \end{cases} \quad (7)$$

It is well known that (7) is equivalent to

$$\begin{aligned} & a(v, w) + b(v, v, w) + b(v, \psi, w) + b(\psi, v, w) - b(\vec{\omega} \times x, v, w) \\ & \quad + (\vec{\omega} \times v, w) + \langle w, \lambda \rangle = (F, w) \quad \forall w \in V_\sigma. \end{aligned} \quad (8)$$

On the other hand, if (v, p) is the solution of (4) in Ω^R , then

$$\begin{cases} -\nu \Delta v + \nabla p = 0 & \text{in } \Omega^R, \\ \operatorname{div} v = 0 & \text{in } \Omega^R, \\ v(x) \longrightarrow 0 & \text{as } x \longrightarrow +\infty. \end{cases}$$

Denote

$$T = \{ \mu \in H^{-\frac{1}{2}}(\partial B_R)^3, \quad \int_{\partial B_R} \mu ds = 0 \}.$$

Then

$$\lambda = \left(\sum_{j=1}^3 \sigma_{ij}(v, p) n_j|_{\partial B_R} \right)_i \in T, \quad i = 1, 2, 3.$$

In fact,

$$\begin{aligned} \int_{\partial B_R} \lambda ds &= \int_{\partial B_R} \sigma(v, p) \cdot n ds \\ &= \int_{\Omega^R} \operatorname{div} \sigma(v, p) dx \\ &= \int_{\Omega^R} (\nu \Delta v - \nabla p) dx = 0. \end{aligned}$$

Multiplying (5) by $\mu \in T$ and integrating over ∂B_R yields

$$2b(\lambda, \mu) - \langle v, \mu \rangle - 2 \langle Gv, \mu \rangle = 0 \quad \forall \mu \in T, \quad (9)$$

where $G = \{G_1, G_2, G_3\}$, and

$$\begin{cases} b(\lambda, \mu) = \sum_{i,k=1}^3 \int_{\partial B_R} \int_{\partial B_R} \mu_k(x) E_{ki}(x-y) \lambda_i(y) ds(y) ds(x), \\ G_k(v) = \int_{\partial B_R} v(y) \cdot \sigma(E_k, P_k)(x-y) \cdot n(y) ds(y). \end{cases}$$

Thus, the variational formulation of the coupled problem by the rotating Navier–Stokes equations (6) in finite domain Ω_R and the boundary integral equation (5) is

$$\begin{cases} \text{Find } (v, \lambda, p) \in V \times T \times M \text{ such that} \\ a(v, w) + b(v, v, w) + b(v, \psi, w) + b(\psi, v, w) - b(\vec{\omega} \times x, v, w) \\ \quad + (\vec{\omega} \times v, w) - d(w, p) + \langle w, \lambda \rangle = (F, w) \quad \forall w \in V, \\ 2b(\lambda, \mu) - \langle v, \mu \rangle - 2 \langle Gv, \mu \rangle = 0 \quad \forall \mu \in T, \\ d(v, q) = 0 \quad \forall q \in M. \end{cases} \quad (10)$$

Moreover, the variational formulation (10) is equivalent to

$$\begin{cases} \text{Find } (v, \lambda) \in V_\sigma \times T \text{ such that} \\ a(v, w) + b(v, v, w) + b(v, \psi, w) + b(\psi, v, w) - b(\vec{\omega} \times x, v, w) \\ \quad + (\vec{\omega} \times v, w) + \langle w, \lambda \rangle = (F, w) \quad \forall w \in V_\sigma, \\ 2b(\lambda, \mu) - \langle v, \mu \rangle - 2 \langle Gv, \mu \rangle = 0 \quad \forall \mu \in T. \end{cases} \quad (11)$$

Observe that the trilinear term satisfies

$$b(u, v, w) \leq c_0 R \|u\|_V \|w\|_V \|w\|_V,$$

where $c_0 > 0$ is independent of R . The coefficient $c_0 R$ brings the difficulty to show the existence of solution of the problem (11). In this case, we can show that the solution exists only if $\nu > cR$, where $c > 0$ is independent of R . However, the condition $\nu > cR$ is too strict for the fluids. To avoid using this condition, we will consider the problem (6) in a closed convex subset of V . Denote by K the closed convex subset of V

$$K = \{v \in V, \quad \|v \cdot n\|_{2, \partial B_R} \leq \kappa_2\},$$

where $\kappa_2 > 0$ is a positive constant. Then the variational formulation of the coupled problem is the following variational inequality:

$$\begin{cases} \text{Find } (v, \lambda, p) \in K \times T \times M \text{ such that} \\ a(v, w - v) + b(v, v, w - v) + b(v, \psi, w - v) + b(\psi, v, w - v) \\ \quad - b(\vec{\omega} \times x, v, w - v) + (\vec{\omega} \times v, w - v) - d(w - v, p) \\ \quad + \langle w - v, \lambda \rangle \geq (F, w - v) \quad \forall w \in K, \\ 2b(\lambda, \mu) - \langle v, \mu \rangle - 2 \langle Gv, \mu \rangle = 0 \quad \forall \mu \in T, \\ d(v, q) = 0 \quad \forall q \in M. \end{cases} \quad (12)$$

Denote by K_σ the solenoidal subset of K , then (12) is equivalent to

$$\begin{cases} \text{Find } (v, \lambda) \in K_\sigma \times T \text{ such that} \\ a(v, w - v) + b(v, v, w - v) + b(v, \psi, w - v) + b(\psi, v, w - v) \\ \quad - b(\vec{\omega} \times x, v, w - v) + (\vec{\omega} \times v, w - v) \\ \quad + \langle w - v, \lambda \rangle \geq (F, w - v) \quad \forall w \in K_\sigma, \\ 2b(\lambda, \mu) - \langle v, \mu \rangle - 2 \langle Gv, \mu \rangle = 0 \quad \forall \mu \in T. \end{cases} \quad (13)$$

5. Existence of solution

In this section, we will show the existence of (13) by the Galerkin method in [23]. Firstly, we introduce the following lemma due to Kaitai Li and Yinnian He [16]:

Lemma 5.1. *The bilinear form $b(\cdot, \cdot)$ is symmetric and continuous on $T \times T$. Moreover, there exists a positive constant $c > 0$ such that*

$$b(\mu, \mu) \geq c \|\mu\|_{H^{-\frac{1}{2}}(\partial B_R)^3}^2 \quad \forall \mu \in T.$$

Then for given $v \in V_\sigma$, the problem

$$\begin{cases} \text{Find } \lambda \in T \text{ such that} \\ 2b(\lambda, \mu) - \langle v, \mu \rangle - 2 \langle Gv, \mu \rangle = 0 \quad \forall \mu \in T \end{cases} \quad (14)$$

admits a unique solution $\lambda = \lambda(v)$. Hence, the problem (13) is equivalent to

$$\begin{cases} \text{Find } v \in K_\sigma \text{ such that} \\ a(v, w - v) + b(v, v, w - v) + b(v, \psi, w - v) + b(\psi, v, w - v) \\ \quad - b(\vec{\omega} \times x, v, w - v) + (\vec{\omega} \times v, w - v) + \langle w - v, \lambda(v) \rangle \\ \quad \geq (F, w - v) \quad \forall w \in K_\sigma. \end{cases} \quad (15)$$

On $\lambda(v)$, we have

Lemma 5.2. *For given $v \in V_\sigma$, there holds*

$$\langle v, \lambda(v) \rangle \geq 0.$$

Proof. Consider the following problem

$$\begin{cases} -\nu \Delta w + \nabla \pi = 0 & \text{in } \Omega_R, \\ \operatorname{div} w = 0 & \text{in } \Omega_R, \\ w|_\Gamma = 0, \quad w|_{\partial B_R} = v|_{\partial B_R}. \end{cases} \quad (16)$$

It is easy to show that the problem (14) exists a unique solution $w \in V_\sigma$. Then

$$\begin{aligned} \langle v, \lambda(v) \rangle &= \int_{\partial B_R} \sigma^{ij}(v, p) n_i v_j ds = \int_{\partial B_R} \sigma^{ij}(w, \pi) n_i w_j ds \\ &= \int_{\Omega_R} \operatorname{div}(\sigma(w, \pi) \cdot w) dx = \int_{\Omega_R} \operatorname{div} \sigma(w, \pi) \cdot w dx \\ &\quad + \int_{\Omega_R} \sigma^{ij}(w, \pi) \varepsilon^{ij}(w) dx \\ &= \int_{\Omega_R} \sigma^{ij}(w, \pi) \varepsilon^{ij}(w) dx = 2\nu \int_{\Omega_R} \varepsilon^{ij}(w) \varepsilon^{ij}(w) dx \geq 0. \end{aligned}$$

□

Define the projection operator from V_σ to K_σ . For each $u \in V_\sigma$, let $\Phi(u) = u - Pu$. Introduce the following lemmas:

Lemma 5.3. [24] For all $v \in V_\sigma$, there exists a unique $Pv \in K_\sigma$ such that

$$\begin{cases} (v - Pv, w - Pv)_1 \leq 0 & \forall w \in K_\sigma, \\ \|Pv_1 - Pv_2\|_V \leq \|v_2 - v_1\|_V & \forall v_1, v_2 \in V_\sigma, \end{cases}$$

where $(\cdot, \cdot)_1$ is the inner product in V .

Lemma 5.4. [23] For all $v \in V_\sigma$, there holds

$$(\Phi(v), v)_1 \geq \|\Phi(v)\|_V^2.$$

Since V_σ is divisible and reflexive, there exists a standard orthogonal basis $\{\phi_i\}_{i=1}^\infty$ which is dense and complete in V_σ . Denote by $V_n = \{\phi_1, \dots, \phi_n\}$, then we define the Galerkin approximation solution of (15) $v^n = \sum_{k=1}^n \alpha_k \phi_k \in V_n$ by

$$\begin{aligned} a(v^n, \phi_k) + b(Pv^n, v^n, \phi_k) + b(v^n, \psi, \phi_k) + b(\psi, v^n, \phi_k) - b(\vec{\omega} \times x, v^n, \phi_k) \\ + (\vec{\omega} \times v^n, \phi_k) + \langle \phi_k, \lambda(v^n) \rangle + n(\Phi(v^n), \phi_k)_1 = (F, \phi_k), \end{aligned} \quad (17)$$

where $k = 1, \dots, n$. Multiplying α_k in (17) and summing for $k = 1, \dots, n$, we obtain

$$\begin{aligned} a(v^n, v^n) + b(Pv^n, v^n, v^n) + b(v^n, \psi, v^n) + b(\psi, v^n, v^n) - b(\vec{\omega} \times x, v^n, v^n) \\ + (\vec{\omega} \times v^n, v^n) + \langle v^n, \lambda(v^n) \rangle + n(\Phi(v^n), v^n)_1 = (F, v^n). \end{aligned}$$

From Lemma 5.4, one has

$$a(v^n, v^n) + b(Pv^n, v^n, v^n) + b(v^n, \psi, v^n) \leq (F, v^n),$$

where we use

$$b(\psi, v^n, v^n) = b(\vec{\omega} \times x, v^n, v^n) = 0, \quad (\vec{\omega} \times v^n, v^n) = 0$$

and

$$\langle v^n, \lambda(v^n) \rangle \geq 0, \quad (\Phi(v^n), v^n)_1 \geq 0.$$

In terms of Lemma 2.1, one has

$$\begin{aligned} a(v^n, v^n) + b(Pv^n, v^n, v^n) + b(v^n, \Psi, v^n) \\ \geq \nu \|v^n\|_V^2 - \frac{\nu}{2} \|v^n\|_V^2 - \frac{1}{2} \left| \int_{\partial B_R} Pv^n \cdot n |v^n|^2 ds \right| \\ \geq \frac{\nu}{2} \|v^n\|_V^2 - \frac{\kappa_2 c_1}{2} \|v^n\|_V^2 = \frac{\nu - \kappa_2 c_1}{2} \|v^n\|_V^2. \end{aligned}$$

Assume $\nu > \kappa_2 c_1$, then

$$\frac{\nu - \kappa_2 c_1}{2} \|v^n\|_V^2 \leq \|F\|_{\frac{6}{5}} \|v^n\|_V \leq \kappa_1 |\vec{\omega}| \|v^n\|_V.$$

That is

$$\|v^n\|_V \leq \frac{2\kappa_1}{\nu - \kappa_2 c_1} |\vec{\omega}|. \quad (18)$$

From Lemma 5.4, we have

$$\begin{aligned}
 n\|\Phi(v^n)\|_V^2 &\leq n(\Phi(v^n), v^n) \\
 &\leq (F, v^n) - a(v^n, v^n) - b(v^n, \psi, v^n) - b(Pv^n, v^n, v^n) \\
 &\leq \|F\|_{\frac{6}{5}} \|v^n\|_V + \frac{3\nu}{2} \|v^n\|_V^2 + \frac{\kappa_2 c_1}{2} \|v^n\|_V^2,
 \end{aligned}$$

which together (18) show

$$\lim_{n \rightarrow +\infty} \|\Phi(v^n)\|_V \longrightarrow 0.$$

Since v^n is uniformly bounded in V_σ , there exist a subsequence of $\{v^n\}$, which is denoted by $\{v^n\}$ again, and a $v \in V_\sigma$ satisfying (18) such that v^n converges weakly to v in V and converges strongly to v in $L^2(\Omega_R)^3$ and $L^p(\partial B_R)^3$ for $2 \leq p < 4$. With the similar method in [23], we have $v \in K_\sigma$ and $Pv = v$. For each $w^m = \sum_{k=1}^m \beta_k \phi_k \in K_\sigma$, $m < n$ and v^n , we have

$$\begin{aligned}
 &a(v^n, w^m - Pv^n) + b(Pv^n, v^n, w^m - Pv^n) + b(v^n, \psi, w^m - Pv^n) \\
 &+ b(\psi, v^n, w^m - Pv^n) - b(\vec{\omega} \times x, v^n, w^m - Pv^n) + (\vec{\omega} \times v^n, w^m - Pv^n) \\
 &+ \langle w^m - Pv^n, \lambda(v^n) \rangle + n(\Phi(v^n), w^m - Pv^n) = (F, w^m - Pv^n).
 \end{aligned}$$

From Lemma 5.3, there holds

$$(\Phi(v^n), w^m - Pv^n) = (v^n - Pv^n, w^m - Pv^n) \leq 0.$$

Hence,

$$\begin{aligned}
 &a(v^n, w^m - Pv^n) + b(Pv^n, v^n, w^m - Pv^n) + b(v^n, \psi, w^m - Pv^n) \\
 &+ b(\psi, v^n, w^m - Pv^n) - b(\vec{\omega} \times x, v^n, w^m - Pv^n) + (\vec{\omega} \times v^n, w^m - Pv^n) \\
 &+ \langle w^m - Pv^n, \lambda(v^n) \rangle \geq (F, w^m - Pv^n).
 \end{aligned} \tag{19}$$

Next, we will show the following relations:

$$\lim_{n \rightarrow +\infty} b(Pv^n, v^n, Pv^n) = b(v, v, v), \tag{20}$$

$$\lim_{n \rightarrow +\infty} b(v^n, \psi, Pv^n) = b(v, \psi, v), \tag{21}$$

$$\lim_{n \rightarrow +\infty} b(\psi, v^n, Pv^n) = b(\psi, v, v), \tag{22}$$

$$\lim_{n \rightarrow +\infty} b(\psi, v^n, w^m) = b(\psi, v, w^m), \tag{23}$$

$$\lim_{n \rightarrow +\infty} \langle Pv^n, \lambda(v^n) \rangle = \langle v, \lambda(v) \rangle. \tag{24}$$

About (20), we have

$$\begin{aligned}
 &|b(Pv^n, v^n, Pv^n) - b(v, v, v)| \\
 &\leq |b(Pv^n, v^n, v^n) - b(v, v, v)| + |b(Pv^n, v^n, \Phi(v^n))| := I_1 + I_2.
 \end{aligned}$$

Since $\Phi(v^n)$ converges to 0 in V as $n \rightarrow +\infty$, it is easy to show

$$I_2 \leq c\|Pv^n\|_V \|v^n\|_V \|\Phi(v^n)\|_V \longrightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

For I_1 , we have

$$\begin{aligned}
 I_1 &= \frac{1}{2} \left| \int_{\partial B_R} P v^n \cdot n |v^n|^2 - v \cdot n |v|^2 ds \right| \\
 &\leq \frac{1}{2} \left| \int_{\partial B_R} P v^n \cdot n (|v^n|^2 - |v|^2) ds \right| + \frac{1}{2} \left| \int_{\partial B_R} (P v^n \cdot n - v \cdot n) |v|^2 ds \right| \\
 &\leq c \|P v^n \cdot n\|_{\alpha, \partial B_R} (\|v^n\|_{\alpha', \partial B_R}^2 - \|v\|_{\alpha', \partial B_R}^2) + c \|v^n - v\|_{2, \partial B_R} \|v\|_{4, \partial B_R}^2 \\
 &\longrightarrow 0 \quad \text{as } n \rightarrow +\infty,
 \end{aligned}$$

where $2 < \alpha \leq 4$ and $\frac{8}{3} \leq \alpha' = \frac{2\alpha}{\alpha-1} < 4$. We complete the proof of (20). Next, we show (21).

$$\begin{aligned}
 &|b(v^n, \psi, P v^n) - b(v, \psi, v)| \\
 &\leq |b(v^n, \psi, v^n) - b(v, \psi, v)| + |b(v^n, \psi, \Phi(v^n))| \\
 &\leq |b(v^n - v, \psi, v^n)| + |b(v, \psi, v^n - v)| + |b(v^n, \psi, \Phi(v^n))| \\
 &\leq c \|v^n - v\|_V^{\frac{1}{2}} \|v^n - v\|_V^{\frac{1}{2}} \|\psi\|_V \|v^n\|_V + c \|v^n\|_V \|\psi\|_V \|\Phi(v^n)\|_V \\
 &\longrightarrow 0 \quad \text{as } n \rightarrow +\infty.
 \end{aligned}$$

For (22), we have

$$\begin{aligned}
 &|b(\psi, v^n, P v^n) - b(\psi, v, v)| \\
 &\leq |b(\psi, v^n, v^n) - b(\psi, v, v)| + |b(\psi, v^n, \Phi(v^n))| \\
 &= |b(\psi, v^n, \Phi(v^n))| \leq c \|\psi\|_V \|v^n\|_V \|\Phi(v^n)\|_V \longrightarrow 0 \quad \text{as } n \rightarrow +\infty.
 \end{aligned}$$

For (23), we have

$$\begin{aligned}
 &|b(\psi, v^n, w^m) - b(\psi, v, w^m)| \\
 &\leq |b(\psi, w^m, v^n) - b(\psi, w^m, v)| \\
 &\leq c \|\psi\|_V \|w^m\|_V \|v - v^n\|_V^{\frac{1}{2}} \|v - v^n\|_V^{\frac{1}{2}} \\
 &\longrightarrow 0 \quad \text{as } n \rightarrow +\infty.
 \end{aligned}$$

For (24), we have

$$\begin{aligned}
 &|\langle P v^n, \lambda(v^n) \rangle - \langle v, \lambda(v) \rangle| \\
 &\leq |\langle v^n, \lambda(v^n) \rangle - \langle v, \lambda(v) \rangle| + |\langle \Phi(v^n), \lambda(v^n) \rangle| \\
 &\leq |\langle v^n - v, \lambda(v^n) \rangle| + |\langle v, \lambda(v^n - v) \rangle| + |\langle \Phi(v^n), \lambda(v^n) \rangle| \\
 &= I_3 + I_4 + I_5.
 \end{aligned}$$

We can easily show

$$I_3, I_5 \longrightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since λ is continuous, then $\lambda(v^n)$ converges to $\lambda(v)$ in $L^2(\partial B_R)^3$ and

$$I_4 \longrightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Hence, we obtain (24). Hence, pass to limit $n \rightarrow +\infty$ in (19), we have

$$\begin{aligned}
 &a(v, w^m - v) + b(v, v, w^m - v) + b(v, \psi, w^m - v) + b(\psi, v, w^m - v) \\
 &- b(\vec{\omega} \times x, v, w^m - v) + (\vec{\omega} \times v, w^m - v) + \langle w^m - v, \lambda(v) \rangle \geq \langle F, w^m - v \rangle.
 \end{aligned}$$

According to the property of ϕ_k , For all $w \in K_\sigma$, The variational inequality (15) holds. Thus, we show the following theorem:

Theorem 5.1. *If for sufficiently large ν such that $\nu > \kappa_2 c_1$, then the coupled problem (13) exists a solution $(v, \lambda) \in K_\sigma \times T$, where $c_1 > 0$ is independent of R , and v satisfies*

$$\|v\|_V \leq \frac{2\kappa_3}{\nu - \kappa_4 c_1} |\vec{\omega}|.$$

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